BULLETIN DE LA S. M. F.

J.L. DOOB

Conditional brownian motion and the boundary limits of harmonic functions

Bulletin de la S. M. F., tome 85 (1957), p. 431-458

<http://www.numdam.org/item?id=BSMF_1957__85__431_0>

Bull. Soc. math. France,
85, 1957, p. 431 à 458.

CONDITIONAL BROWNIAN MOTION AND THE BOUNDARY LIMITS OF HARMONIC FUNCTIONS;

by

J. L. Doob (1).

1. Preliminary remarks. — In the present paper, we shall consider functions and stochastic processes on a Green space R, as defined by Brelot and Choquet [3], except that the dimensionality 2 is not made exceptional (so that Riemann surfaces are excluded) and that points at ∞ are excluded. That is, in the language of [6], R is a Green space which is connected and has a positive boundary. Since a Riemann surface is the conformal image of a Green space in this sense, the results can be interpreted to be applicable to Riemann surfaces also.

We shall say that a sequence of points of R converges to ∞ if only finitely many points of the sequence are in any compact subset of R. The corresponding definition is made for convergence of a curve to ∞. The boundary R' of R will not however consist in general of a single point ∞ corresponding to this definition, but will be taken as the Martin boundary. The boundary of any set A will be denoted by A'.

We shall use repeatedly the fact that, if u is superharmonic and positive on R, there is a finite measure μ of Borel subsets of $R \cup R'$ such that

$$u(\eta) = \int_{R \cup R'} K_{\xi_0}(\xi, \eta)\, \mu(d\xi). \tag{1.1}$$

Here K is defined as follows. Some point ξ_0 at which u is finite is chosen and then, if g is the Green's function of R,

$$\begin{aligned} K_{\xi_0}(\xi, \eta) &= \frac{g(\xi, \eta)}{g(\xi, \xi_0)}, \qquad \xi \in R, \\ &= \lim_{\substack{\xi' \to \xi \\ \xi' \in R}} K(\xi', \eta), \qquad \xi \in R'. \end{aligned} \tag{1.2}$$

The limit exists, by definition of the Martin boundary.

(1) This research was supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command.

According to BRELOT [2], the Perron-Wiener-Brelot (PWB) method, when applied to the solution of the Dirichlet problem on R, yields the conclusion that all continuous boundary functions are PWB resolutive. Hence there is a harmonic measure of subsets of R' (a measure which is the completed measure of a measure of Borel sets) relative to each point of R. The class of measurable sets, and the class of sets of measure o are independent of the reference point, so that we shall write " measurable " and " almost everywhere " in discussing harmonic measure without specifying the reference point.

In particular [2] the function $u = 1$ is given by (1.1) if the measure μ is harmonic measure relative to ξ_0 and, more generally, if harmonic functions are replaced by h-harmonic functions, that is, by the quotients of harmonic functions divided by a strictly positive harmonic function h, so that the PWB^h Dirichlet problem solution leads to h-harmonic measure on R', the measure in (1.1) which yields the function h is simply the h-harmonic measure relative to ξ_0.

A minimal harmonic function on R is a strictly positive harmonic function which is proportional to any smaller positive harmonic function. We shall use the fact that, if $\zeta \in R$, $g(\zeta, .)$ is a minimal harmonic function for the Grenn space $R-\{\zeta\}$. Using the above notation, for almost all (harmonic measure) points ξ of R', the function $K_{\xi_0}(\xi, .)$ is minimal. If $K_{\xi_0}(\xi, .)$ is minimal, ξ is called the pole of any strictly positive multiple of $K_{\xi_0}(\xi, .)$, and ξ is called a minimal boundary point. In (1.1), μ can be chosen to assign measure o to the set of non-minimal points of R', and is then uniquely determined by u and ξ_0. If μ is so chosen, it is called " canonical ".

Brownian motion on R was defined in [6]. In [7], a procedure of relativizing generalized harmonic functions was discussed in its implications for the study of the Dirichlet problem by probability methods. The purpose of the present paper is to apply and carry further the results of [7] to the present more special situation. This leads to the study of conditional Brownian motion processes, and leads in a natural way to NAÏM's concept [11] of a fine limit at a point of R'. By means of the probabilistic interpretation of this fine limit, we are enabled to show that our probabilistic theorem 9.1 is equivalent to the theorem that, if u is a positive superharmonic function on R, $\frac{u}{h}$ has a finite fine limit at almost every point of R' (h-harmonic measure). It is interesting that no non-probabilistic proof of this thorem is known.

2. Conditional Brownian motion. — Let p be the transition density of Brownian motion on R. Let h be a strictly positive superharmonic function. The set of infinities of h is the intersection of a sequence of open sets, and has capacity o. Hence this set has zero Lebesgue measure of the dimensionality of R. Moreover, as noted by HUNT [10], h is an excessive function,

that is, h satisfies the inequality

$$\int_R p(t, \xi, \eta)\, h(\eta)\, d\eta \leqq h(\xi), \tag{2.1}$$

and the integral on the left defines a monotone finite-valued function of t, increasing to the value on the right when t decreases to 0. [The fact that the left side is finite even if the right side is not can be seen by replacing h in a small sphere of center ξ by the Dirichlet solution in the sphere for h on the perimeter as boundary function, and applying (2.1) to the modified function] Hence, if we define p^h by

$$p^h(t, \xi, \eta) = p(t, \xi, \eta)\frac{h(\eta)}{h(\xi)}, \tag{2.2}$$

except when $h(\xi)$ and $h(\eta)$ are both infinite, and with the obvious conventions when only one is infinite, p^h satisfies the inequality

$$\int_R p^h(t, \xi, \eta)\, d\eta \leqq 1. \tag{2.3}$$

Moreover it is trivial to verify that p^h satisfies the Chapman-Kolmogorov equation

$$p^h(s+t, \xi, \eta) = \int_R p^h(s, \xi, \zeta)\, p^h(t, \zeta, \eta)\, d\zeta \tag{2.4}$$

unless h is infinite at both ξ and η. If P^h is defined by

$$P^h(s, \xi, A) = \int_A p^h(s, \xi, \eta)\, d\eta,$$

where A is a Borel subset of R, P^h satisfies the usual integrated form of the Chapman-Kolmogorov equation, without any exceptional values of ξ, and

$$\lim_{t \to 0} P^h(t, \xi, R) = 1 \tag{2.5}$$

if $h(\xi)$ is finite. Moreover, if B is the set of infinities of h, B has Lebesgue measure 0, so that $P^h(t, \xi, B) = 0$. Hence, generalizing trivially some remarks in [6], a distribution assigning probability 0 to B, together with the transition probability P^h, determines a Markov process with state space R, and lifetime which may be finite. The superscript h will be used to identify the random variables associated with such a process, except that no superscript will be used when $h = 1$. Thus the process will be denoted by $\{z^h(t),\ t \geqq 0\}$, the lifetime of the process by τ^h. A point of the measure space on which the process is defined will be denoted by ω, and the absence of a superscript here should cause no confusion. If the initial distribution is confined to the single point ξ, we shall sometimes write z_ξ^h, τ_ξ^h and so on,

when the subscript may clarify the work. It will always be supposed that the process is separable relative to the closed sets. The process paths will be called h-paths, or Brownian paths if $h=1$. When the results will have justified the term, the h-paths will sometimes be called conditional Brownian paths.

We shall prove that almost all h-paths from a point are continuous. If R_1 is an open subset of R, and if ξ is a point of R_1) we shall denote by $\tau_\xi^h(R_1)$ the minimum of τ_ξ^h and the first parameter value (or τ_ξ^h if there is none) at which an h-path from ξ meets the boundary R_1' of R_1. We adopt the notation

$$z_\xi^h[\tau_\xi^h(\omega)] = \lim_{t \to \tau_\xi^h(\omega)} z_\xi^h(t, \omega),$$

when this limit exists, and we denote the random variable so defined by $z_\xi^h(\tau_\xi^h)$. The distribution of this random variable is the h-harmonic measure on R', relative to ξ, if h is harmonic (*see* section 7).

3. **h-path properties.** — It will turn out that the general h-path process can be reduced to the special case in which h is a minimal harmonic function. Thus we could simplify some of the preliminary work by always imposing the restriction that h be harmonic. Since the simplification is not significant for our purposes, however, we shall not impose this restriction.

Suppose then that h is strictly positive and superharmonic on R, and let ξ be a point at which h is finite. Let $\hat{R}_t$ be the space of functions from $[0, t]$ to R with value ξ at 0. Let t be a strictly positive number, suppose that $0 < t_1 < \ldots < t_k = t$, that A is a Borel set of the product space R^k, and let $\hat{A}_t$ be the set of those functions in $\hat{R}_t$ with

$$[f(t_1), \ldots, f(t_k)] \in A.$$

Then, according to our definition of $\mathbf{P}$ for h-path processes, and remembering that we have agreed to write $z(t)$ instead of $z^1(t)$,

$$h(\xi)\,\mathbf{P}\{z_\xi^h(\cdot, \omega) \in \hat{A}_t\} = \int_{\{z_\xi(\cdot, \omega) \in \hat{A}_t\}} h[z_\xi(t)]\, d\mathbf{P}. \tag{3.1}$$

It follows that the same equation is correct if (keeping t fixed) $\hat{A}_t$ is now any subset of $\hat{R}_t$ in the Borel field of sets generated by those just described, as k, $\{t_j\}$, A vary. We have thus a way of evaluating h-path probabilities in terms of Brownian path probabilities.

Using this evaluation, and the separability of the $z_\xi^h(t)$ process, we can evaluate the probability that $\tau_\xi^h(\omega) > t$ and that (simultaneously) h-paths are continuous on the interval $[0, t]$. This amounts to choosing $\hat{A}_t$ properly. The evaluation (3.1) does not change if we omit the continuity condition,

because almost all Brownian paths from ξ are continuous. Hence almost all h-paths from ξ with lifetime $> t$ are continuous on $[0, t]$. Since t is arbitrary, almost all h-paths from ξ are continuous throughout their lifetimes. A similar argument yields the fact that, if A is any subset of R of zero capacity, almost no h-path from ξ passes through a point of A. In particular this means that, if u is superharmonic, u is finite-valued on almost all h-paths. An important special case we shall use repeatedly is the case $u = h$: almost no h-path from ξ passes through an infinity of h. Going somewhat further, and using the fact that Brownian paths have small probability of meeting sets of small capacity, and Cartan's theorem that a superharmonic function is continuous relative to a compact set whose complement has small capacity, it follows as in [4] that, if u is superharmonic, u is continuous on almost all h-paths from ξ.

Let R_1 be any open subset of R. Then R_1 is also a Green space, and we can consider h-paths in R_1. Using the same notation as above except for a prescript 1 when paths relative to R_1 are involved, we obtain the evaluation

$$(3.2)\qquad \mathbf{P}\{{}_1z^h(\cdot,\omega)\in {}_1\hat{A}_t\} = \int_{\{{}_1z_\xi(\cdot,\omega)\in {}_1\hat{A}_t\}} h[{}_1z_\xi(t)]\frac{d_1\mathbf{P}}{h(\xi)} = \int_{\{z_\xi(\cdot,\omega)\in {}_1\hat{A}_t\}} h[z_\xi(t)]\frac{d\mathbf{P}}{h(\xi)}.$$

The last expression is the probability that an h-path on R from ξ coincides with an element of ${}_1\hat{A}_t$ to time t. Thus the h-paths from ξ relative to R_1 are those relative to R, with lifetime shortened from τ_ξ^h to $\tau_\xi^h(R_1)$.

We now add the hypothesis that the closure of R_1 is a compact subset of R, and use the fact [6] that the $h[z(t)]$ process, stopped at time $\tau(R_1)$ is a lower semimartingale. Let ${}_1\hat{A}_t$ be defined like $\hat{A}_t$ except that in the definition R is replaced by $R_1 \cup R_1'$, and let $A(R_1)$ be any set in the Borel field of sets generated by the sets ${}_1\hat{A}_t$ for $0 < t < \infty$. Then the inequality

$$(3.3)\quad h(\xi)\,\mathbf{P}\{z_\xi^h(\cdot,\omega)\in A(R_1)\} \geqq \int_{\{z_\xi(\cdot,\omega)\in A(R_1)\}} h[z_\xi(\tau_\xi(R_1))]\,d\mathbf{P},$$

becomes, in view of (3.1), the standard lower semimartingale inequality if $A(R_1)$ is a set ${}_1\hat{A}_t$. Hence the inequality is true for general $A(R_1)$. Moreover, if h is harmonic, the inequality becomes an equality, because in that case the stopped $h[z_\xi(t)]$ process is a martingale.

If h is harmonic, (3.3) (with equality) allows us to conclude that almost no h-path from a point of R has a closure compact relative to R. In fact using the above notation, it is sufficient to prove that almost every h-path from ξ meets R_1', and this is effected by choosing $A(R_1)$ properly, remembering

that (martingale property)

$$\mathbf{E}\{h[z_\xi(\tau_\xi(R_1))]\} = h(\xi)$$

and that this assertion is true for $h = 1$ [6].

Finally, if h is harmonic, we show that almost every h-path from a point of R approaches ∞ as the path parameter increases to the path lifetime. This is a slight strengthening of the previous result. To prove this, let $R_1 \subset R_2 \subset \ldots$ be open subsets of R, such that $\xi \in R_1$, that the closure of R_n is a compact subset of R_{n+1}, and that $\bigcup_1^\infty R_n = R$. Consider the probability that an h-path from ξ meets R'_1 after meeting R_m. It is sufficient to prove that this probability p_m approaches o when $m \to \infty$. Now $p = \lim_{n \to \infty} p_{mn}$, where p_{mn} is the probability that an h-path from ξ meets R'_1 after meeting R'_m but before meeting R'_n. Let Λ_{mn} be the ω set corresponding to those Brownian paths from ξ which meet R'_1 after meeting R'_m but before meeting R'_n, and let τ_{mn} be the first such intersection time. According to our evaluations of h-path probabilities,

$$p_{mn} = \int_{\Lambda_{mn}} h[z(\tau(R_n))] \frac{d\mathbf{P}}{h(\xi)}. \tag{3.4}$$

By a standard martingale theorem on systems, we then find that

$$\begin{aligned} p_{mn} &= \int_{\Lambda_{mn}} h[z(\tau_{mn})] \frac{d\mathbf{P}}{h(\xi)} \\ &\leqq K\mathbf{P}\{\Lambda_{mn}\}, \qquad K = \sup_{\eta \in R'_m} h(\eta). \end{aligned} \tag{3.5}$$

When n increases, Λ_{mn} increases to the set Λ_m corresponding to Brownian paths from ξ which meet R'_1 after meeting R_m, so that

$$p_m \leqq K\mathbf{P}\{\Lambda_m\}. \tag{3.6}$$

Finally, when $m \to \infty$ the right side of this inequality goes to o because almost all Brownian paths from ξ go to ∞ (that is, because, according to [6], the result in question is true for $h = 1$). This completes the proof of the lemma.

If h is continuous, h-path processes are strongly Markov in the sense of [1], as appropriately modified to apply to processes with finite lifetimes. In fact, even if h is discontinuous, the statement remains true, and Blumenthal's discussion, somewhat more delicately handled, is applicable.

4. h-harmonic functions. — If h is superharmonic and strictly positive on R, we shall call a function u on R h-superharmonic [h-harmonic] if uh, considered only on the set where h is finite-valued, can be extended to R in

such a way that the extended function is superharmonic [harmonic]. An h-subharmonic function is one whose negative is h-superharmonic. Note that, according to this definition, $\frac{1}{h}$, defined arbitrarily at the infinities of h, is h-harmonic. An h-harmonic or h-superharmonic function can be changed arbitrarily at the infinities of h without affecting the applicability of the above definitions. We have already remarked that almost no h-path from a point at which h is finite ever passes through an infinity of h, and this fact illustrates the point that the set of infinities of h is negligible for many of our considerations. The function 1 is h-superharmonic for all h, h-harmonic if h is harmonic.

We shall use lower semimartingales in many places in this paper, always accepting as part of the definition that the random variables of such a process have finite expectations. Our first application of martingale theory yields the following lemma, to be strengthened later.

LEMMA 4.1. — *Let u and h be strictly positive superharmonic functions. Let ξ be a point of finiteness of h, and define $x(t)$ by*

$$x(t,\omega)=\left(\frac{u}{h}\right)[z_\xi^h(t,\omega)], \qquad t<\tau_\xi^h(\omega), \tag{4.1}$$
$$=0, \qquad t\geqq\tau_\xi^h(\omega).$$

Then the $x(t)$ process is a lower semimartingale for $0<t\leqq\infty$, and also for $0\leqq t\leqq\infty$ if $u(\xi)<\infty$.

Since almost no h-path from ξ passes through an infinity of h, and since the $z_\xi^h(t)$ process is Markov, it is sufficient to prove that $\mathbf{E}\{x(t)\}<\infty$ and that, if $h(\xi)<\infty$ and if $u(\xi)<\infty$, then

$$x(0)=\frac{u(\xi)}{h(\xi)}\geqq\mathbf{E}\{x(t)\}. \tag{4.2}$$

Now it is known [6] that u has a limit on almost every Brownian path from ξ, as $t\uparrow\tau_\xi$, and that, if $u(\xi)<\infty$, and if $u[z_\xi(t)]$ is defined as this limit when $t\geqq\tau_\xi$, then the $u[z_\xi(t)]$ process is a lower semimartingale for $0\leqq t\leqq\infty$. Hence

$$u(\xi)\geqq\mathbf{E}\{u[z_\xi(t)]\}\geqq\int_{\{\tau_\xi(\omega)>t\}}u[z_\xi(t)]\,d\mathbf{P}. \tag{4.3}$$

Moreover the last integral can be written in terms of h-path process integrals, in the form

$$h(\xi)\int_{\{\tau_\xi^h(\omega)>t\}}\left(\frac{u}{h}\right)[z_\xi^h(t)]\,d\mathbf{P}=h(\xi)\,\mathbf{E}\{x(t)\}. \tag{4.4}$$

The inequality (4.3) is thus epuivalent to (4.2). The inequality (4.2) shows

that $\mathbf{E}\{x(t)\} < \infty$ if $u(\xi) < \infty$. If $u(\xi) = \infty$, a direct evaluation of $\mathbf{E}\{x(t)\}$ is still possible, and yields a finite number, in view of the remarks on excessive functions in section 1.

THEOREM 4.2. — *If u and h are strictly positive superharmonic functions on R, $\frac{u}{h}$ has a finite limit along almost every h-path from a point of finiteness of r. If h is minimal harmonic, this limit is $\inf_{\xi \in R}\left(\frac{u}{h}\right)$ on almost every such path.*

In this and similar theorems, when we write of a limit along a probability path, without further qualification, we always mean limit at the path lifetime. Since the $x(t)$ process of the lemma is a positive lower semimartingale, which is trivially separable, almost all its sample functions have right and left hand finite limits at all parameter values. This fact, for the parameter value τ_ξ^h, gives the theorem. In particular, if h is minimal harmonic, the evaluation given is a consequence of the general theory in [7].

5. **The case when h is harmonic.** — If h is harmonic, we have seen that almost all h-paths from a point of R approach ∞. Moreover, if u is a positive and h-superharmonic function, if ξ is the initial point of an h-path process, and if R_1 is an open set containing ξ, with closure a compact subset of R, then the $u[z_\xi^h(t)]$ process stopped at time $\tau_\xi^h(R_1)$, that is, this process made constant for times at least equal to this stopping time, is a lower semi-martingale. In fact the stopped process is the lower semimartingale of lemma 4.1 stopped at time $\tau_\xi^h(R_1)$. With this much, the apparatus of [7] becomes available. Let $\{R_n, n \geqq 1\}$ be a sequence of open subsets of R, with union R, and such that the closure of R_n is a compact subset of R_{n+1}. If ξ is a point of R, and if n is the first integer with $\xi \in R_n$, the sequence of random variables

$$\xi, \quad z_\xi^h[\tau_\xi^h(R_n)], \quad z_\xi^h[\tau_\xi^h(R_{n+1})], \quad \ldots$$

defines a system of discrete paths from ξ to R'. These are precisely the paths used in [7], when brought into present context. However we need not restrict ourselves to $z_\xi^h(t)$ for t ranging through o, $\tau_\xi^h(R_n)$, $\tau_\xi^h(R_{n+1})$, ... In fact, with the background of h-paths we have now developed, the theorems of [7] involving the discrete paths (there called h-paths) go over into the exactly corresponding ones for continuous h-paths considered here. The changes necessary in the proofs, if any, are always obvious. For example, according to [7], if u is h-superregular (h-superharmonic in the present context) and positive, u has a limit along almost all h-paths from a point of R. (In [7] h was always assumed regular.) According to theorem 4.2 this is true here, for our continuous h-paths.

Now consider the Dirichlet problem for h-harmonic functions on R, using the Martin boundary R', and supposing that h is harmonic. BRELOT [2] has proved that all continuous functions on R' are PWB^h resolutive. According to [7], this fact implies the truth of the following theorem, to be generalized by theorem 7.1.

THEOREM 5.1. — *If h is harmonic and strictly positive, almost every h-path from a point of R converges to a point of R'. In particular, if h is minimal, almost all h-paths from a point of R converge to the same point, the pole of h.*

This is of course a much stronger result than that proved earlier, that almost all h-paths from a point of R tend to ∞.

For later reference, we extract from [7], translated to our present context, the following theorem, giving the relation between PWB^h resolutive boundary functions and their corresponding Dirichlet solutions, which we shall call PWB^h solutions.

THEOREM 5.2. — *If u is the* PWB^h *solution of the Dirichlet problem for h-harmonic functions, corresponding to the* PWB^h *resolutive boundary function f, then u is given by*

$$u(\xi) = \mathbf{E}\{f[z_\xi^h(\tau_\xi^h)]\}, \tag{5.1}$$

and u has f as a limit along almost all h paths from any point of R, in the sense that

$$\lim_{t \to \tau_\xi^h} u[z_\xi^h(t)] = f[z_\xi^h(\tau_\xi^h)], \tag{5.2}$$

with probability 1.

If h is positive and superharmonic on R, and if $A \subset R \cup R'$, we define h_A, following BRELOT, as the lower envelope of the positive superharmonic functions on R which exceed h near A. Then, if A is a Borel subset of R', and if h is harmonic, it was proved in [7] that $\frac{h_A(\xi)}{h(\xi)}$ is the probability that an h-path from ξ converges to a point of A. A minor development of this discussion shows that, if A is a Borel subset of $R \cup R'$, $\frac{h_A(\xi)}{h(\xi)}$ is the probability that an h-path from ξ either meets a point of AR at a strictly positive parameter value, or converges to a point of AR'. If A is closed, $\frac{h_A}{h}$ in R-A is the PWB^h Dirichlet solution for h-harmonic functions given the boundary function 1 at the boundary points of R-A in A, 0 at the other boundary points.

6. A simple example. — Let ξ_1 be a point of R, and consider h-paths from a point $\xi \neq \xi_1$, for $h = g(\xi_1, .)$. Since almost no such path passes

through ξ_1, we can replace R by $R_1 = R - \{\xi_1\}$ without changing the h-paths. The Martin boundary R_1' consists of R' together with the point ξ_1, which is a minimal point, the pole of h. According to theorem 5.1, almost every h-path converges to the point ξ_1. That is, almost every h-path on R converges to this point as the parameter value approaches the path lifetime. If we insert in the equality

$$\mathbf{P}_1\{\tau_\xi^h(\omega) > t\} = \int_R p(t, \xi, \eta) \frac{g(\xi_1, \eta)}{g(\xi_1, \xi)}\, d\eta, \tag{6.1}$$

the evaluation of g in terms of p in [6]

$$g(\xi, \eta) = \int_0^\infty p(t, \xi, \eta)\, dt, \tag{6.2}$$

the right side of (6.1) becomes

$$\frac{\int_t^\infty p(s, \xi, \xi_1)\, ds}{g(\xi, \xi_1)}, \tag{6.3}$$

so that there is convergence to 0 when $t \to \infty$ in (6.1). This means that τ_ξ^h is almost certainly finite. Note that the quantity (6.3) is symmetric in ξ, ξ_1.

If u is superharmonic, it may not be continuous. If $u(\xi_1) = +\infty$, however, u is continuous at ξ_1, because u is everywhere lower semicontinuous. Hence, trivially, u has the limit $u(\xi_1)$ along almost all $g(\xi_1, .)$-paths to ξ_1. If $u(\xi_1)$ is finite, it remains true, but is no longer trivial, that u has the limit $u(\xi_1)$ along almost all $g(\zeta_1, .)$ paths to ξ_1. We shall prove this result in section 14.

7. **h-paths in the general case.** — Suppose now that h is an arbitrary strictly positive superharmonic function on R, with canonical mass distribution μ:

$$h(\eta) = \int_{R \cup R'} K_{\xi_0}(\xi, \eta)\, \mu(d\xi), \tag{7.1}$$

as discussed in section 1. We can write the transition density p^h in the form

$$p^h(t, \xi, \eta) = \int_{R \cup R'} \frac{K_{\xi_0}(\zeta, \xi)\, \mu(d\zeta}{h(\xi)} \left[\frac{p(t, \xi, \eta)\, K_{\xi_0}(\zeta, \eta)}{K_{\xi_0}(\zeta, \xi)}\right]. \tag{7.2}$$

This form can be interpreted as follows. To construct h-paths from a point ξ of R at which h is finite, first choose a point z on $R \cup R'$, where the probability that z lies in A is given by

$$\int_A \frac{K_{\xi_0}(\zeta, \xi)}{h(\xi)}\, \mu(d\zeta). \tag{7.3}$$

Then either the value ζ found for z is a point of R or a point of R'. In the latter case we can and shall assume that ζ is minimal, for the contrary possibility has probability o. In either case, choose a $K_{\xi_0}(\zeta, .)$-path process from ξ. Almost all paths of the latter process, which is a $g(\zeta, .)$-path process if $\zeta \in R$, go from ξ to ζ. Then the composite process is an h-path process from ξ. To justify this interpretation, we must of course (trivially) write

$$\prod_1^n p^h(t_j, \zeta_j, \zeta_{j+1})$$

in a form corresponding to that in (7.2), but we shall forbear to do so. Another way of stating the same conclusion is given by the following theorem.

THEOREM 7.1. — *Let h be given by the canonical form* (7.1). *Then almost all h-paths from a point ξ at which h is finite converge, at time τ_ξ^h, to a point of $R \cup R'$. The probability that this point, $z_\xi^h(\tau_\xi^h)$, lies in the Borel set A is given by* (7.3) *The process can be defined in such a way that the conditional probability distribution of h-paths, given that $z_\xi^h(\tau_\xi^h)$ has the value ζ, is the distribution of $K_{\xi_0}(\zeta, .)$-paths.*

The reservation " The process can be defined in such a way that " is due to the following consideration. The decomposition of the process described suggests a way of constructing it, simply by making the above outline more precise, and adopting as the basic measure space the space of functions from $[0, \infty)$ to the union of R with a point δ, the functions, being continuous and having values in R before a value of the parameter depending on the function, approaching a limit in $R \cup R'$ from the left, at that value, and identically δ thereafter. Carrying these details through, one obtains what the theorem states. It is well-known, however that, if the h-path process is given arbitrarily, the conditional distributions involved may not exist except in an extended sense.

The structure of an h-path process is now clear. For example, the path lifetime is finite for almost all paths to points of R. If R has the property that almost all Brownian paths have finite lifetimes, it will follow that, for almost every (harmonic measure) minimal point ζ of R', almost all $K_{\xi_0}(\zeta, .)$-paths have finite lifetimes.

In view of our analysis, it is not unreasonable to describe h-paths, for h minimal, as Brownian paths conditioned to converge to the pole of h, and to describe $g(\xi, .)$-paths as Brownian paths conditioned to converge to ξ. For general h, h-paths will be called conditional Brownian paths.

8. Fine boundary functions. — We recall that, if h is a strictly positive harmonic function on R, h-harmonic measure on R' relative to a specified

point of R plays the same role for h-harmonic functions as harmonic measure for harmonic functions. It is the distribution of the endpoints of h-paths from the specified point of R.

Let $\{R_n, n \geqq 1\}$ be a monotone sequence of open subsets of R, with union R, whose closures are compact subsets of R. Let h be a strictly positive harmonic function on R. Let u be a Baire function defined on $\bigcup_n R'_n$. Suppose that, for some point ξ of R, u has a limit on approach to R' along almost every h-path from ξ, considering u only on the sequence of first meetings of the path with $R'_1, R'_2, \ldots$. That is, we suppose that

$$\lim_{n \to \infty} u[z_\xi^h(\tau_\xi^h(R_n))] = u_\xi^* \tag{8.1}$$

exists with probability 1. Then u_ξ^* is a random variable. Applying what we know of the structure of h-paths, we find that, for almost all ζ (h-harmonic measure) on R', u has a limit along the first meetings of almost every $K_{\xi_0}(\zeta, .)$-path with $R'_1, R'_2, \ldots$. We denote this limit by $u_\xi^*(\zeta)$, again a random variable, but not defined on the same measure space as u_ξ^*.

Conversely, if, for almost all ζ in this sense, the limit $u_\xi^*(\zeta)$ exists as indicated, then the limit u_ξ^* must also exist, with probability 1.

The probability that the limit in (8.1) exists defines an h-harmonic function of the initial point ξ, with values between 0 and 1. Hence, if the value is 1 at a point, it is identically 1, so that u_ξ^* is defined with probability 1 for every ξ if for any ξ. If u_ξ^* is defined with probability 1, this very argument applied to $h = K_{\xi_0}(\zeta, .)$ with $\zeta \in R'$, shows that $u_\xi^*(\zeta)$ is defined with probability 1 for each ξ and almost every ζ (h-harmonic measure), and the exceptional ζ set does not depend on ξ.

Now suppose that h is a minimal harmonic function, with pole ζ, so that h-harmonic measure is concentrated at ζ. Then almost all h-paths that we are considering approach ζ. The class of limits $\{u_\xi^*, \xi \in R\}$ is what we have called a stochastically ramified boundary function in [6] and [7], and such functions are identically constant if h is minimal, according to [7], because the class of stochastically ramified Dirichlet solutions is the class of constant functions in that case. That is, if h is minimal, there is a constant c, independent of ξ, such that $u_\xi^* = c$ with probability 1, for every ξ in R.

Going back to the case of general harmonic h, we see that, in view of the proceeding paragraphs, *if u_ξ^* exists with probability 1 for a single value of ξ, it does for all ξ, and there is a function f defined on R' with the property that, if ζ is not in some subset of R' of h-harmonic measure 0, $u_\xi^*(\zeta) = f(\zeta)$ with probability 1, for each ξ in R.* Moreover, as we now prove, *f is measurable with respect to the h-harmonic measure.*

It is no restriction in the proof to suppose that u is bounded, and we shall do so. To prove measurability, we remark first that, if φ_n is the

function on R' defined by

$$\varphi_n(\zeta) = \mathbf{E}\{u[z_\xi^\nu(\tau_\xi^\nu(R_n))]\}, \qquad \nu = K_{\xi_0}(\zeta, .), \tag{8.2}$$

then φ_n is a Baire function, and

$$\mathbf{E}\{u[z_\xi^h(\tau_{\xi]}^h(R_n))] \mid z_\xi^h(\tau_\xi^h)\} = \varphi_n[z_\xi^h(\tau_\xi^h)] \tag{8.3}$$

with probability 1, according to our analysis of h-paths. Hence, using our hypotheses on u,

$$\lim_{n\to\infty} \varphi_n[z_\xi^h(\tau_\xi^h)] = \mathbf{E}\{u_\xi^* \mid z_\xi^h(\tau_\xi^h)\} = \varphi^* \tag{8.4}$$

exists with probability 1. That is $\lim_{n\to\infty} \varphi_n = \varphi$ exists almost everywhere on R' (h-harmonic measure). In proving the measurability result, we can and shall assume that the conditional probability distributions of the h-path process exist as described in theorem 7.1. Actually we only need them for the sequence

$$\{z_\xi^h[\tau_\xi^h(R_n)], n \geqq 1\}. \tag{8.5}$$

Then, for fixed $z_\xi^h(\tau_\xi^h) = \zeta$, the distribution of φ^* is concentrated at $f(\zeta)$. In other words,

$$\mathbf{E}\{\varphi^* \mid z_\xi^h(\tau_\xi^h)\} = f[z_\xi^h(\tau_\xi^h)] \tag{8.6}$$

with probability 1, so that the quantity on the right is measurable. Now if, as we can suppose, the basic measure space of the h-path process is a perfect measure space in the sense of KOLMOGOROV [9], or even only if we restrict ourselves to the sequence (8.5) and suppose that this sequence is defined on a perfect measure space, it follows that $f[z_\xi^h(\tau_\xi^h)]$ cannot be measurable unless f itself is measurable with respect to h-harmonic measure, as was to be proved.

Now let u be a Baire function on R, let h be a strictly positive harmonic function on R, and let f be a function on R'. We shall say that u has the function f as its h-fine boundary function if there is a subset of R', of h-harmonic measure 0, such that, if ζ is a point of R' not in this set, and if $\xi \in R$, u has the limit $f(\zeta)$ along almost all $K_{\xi_0}(\zeta, .)$-paths from ξ to ζ. We write simply " fine boundary function " if $h = 1$. According to what we have proved above, f is necessarily measurable with respect to h-harmonic measure. The adjective " fine " will be put into a topological context in section 14, by an identification of the boundary limit concept involved here with Naïm's fine topology limit at the boundary.

We observe that, according to what we have proved above, if u is a Baire function which has a limit along almost all h-paths to R' from each point of R, and if u is say right continuous on almost all such paths, so that there are no measure difficulties in translating all the results obtained above

for discrete h-paths into the case of the full h-paths, then u has an h-fine boundary function.

9. Fatou's boundary value theorem. — If u is positive and superharmonic on R, and if h is strictly positive and harmonic on R, $\frac{u}{h}$ has a finite limit on almost all h-paths from any point ξ of R to R'. This theorem, a special case of theorem 4.2, is a probability version of a generalization of Fatou's classical boundary value theorem. It suffers from the unsatisfactory feature that it stresses the paths from ξ rather than the path endpoints, and it is not clear how the limits along paths to the same boundary point are related. The corresponding advantage is that the theorem does not even involve a boundary. All it really states in that $\frac{u}{h}$ has a finite limit on almost all h-paths, at the time τ_ξ^h. Theorem 9.1 does away with the stated disadvantage, at the price of involving the Martin boundary explicitly.

THEOREM 9.1. — *Let u be a positive superharmonic function on R, and let h be a strictly positive harmonic function on R. Then $\frac{u}{h}$ has an h-fine boundary function f on R', which is finite almost everywhere (h-harmonic measure), and*

$$\frac{u(\xi)}{h(\xi)} \geq \mathbf{E}\{f[z_\xi^h(\tau_\xi^h)]\}. \tag{9.1}$$

The existence of f follows from theorem 4.2 and the last paragraph of section 8. The inequality (9.1) is a special case of the general results in [7].

According to theorem 9.1, a positive h-superharmonic function, and hence any h-superharmonic function greater than the negative of some other positive h-superharmonic function, have h-fine boundary functions. The expectation on the right in (9.1) defines an h-harmonic function v of ξ. The function v is the PWB^h Dirichlet solution corresponding to the boundary function f, and has f as h-fine boundary function, according to theorem 5.2. Thus $\frac{u}{h}$ is itself a PWB^h solution if and only if there is equality in (9.1).

THEOREM 9.2. — *If h is a strictly positive harmonic function, it has an h-fine boundary function which is strictly positive almost everywhere (h-harmonic measure) on R'.*

If $u \equiv 1$ in theorem 9.1, we see that $\frac{1}{h}$ has an h-fine boundary function which is finite almost everywhere (h-harmonic measure), and the theorem then follows immediately. Note that the fine boundary function of h may

not be finite-valued, and in fact may be $+\infty$ almost everywhere (h-harmonic measure) on R'.

In the language of [6] and [7], what we have proved is that every stochastically ramified boundary function in the study of h-superharmonic and h-harmonic functions is derived from an ordinary boundary function on the Martin boundary, measurable with respect to h-harmonic measure. Thus the use of the Martin boundary makes unnecessary the use of stochastically ramified boundary functions.

Note that our results relate to the same kind of limit behavior at the boundary points in question for h-superharmonic functions as for h-harmonic functions. This is curious in the light of the following fact. If R is an N-dimensional sphere (ball) with $N>1$, R' is the ordinary sphere boundary. If u is positive and superharmonic, it is classical that u has a limit along almost every radius to the surface R', that is along radii going to almost every [Lebesgue $(N-1)$-dimensional measure on R', or equivalently, harmonic measure on R'] point of R'. If u is harmonic, but not in the general superharmonic case, the theorem remains correct if approach to a boundary point ζ along the radius to ζ is replaced by non-tangential approach, that is, by approach in any cone with vertex ζ and lying in R near ζ.

In order to analyze the class of PWB^h Dirichlet solutions, we shall recall a known definition. Let ξ be a point of R, and let $\{R_n, n\geqq 1\}$ be a monotone sequence of open subsets of R, containing ξ, with union R, and such that the closure of each set R_n is a compact subset of R. Let μ_n be the h-harmonic measure of subsets of R'_n relative to ξ. In probability language, μ_n is the distribution induced on R'_n by $z_\xi^h[\tau_\xi^h(R_n)]$. If u is a Baire function on R, it induces on R'_n, along with the measure μ_n, a sequence of measurable functions, as n varies. The class $\mathbf{D}^h$ is the class of h-harmonic functions u for which this sequence is uniformly integrable. This class is independent of the choice of ξ and $\{R_n, n\geqq 1\}$ [6], [7]. We can now summarize our results on h-fine boundary functions as applied to the Dirichlet problem as follows.

THEOREM 9.3. — *If f is a* PWB^h *resolutive boundary function, the corresponding Dirichlet solution is in the class* $\mathbf{D}^h$. *Conversely, if* $u\in\mathbf{D}^h$, *it has an h-fine boundary function f which is* PWB^h *resolutive and has u as its* PWB^h *Dirichlet solution.*

The direct half of this theorem was proved in [5]. The converse half follows from the work of this section and theorem 5.2.

We conclude this discussion by noting a few cases in which the h-fine boundary function is known. If u is the potential of a positive mass distribution, it is known that u has the limit o on almost all Brownian paths from each point of R [6], and the proof referred to proves also, with the obvious changes, that, if h is strictly positive and harmonic on R, $\frac{u}{h}$ has the

limit o on almost all h-paths from each point of R. Hence $\frac{u}{h}$ has the h-fine boundary function o, in the sense that, for almost every minimal boundary point (h-harmonic measure), $\frac{u}{h}$ has the limit o along almost every conditional Brownian path from a point of R to the boundary point. An equivalent result has been obtained by NAÏM [11] using non-probabilistic methods. If u is a minimal harmonic function on R, there are two possibilities, according to [7]. Either u is unbounded (and in fact not even in the class $\mathbf{D}^1$), and in that case u has the fine boundary function o, or u is bounded, and in that case either u is identically constant or u has a fine boundary function which (neglecting a subset of R' of harmonic measure o in the following) only takes on two values, o and a strictly positive value a. Moreover $\{f(\zeta)=0\}$, $\{f(\zeta)=a\}$ are sets of positive harmonic measure, and the second contains only a single point, the pole of u.

10. Absolute probability systems. — Suppose that, for each strictly positive number t, there is a positive finite-valued Lebesgue measurable function $q^h(t, .)$ on R, satisfying the equation

$$q^h(s+t,\eta)=\int_R q^h(s,\xi)\,p^h(t,\xi,\eta)\,d\xi, \tag{10.1}$$

for strictly positive s, t. Then q^h is continuous, and in fact $\frac{q^h}{h}$ is a parabolic function of its arguments. Moreover the integral

$$\int_R q^h(t,\eta)\,d\eta$$

defines a monotone non-increasing function of t, and we suppose that the limit of this function when $t\to 0$ is 1. Then we shall say that $\{q^h(t, .), t>0\}$ is an absolute probability density system for h-paths. By the usual argument, it is seen that q^h together with p^h determines a stochastic process $\{z^h(t),\ t>0\}$ with state space R, separable relative to the compact sets. This process is an h-path process with no initial probability distribution, since $z^h(0)$ is not defined. It is a Markov process with continuous sample functions, with transition density p^h, and

$$\mathbf{P}^h\{z^h(t,\omega)\in A\}=\int_A q^h(t,\xi)\,d\xi. \tag{10.2}$$

It is easily seen that there is no increase in generality if one goes from *density* systems to *distribution* systems. In particular, if q^h is given by

$$q^h(t,\xi)=p^h(t,\xi_0,\xi), \tag{10.3}$$

for some point ξ_0, it is clear that the $z^h(t)$ process is simply an h-path

process with initial point ξ_0, considered only for strictly positive parameter values. Another way of looking at this is to say that, with this choice of q^h, the sample paths almost all converge to ξ_0 when $t \to 0$, and $z^h(0)$ can then be defined as ξ_0 to obtain a process almost all of whose sample functions are continuous at the parameter value 0.

From now on, when we write " h-path process ", we mean a process $\{ z^h(t),\ t > 0 \}$ determined by p^h and some q^h, with $z^h(0)$ not defined *a priori* unless the definition is specified explicitly. However we shall always write $z^h(0)$ for $\lim_{t \to 0} z^h(t)$ when this limit exists and shall call the value of the limit for a path the initial point of the path. If $t_0 > 0$. the process $\{ z^h(t_0 + t),\ t \geqq 0 \}$ is an h-path process with initial distribution that of $z^h(t_0)$, [dividing all probabilities by the probability that $z^h(t_0)$ is defined]. Hence it is clear how to apply the theorems of the preceding sections to the present processes. For most of our theorems the new point of view is irrelevant. For others the change is a triviality. Our only hope to get something new is to consider parameter values near 0. The first result in this direction is the following theorem, the dual of theorem 4.2.

THEOREM 10.1. — *Let u and h be strictly positive superharmonic functions on the Green space R. Define $x(t)$ by (4.1) for $t > 0$, where the $z^h_\xi(t)$ process is replaced by a $z^h(t)$ process determined by p^h and an absolute probability density system. Then $\lim_{t \to 0} x(t)$ exists with probability 1. If $x(0)$ is defined as this limit, $0 < x(0) \leqq \infty$ with probability 1, and*

$$\lim_{t \to 0} \mathbf{E}\{ x(t) \} = \mathbf{E}\{ x(0) \}. \tag{10,4}$$

If the right hand side of (10.4) is finite, the process $\{ x(t),\ 0 \leqq t \leqq \infty \}$ is a lower semimartingale.

By a trivial extension of lemma 4.1, if $\mathbf{E}\{ x(t) \} < \infty$ for all $t > 0$, the process $\{ x(t),\ 0 < t < \infty \}$ is a lower semimartingale. It follows that, if we replace $x(t)$ by

$$x_n(t) = \min[x(t),\ n],$$

the $x_n(t)$ process is a positive lower semimartingale. This change amounts to replacing u by $\min[u,\ nh(t)]$. Hence, by a standard martingale convergence theorem, $\lim_{t \to 0} x_n(t)$ exists with probability 1. We conclude that

$$\lim_{t \to 0} x(t) = x(0) \quad (\leqq \infty)$$

exists with probability 1, and, applying Fatou's theorem,

$$\lim_{t \to 0} \mathbf{E}\{ x(t) \} \geqq \mathbf{E}\{ x(0) \}. \tag{10.5}$$

Here the expectation on the left defines a monotone non-increasing function of t, because this is true when $x(t)$ is replaced by $x_n(t)$. If $\mathbf{E}\{x(t)\}$ is finite, the $x(t)$ process is a lower semimartingale for $t \geqq 0$ because the $x_n(t)$ process is, and then (10.4) is a consequence of (10.5) and the lower semimartingale inequality. If $\mathbf{E}\{x(0)\} = \infty$, (10.4) is a consequence of (10.5). If we had allowed infinite-valued integrals in our definition of lower semimartingales, the $x(t)$ process would have been a lower semimartingale in all cases. Finally, because of the lower semimartingale properties of the $x_n(t)$ process, $x(t)$ vanishes simultaneously for all t for almost all h-paths with $x(0) = 0$. Hence $x(0)$ is strictly positive with probability 1.

The following theorem is the dual of the theorem 7.1.

THEOREM 10.2. — *If* $\{z^h(t), t > 0\}$ *is an h-path process,* $\lim_{t \to 0} z^h(t)$ *exists with probability* 1.

We prove the theorem by showing that almost no h-path can have two limit points in $R \cup R'$ as $t \to 0$. Let B be a denumerable subset of R, dense in R. Then by theorem 10.1, almost all h-paths of the process in question have the property that on them $\frac{g(\xi, \cdot)}{h}$ has a strictly positive limit as $t \to 0$ simultaneously for all ξ in B. For a non-exceptional h-path, let $\nu(\xi)$ be this limit, and suppose that this h-path has limit points ζ_1, ζ_2 when $t \to 0$. To prove the theorem, we show that $\zeta_1 = \zeta_2$. If B_1 is B less the points ζ_1, ζ_2 (if either is in B), and if ξ_1, ξ_2 are points of B_1, then

$$\nu(\xi_2) = \nu(\xi_1) K_{\xi_1}(\zeta_1, \xi_2) = \nu(\xi_1) K_{\xi_1}(\zeta_2, \xi_2). \tag{10.6}$$

Hence

$$K_{\xi_1}(\zeta_1, \xi_2) = K_{\xi_1}(\zeta_2, \xi_2) \tag{10.7}$$

for ξ_1 and ξ_2 in B_1, so that

$$K_{\xi_1}(\zeta_1, \cdot) = K_{\xi_1}(\zeta_2, \cdot), \tag{10.8}$$

which implies that $\zeta_1 = \zeta_2$, as was to be proved.

11. Paths from the boundary. — In the preceding sections we have stressed h-paths whose initial points are in R. It is natural to try to choose h and q^h in such a way that the initial points are in R', and, in view of the Markov property, this depends on being able to choose h and q^h to make almost all h-paths have the same initial point, a point of R'. If $h = 1$, it is impossible, in general, to do this, unless the initial point is exceptional. To see this, suppose that R is an open subset of a Euclidean space, so regular that the Martin boundary R' is the relative boundary. Then a Brownian

$z(t)$ process with $z(0)$ identically a point ζ of R' reduces to an ordinary Brownian motion process with initial point ζ, with the property (*) that the sample paths are initially in R, except for the initial point ζ. The duration of the process is the time it takes the paths to reach R'. Since the property (*) is known [4] to be necessary and sufficient that ζ be an irregular boundary point, we see that we are considering an exceptional situation. The problem is to choose h so that h-paths near ζ will not necessarily go to R' near ζ. This suggests using conditional Brownian paths with endpoint a point of R, that is, choosing $h = g(\xi_0, .)$, for some point ξ_0 of R. Another, somewhat less promising choice, is $h = K_{\xi_0}(\eta, .)$, where η is any boundary point which is minimal and not ζ. We shall use the first choice in section 12.

It is not known, even with such choices of h, when q^h can be chosen to make $z^h(0)$ identically a point of R', except in simple cases (*see* [8] for the case when R is a half-space of Euclidean N-space). However in the following we shall obtain results almost as useful, without any further hypotheses on R, and which are applicable whenever the desired processes exist, not only to those processes, but also to the slightly distorted processes which we shall show always exist. The key is the following idea. Suppose that h and q^h are chosen in such a way that almost every initial point of h-paths is on R'. Let $\{ R_n, n \geqq 1 \}$ be a monotone sequence of open subsets of R, with union R, whose closures are compact subsets of R. Let $\mu_n(A)$ be the probability that the first meeting of an h-path of the given process with R'_n lies in the subset A of R'_n. Then μ_n is a measure of Borel subsets of R'_n, and

(M 1) $$\mu_n(R'_n) \leqq 1, \qquad \lim_{n \to \infty} \mu_n(R'_n) = 1.$$

Moreover (M 2) if one considers an h-path process with an initial distribution μ_n on R'_n [note that this may not be "true" probability if there is strict inequality in (M 1)], then the distribution of first meetings of paths of this process with R'_m for $m < n$ will be μ_m.

Suppose now that there is a positive superharmonic h for which there is a family of measures $\{ \mu_n, n \geqq 1 \}$ on the boundaries of the sequence $\{ R_n, n \geqq 1 \}$ satisfying (M 1) and (M 2). Then there is what we shall call an h-walk on these boundaries. The h-walk is a stochastic process $\{ z_n^h, n \geqq 1 \}$ such that z_n^h has range in R'_n and has distribution μ_n, that the sequence (note the order)

$$\ldots, \quad z_2^h, \quad z_1^h$$

is a Markov process, with the usual conventions if there is strict inequality in (M 1), and that the transition probability of going from point ξ in R_{n+1} into a point of the subset A of R_n is the probability that an h-path from ξ first meets R'_n if at all in a point of A.

With the above definitions, it is trivial that $z_n^h \to \infty$ when $n \to \infty$. It is less trivial that almost all h-walk paths are convergent. This is proved as

follows. Let u be strictly positive and superharmonic, and consider, for some fixed n, an h-path process $\{ z^h(t), t \geqq 0 \}$ with initial " distribution " μ_n. [If one dislikes improper distributions, one can replace μ_n by $\frac{\mu_n}{\mu_n(R_n')}$.] Then, applying lemma 4.1, we find that the process

$$\left\{ \frac{u[z^h(t)]}{h[z^h(t)]}, t \geqq 0 \right\}$$

is a lower semimartingale, if its random variables are integrable, and if this quotient is defined as 0 when $z^h(t)$ is undefined and $z^h(0)$ is defined. It follows that the last n members of the sequence

$$\ldots, \quad \frac{u(z_2^h)}{h(z_2^h)}, \quad \frac{u(z_1^h)}{h(z_1^h)}$$

form a lower semimartingale, under the same conventions, and hence that the whole sequence is a lower semimartingale, in the order exhibited. The proof that almost all paths of an h-path process have initial points now goes through with unessential changes to prove that the sequence $\{ z_n^h, n \geqq 1 \}$ is convergent with probability 1. The limit, for a given h-walk path, will be called the initial point of the path.

It is natural to try to fill in an h-path walk with arcs from z_{n+1} to z_n, $n \geqq 1$, and from z_1 on. This can be done as follows, to have the desired properties to be discussed below. Let I_n be the interval $[2^{-n}, 2^{-n+1})$, and let φ_n be a monotone strictly increasing continuous function, taking the interval I_n onto the interval $[0, \infty)$. We define a process $\{ Z^h(t), t > 0 \}$, or rather we define the joint distributions to be assigned to the random variables of this process, by the following conventions. The process is to be Markov. The joint distributions of the sequence

$$\ldots, \quad Z^h(2^{-n}), \quad Z^h(2^{-n+1}), \quad \ldots, \quad Z^h(2^{-1})$$

are to be those of the sequence

$$\ldots, \quad z_n^h, \quad z_{n-1}^h, \quad \ldots, \quad z_1^h.$$

For $t \in I_n$, the $Z^h(t)$ process random variables are to have the same joint distributions as the random variables of the process $\{ z_n^h[\varphi_n(t)], t \in I_n \}$, where the $z_n^h(t)$ process is an h-path process with initial distribution μ_n, stopped and made constant when the paths meet R_{n-1}', if ever. Finally, for $t \geqq 2^{-1}$, and $Z^h(2^{-1}) = \xi$ given, the $Z^h(t)$ process is to have the distribution of an h-path process from ξ. Then the martingale theorem that, roughly, if u is positive and superharmonic, $\frac{u}{h}$ on the $Z^h(t)$ process paths defines a lower semimartingale, holds just as it did for an ordinary h-path process

and for an h-path walk. That is, the analogues of theorem 10.1 and 10.2 hold for the $Z^h(t)$ process. We omit the trivial adjustments of proofs already given. Thus almost all $Z^h(t)$ process paths have initial points, in the usual sense.

12. An existence theorem. — In this section, we show how to obtain an h-path walk from any minimal boundary point ζ. This walk can then be filled in, as described in section 11. The sequence $\{R_n, n \geqq 1\}$, on whose boundaries the walk is to be defined, is supposed specified. Let ξ_0 be any point of R_1, and define $h = g(\xi_0, .)$. This choice of h will be held fast throughout this section. The walk will have initial point ζ. The filled in walk will therefore have initial point ζ, and almost all paths will have endpoint ξ_0. Let $\{\zeta_k, k \geqq 1\}$ be a sequence of points of R, converging to ζ, and with the property that, il μ_{nk} is the distribution of the first point z^h_{nk} in which an h-path from ζ_k meets R'_n, for k so large that ζ_k is not in the closure of R_n, then the sequence of measures $\{\mu_{nk}, k \geqq 1\}$ converges ($k \to \infty$) in the usual weak sense. If the limit distribution is μ_n, then

$$\mu_n(R'_n) = \mu_{nk}(R'_n) = 1,$$

and it is clear that the sequence $\{\mu_n, n \geqq 1\}$ is an absolute probability system for an h-path walk $\{z^h_n, n \geqq 1\}$. We prove that almost all paths of this walk have initial point ζ as follows.

We note first that

$$\lim_{k \to \infty} \frac{g(\zeta_k, \eta)}{g(\zeta_k, \xi_0)} = K_{\xi_0}(\zeta, \eta). \tag{12.1}$$

Secondly, since $\dfrac{g(., \eta)}{g(., \xi_0)}$ defines a lower semimartingnale on h-paths from ξ_k, under the conventions of lemma 4.1, the process remains a lower semimartingale if stopped and held fast when the paths meet R'_n, so that, if ζ_k is not in the closure of R_n, the lower semimartingale inequality yields

$$\frac{g(\zeta_k, \eta)}{g(\zeta_k, \xi_0)} \geqq \mathbf{E}\left\{\frac{g(z^h_{nk}, \eta)}{g(z^h_{nk}, \xi_0)}\right\}. \tag{12.2}$$

Hence

$$K_{\xi_0}(\zeta, \eta) \geqq \mathbf{E}\left\{\frac{g(z^h_n, \eta)}{g(z^h_n, \xi_0)}\right\}. \tag{12.3}$$

When $n \to \infty$ here we find

$$K_{\xi_0}(\zeta, \eta) \geqq \mathbf{E}\{K_{\xi_0}(\zeta^*, \eta)\}, \tag{12.4}$$

where ζ^* is the initial point of the h-path walk, a random variable. Now when $\eta = \xi_0$, the terms in this inequality are both 1, and both terms define harmonic functions of η. Hence the terms are identical. Since $K_{\xi_0}(\zeta, .)$

is a minimal function, this means that the distribution of ζ^* must be concentrated at the point ζ, as was to be proved.

THEOREM 12.1. — *Let u be a strictly positive superharmonic function on R, and let ξ_0 be a point of R. Then $\frac{u}{g(\xi_0, \cdot)}$ has the strictly positive limit*

$$\liminf_{\xi \to \zeta} \frac{u(\xi)}{g(\xi_0, \xi)} \quad (\leqq \infty)$$

along almost all $g(\xi_0, \cdot)$ paths (as defined above) back to ζ.

The existence and strict positivity of the limit along the given paths is assured by theorem 10.1. A zero-one law argument can be used to show that the limit is constant with probability 1, but rather than go on in this way, we shall wait and reduce the result to one due to NAÏM, which has a simple direct proof.

13. The probability of meeting a set. — If A is a compact subset of R, the probability $u_A(\xi)$ that an h-path from ξ will pass through a point of A, at a strictly positive parameter value, is well-defined. According to the remarks in section 5, $u_A = \frac{h_A}{h}$. In probability language, if τ_ξ is the infimum of the strictly positive times at which a Brownian path from ξ meets A, $h_A(\xi)$ is the expected value of h on the Brownian path at this time (taking the value of h involved to be o if there is no meeting at a strictly positive time). The function h_A is called by BRELOT the extremalization of h on R-A.

Let ξ_1, ξ_2 be points of R, and let A be a compact subset of R. Let $u_A(\xi_1, \xi_2)$ be the probability that a $g(\xi_2, \cdot)$-path from ξ_1 meets A at a strictly positive parameter value. Then the evaluation of u_A just described becomes

$$u_A(\xi_1, \xi_2) = \frac{U_A(\xi_1, \xi_2)}{g(\xi_1, \xi_2)}. \tag{13.1}$$

Here the numerator on the right is, in the usual language of potential theory, the value at ξ_1 of the potential of the mass obtained by sweeping the unit mass at ξ_2 onto A. If, instead of describing these paths as $g(\xi_2, \cdot)$-paths, we describe them as $K_{\xi_1}(\xi_2, \cdot)$-paths, nothing is changed, but this definition now is applicable, and we adopt it, even if ξ_2 is a point of R'.

Finally, if ξ_1 is a minimal point of R', and if $\xi_2 \in R$, a $g(\xi_2, \cdot)$-path process from ξ_1 to ξ_2 can be defined, by filling in the $g(\xi_2, \cdot)$-path walk from ξ_1 derived in section 12, and we define $u_A(\xi_1, \xi_2)$ as the pobability that a $g(\xi_2, \cdot)$-path from ξ_1 meets a point of A. The following theorem will show, among other things, that $u_A(\xi_1, \xi_2)$ as so defined depends only on A, ξ_1, ξ_2. This is by no means obvious, since there was no obvious uniqueness in our definition of the process from ξ_1 to ξ_2. We stress that all results

obtained for our rather artificial h-path process from a minimal boundary point to an interior point hold also, with no change in proof, for an ordinary h-path process from the first to the second point, if there is such a process.

THEOREM 13.1. — *If $\xi_1 \in R$, and if ξ_2 is either a point of R or a minimal boundary point, then*

$$u_A(\xi_1, \xi_2) = u_A(\xi_2, \xi_1). \tag{13.2}$$

If ξ_1 and ξ_2 are in R, it is well-known that U_A is a symmetric function of its variables. From the probability point of view, the symmetry is obvious from the following evaluation of U_A, in this case. Let $z(\xi)$ be the location of a Brownian path from ξ at the infimum of the strictly positive times at which the path meets A. Then $z(\xi)$ is undefined if the path never hits A. It is easily seen that U_A is given by

$$U_A(\xi_1, \xi_2) = \mathbf{E}\{g[z(\xi_1), z(\xi_2)]\}, \tag{13.3}$$

where $z(\xi_1)$ and $z(\xi_2)$ are obtained from Brownian processes, from ξ_1 and ξ_2 respectively, which are independent of each other, and g is interpreted as o if either argument is undefined, There remains the case when $\xi_2 = \zeta$ is a minimal boundary point. We shall use the notation introduced in section 12 in discussing the $g(\xi_1, .)$-paths from ζ. Let n be so large that $A \subset R_n$. Then

$$u_A(\zeta, \xi_1) = \int_{R_n'} u_A(\xi, \xi_1)\, \mu_n(d\xi), \tag{13.4}$$

and, for large k,

$$u_A(\zeta_k, \xi_1) = \int_{R_n'} u_A(\xi, \xi_1)\, \mu_{nk}(d\xi). \tag{13.5}$$

When $k \to \infty$ we find that, since $\mu_{nk} \to \mu_n$ weakly,

$$\lim_{k \to \infty} u_A(\zeta_k, \xi_1) = u_A(\zeta, \xi_1). \tag{13.6}$$

Now we can write $u_A(\zeta_k, \xi_1)$ in the form

$$u_A(\zeta_k, \xi_1) = \frac{U_A(\zeta_k, \xi_1)\, g(\zeta_k, \xi_0)}{g(\zeta_k, \xi_0)\, g(\zeta_k, \xi_1)}, \tag{13.7}$$

Define $h = K_{\xi_0}(\zeta, .)$. When $k \to \infty$ in (13.7), we see from (13.4) and (13.7) that the limit on the left, $u_A(\zeta, \xi_1)$, defines, for fixed ζ, an h-superharmonic function of ξ_1, h-harmonic on R-A. The function $u_A(\zeta, .)$ on R-A has, using (13.4), the boundary limits 1 on A and o on R', along almost all h-paths from a point of R-A. Since $u_A(., \zeta)$ has exactly these same properties on R-A, these two bounded h-harmonic functions on R-A are equal, as PWB^h Dirichlet solutions with the same h-fine boundary

function. Thus

$$u_A(\zeta, \xi) = u_A(\xi, \zeta) \tag{13.8}$$

if $\xi \in R\text{-}A$. If $\xi \in A$, let A_n be A less a spherical neighborhood of A of diameter $\frac{1}{n}$. Then (13.8) is true if A is replaced by A_n. When $n \to \infty$ we find that (13.8) is true for all ξ in R, as was to be proved.

Following Hunt's application [10] of Choquet's capacity theory, one sees immediately that our definitions of $u_A(\xi)$ and $u_A(\xi_1, \xi_2)$ are applicable, and theorem 13.1 is valid, for every analytic subset A of R.

Our method of defining $u_A(\zeta, \xi_2)$, for ζ a minimal boundary point and ξ_2 a point of R has produced the uniquely defined number $u_A(\xi_2, \zeta)$ in spite of the fact that the method seemed to lack uniqueness. The point is that in (13.6) the right side is now seen to be uniquely defined, regardless of the choice of sequence $\{\zeta_k, k \geqq 1\}$ on the left. This means that

$$\lim_{\eta \to \zeta} u_A(\eta, \xi_2) = u_A(\xi_2, \zeta). \tag{13.9}$$

In other words the function $\frac{U_A(\cdot, \xi_2)}{g(\cdot, \xi_2)}$ has the right side of (13.9) as its ordinary limit at ζ. The existence of this limit, which implies the independence of our final result of the choice of $\{\zeta_k, k \geqq 1\}$, is easy to prove directly.

14. **The fine topology.** — Let ξ be a point of R, and let A be a subset of R. Then Brelot's concept of A being thin (effilé) at ξ has been given the following probabilistic interpretation [4]. If A is closed or the union of a sequence of closed sets, either almost every or almost no Brownian path from ξ meets A for arbitrarily small strictly positive parameter values. In the second case, and only in that case, A is thin at ξ. Following Hunt again, the result remains correct if A is only analytic. The following theorem generalizes this probabilistic interpretation on the one hand by allowing more general paths, and on the other hand by allowing ξ to be a minimal boundary point. The definition of thinness at a boundary point is due to NAÏM [11].

THEOREM 14.1. — *Let A be an analytic subset of R. Let ξ be a point of R, and if so let h be any strictly positive superharmonic function on R, or let ξ be a minimal boundary point, and if so let $h = g(\eta, .)$, for any η in R.*

(a) Either almost every h-path from ξ meets A at arbitrarily small strictly positive parameter values or almost none does.

(b) If the second possibility in (a) holds for some choice of h, it holds for every choice, and A is thin at ξ if and only if this is true.

This theorem is slightly unbalanced, because of the greater choice of h

when ξ lies in R than when this point is a minimal boundary point. It will be clear from the discussion that in the latter case h could also have been taken as any strictly positive superharmonic function for which there is an h-path process from ξ, but the choice described in the theorem is the only one which we know assures the existence of the desired h-paths, even though only in the distorted form obtained in section 12.

Before proving theorem 14.1, we state its dual.

THEOREM 14.2. — *Let A be an analytic subset of R. Let ξ be a point of R, and if so let $h = g(\xi, .)$, or let ξ be a minimal boundary point, and if so let $h = K_{\xi_0}(\xi, .)$.*

(a) Either almost every h-path from a point of R or minimal point of R' meets A arbitrarily near ξ or almost none does.

(b) If the second possibility in (a) holds for one choice of h and initial point of paths, it holds for all choices, and A is thin at ξ if and only if this is true.

It is clear that there is no real increase of generality obtainable by allowing a general initial distribution of h-paths, or, dually, by choosing h to allow h-paths to have endpoint ξ with strictly positive probability less than 1. In (a), if $\xi \in R$, we are only assured of the existence of the indicated h-path process if the initial point is a point of R other than ξ, or a minimal boundary point, whereas if ξ is a minimal boundary point, we are only assured of the existence of the indicated h-path process if the initial point is a point of R.

In view of our symmetry theorem, theorem 13.1, and of our analysis of the structure of h-paths, it will be sufficient to prove only one of the two preceding theorems, and we shall find it convenient to prove the second. To prove it, it is sufficient to treat only the case when ξ is a minimal boundary point, because if ξ is a point of R it is a minimal boundary point of $R - \{\xi\}$. Suppose then that ξ is a minimal boundary point, and let $\nu_A(\eta)$ be the probability that an h-path from η meets A arbitrarily near ξ, for $h = K_{\xi_0}(\xi, .)$. Then ν_A is a bounded h-harmonic function, and, as such, is a constant function, since h is minimal [7]. Moreover, by familiar reasoning, ν_A has the limit 1 on almost every h-path (from any point of R) which meets A arbitrarily near ξ, the limit 0 on almost every other h-path from the point. Hence ν_A is identically 1 or identically 0. Thus theorem 14.2 (a) and the first part of (b) are true in the case considered. Finally, according to NAÏM [11], A is thin at ξ if and only if h_{AG} can be made arbitrarily small at any specified point η by choosing a sufficiently small neighborhood G of ξ. The theorem follows from the fact that $\frac{h_{AG}(\eta)}{h(\eta)}$ is the probability that an h-path from η ever meets AG, so that this ratio decreases to $\nu_A(\eta)$ as G shrinks to ξ.

The fine topology on R is defined as the least fine topology in terms of which superharmonic functions on R are continuous. Equivalently it is the topology in which the neighborhoods of a point are the complements of the sets thin at the point. The latter definition has been used by NAÏM [11] to obtain the fine topology on $R \cup R'$. The points of R' which are fine limit points of R are the minimal points. If ξ is a point of R or a minimal boundary point, and if G is a fine neighborhood of ξ, there is a smaller fine neighborhood G_0 of ξ which is a closed set in the Martin topology. The set R-G_0 is thin at ξ, so almost every conditional Brownian path from a point of R to ξ lies entirely in G_0, and so in G, sufficiently near ξ, according to theorem 14.2. The corresponding statement holds for paths with initial point ξ.

If ξ is as in the preceding paragraph, we shall say that a function u, defined on a set having ξ as a fine limit point, has the fine limit b at ξ, written $F\lim_{\eta \to \xi} u(\eta) = b$, if u has this limit at ξ in terms of approach to ξ in the fine topology. NAÏM proved that, with this definition, u has the fine limit b at ξ if and only if there is a subset B of the domain of definition of u, such that the domain less B is thin at ξ, and that u considered only on B has the limit b at ξ, in the usual sense. This fact, combined with the remark in the preceding paragraph, yields the following theorem.

THEOREM 14.3. — *Let ξ be a point of R or a minimal boundary point, and let u be a Baire function defined on a subset of R having ξ as a fine limit point. Then u has the fine limit b at ξ if and only if it has the limit b along almost every conditional Brownian path from a point of R (and if so, from every point of R) to ξ. (Corresponding statement for paths from ξ.)*

Note that, on almost every path indicated, there will be points of the domain of u arbitrarily near ξ, and the theorem states the condition that u approaches b along these points, for almost every path.

According to the criterion of this theorem, a Baire function u, defined on R, has the function f, defined on R', as h-fine boundary function, in the sense of section 8, if and only if u has $f(\xi)$ as a fine limit at ξ, for almost every (h-harmonic measure) minimal point ξ of R'. This fact explains the term " h-fine boundary function ".

15. Some fine limit theorems. — Let h be a positive superharmonic function, given by a canonical measure μ^h,

$$h(\eta) = \int_{R \cup R'} K_{\xi_0}(\xi, \eta)\, \mu^h(d\xi), \qquad h(\xi_0) < \infty. \tag{15.1}$$

Then we have the decomposition $h = h_{\mathrm{I}} + h_{\mathrm{II}}$, where h_{I} is the potential

obtained by replacing μ^h by its restriction to R and h_{II} is the harmonic function obtained by replacing μ^h by its restriction to R'. The function h_{II} is the greatest harmonic minorant of h.

Let u, h be positive and superharmonic on R, with h strictly positive. We shall suppose from now on that $\frac{u}{h}$ is defined, even at an infinity of h or u, as the fine limit of the ratio at the infinity, if the fine limit exists.

Since u and h are both continuous in the fine topology, $\frac{u}{h}$ is thereby defined as o at an infinity of h but not of u, ∞ at an infinity of u but not of h. There remains the set of common infinities. We have already interpreted theorem 4.2 as a generalized Fatou boundary limit theorem, to mean that, if h is harmonic, $\frac{u}{h}$ has an h-fine boundary function. The exact same reasoning, which there is no need to repeat, now yields the fact that, *if u and h are positive superharmonic functions, with h strictly positive, $\frac{u}{h}$ has an h_{II} fine boundary function, and $\frac{u}{h}$ has a finite fine limit at almost all points of R (μ^h measure)*. In other words $\frac{u}{h}$ is defined almost everywhere on R (μ^h measure). This is a kind of internal Fatou theorem! Note that, although almost no h-path from a point of R passes through an infinity of h, almost every such path may have an infinity of h as its endpoint. This is true, for example, if $h = g(\xi, .)$.

NAÏM [11] proved that, if u is strictly positive and superharmonic on R, if ξ_0 is an arbitrary point of R, and if η is a minimal boundary point, then

$$F\lim_{\xi \to \eta} \frac{u(\xi)}{g(\xi_0, \xi)} = \liminf_{\xi \to \eta} \frac{u(\xi)}{g(\xi_0, \xi)}. \tag{15.2}$$

(The quantity on the right is strictly positive but may be infinite.) According to our probabilistic interpretation of fine limits, this result is equivalent to theorem 12.1 (but we recall that we did not actually evaluate the limit in the discussion of that theorem).

Theorem 4.2 can be considered a dual form of theorem 12.1. Its interpretation in terms of fine limits has already been discussed. In particular, when the reference function h is minimal harmonic, the fine limit statement of the theorem is due to NAÏM [11].

Our probability approach to potential theory allows the application of a Fubini theorem argument which has as yet no analogue in non-probabilistic potential theory. One such application led to the conclusion, not yet obtained by non-probabilistic methods, that every positive h-superharmonic function has an h_{II}-fine boundary function. A second such application leads to the following theorem.

THEOREM 15.1. — *Let u be a positive superharmonic function on R, let ξ_0 be a point of R, and let h be a strictly positive superharmonic function on R. Then $\frac{u}{g}(\xi_0, .)$ has a strictly positive limit ($\leq \infty$) on almost every h-path from any point of finiteness of h.*

If h is a minimal harmonic function, this result is equivalent to the existence of the limit in (15.2). If h is not minimal, the result follows from our analysis of h-paths. This result is really only speciously more general for h superharmonic than for h harmonic.

BIBLIOGRAPHIE.

[1] R. BLUMENTHAL, *An extended Markov property* (*Trans. Amer. math. Soc.*, t. 85, 1957, p. 52-72).

[2] M. BRELOT, *Le problème de Dirichlet. Axiomatique et frontière de Martin* (*J. Math. pures et appl.*, 9e série, t. 35, 1956, p. 297-335).

[3] M. BRELOT et G. CHOQUET, *Espaces et lignes de Green* (*Ann. Inst. Fourier*, Grenoble, t. 3, 1951-1952, p. 199-263),

[4] J. L. DOOB, *Semimartingales and subharmonic functions* (*Trans. Amer. math. Soc.*, t. 77, 1954, p. 86-121).

[5] J. L. DOOB, *Probability methods applied to the first boundary value problem* [*Proc. Third Berkeley Symp. math. Stat. and Prob.*, t. 2, 1956, p. 49-80 (1955, Berkeley, Calif.) Berkeley et Los Angeles, University of California Press].

[6] J. L. DOOB, *Brownian motion on a Green space* (*Teoriya Veroyatnosti*, t. 2, 1957, p. 3-33).

[7] J. L. DOOB, *Probability theory and the first boundary value problem* (*Illinois J. Math.* t. 2, 1958, p. 19-36).

[8] J. L. DOOB, *Boundary limit theorems for a half-space* [*J. Math. pures et appl.* (to appear)].

[9] B. V. GNEDENKO and A. N. KOLMOGOROV, *Limit distributions for sums of independent random variables* (in russian), Moscow, 1949, (in translation) Cambridge, Mass, 1954.

[10] G. A. HUNT, *Markoff processes and potentials* I (*Illinois J. Math.*, t. 1, 1957, p. 44-93).

[11] L. NAÏM, *Sur le rôle de la frontière de R. S. Martin dans la théorie du potentiel*, Chartres, Durand, 1957 (*Thèse Sc. math.*, Paris, 1957).

(Manuscrit reçu le 7 octobre 1957.)