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TAKESHI KOTAKE
MUDUMBAI S. NARASIMHAN

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REGULARITY THEOREMS FOR FRACTIONAL POWERS OF A LINEAR ELLIPTIC OPERATOR ;

BY

TAKESHI KOTAKE AND MUDUMBAI S. NARASIMHAN.

1. Introduction. — Let L be a linear elliptic operator with C^∞ coefficients in an open subset Ω of $\mathbf{R}^n (n \geq 2)$. We suppose that L admits a (strictly) positive self-adjoint realisation $\tilde{L}$ in $L^2(\Omega)$. Let $\{E_\lambda\}$ be the spectral resolution of $\tilde{L}$ so that

$$\tilde{L} = \int \lambda \, dE_\lambda.$$

We consider the family of operators $\tilde{L}^s$, depending on a complex parameter s, defined by

$$\tilde{L}^s = \int \lambda^s \, dE_\lambda.$$

The operators $\tilde{L}^s$ may be viewed as " fractional powers " of L. For $s = -1, -2, \ldots$, we obtain the Green's operator and its iterates.

We study in this paper the regularity properties of the operators $\tilde{L}^s$. For integral values of s, it is known that the operators $\tilde{L}^s$ define kernels which are " very regular " in the sense of SCHWARTZ ([17], chap. V, § 6) and that if further the coefficients of L are analytic the kernels of $\tilde{L}^s$ are analytically very regular. For positive integral values of s the results are trivial, for negative integral values of s these follow from well-known regularity theorems for elliptic operators [11]. The question arises whether these results are true for all values of s. We prove in this paper that this is in fact the case (Theorems 2 and 3). The case of elliptic operators with constant coefficients on a torus and on $\mathbf{R}^n$ has already been dealt with respectively by S. BOCHNER [3] and L. SCHWARTZ ([16], chap. VII, § 10, ex. 7).

That the operators $\tilde{L}^s$ possess kernels follows from regularity theorems for elliptic operators. In order to prove that the kernels are very regular, we represent the kernels, for $Rl(-s)$ sufficiently large, in terms of the Green's function $G(t, x, y)$ of the associated parabolic operator. By using some results of G. BERGENDAL [1] and S. D. EIDELMAN [6] and showing that $G(t, x, y)$ and its derivatives fall off exponentially as $t \to \infty$, we then prove that the kernel $\tilde{L}^s$ is very regular.

The proof of analytic regularity, when the coefficients are analytic, is more difficult. It involves in the first instance estimates for the norms $\| A^k u \|_{L^2}$, where u is a function that is to be proved to be analytic and A a linear elliptic operator with analytic coefficients. Next we need to prove a general theorem (Theorem 1) to the effect that if A is a linear elliptic operator of order m with analytic coefficients in an open set Ω' of $\mathbf{R}^n$, and u is a function satisfying the inequalities

$$\| A^k u \|_{L^2(\Omega')} \leq (km)!\, c^{k+1}$$

for every integer $k \geq 0$, with a positive constant c independent of k, then u is analytic in Ω'.

This theorem is a natural one in as much as the conditions

$$\| A^k u \| \leq (km)!\, c^{k+1}$$

on every compact set are necessary for u to be analytic. We notice also that this theorem contains the well-known result : if A is linear elliptic operator and has analytic coefficients, and if $Au = f$ with f analytic, then u is analytic.

A weaker version of Theorem 1 has been proved by E. NELSON ([14], th. 7); he proves the analyticity of u under the stronger assumption

$$\| A^k u \| \leq k!\, c^{k+1}.$$

Theorem 1 is proved by suitably estimating the L^2-norms of derivatives of order km of u in terms of L^2-norms of u, Au, ..., $A^k u$. The proof of this theorem uses some ideas of a paper of C. B. MORREY and L. NIRENBERG [13].

The use of the parabolic equation in the proofs of Theorems 2 and 3 was suggested by a paper of S. MINAKSHISUNDARAM [12].

For spaces of distributions we use the usual notation [17].

The results of this paper have been announced in [10].

2. Statement of the theorems. — Let Ω be an open subset of $\mathbf{R}^n$. Let $\mathcal{D}(\Omega)$ be the space of complex-valued C^∞ functions with compact support in Ω. $L^2(\Omega)$ is the Hilbert space of complex-valued square summable functions on Ω, with scalar product (φ, ψ) defined by

$$(\varphi, \psi) = \int_\Omega \varphi . \overline{\psi}\, dx$$

for $\varphi, \psi \in L^2(\Omega)$; $\| \varphi \|_{L^2(\Omega)}$ means $(\varphi, \varphi)^{1/2}$.

Let A be a linear differential operator of order m,

$$A = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$$

with sufficiently differentiable complex-valued coefficients $a_\alpha(x)$ defined in Ω, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, α_i being integer ≥ 0 and we put :

$$|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n,$$

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

We say now A is an elliptic operator in Ω, if the homogeneous form of order m

$$\sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha \neq 0$$

for every $x \in \Omega$ and for every non vanishing real vector $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$.

THEOREM 1. — *Let Ω be an open subset of $\mathbf{R}^n$. Let A be a linear elliptic operator of order m with analytic coefficients in Ω. Let A^k be the k^{th} iterate of A. Suppose that a function u (of class C^∞) satisfies the inequality*

$$\| A^k u \|_{L^2(\Omega)} \leq (km)!\, c^{k+1}$$

for every integer $k \geq 0$ with a positive constant c independent of k. Then the function u is analytic in Ω.

REMARK. — The above theorem is also valid for elliptic systems; the demonstration is the same as for the scalar case.

As for the following theorems, we consider a linear elliptic operator L defined on Ω such that

$$(L\varphi, \psi) = (\varphi, L\psi)$$

for every $\varphi, \psi \in \mathcal{D}(\Omega)$.

Suppose further that L when defined on $\mathcal{D}(\Omega)$ ($\subset L^2$), where it is symmetric, has a strictly positive self-adjoint tion extension $\tilde{L}$.

Remark that these conditions entail that the form

$$L(x, \xi) = \sum_{|\alpha| = m} b_\alpha(x) \xi^\alpha$$

is real and definite for every $x \in \Omega$ and ξ real vector, when $L = \sum_{|\alpha| \leq m} b_\alpha(x) D^\alpha$ has sufficiently smooth coefficients.

Let $\{E_\lambda\}$ be the spectral resolution of $\tilde{L}$. By the hypothesis on $\tilde{L}$, we have $\lambda > c_0 > 0$ on the spectrum.

We can now define a family of operators $\tilde{L}^s$ depending on the complex parameter s, by

$$\tilde{L}^s = \int \lambda^s \, dE_\lambda.$$

As we shall see in section 5, $\tilde{L}^s$ thus defined is a continuous linear map of $\mathcal{D}(\Omega)$ into the space of distributions $\mathcal{D}'(\Omega)$ for every s, so that $\tilde{L}^s$ defines a kernel $L^s(x, y)$ ([17], [19]); the theorems to be proved concern the regularity of the kernel $L^s(x, y)$.

THEOREM 2. — *Let L be a linear elliptic differential operator with* C^∞ *coefficients in an open set* Ω *of* $\mathbf{R}^n$. *We suppose further that L admits a strictly positive self-adjoint realisation*

$$\tilde{L} = \int \lambda \, dE_\lambda.$$

in $L^2(\Omega)$. *Let s be a complex number. Then the operator*

$$\tilde{L}^s = \int \lambda^s \, dE_\lambda.$$

defines a kernel which is very regular.

THEOREM 3. — *Let L be a linear elliptic differential operator with analytic coefficients in an open set* Ω *of* $\mathbf{R}^n$, *admitting a strictly positive self-adjoint realisation* $\tilde{L}$ *in* $L^2(\Omega)$. *Then, for every complex number s, the kernel of the operator*

$$\tilde{L}^s = \int \lambda^s \, dE_\lambda$$

is analytically very regular.

For the definition of very regular kernels and analytically very regular kernels *see* ([17], chap. V, § 6).

As a consequence of the above theorems, $\tilde{L}^s(T)$ can be defined for T, a distribution with compact support and when L has the C^∞ (analytic) coefficients, $\tilde{L}^s(T)$ is an infinitely differentiable (resp. analytic) function in an open set of Ω where T is an infinitely differentiable (resp. analytic) function.

3. Preliminary lemmas. — We consider in this section some lemmas which are required in the proof of Theorem 1.

Let Ω' be any open subset of Ω. Let u be of class C^∞ on the closure $\overline{\Omega'}$ of Ω'. Let k be an integer $\geqq 0$. We define the k-norm of $u \in C^\infty(\overline{\Omega'})$ by

$$\| u \|_{k,\Omega'} = \sum_{|\alpha| = k} \frac{k!}{\alpha!} \| D^\alpha u \|_{L^2(\Omega')},$$

where we put $\alpha ! = \alpha_1 ! \, \alpha_2 ! \ldots \alpha_n !$ for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$.

Lemma 3.1. — *Let k, k', be given integers $\geqq 0$. Then we have*

$$\| u \|_{k+k',\Omega'} = \sum_{|\alpha|=k} \frac{k!}{\alpha!} \| D^\alpha u \|_{k',\Omega'},$$

Proof. — We have

$$\frac{(k+k')!}{\gamma!} = \sum_{\substack{|\alpha|=k \\ |\beta|=k' \\ \alpha+\beta=\gamma}} \frac{k!}{\alpha!} \frac{k'!}{\beta!}.$$

The lemma follows immediately from this equality.

The next lemma is a refined version of Friedrichs' inequality [7]. The proof is a modification of Friedrichs' proof as in [13].

We denote by Ω_r the ball $|x| < r$ of radius r in $\mathbf{R}^n$.

Lemma 3.2. — *Let A be a linear elliptic operator of order m with C^∞ coefficients in Ω. Let r, δ be positive numbers such that $\delta < r$ and $\Omega_{r+\delta} \subset \Omega$. Then there exists a constant $c > 0$ independent of δ such that for every $u \in C^\infty(\Omega)$ we have*

$$\| u \|_{m,\Omega_r} \leqq c \{ \| Au \|_{0,\Omega_{r+\delta}} + \delta^{-m} \| u \|_{0,\Omega_{r+\delta}} \}.$$

Proof. — Let $\zeta \in \mathcal{D}(\Omega)$ have its support in $\Omega_{r+\delta}$ and be such that $\zeta \equiv 1$ on Ω_r and satisfies

$$\sup_{\Omega_{r+\delta}} | D^\alpha \zeta(x) | \leqq c_\alpha \delta^{-|\alpha|} \qquad (\delta < r) \tag{3.1}$$

with $c_\alpha > 0$ depending only on α.

For any $u \in C^\infty(\Omega)$, we shall consider $\zeta^m u$, which is of class C^∞ having its support in $\Omega_{r+\delta}$. Since A is an elliptic operator with C^∞ coefficients, we have the well-known inequality [10]

$$\| \zeta^m u \|_{m,\Omega_{r+\delta}} \leqq c \{ \| A(\zeta^m u) \|_{0,\Omega_{r+\delta}} + \| \zeta^m u \|_{0,\Omega_{r+\delta}} \} \tag{3.2}$$

with a constant $c > 0$ depending only on A and $\Omega_{r+\delta}$.

By using the estimate (3.1), we obtain

$$\| A(\zeta^m u) \|_{0,\Omega_{r+\delta}} \leqq c' \left\{ \| \zeta^m A u \|_{0,\Omega_{r+\delta}} + \sum_{k=0}^{m-1} \delta^{-m+k} \| \zeta^k u \|_{k,\Omega_{r+\delta}} \right\},$$

$$\sum_{|\alpha|=m} \| \zeta^m D^\alpha u \|_{0,\Omega_{r+\delta}} \leqq c'' \left\{ \| \zeta^m u \|_{m,\Omega_{r+\delta}} + \sum_{k=0}^{m-1} \delta^{-m+k} \| \zeta^k u \|_{k,\Omega_{r+\delta}} \right\}.$$

It follows then from (3.2),

$$(3.3)\qquad \sum_{k=0}^{m} \delta^{-m+k} \| \zeta^k u \|_{k,\Omega_{r+\delta}} \leq c \left\{ \| \zeta^m A u \|_{o,\Omega_{r+\delta}} + \sum_{k=0}^{m-1} \delta^{-m+k} \| \zeta^k u \|_{k,\Omega_{r+\delta}} \right\}$$

with $c > 0$ independent of k.

To complete the proof of the lemma, we need the following fact : for every ε, $\delta > 0$, there exists a constant c independent of ε, δ and u such that

$$(3.4)\qquad \sum_{|\alpha|=k} \| \zeta^k D^\alpha u \|_{o,\Omega} \leq \varepsilon \sum_{|\alpha|=k+1} \| \zeta^{k+1} D^\alpha u \|_{o,\Omega} + c(\varepsilon^{-1} + \delta^{-1}) \sum_{|\alpha|=k-1} \| \zeta^{k-1} D^\alpha u \|_{o,\Omega}$$

where $k \geq 1$.

In fact we have the equality

$$-(\zeta^k D^\alpha u, \zeta^k D^\alpha u) = (\zeta^{k-1} D^{\alpha'} u, \zeta^{k+1} D_1 D^\alpha u) + 2k((D_1 \zeta) \zeta^{k-1} D^{\alpha'} u, \zeta^k D^\alpha u),$$

where $\alpha' = (\alpha_1 - 1, \alpha_2, \ldots, \alpha_n)$ (we suppose $\alpha_1 \neq 0$) and $D_1 = \partial/\partial x_1$.

Now we can obtain the inequality (3.4) by Schwarz's inequality and by taking into account the estimate (3.1) for ζ.

In (3.4) we take $k = m - 1$ and choose ε as $\varepsilon = \delta/2c$. Bringing the inequality thus obtained in the right side of (3.3), we have

$$(3.5)\qquad \sum_{k=0}^{m} \delta^{-m+k} \| \zeta^k u \|_{k,\Omega_{r+\delta}} \leq c \left\{ \| \zeta^m A u \|_{o,\Omega_{r+\delta}} + \sum_{k=0}^{m-2} \delta^{-m+k} \| \zeta^k u \|_{k,\Omega_{r+\delta}} \right\}$$

with $c > 0$ independent of k. Thus in the right side of (3.3), the terms corresponding to $k = m - 1$ can be absorbed in the left side. Repeating this procedure by using (3.4) with appropriate ε, we arrive finally at the desired inequality stated in the lemma.

LEMMA 3.3. — *Let q be positive integer such that $q < m$. Let $r < r_0$, r_0 being fixed. Then there exists a constant $c_m > 0$ depending only on m and r_0 such that for every $\varepsilon > 0$ and $u \in C^\infty(\Omega)$ one has*

$$\| u \|_{q,\Omega_r} \leq \varepsilon \| u \|_{m,\Omega_r} + c_m \varepsilon^{-q/(m-q)} \| u \|_{o,\Omega_r}.$$

A proof of this lemma can be given by using Fourier transforms after extending the functions suitably to $\mathbf{R}^n$. Another proof can be found in [15] (Appendix).

REMARK. — Let p be any integer ≥ 0. By applying the above inequality to $D^\alpha u$ and by summing up the inequality thus obtained with respect to α such that $|\alpha| = mp$, we obtain from Lemma 3.1,

$$\| u \|_{pm+q,\Omega_r} \leq \varepsilon \| u \|_{(p+1)m,\Omega_r} + c_m \varepsilon^{-q/(m-q)} \| u \|_{pm,\Omega_r}$$

with the same constant c_m as in the above lemma.

4. Proof of theorem 1. — In this section we shall prove Theorem 1. The proof is preceeded by several lemmas which permit one to estimate suitably $\| u \|_{km}$ in terms of zero-norms of u, Au, ..., $A^k u$.

We suppose throughout this section that A has analytic coefficients. In this section, $c(c_1, c_2, \ldots$, etc.) will denote a positive constant, always independent of k, which may vary from place to place.

The first lemma gives an estimate for the commutator of the operator D^α and the operator of multiplication by an analytic function.

LEMMA 4.1. — *Let a be an analytic function in $\overline{\Omega'}$. We define the commutator $[a, D^\alpha]$ by $[a, D^\alpha]u = a.D^\alpha u - D^\alpha(au)$, then we have for every integer $k > 0$.*

$$(4.1) \qquad \sum_{|\alpha|=k} \frac{k!}{\alpha!} \| [a, D^\alpha] u \|_{0,\Omega'} \leq k!\, c^k \sum_{p=0}^{k-1} (p!)^{-1} c^{-p} \| u \|_{p,\Omega'}$$

with $c > 0$ independent of k.

PROOF. — Since a is analytic in $\overline{\Omega'}$, we have

$$(4.2) \qquad \sup_{\Omega'} | D^\alpha a | \leq \alpha!\, c^{|\alpha|+1}.$$

The Leibniz formula gives

$$D^\alpha(au) = \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} (D^\beta a)(D^{\alpha-\beta} u)$$

where $\alpha - \beta = (\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n)$ and $\beta \leq \alpha$ means $\beta_i \leq \alpha_i$ for each $i (i = 1, 2, \ldots, n)$.

From (4.2) and the definition of $[a, D^\alpha]$, it follows immediately

$$\sum_{|\alpha|=k} \frac{k!}{\alpha!} \| [a, D^\alpha] u \|_{0,\Omega'} \leq \sum_{|\alpha|=k} \sum_{\substack{\gamma \leq \alpha \\ \gamma \neq \alpha}} \frac{k!}{\gamma!} c^{k-|\gamma|} \| D^\gamma u \|_{0,\Omega_r}.$$

Now the number of α's such that $\alpha > \gamma$ for fixed γ is at most of order $n^{k-|\gamma|}$, so that the right side is majorised by

$$\sum_{p=0}^{k} \frac{k!}{p!} (nc)^{k-p} \sum_{|\gamma|=p} \frac{p!}{\gamma!} \| D^\gamma u \|_{0,\Omega'};$$

this proves the lemma.

LEMMA 4.2. — *Let r, δ be as in the lemma 3.2. Let ε be any positive number. Then there exist constants c, c_1 (c depending only on A and c_1 depending on A and ε) such that one has for every k and $u \in C^\infty(\Omega)$,*

$$(4.3) \quad \| u \|_{(k+1)m,\Omega_r} \leq c \Big\{ \| Au \|_{km,\Omega_{r+\delta}} + \delta^{-m} \| u \|_{km,\Omega_{r+\delta}} + \varepsilon \| u \|_{(k+1)m,\Omega_{r+\delta}} + ((k+1)m)!\, c_1^{k+1} \sum_{p=0}^{k} ((pm)!)^{-1} c_1^{-p} \| u \|_{pm,\Omega_{r+\delta}} \Big\}.$$

PROOF. — From Lemma 3.1 and the Friedrichs' inequality (Lemma 3.2), we have

$$(4.4) \quad \| u \|_{(k+1)m,\Omega_r} = \sum_{|\alpha|=km} \frac{(km)!}{\alpha!} \| D^\alpha u \|_{m,\Omega_r}$$
$$\leq c \Big\{ \| Au \|_{km,\Omega_{r+\delta}} + \delta^{-m} \| u \|_{km,\Omega_{r+\delta}} + \sum_{|\alpha|=km} \frac{(km)!}{\alpha!} \| [A, D^\alpha] u \|_{0,\Omega_{r+\delta}} \Big\}.$$

Now, writting A explicitly as $A = \sum_{|\beta| \leq m} a_\beta D^\beta$ with analytic coefficients a_β and applying the Lemma 4.1 for $[A, D^\alpha] u = \sum_{|\beta| \leq m} [a, D^\alpha] D^\beta u$, we obtain

$$(4.5) \quad \sum_{|\alpha|=km} \frac{(km)!}{\alpha!} \| [A, D^\alpha] u \|_{0,\Omega_{r+\delta}} \leq \sum_{p=0}^{km-1} \sum_{q=0}^{m} \frac{(km)!}{p!} c_1^{km-p} \| u \|_{p+q,\Omega_{r+\delta}}.$$

Since we may suppose $c_1 > 1$ in (4.5), it follows immediately that there exists a constant $c_2 > 0$ independent of k such that

$$(4.6) \quad \sum_{|\alpha|=km} \frac{(km)!}{\alpha!} \| [A, D^\alpha] u \|_{0,\Omega_{r+\delta}} \leq \sum_{s=0}^{(k+1)m-1} \frac{((k+1)m)!}{s!} c_2^{(k+1)m-s} \| u \|_{s,\Omega_{r+\delta}}.$$

We wish now to majorize the right side of (4.6), containing terms $\| u \|_s$, for $s = 0, 1, \ldots, (k+1)m - 1$, by an expression which contains only $\| u \|_{pm}$, for $p = 0, 1, \ldots, (k+1)$.

For this purpose, we write s as $s = pm + q$ with $0 \leq p \leq k$, and $0 \leq q < m$. Then the remark of Lemma 3.3 gives

$$(4.7) \quad \| u \|_{pm+q,\Omega_{r+\delta}} \leq \varepsilon' \| u \|_{(p+1)m,\Omega_{r+\delta}} + c_m \varepsilon'^{-q/(m-q)} \| u \|_{pm,\Omega_{r+\delta}}$$

with c_m independent of ε' and δ.

In (4.7), we choose ε' as

$$\varepsilon' = \varepsilon \frac{(pm+q)!}{((p+1)m)!} c_2^{-(m-q)}$$

where $\varepsilon\,(0 < \varepsilon < 1)$ is given.

Then we have

$$\varepsilon'^{-q/(m-q)} \leqq \left(\frac{m}{\varepsilon}\right)^m \frac{(pm+q)!}{(pm)!} c_2^q$$

so that we obtain for $s = pm + q$,

$$(4.8) \qquad \frac{c^{-s}}{s!} \| u \|_{s,\Omega_{r+\delta}} \leqq \varepsilon \frac{c_2^{-(p+1)m}}{((p+1)m)!} \| u \|_{(p+1)m,\Omega_{r+\delta}} + c_m \left(\frac{m}{\varepsilon}\right)^m \frac{c_2^{-pm}}{(pm)!} \| u \|_{pm,\Omega_{r+\delta}}.$$

Bringing this in the expression (4.6), we have

$$(4.9) \quad \sum_{|\alpha|=km} \frac{(km)!}{\alpha!} \| [A, D^\alpha] u \|_{0,\Omega_{r+\delta}} \leqq m\varepsilon \| u \|_{(k+1)m,\Omega_{r+\delta}} + c'(\varepsilon) \sum_{p=0}^{k} \frac{((k+1)m)!}{(pm)!} c_2^{(k+1)m-pm} \| u \|_{pm,\Omega_{r+\delta}}$$

where we put $c'(\varepsilon) = 1 + m\varepsilon + \left(\frac{m}{\varepsilon}\right)^m c_m$. We take now in (4.9) the constant c_2 large enough to absorb the constant $c'(\varepsilon)$ which is independent of k. Then, from (4.4), the desired inequality follows.

Definition (*see* [13]). — Let λ be a positive number. For each integer $k \geqq 0$, we define

$$\sigma^k(u, \lambda, R) = ((km)!)^{-1} \lambda^{-k} (R-r)^{km} \sup_{R/2 \leqq r < R} \| u \|_{km,\Omega_r}.$$

Lemma 4.3. — *Let $R < 1$. There exists a constant λ depending only on A and R such that for every k and $u \in C^\infty(\Omega)$ we have*

$$(4.10) \quad \sigma^{k+1}(u, \lambda, R) \leqq [(km+1)\ldots((k+1)m)]\, \sigma^k(Au, \lambda, R) + \sum_{p=0}^{k} \sigma^p(u, \lambda, R).$$

Proof. — Multipliying by $[((k+1)m)!]^{-1} \lambda^{-(k+1)} (R-r)^{(k+1)m}$ on both sides of the inequality of Lemma 4.2 and taking the supremum for $R/2 \leqq r < R$, we obtain

$$(4.11) \qquad \sigma^{k+1}(u, \lambda, R) \leqq \sup_{R/2 \leqq r < R} (I_1 + \varepsilon I_2 + I_3 + I_4),$$

where

$$(4.12)\quad \begin{cases} I_1 = c[((k+1)m)!]^{-1} \lambda^{-(k+1)} (R-r)^{(k+1)m} \| Au \|_{km,\Omega_{r+\delta}}, \\ I_2 = c[((k+1)m)!]^{-1} \lambda^{-(k+1)} (R-r)^{(k+1)m} \| u \|_{(k+1)m,\Omega_{r+\delta}}, \\ I_3 = c[((k+1)m)!]^{-1} \lambda^{-(k+1)} (R-r)^{(k+1)m} \delta^{-m} \| u \|_{km,\Omega_{r+\delta}}, \\ I_4 = c\,\lambda^{-(k+1)} (R-r)^{(k+1)m} \sum_{p=0}^{k} \frac{c_1^{k+1-p}}{(pm)!} \| u \|_{pm,\Omega_{r+\delta}}. \end{cases}$$

We choose in what follows $\delta = \frac{R-r}{k+1}$; then we have

$$\left(\frac{R-r}{R-r-\delta}\right)^{km} = \left(1 - \frac{1}{k+1}\right)^{-km} < c_2$$

with c_2 independent of k. It follows now from the definition of $\sigma^k(u, \lambda, R)$,

$$(4.13)\qquad I_1 \leqq [(km+1)\ldots((k+1)m)]^{-1} \left(\frac{cc_2}{\lambda}\right) \sigma^k(Au, \lambda, R).$$

Similarly

$$(4.14)\qquad I_2 \leqq (cc_2)\, \sigma^{k+1}(u, \lambda, R).$$

For I_3, we have

$$(4.15)\qquad I_3 \leqq \frac{c}{\lambda} \left(\frac{R-r}{R-r-\delta}\right)^{km} \left(\frac{R-r}{\delta}\right)^{m} \frac{(km)!}{((k+1)m)!} \sigma^k(u, \lambda, R).$$

Since we have from the definition of δ.

$$\left(\frac{R-r}{\delta}\right)^{m} = (k+1)^m$$

it follows from (4.15)

$$(4.16)\qquad I_3 \leqq \left(\frac{cc_2}{\lambda}\right) \sigma^k(u, \lambda, R).$$

Finally we obtain for I_4,

$$(4.17)\qquad I_4 \leqq \left(\frac{cc_1c_2}{\lambda}\right) \sum_{p=0}^{k} \left(\frac{c_1}{\lambda}\right)^{k-p} \sigma^p(u, \lambda, R) \qquad (\lambda \geqq 1).$$

It follows now for every $k \geqq 0$,

$$(4.18)\qquad (1-\varepsilon c)\, \sigma^{k+1}(u, \lambda, R) \leqq [(km+1)\ldots((k+1)m)]^{-1} \left(\frac{c_1}{\lambda}\right) \sigma^k(Au, \lambda, R) + \left(\frac{c_1}{\lambda}\right) \sum_{p=0}^{k} \left(\frac{c_1}{\lambda}\right)^{k-p} \sigma^p(u, \lambda, R)$$

for sufficiently large constants c, $c_1 > 0$, c being independent of ε, while c_1 depends on ε. After we have chosen $\varepsilon = 1/2c$ in (4.18) c_1 is a constant dependent only on A and R so that it is possible to find λ independent of k such that $\lambda > 2c_1$; thus we obtain the inequality (4.10).

LEMMA 4.4. — *Let λ be the same constant as in lemma 4.3; we have then*

$$(4.19)\quad \sigma^{k+1}(u, \lambda, R) \leq \sum_{p=0}^{k+1} 2^{k-p+1}\binom{k+1}{p}((mp)!)^{-1}\sigma^0(A^p u, \lambda, R).$$

PROOF. — The proof is by induction on k. For $k = 0$, the Lemma is valid (*see* Lemma 4.3). Suppose that the lemma is valid upto $k-1$. Applying the induction hypothesis to the function Au, we have

$$(4.20)\quad \sigma^k(Au, \lambda, R) \leq \sum_{p=0}^{k} 2^{k-p}\binom{k}{p}((pm)!)^{-1}\sigma^0(A^{p+1}u, \lambda, R).$$

Also, we have for $q \leq k$,

$$(4.21)\quad \sigma^q(u, \lambda, R) \leq \sum_{p=0}^{q} 2^{q-p}\binom{q}{p}((pm)!)^{-1}\sigma^0(A^p u, \lambda, R).$$

From Lemma 4.3, we get

$$(4.22)\quad \sigma^{k+1}(u, \lambda, R) \leq [(km+1)\dots((k+1)m)]^{-1} \times \sum_{p=0}^{k} 2^{k-p}\binom{k}{p}((pm)!)^{-1}\sigma^0(A^{p+1}u, \lambda, R) + \sum_{q=0}^{k}\sum_{p=0}^{q} 2^{q-p}\binom{q}{p}((pm)!)^{-1}\sigma^0(A^p u, \lambda, R).$$

Now, let c_p be the coefficient of $\sigma^0(A^p u, \lambda, R)$. Then for $0 \leq p \leq k$

$$c_p = [(km+1)\dots((k+1)m)]^{-1}2^{k-p+1}\binom{k}{p-1}[((p-1)m)!]^{-1} + \sum_{q=p}^{k} 2^{q-p}\binom{q}{p}[(mp)!]^{-1}.$$

Since

$$\sum_{q=p}^{k} 2^{q-p}\binom{q}{p} \leq 2^{k-p+1}\binom{k}{p}$$

we get

$$c_p \leq 2^{k-p+1}\binom{k+1}{p}((pm)!)^{-1}.$$

On the other hand, for $p = k+1$, we have evidently,

$$c_{k+1} = [((k+1)m)!]^{-1}.$$

Hence, it follows

$$(4.23) \qquad \sigma^{k+1}(u, \lambda, R) \leq \sum_{p=0}^{k+1} 2^{k-p+1} \binom{k+1}{p} [((pm)!)]^{-1} \sigma^0(A^p u, \lambda, R);$$

this is the inequality which we wanted to prove; thus the induction is completed.

Proof of theorem 1. — Let $u \in C^\infty(\Omega)$ such that

$$(4.24) \qquad \| A^k u \|_{L^2(\Omega')} \leq (km)!\, c^{k+1}$$

for Ω' an open set of Ω and for all $k \geq 0$ with a constant c independent of k.

Since the analyticity is a local property, we may suppose that the origin of $\mathbf{R}^n$ belongs to Ω' and it is sufficient to prove analyticity at the origin. Take $R < 1$ with $\Omega_R \subset \Omega'$, then

$$(4.25) \qquad \sigma^0(A^k u, \lambda, R) = \| A^k u \|_{L^2(\Omega_R)} \leq km!\, c^{k+1}.$$

Now from Lemma 4.4, we have

$$(4.26) \qquad \sigma^{k+1}(u, \lambda, R) \leq \sum_{p=0}^{k+1} 2^{k-p+1} \binom{k+1}{p} [((pm)!)]^{-1} \sigma^0(A^p u, \lambda, R)$$

$$\leq \sum_{p=0}^{k+1} 2^{k-p+1} c^{p+1} \binom{k+1}{p} = c(c+2)^{k+1}.$$

From the definition of $\sigma^{k+1}(u, \lambda, R)$ we obtain

$$\| u \|_{(k+1)m, \Omega_{R/2}} \leq ((k+1)m)!\,.\, c^{k+1}$$

with a certain constant c independent of k.

Then, Lemma 3.3 permits us to estimate $\| u \|_p$ for $p = 0, 1, \ldots$ by $\| u \|_{(k+1)m}$ for $k = 0, 1, \ldots$ and we have

$$(4.27) \qquad \| u \|_{p, \Omega_{R/2}} \leq p!\, c^{p+1}$$

for all $p\ (= 0, 1, \ldots)$, where c is a constant depending only on A and Ω_R. Now, by Sobolev's lemma [13], we see that u is analytic at the origin. Hence, the proof of Theorem 1 is completed.

5. **Regurarity of the kernel of $\tilde{L}^s$.** — We denote by $D(\tilde{L}^s)$ the domain of $\tilde{L}^s$, that is the set of elements $f \in L^2(\Omega)$ such that $\int |\lambda^s|^2 d\|E_\lambda f\| < \infty$.

Then, under our hypothesis on $\tilde{L}$, it is easy to see that

(5.1) $$D(\tilde{L}^s) \subseteq D(\tilde{L}^{s'}) \quad \text{if} \quad Rl\, s \geqq Rl\, s',$$

(5.2) for every complex number s,

$$\tilde{L}^s f \in \bigcap_{k=0}^{\infty} D(\tilde{L}^k) \quad \text{if} \quad f \in \bigcap_{k=0}^{\infty} D(\tilde{L}^k).$$

Let $f \in \bigcap_{k=0}^{\infty} D(\tilde{L}^k)$. It follows from (5.1), (5.2) that $f \in D(\tilde{L}^{s+k})$

and $\tilde{L}^s f \in D(\tilde{L}^k)$ for every complex number s and integer $k \geqq 0$. We have then

(5.3) $$\tilde{L}^k \tilde{L}^s f = \tilde{L}^s \tilde{L}^k f = \tilde{L}^{s+k} f$$

(for these properties, *see* [16], § 228; [18], p. 222).

PROPOSITION 5.1. — *For any complex number s, $\tilde{L}^s$ defines a kernel $L^s(x, y)$, that is, a distribution in the product space $\Omega \times \Omega$.*

PROOF. — We first consider the case $Rl\, s < 0$. In this case, $\tilde{L}^s$ is a continuous map of $L^2(\Omega)$ into itself. For, by hypothesis on $\tilde{L} = \int \lambda \, dE_\lambda$, we have a positive constant c_0 such that $\lambda > c_0$ on the spectrum, hence $\lambda^{Rl\, s} \leqq c_0^{Rl\, s}$ for $\lambda \leqq 1$ and $\lambda^{Rl\, s} \leqq 1$ for $\lambda > 1$, since $Rl\, s < 0$.

Thus, λ^s is bounded on the spectrum of $\tilde{L}$. Hence $\tilde{L}^s$ is a continuous linear map of $L^2(\Omega)$ into itself. *A fortiori*, $\tilde{L}^s$ is a continuous linear map of $\mathcal{D}(\Omega)$ into $\mathcal{D}'(\Omega)$. By the kernel theorem of L. SCHWARTZ [19], $\tilde{L}^s$ defines a kernel.

For general s, we take a positive integer m such that $Rl(s - m) < 0$. Then, as seen above, $\tilde{L}^{s-m}$ is a continuous map of $\mathcal{D}(\Omega)$ into $\mathcal{D}'(\Omega)$ while $\tilde{L}^m$, m^{th} iterate of L with C^∞ coefficients, is evidently a continuous map of $\mathcal{D}(\Omega)$ into itself.

Now, the proposition follows from (5.3), by remarking that

$$\tilde{L}^s \varphi = \tilde{L}^{s-m} \tilde{L}^m \varphi \quad \text{for} \quad \varphi \in \mathcal{D}(\Omega) \quad \text{since} \quad \mathcal{D}(\Omega) \subseteq \bigcap_{k=0}^{\infty} D(\tilde{L}^k).$$

From now on, we denote by $L^s(x, y)$ the kernel of $\tilde{L}^s$.

PROPOSITION 5.2. — *For every complex number s, the kernel $L^s(x, y)$ is regular.*

PROOF. — We have to prove that $\tilde{L}^s$ maps continuously $\mathcal{D}(\Omega)$ into $\mathcal{E}(\Omega)$ and can be extended to a continuous linear map of $\mathcal{E}'(\Omega)$ into $\mathcal{D}'(\Omega)$.

Suppose that for every s, $\tilde{L}^s$ maps continuously $\mathcal{D}(\Omega)$ into $\mathcal{E}(\Omega)$. Let φ, ψ be in $\mathcal{D}(\Omega)$. We have then $(\tilde{L}^s\varphi, \psi) = (\varphi, \tilde{L}^{\bar{s}}\psi)$, $\bar{s}$ denoting the conjugate complex of s, this implies that $\tilde{L}^s$ can be identified on the dense subspace $\mathcal{D}(\Omega)$ of $\mathcal{E}'(\Omega)$ with the transpose of $\tilde{L}^{\bar{s}}$, while the transpose of $\tilde{L}^{\bar{s}}$ is a continuous map of $\mathcal{E}'(\Omega)$ into $\mathcal{D}'(\Omega)$ when $\tilde{L}^{\bar{s}}$ is a continuous map of $\mathcal{D}(\Omega)$ into $\mathcal{E}(\Omega)$. Hence, $\tilde{L}^s$ can be extended to a continuous map of $\mathcal{E}'(\Omega)$ into $\mathcal{D}'(\Omega)$.

It remains now to prove that $\tilde{L}^s$ maps continuously $\mathcal{D}(\Omega)$ into $\mathcal{E}(\Omega)$.

Remark first that the image of $\mathcal{D}(\Omega)$ by $\tilde{L}^s$ is contained in $\mathcal{E}(\Omega)$. For, if $\varphi \in \mathcal{D}(\Omega)$, then $\varphi \in \bigcap_{k=0}^{\infty} D(\tilde{L}^k)$, so that by (5.2) we have $\tilde{L}^s\varphi \in \bigcap_{k=0}^{\infty} D(\tilde{L}^k)$. From the regularity theorem for a linear elliptic operator with C^∞ coefficients ([7], [15]), it follows that $\tilde{L}^s\varphi$ is of class C^∞.

As for the continuity of the mapping $\tilde{L}^s$, it is sufficient [17] to verify that the image of every bounded set in $\mathcal{D}(\Omega)$ by $\tilde{L}^s$ is also a bounded set in $\mathcal{E}(\Omega)$.

Let s be such that $Rl\, s < 0$. Let B be a bounded set in $\mathcal{D}(\Omega)$. Then, by definition [17], the image $\tilde{L}^k(B)$ of B by $\tilde{L}^k$ is bounded in $\mathcal{D}(\Omega)$, *a fortiori*, bounded in $L^2(\Omega)$. Now $\tilde{L}^s$ is a continuous map of $L^2(\Omega)$ into itself, so that $\tilde{L}^s\tilde{L}^k(B)$ is bounded in $L^2(\Omega)$. On the other hand, $\tilde{L}^s(B)$ is a family of C^∞ functions belonging to the domain of $\tilde{L}^k$; hence it follows from (5.3) that $\tilde{L}^k\tilde{L}^s(B)$ is bounded in $L^2(\Omega)$, from this, we see, according to Lemma 3.2 and Sobolev's lemma [13], that $\tilde{L}^s(B)$ is a family of C^∞ functions whose derivatives of orders $mk - \left[\frac{n}{2}\right] - 1$ are uniformly bounded on every compact of Ω. Since k is arbitrary, this proves that $\tilde{L}^s(B)$ is bounded in $\mathcal{E}(\Omega)$.

For general s, as in the proof of Proposition 5.1, choose m so large that $Rl(s-m) < 0$ and remark that $\tilde{L}^s\varphi = \tilde{L}^{s-m}\tilde{L}^m\varphi$ for $\varphi \in \mathcal{D}(\Omega)$, then $\tilde{L}^m$ and $\tilde{L}^{s-m}$ map respectively $\mathcal{D}(\Omega)$ into $\mathcal{D}(\Omega)$ and $\mathcal{E}(\Omega)$ continuously. This completes the proof.

6. Estimates for the Green's function of the associated parabolic operator. — Consider the family of operators $G_t = \int e^{-\lambda t}\, dE_\lambda$ for $t > 0$.

G_t is a bounded and Hermitian operator in $L^2(\Omega)$. Associated with these operators we have a C^∞ function in $\mathbf{R}\times\Omega\times\Omega$,

$$G(t, x, y) = \int e^{-\lambda t}\, de(\lambda, x, y),$$

where $e(\lambda, x, y)$ denotes the spectral function of $\tilde{L}$ [8].

We have then

$$\left(\frac{\partial}{\partial t} + L_x\right) G(t, x, y) = 0 \qquad \text{and} \qquad \left(\frac{\partial}{\partial t} + L_y\right) \overline{G(t, x, y)} = 0$$

for $t > 0$.

The next lemma shows that the function $G(t, x, y)$ and its derivatives fall off exponentially as $t \to \infty$.

LEMMA 6.1. — *Let H be a compact in $\Omega\times\Omega$. Under our assumption that $\tilde{L}$ is strictly positive operator ($\lambda > c_0 > 0$ on the spectrum), we have*

$$\left|\left(\frac{\partial}{\partial t}\right)^p D_x^\alpha D_y^\beta G(t, x, y)\right| \leq c\, e^{-c_0 t/2}$$

for $t > 1$ and uniformly for $(x, y) \in H$, where c depends on p, α, β and H.

PROOF. — Denote by $\bar{L}$ the elliptic operator with conjugate complex coefficients of L.

Consider the operator :

$$L_x + \bar{L}_y = L\left(x, \frac{\partial}{\partial x}\right) + \bar{L}\left(y, \frac{\partial}{\partial y}\right)$$

which is evidently elliptic with C^∞ coefficients in the product space $\Omega\times\Omega$.

Now, by Lemma 3.2 and Sobolev's lemma [13] applied to $(L_x + \bar{L}_y)$, it is easy to see that the desired estimate is a simple consequence of the following : let U be a relatively compact open subset in Ω such that $H \subset U \times U$. Then for every positive integers k', k'', we have

$$\left|(L_x + \bar{L}_y)^{k'} \left(\frac{\partial}{\partial t}\right)^{k''} G(t, x, y)\right| \leq c\, e^{-c_0 t/2}$$

for $t > 1$ and for $(x, y) \in U \times U$. Since

$$L_x G(t, x, y) = \bar{L}_y G(t, x, y) = -\frac{\partial}{\partial t} G(t, x, y) \qquad \text{for} \quad t > 0,$$

it is sufficient to estimate $\left(\frac{\partial}{\partial t}\right)^k G(t, x, y)$ for every positive integer k.

Let m be a sufficiently large positive integer such that $\tilde{L}^{-m}$ has a kernel $K(x, y)$ of the Carleman type ([4], [5], [8]). For $x \in \Omega$, let $K_x \in L^2$ denote the function $K(x, \hat{\ })$.

Now

$$\left|\left(\frac{\partial}{\partial t}\right)^k G(t, x, y)\right| = \left|\left(\frac{\partial}{\partial t}\right)^k \int e^{-\lambda t} de(\lambda, x, y)\right|$$
$$= \left|\int e^{-\lambda t}(-\lambda)^k de(\lambda, x, y)\right|$$
$$= \left|\int e^{-\lambda t}(-\lambda)^k \lambda^{2m} de\ (E_\lambda K_x, K_y)\right|$$
$$\leqq e^{-c_0 t/2} \int e^{-\lambda/2} \lambda^{2m+k} |\, d(E_\lambda K_x, K_y)\, |$$

since $\lambda > c_0$ and $t \geqq 1$. Now the variation of $(E_\lambda K_x, K_y)$ in $\mathbf{R}$ is majorised by $\| K_x \|_{L^2} \| K_y \|_{L^2}$ ([16], § 126) and $\| K_x \|_{L^2} \| K_y \|_{L^2} \leqq c(U)$ for $(x, y) \in U \times U$, where $c(U)$ is a constant depending only on U and $\tilde{L}$.

It follows that

$$\left|\left(\frac{\partial}{\partial t}\right)^k G(t, x, y)\right| \leqq c\, e^{-c_0 t/2}$$

for $t > 1$ and $(x, y) \in H$ with a constant c depending on k, H and $\tilde{L}$. Thus Lemma 5.1 is proved.

We next consider the behaviour of $G(t, x, y)$ and its derivatives as $t \to 0$. The required information is given by the results of G. Bergendal [1] and S. D. Eidelman [6].

Let K be a relatively compact open subset of Ω. Consider now the parabolic operator $\left(\frac{\partial}{\partial t} + L\right)$ on $\mathbf{R} \times K$ assiociated with L. According to S. D. Eidelman, we have a fundamental solution $E(t, x, y)$ of $\left(\frac{\partial}{\partial t} + L_x\right)$. It is of class C^∞ in (t, x, y) when $t > 0$ and satisfies near $t = 0$ the following estimate.

Lemma 6.2 (S. D. Eidelman). — *For* $0 < t < 1$ *and* $(x, y) \in K \times K$, *we have*

$$\left|\left(\frac{\partial}{\partial t}\right)^p D_x^\alpha D_y^\beta E(t, x, y)\right| \leqq c t^{-(pm+|\alpha|+|\beta|+n)/m} e^{-c_1 |x-y|^{1+\mu} t^{-\mu}},$$

where $\mu = 1/(m-1)$ *and* c_1 *depends only on* L, K, *while* c *depends also on* p, α, β.

As for the behaviour of $G(t, x, y)$ we have

Lemma (6.3) (G. Bergendal). — *Let* H *be a compact subset of* $\Omega \times \Omega$ *such that* $H \subset K \times K$. *Let* $E(t, x, y)$ *be the same as in lemma* 6.2. *Then there exist positive constants* c, c_1 *such that*

$$\left|\left(\frac{\partial}{\partial t}\right)^p D_x^\alpha D_y^\beta [\, G(t, x, y) - E(t, x, y)\,]\right| \leqq c\, e^{-c_1 t^{-\mu}}$$

for $0 < t < 1$ *and for* $(x, y) \in H$, *where* c_1 *depends only on* L *and* H, *while* c *depends also on* p, α, β.

For $p + |\alpha| + |\beta| = 0$, this is proved in [1]. The general case can be proved in a similar fashion (*see* [2], § 2.3).

7. A representation for the kernel $L^s(x, y)$ in terms of the Green's function $G(t, x, y)$.

PROPOSITION 7.1. — *Let* s *be a complex number such that* $Rl\, s < -n/m$. *Then we have*

$$L^s(x, y) = \frac{1}{\Gamma(-s)} \int_0^\infty t^{-s-1} G(t, x, y)\, dt. \tag{7.1}$$

The integral on the right converges uniformly on every compact subset of $\Omega \times \Omega$ *and represents a continuous function of* (x, y) *in* $\Omega \times \Omega$, *where we denote by* $\Gamma(-s)$ *the Gamma function.*

PROOF. — From Lemma 6.1 we have for $t \geq 1$ and for $(x, y) \in H$,

$$|G(t, x, y)| \leq c\, e^{-c_0 t/2} \tag{7.2}$$

while for $0 < t < 1$ and for $(x, y) \in H$, it follows from Lemma 6.2 and Lemma 6.3,

$$\begin{aligned} |G(t, x, y)| &\leq |E(t, x, y)| \\ &+ |(E - G)(t, x, y)| \leq c\, t^{-n/m} + c\, e^{-c_1 t^{-\mu}} \end{aligned} \tag{7.3}$$

with positive constants c, c_1 depending on H.

From these estimates, it is easy to see that the integral converges uniformly for $(x, y) \in H$ when $Rl\, s < -n/m$ and represents a continuous function of (x, y) since $G(t, x, y)$ is of class C^∞ for $t > 0$.

We shall prove now the equality stated in proposition 7.1. For φ, $\psi \in \mathcal{D}(\Omega)$, consider

$$P = \frac{1}{\Gamma(-s)} \left\langle \int_0^\infty t^{-s-1} G(t, x, y)\, dt,\ \varphi(x)\, \overline{\psi}(y) \right\rangle,$$

where $\langle\ ,\ \rangle$ denote the scalar product between $\mathcal{D}'(\Omega \times \Omega)$ and $\mathcal{D}(\Omega \times \Omega)$. By what has been seen,

$$\begin{aligned} P &= \frac{1}{\Gamma(-s)} \int_0^\infty t^{-s-1}\, dt \int_{\Omega \times \Omega} G(t, x, y)\, \varphi(x)\, \overline{\psi}(y)\, dx\, dy \\ &= \frac{1}{\Gamma(-s)} \int_0^\infty t^{-s-1}\, dt \int_{c_0}^\infty e^{-\lambda t}\, d(E_\lambda \varphi, \psi), \end{aligned}$$

where the integration $\int e^{-\lambda t}\, d(E_\lambda \varphi, \psi)$ is taken in the sense of the Radon-Stieltjes integral with respect to the complex-valued function of bounded variation $(E_\lambda \varphi, \psi)$ in $-\infty < \lambda < \infty$.

Let

$$(E_\lambda \varphi, \psi) = [\rho_1(\lambda) - \rho_2(\lambda)] + i[\rho_3(\lambda) - \rho_4(\lambda)]$$

be the canonical resolution of $(E_\lambda \varphi, \psi)$ with the real valued monotone increasing functions of bounded variation $\rho_k(\lambda)$, $k = 1, 2, 3, 4$ ([20], p. 202).

Then we have

$$\int_{c_0}^{\infty} e^{-\lambda t}\, d(E_\lambda \varphi, \psi) = \sum_{k=1}^{4} \varepsilon_k \int_{c_0}^{\infty} e^{-\lambda t}\, d\rho_k(\lambda)$$

where $\varepsilon_1 = -\varepsilon_2 = -i\varepsilon_3 = i\varepsilon_4 = 1$.

Consider now

$$\int_{c}^{\infty} t^{-s-1}\, dt \int_{c_0}^{\infty} e^{-\lambda t}\, d\rho_k(\lambda).$$

Since $t^{-s-1} e^{-\lambda t}$ is a continuous function of (t, λ) in the integration domain: $0 < t < \infty$, $c_0 < \lambda < \infty$ and the ovbious estimate $|t^{-s-1} e^{-\lambda t}| \leq t^{-Rl\, s-1} e^{-c_0 t}$ implies that it is integrable there with respect to the product measure $dt\, d\rho_k(\lambda)$ when $Rl\, s < 0$.

By *Fubini's theorem*, we have,

$$\int_{0}^{\infty} t^{-s-1}\, dt \int_{c_0}^{\infty} e^{-\lambda t}\, d\rho_k(\lambda) = \int_{c_0}^{\infty} d\rho_k \int_{0}^{\infty} t^{-s-1} e^{-\lambda t}\, dt.$$

Noting that $\int_{0}^{\infty} t^{-s-1} e^{-\lambda t}\, dt = \Gamma(-s)\lambda^s$ and summing up the above integral with respect to k, we have

$$P = \sum_{k=1}^{4} \varepsilon_k \int \lambda^s\, d\rho_k(\lambda)$$

which is equal to

$$\int \lambda^s\, d(E_\lambda \varphi, \psi) = (\tilde{L}^s \varphi, \psi).$$

This completes the proof.

8. Proof of theorem 2. — As in paragraph 5, we see that it is sufficient to prove Theorem 2 for $Rl\, s < -\frac{n}{m}\cdot$ Since we have already proved that $L^s(x, y)$ is regular, it is sufficient to prove that $L^s(x, y)$ is of class C^∞ outside the diagonal [17].

For $Rl\, s < -\frac{n}{m}$, we have by Proposition 7.1,

$$L^s(x, y) = \frac{1}{\Gamma(-s)} \int_0^\infty t^{-s-1} G(t, x, y)\, dt. \tag{8.1}$$

If (x, y) belongs to a compact set H in the complement of the diagonal we see from Lemmas 6.1, 6.2 and 6.3 that

$$\left| \left(\frac{\partial}{\partial t}\right)^k D_x^\alpha D_y^\beta G(t, x, y) \right| \leq c\, e^{-c_1(t+t^{-\mu})} \qquad (0 < t < \infty) \tag{8.2}$$

with positive constants c, c_1, where c_1 is independent of k, α, β.

It now follows from (8.1) and (8.2) that $L^s(x, y)$ is of class C^∞ outside the diagonal, since we may differentiate under the integral sign any number of times.

9. **Proof of theorem 3.** — In this section c, $c_i (i = 1, 2, \dots)$ will denote positive constants independent of k. We suppose that L has analytic coefficients.

To prove Theorem 3, it is sufficient to prove the following two statements:

(i) $L^s(x, y)$ is an analytic function in the complement of the diagonal in $\Omega \times \Omega$.

(ii) For each $\varphi \in \mathcal{D}(\Omega)$, $\tilde{L}^s \varphi$ is an analytic function in every open set where φ is analytic.

Proof of (i). — $(L_x + \bar{L}_y)^k$ is a linear elliptic operator of order m with analytic coefficients in $\Omega \times \Omega$. Applying Theorem 1, we see that to prove (i) it is sufficient to prove the following: for each compact set H in the complet ment of the diagonal, there exists a constant c independent of k such tha-

$$\sup_{(x,y) \in H} |(L_x + \bar{L}_y)^k L^s(x, y)| \leq (mk)!\, c^{k+1}. \tag{9.1}$$

It is sufficient to consider the case $Rl\, s < -\frac{n}{m}$.

As in paragraph 8, we start from the integral representation of $L^s(x, y)$:

$$L^s(x, y) = \frac{1}{\Gamma(-s)} \int_0^\infty t^{-s-1} G(t, x, y)\, dt. \tag{9.2}$$

If $(x, y) \in H$, we have the estimate (8.2) which permits us to differentiate under the integral sign, so that we have

$$(L_x + \bar{L}_y)^k L^s(x, y) = \frac{(-1)^k 2^k}{\Gamma(-s)} \int_0^\infty t^{-s-1} \left(\frac{\partial}{\partial t}\right)^k G(t, x, y)\, dt. \tag{9.3}$$

For we have

$$\left(\frac{\partial}{\partial t}+L_x\right)G(t,\,x,\,y)=\left(\frac{\partial}{\partial t}+\bar{L}_y\right)G(t,\,x,\,y)=0$$

for $t>0$. Let us first suppose that s is not a negative integer. By integration by parts in (9.3) [which is permitted by (8.2)] we obtain

$$(9.4)\quad (L_x+\bar{L}_y)^k L^s(x,\,y)$$
$$=\frac{2^k}{\Gamma(-s)}(-s-1)(-s-2)\ldots(-s-k)\int_0^\infty t^{-s-k-1}\,G(t,\,x,\,y)\,dt.$$

Now as a special case of (8.2) we have

$$|\,G(t,\,x,\,y)\,|\leqq c\,e^{-c_1(t+t^{-\mu})}$$

uniformly for $(x,\,y)\in H$ with positive constants c, c_1 depending on H.

Remembering that $\mu=(m-1)^{-1}$, it follows from a simple calculation that

$$(9.5)\qquad \sup_{(x,y)\in H}\left|\int_0^\infty t^{-s-k-1}\,G(t,\,x,\,y)\,dt\right|\leqq((m-1)k)!\,c^{k+1},$$

c being independent of k, which gives evidently, from (9.4),

$$\sup_{(x,y)\in H}|\,(L_x+\bar{L}_y)^k L^s(x,\,y)\,|$$
$$\leqq\frac{2^k}{|\,\Gamma(-s)\,|}\,|\,(-s-1)(-s-2)\ldots(-s-k)\,|\,((m-1)k)!\,c^{k+1}\leqq(mk)!\,c_1^k.$$

If s is a negative integer, we see that the integral

$$\int_0^\infty t^{-s-1}\left(\frac{\partial}{\partial t}\right)^k G(t,\,x,\,y)\,dt\qquad (x,\,y)\in H$$

vanishes for all large k and (9.1) is trivially valid. So (i) is proved.

PROOF OF (ii). — Let $\varphi\in\mathcal{D}(\Omega)$. We suppose φ is analytic in an open subset Ω_0 of Ω. We shall show that $\tilde{L}^s\varphi$ is analytic in Ω_0.

Let Ω_1, Ω_2 be any relatively compact open subsets of Ω_0 such that

$$\bar{\Omega}_1\subset\Omega_2\subset\bar{\Omega}_2\subset\Omega_0.$$

Let $\alpha\in\mathcal{D}(\Omega_0)$ and $\alpha\equiv 1$ on Ω_2. One has then

$$\tilde{L}^s(\varphi)=\tilde{L}^s(\alpha\varphi)+\tilde{L}^s((1-\alpha)\varphi).$$

Now, $(1-\alpha)\varphi\in\mathcal{D}(\Omega)$ and its support does not inersect Ω_1; by what has been seen in (i), $L^s(x,\,y)$ is an analytic function of $(x,\,y)$ outside the diagonal in $\Omega\times\Omega$, so that it follows immediately from the integral representation of $L^s(x,\,y)$ that $\tilde{L}^s((1-\alpha)\varphi)$ is analytic in Ω_1.

It remains to show that $\tilde{L}^s(\alpha\varphi)$ is analytic in Ω_1. It is sufficient to consider the case $Rl\, s < -\frac{n}{m}\cdot$ Then we have for each integer $k \geqq 0$

$$(9.6)\qquad L^k \tilde{L}^s(\alpha\varphi)(x) = \frac{1}{\Gamma(-s)}\int_0^\infty t^{-s-1}\,dt \int_\Omega G(t, x, y)\,(\alpha L^k \varphi)_y\,dy$$

$$+ \frac{1}{\Gamma(-s)} \sum_{p=0}^{k=1} L_x^p \int_0^\infty t^{-s-1}\,dt$$

$$\times \int_\Omega G(t, x, y)\,([L, \alpha] L^{k-p-1}\varphi)_y\,dy,$$

where $[L, \alpha]$ is the commutator of L and α.

Consider the second term in the above expression, which we write as

$$(9.7)\qquad \frac{1}{\Gamma(-s)} \sum_{p=0}^{k-1} F_p(x),$$

where

$$F_p(x) = L_x^p \int_0^\infty t^{-s-1}\,dt \int_\Omega G(t, x, y)\,([L, \alpha] L^{k-p-1}\varphi)_y\,dy.$$

Now $[L, \alpha]$ is a differential operator of order $(m-1)$ whose coefficients have their supports in $(\Omega_0 - \Omega_2)$, so that if we consider x in Ω_1 we may perform the differentiation L_x^p under the integral sign as in paragraph 8 and we obtain,

$$(9.8)\qquad F_p(x) = (-s-1)(-s-2)\ldots(-s-p)\int_0^\infty t^{-s-p-1}\,dt$$

$$\times \int_\Omega G(t, x, y)\,([L, \alpha] L^{k-p-1}\varphi)_y\,dy, \quad \text{for } s \text{ non-integral}$$

$= 0$ for all large p if s is a negative integer.

Since the coefficients of $[L, \alpha]$ have their supports in $(\Omega_0 - \Omega_2)$ and φ is analytic in Ω_0 by hypothesis, we have

$$(9.9)\qquad \sup_{x \in \Omega_0} |[L, \alpha] L^{k-p-1}\varphi| \leqq ((k-p)m)!\, c^{k-p+1}$$

with c independent of k and p. Further we have (*see* § 8)

$$(9.10)\qquad \sup_{(x, y) \in \Omega_1 \times (\Omega_0 - \Omega_2)} |G(t, x, y)| \leqq c\, e^{-c_1(t + t^{-\mu})};$$

we obtain from (9.8), (9.9) and (9.10)

$$\sup_{x \in \Omega} |F_p(x)| \leqq (pm)!\, ((k-p)m)!\, c^{k+1}$$

with a constant c independent of k, p.

Consequently, we have

$$(9.11) \qquad \sup_{x \in \Omega_1} \left| \frac{1}{\Gamma(-s)} \sum_{p=0}^{k-1} F_p(x) \right| \leq (km)!\, c^{k+1}.$$

On the other hand, since φ is analytic in Ω_0, we have

$$\sup_{x \in \Omega_1} | \alpha L^k \varphi | \leq (km)!\; c^{k+1}$$

and from the results of paragraph 6 [*see* (7.2), (7.3)], we have

$$\sup_{(x,y) \in \Omega_1 \times \Omega_0} | G(t, x, y) | \leq c\, t^{-n/m} e^{-c_1 t}$$

so that it follows for $Rl\, s < -n/m$,

$$(9.12) \quad \sup_{x \in \Omega_1} \left| \frac{1}{\Gamma(-s)} \int_0^\infty t^{-s-1}\, dt \int_{\Omega_0} G(t, x, y)\, (\alpha L^k \varphi)_y\, dy \right| \leq (km)!\; c^{k+1}.$$

From (9.6), (9.11) and (9.12) we obtain finally

$$\sup_{x \in \Omega_1} | L^k \tilde{L}^s (\alpha\varphi) | \leq (km)!\, c^{k+1}.$$

with c independent of k; now from Theorem 1 we see that $\tilde{L}^s(\alpha\varphi)$ is analytic in Ω_1, which was an arbitrary open subset of Ω_0. This proves (ii) and the proof of Theorem 3 is thus completed.

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Takeshi KOTAKE,
Attaché de Recherches C. N. R. S., Paris,
Yale University,
New Haven (Conn.) U. S. A.

Mudumbai S. NARASIMHAN,
Tata Institute of fundamental Research,
Bombay (Inde).