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INDUCED EXPANSION FOR QUADRATIC POLYNOMIALS

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ABSTRACT. – We prove that non-hyperbolic non-renormalizable quadratic polynomials are expansion inducing. For renormalizable polynomials a counterpart of this statement is that in the case of unbounded combinatorics renormalized mappings become almost quadratic. The reason for both results is in the properties of "box mappings". This class of dynamical systems is systematically studied and the decay of the box geometry is the reason for both results. Specific estimates of the rate of this decay are shown which are sharp in a class of S-unimodal mappings combinatorially related to rotations of bounded type. For real box mappings we use known methods based on cross-ratios and Schwarzian derivative. To study holomorphic box mapping we introduce a new type of estimates in terms of moduli of certain annuli which control the box geometry.

RÉSUMÉ. – Nous démontrons que les polynômes non-hyperboliques, non-renormalisables ont la propriété des induites dilatantes. Pour les polynômes renormalisables le résultat correspondant est que dans le cas d'une combinatoire non-bornée, les transformations renormalisables deviennent presque quadratiques. Ces deux résultats se déduisent des propriétés de « box mappings ». Cette catégorie de systèmes dynamiques est étudiée et la décroissance progressive de la taille de boîtes est la raison de ces deux résultats. Les estimations détaillées du taux de décroissance sont présentées dans cet article. Ces estimations sont précises pour la catégorie de transformations S-unimodales liées aux rotations de type bornées. Pour les transformations réelles nous appliquons les méthodes bien connues basées sur les techniques de birapport et dérivée schwarzianne. Afin d'étudier les « box mappings » qui sont holomorphes nous introduisons un nouveau type d'estimations en fonction de module de certains anneaux qui contrôle la taille de boîtes.

Part I

Introduction

1. Main Results

1.1. Overview

In recent years, rather dramatic progress occurred in the study of real quadratic polynomials, or, more broadly, S-unimodal mappings with a singularity of quadratic type. New results include better understanding of measurable dynamics and a proof of the monotonicity conjecture in the real quadratic family.

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This progress was partly based on new estimates. The main breakthrough, however, was in achieving a better understanding of the rich dynamics of unimodal mappings in conjunction with their geometry, and thus being able to apply appropriate tools in different cases. Most of the progress in this direction seems to be due to the application of the idea of *inducing*. The first application of inducing to the study of unimodal maps was in the work [9]. In that work useful geometrical and analytic estimates were obtained only in special cases. Another notable step was the work of [7]. An attempt was made there to handle all cases, though some patterns emerged as analytically unmanageable. Another work which ought to be noted is [16] where a somewhat simpler inducing construction was developed leading to many conclusions, in part the equivalence between the non-existence of weird Cantor attractors and induced expansion for non-renormalizable maps of quadratic type. For the authors, the inducing construction of [10], became the underlying approach of the present paper. Here a complete topological model of unimodal dynamics was obtained. Another achievement of [10] was the discovery of the phenomenon of *decaying box geometry*. It was observed that the occurence of the decay of box geometry leads to induced hyperbolicity. An estimate called the *starting condition* was provided which, if satisfied, allowed one to prove the decaying geometry in general for S-unimodal maps. The starting condition was shown to hold on a robust set mappings in the quadratic family, but not all. From this point of view, our work on is a completion of [10] and our achievement is showing that the starting condition always holds. Moreover, we extend the method to some renormalizable maps.

It should be noted that about the same time a similar case of decaying geometry was observed independently by [24] and [4] for circle mappings with a flat piece. It is not known whether there is more than an analogy between both cases.

The direction for getting rid of the starting condition was shown by [23], *see* [17] for a description. The work was done for non-renormalizable quadratic polynomials, complex as well as real. Inducing was not directly mentioned, but implicitly present in the construction of a Markov partition. It is believed that the approaches of [10] and [23] give equivalent sequences of partitions for real polynomials. In [23] a "hard case" emerged (and was called *persistently recurrent*). For real polynomials, the persistently recurrent case contained some examples which the approach of [10] could only treat under the assumption of the starting condition. The estimates of [23] were done by watching how pieces of the partition nest in one another thus defining certain annuli. Then a computation involving the moduli of these snnuli, quite similar to one done in [2], implied that the sequence of partitions was converging to the partition into points. The work of Yoccoz was not directly concerned with the box geometry and it was not clear for a while that the decaying geometry was implied. However, the idea of watching the annuli was another important inspiration for our paper.

In the real situation, the proof of decaying box geometry was obtained for the so-called *Fibonacci polynomial*. This map was proposed in [8] as an interesting example to study. From the point of view of [10], the Fibonacci polynomial required the starting condition to be assumed in order to prove the decay of geometry and it was persistently recurrent in the sense of [23]. [15] was an important step since it solved one of the "hard examples".

The proof was obtained using ideas of holomorphic dynamics, and a conjugacy with cubic polynomials was invoked. The same approach based on piecewise linear models and conjugation with higher degree polynomials was further applied in [14]. This work demonstrated decaying box geometry in all non-renormalizable quadratic polynomials. The main result of [14] was non-existence of non-Feigenbaum type Cantor attractors.

However, we give a completely different proof of the decaying geometry. The main draw-back of the approach of [15] or [14] to proving the decaying geometry was that it only applied to non-renormalizable maps. On the other hand, the renormalizable case emerged as very important in the attempts to construct a quasiconformal conjugacy between any two topologically conjugate infinitely renormalizable polynomials. After the achievement of [23], this was a missing link in the proof that hyperbolic polynomials are dense in real quadratic family. The decay of box geometry was an important premise of the first proof of this in [21]. In that paper the proof of decaying geometry was extremely involved. However, a certain important step of [21], the construction of complex box mappings by inducing, is followed in our paper. The difficulty of the proof by the method of [21] made us a look for a more robust argument more in the spirit of [23]. We show this proof here as Theorem C. Theorem C states the decay of box geometry in an abstract class of "box mappings". Box mappings appear naturally in inducing. Theorem C works without a distinction in non-renormalizable and renormalizable cases. The proof is based on considering the moduli of certain annuli. In this, it is quite similar to [23] or [2]. However, our estimates are stronger and imply the decay of box geometry. Our Theorems 1 and 2 are applications of Theorem C.

The proof of Theorem C proceeds most of the time by considerations of holomorphic dynamics. However, here a difficult case emerges (called "rotation-like") in which we need to apply real estimates to get the starting condition. We provide a proof in all rotation-like cases (in the case of the Fibonacci pattern this result is implicit in [12]). This gives us the decay of box geometry in a class of real box mappings (only negative Schwarzian derivative needs to be assumed.) In certain situations, these estimates are sharp. This is the content of our Theorems A and B.

In spite of connections and partial overlaps with so many works, the paper is self-contained. No technical references are made to any of the recent works quoted above. We thought that this was a reasonable trade-off for the paper's large size.

Acknowledgements

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1.2. Statement of results

Definition of our class of mappings

DEFINITION 1.1. – *A mapping f of the interval $[-1, 1]$ into itself is called unimodal provided that:*

$f(-1) = -1$

• *the mapping f can be written as $h(x^2)$ where h is an orientation-reversing diffeomorphism from $[0, 1]$ onto its image $[-1, a]$ with $0 < a < 1$.*

We can classify unimodal mappings according to the smoothness of h. So, we get the following definition.

DEFINITION 1.2. – *Let $\eta > 0$. We define a class $\mathcal{F}_\eta$ of unimodal mappings by writing $f(x) = h(x^2)$ (compare Definition 1.1) and imposing the following conditions on h.*

• *h^{-1} has an analytic continuation to a univalent mapping from the upper half-plane into the upper half-plane,*

• *h has an analytic continuation to some open interval $U \supset [0, 1]$ as a diffeomorphism onto $(-1 - \eta, 1 + \eta)$.*

We also define

$$\mathcal{F} := \bigcup_{\eta > 0} \mathcal{F}_\eta .$$

It should be observed that Definition 1.2 implies that the Schwarzian derivative of h continued to U is non-positive. This observation is due to [20]. Indeed, look at any four points $a < b < c < d$ in the image of the continuation of h. Let A, B, C, D denote the respective preimages in U. Then h^{-1} can be continued to a univalent mapping from

$$G = (\mathbf{C} \setminus \mathbf{R}) \cup (a, d).$$

Compose this with a Möbius map which sends A to a, D to d and B to b. This composition maps G into itself, and hence it does not expand the Poincaré metric on G. Since (a, d) is a Poincaré geodesic, then c has to move towards b. Since points a, b, c, d were arbitrary, this means that the Scharzian derivative of the composition, hence of h^{-1} itself, is non-negative.

Theorem about non-renormalizable mappings

THEOREM 1. – *Let $f \in \mathcal{F}_\eta$ be non-renormalizable. Then on an open, dense and having full measure subset of the fundamental inducing domain one can define a continuous function $t(x)$ with values in positive integers so that $f^{t(x)}$ is an expanding Markov mapping. That is, restricted to a maximal interval on which $t(x)$ is defined an constant, $f^{t(x)}$ is a diffeomorphism onto $(1 - q, q)$, expanding, and with distortion (measured as the variation of the logarithm of the Jacobian) bounded by depending on η only. Here, q is a fixed point.*

Theorem 1 has a number of consequences (*see* [10]). It gives an alternative proof of the non-existence of "exotic" attractors in class $\mathcal{F}$ (already known from [14]). It also gives an approach to constructing invariant measures.

Theorem in the renormalizable case

DEFINITION 1.3. – *Let $f \in \mathcal{F}$. A point x in the domain of f is called almost parabolic with period m and depth k provided that:*

- *the derivative of f^m at x is one,*
- *f^m is monotone between x and the critical point,*
- *k consecutive images $f^m(1/2), \ldots, f^{km}(1/2)$ are between x and $1/2$.*

THEOREM 2. – *Let $f \in \mathcal{F}_\eta$ be renormalizable, and let n be the return time of the maximal restrictive interval into itself. Denote by $k(n)$ the maximum of depths of almost parabolic points with periods less than n. Specify a number $D > 0$. For every given k, a number $N(\eta, D, k)$ exists independent of f so that if $n > N(\eta, D, k)$ and $k(n) \leq k$, then f^n on a neighborhood of $1/2$ is conjugate to a mapping from $\mathcal{F}_D$ by an affine transformation.*

Theorem 2 "almost complements" the theory of renormalizable mappings developed in [20]. In fact, it says that such a theory at least in some aspects is much simpler for renormalizable mappings of unbounded type. The exclusion of the unbounded case with almost parabolic returns is the only gap. Theorem 2 is a critical step in the proof of monotonicity in the real quadratic family, see [21] (where, by the way, the theorem is stated wrongly without excluding the almost parabolic case).

Technical theorems

The strongest results of our paper are contained in technical theorems A,B and C. They imply theorems 1 and 2. In addition, Theorem B gives exact bounds on the exponential rate of decay of box geometry for S-unimodal rotation-like maps. Theorem C concerns the decay of *complex* box geometry and can serve as an important step in proving the density of hyperbolicity in the real quadratic family, see [21]. The technical theorems are stated for objects that we call *box mappings*.

Plan of the work

The paper is divided into four parts. The first part discusses results and ideas of the work in general. The first section is introductory and will be completed by this description of the work. The second section introduces box mappings and inducing. The procedure of inducing is based on the algorithm of [10], but is somewhat different. The main formal difference is that we work in the abstract setting of "box mappings" which can be studied completely apart from unimodal dynamics. At the end we state our technical Theorems A, B and C.

Part II of the work deals with inducing on real box mappings in the so-called "rotation-like" sequences. The purpose is to prove the decay of box geometry without assuming the starting condition. The term "rotation-like" refers to a very particular combinatorial pattern of the dynamics. In the rotation-like case the decay of box geometry is particularly slow and not so easy to get with purely holomorphic methods. The advantage is a the combinatorial simplicity. First, we prove the decay of box geometry under a specific and rather weak "starting condition" (the box ratio less or equal to 0.37). Then, we prove that this starting condition is satisfied after a number of inducing steps depending only on the initial geometry.

Part III is probably the most important. We introduce the concept of separating annuli and show a quantity which is weakly increasing in the inducing process and whose growth implies the decay of box geometry. This is an important achievement, since the simplest measure of box geometry, the modulus of the nesting of the related quadratic-like mapping, does not increase monotonically. Then we have to work to show that this "separation index" actualy increases at a linear rate. By purely holomorphic methods we show that the separation index is strictly inreasing except in rotation-like sequences of bounded type. The proofs involve a lot of simple-minded estimates which use no more than the superaddivity and conformal invariance of moduli, though may be involved combinatorially. To see the actual increase of the separation index stronger tools are applied based on Techmüller's theorem that gives a "sharp" inequality for the superadditivity of moduli. Finally, in order to see the increase of separation in the remaining rotation-like case, results of the real-variable work in Part II are applied.

Part IV shows how the technical Theorems, and more precisely Theorem C proved in Part III, imply Theorems 1 and 2. The main step here is the construction of a holomorphic box mapping by inducing on a unimodal mapping $f \in \mathcal{F}$. If the mapping f is a polynomial, the problem is very easy because a box mapping can be obtained from Yoccoz partitions. But even for renormalizable polynomials, if one is interested in uniform bounds of the initial box geometry, the consideration of renormalized mappings leads to the class $\mathcal{F}_\eta$ and not just polynomials. The construction of box mapping is quite different from the method originally used in [21], as well as from the work of [14] done in non-renormalizable cases.

2. Induced Dynamics

2.1. Box mappings

Real box mappings

The method of inducing was applied to the study of unimodal maps first in [9], then in [7]. In [11] and [10] an elaborate approach was developed to study induced maps, that is, transformations defined to be iterations of the original unimodal map restricted to pieces of the domain. We define a more general and abstract notion in this work, namely *box mappings*. Box mappings can occur in two contexts, as transformations of the real line with certain smoothness and as holomorphic dynamical systems in the complex plane. We define the notion of a *real box mapping* first.

DEFINITION 2.1. – *Consider a transformation ϕ defined on an open subset U of the real line into the real line. Restrictions of ϕ to the connected components of U will be referred to as branches of ϕ. For ϕ to be a box mapping, it must satisfy the following assumptions:*

• $B = (-a, a)$ *for some* $a > 0$ *and* B *will be further called the central domain, while* $\psi := \phi_{|B}$ *receives the name of the central branch,*

• *B' is chosen to be the smallest interval symmetric with respect to* 0 *which contains the range of ψ and it is assumed that B' contains the closure of B,*

• *$\psi = h(x^2)$ where h is a diffeomorphism onto its image B' with non-positive Schwarzian derivative,*

• *all branches of ϕ different from the central one are diffeomorphisms onto their respective images and have non-positive Schwarzian derivative,*

• *if D is any connected component of U, then D is disjoint from the border of B',*

• *if V is the range of some monotone* (i.e. *non-central) branch of ϕ and D is a connected component of U, then $D \cap \partial V = \emptyset$.*

A real box mapping is shown on Figure 1.

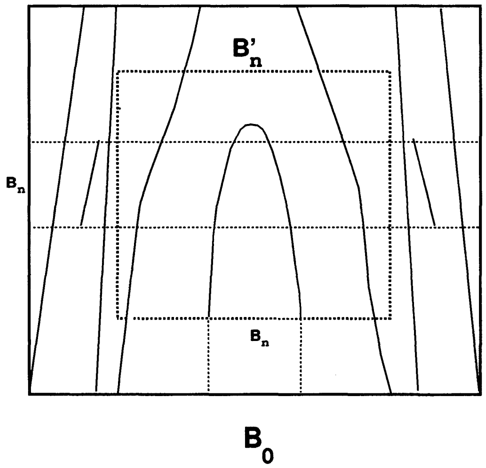

Fig. 1. – A sample graph of a box mapping. "Boxes" B_n and B'_n are shown. Be aware that typically a box mapping has infinitely many branches.

The definition of a *holomorphic* box mapping is the same conceptually and formally obtained by changing a few words:

DEFINITION 2.2. – *Take a holomorphic transformation ϕ defined on an open subset U of the complex plane into the complex plane. Restrictions of ϕ to the connected components of U will be referred to as branches of ϕ. We introduce the following assumptions and notations:*

• *there is certain open topological disc B which is a connected component of U and is mapped onto itself by the transformation $z \longrightarrow -z$; this B will be further called the central domain, while $\psi := \phi_{|B}$ receives the name of the central branch,*

• *B' is the range of ψ,*

• *$\psi = h(x^2)$ where h is univalent onto its image B',*

• *all branches of ϕ different from the central one are univalent onto their respective images,*

• *if D is any connected component of U, then D is disjoint from the border of B',*

• *if V is the range of some univalent* (i.e. *non-central) branch of ϕ and D is a connected component of U, then $D \cap \partial V = \emptyset$.*

We will use the expression *box mapping* where both real and holomorphic box mapping can be substituted.

Particular types of box mappings

We distinguish two special types of box mappings, both real and holomorphic. A *type I* box mapping is determined by the condition that all non-central branches have range B. A *type II* is chracterized by the property that all non-central branches have range B'. A complex box mapping of type I is shown on Figure 2.

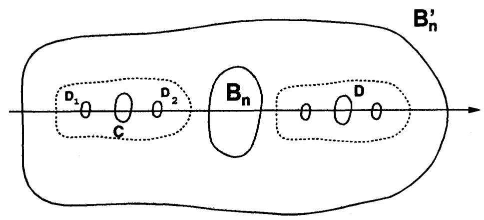

Fig. 2. – A type I complex box mapping. Dotted lines show domains of canonical extensions. Domains C and D look like they are maximal. Then D_1 and D_2 are subordinate to C, but apparently independent from one another as well as from D. C and D are also independent. There may be univalent domains outside of B'_n, not shown here.

DEFINITION 2.3. – *A box mapping is called terminal if there is an open interval $I \subset B$ containing the critical point of ϕ so that $\phi(I) \subset I$ and $\phi(\partial I) \subset \partial I$. The interval I (which must be unique) will then be called the restrictive interval of ϕ.*

2.2. Inducing algorithm

Step A - filling in

Suppose that ϕ is either a real box mapping in the meaning of Definition 2.1, or a holomorphic box mapping according to Definition 2.2. Choose a set S of monotone (univalent) branches of ϕ all with the same range R which must contain the closure of B.

Then define a sequence of box mappings ϕ_i as follows. ϕ_0 is equal to ϕ outside of B and the identity on B. S_0 is S. Given ϕ_i, $i \geq 0$, and a set S_i of monotone (univalent) branches construct ϕ_{i+1} as follows. Set $\phi_{i+1} = \phi_i$ except on the union of domains of branches of S_i, and $\phi_{i+1} = \phi_0 \circ \phi_i$ on the union of domains of branches from S_i. At the same time, S_{i+1} becomes the set of all branches of ϕ_{i+1} in the form $\zeta_1 \circ \zeta_2$ where ζ_1, ζ_2 belong to S_i.

The box mapping Φ which is the outcome of Step A is defined on the set of points z such that the sequence $\phi_i(z)$ is defined for all i and eventually constant. Then $\Phi(z) := \lim_{i \to \infty} \phi_i(z)$ if $z \notin B$ and $\Phi(z) := \phi(z)$ if $z \in B$.

When Φ is compared with ϕ, we see that all branches except those with range R have been left undisturbed, while those branches onto R have all vanished and been replaced with compositions among themselves and with other branches with different images.

Filling-in of a type II map

A typical example of filling-in occurs if ϕ is a type II box mapping. In that case there is only one possibility for R, namely $R = B'$ and the outcome is a type I (holomorphic) box mapping with the same B and B'. Each branch ζ of Φ is a restriction of

$$\overline{\zeta} = \zeta_n \circ \cdots \circ \zeta_1$$

where ζ_i are branches of ϕ. In that context, ζ_1 is called the *parent branch* of ζ, and the domain of ζ_1 is called the *parent domain*. Certainly, the domain of ζ is compactly contained in its parent domain. Notice also that $\overline{\zeta}$ naturally maps onto B' even though ζ by definition maps onto B. Hence, every monotone (univalent) of a type I (holomorphic) box mapping arising from a type II (holomorphic) box mapping has a monotone (univalent) *dynamical extension* onto B'. If we pick another branch of Φ, say η, it may be that $\overline{\eta} = \zeta_{n+k} \circ \cdots \circ \overline{\eta}$ or that $\overline{\eta} = \zeta_{n-k} \circ \cdots \circ \zeta_1$. In the first case, we say that η is *subordinate* to ζ, in the second case ζ is *subordinate* to η, and in the remaining case will say that they are *independent*.

Step B - critical filling

Suppose that a box mapping ϕ is given. For Step B to be feasible, the critical value of ϕ has to belong to the domain of ϕ. Construct ϕ_0 by changing ϕ on the central domain only, and making it the identity there. Then define Φ again by changing ϕ on the central domain only, where we set $\Phi = \phi_0 \circ \phi$. This Φ is the outcome of the Step B applied to ϕ.

Observe that for Φ the range B' is the central domain of ϕ. The central branch of Φ has the form $\zeta \circ \psi$ where ψ is the central branch of ϕ and ζ is either a monotone (univalent) branch of ϕ, or the identity restricted to B.

Again, the particular case of most interest to us is when ϕ is a type I box mapping. In that case, Φ is a type II box mapping. According to whether the critical value of ϕ is in the central domain of ϕ or not, we describe the situation as either a *close* or a *non-close return*.

Inducing steps for type I box mappings

We will now define a *simple inducing step* for type I box mappings. If ϕ is such a mapping, the simple inducing step is defined to be Step B followed by Step A. As remarked above, the outcome will be a type I box mapping. We make a distinction between a *close* and *non-close* return for ϕ, depending on whether the critical value of ϕ is in the central domain of ϕ. The simple inducing step is defined provided that Step B is defined, *i.e.* the critical value of ϕ is in the domain of ϕ.

Now we define the *type I inducing step*. It takes a type I box mapping ϕ. The type I inducing step is defined recursively so that it is equal to the simple inducing step if ϕ makes a non-close return, and is equal to the type I inducing step applied to ϕ_1 obtained by the simple inducing step for ϕ otherwise. In other words, the type I inducing step is an iteration of simple inducing steps continued until the first non-close return occurs. This definition may fail if at some point the simple inducing step is no longer defined, or, more interestingly, if a non-close return is never achieved. If the last possibility occurs for a box mapping ϕ, and $\phi(0) \neq 0$, then ϕ is terminal in the sense of Definition 2.3. Indeed, let ψ denote the central branch of ϕ, and B its central domain. If a non-close return never occurs, then the critical value must be contained in $\psi^{-n}(B)$ for any $n \geq 0$. These intervals form a descending sequence, and the intersection must be more than a point, since otherwise 0 would be fixed by ψ. So the intersection is an non-degenerate interval symmetric with respect to 0 and invariant under ψ which meets the criterion of Definition 2.3.

The construction of the type I inducing step in the case of a close return is shown on Figure 3. In this case the type I inducing step consists of two simple inducing steps. The picture shows the mapping $\tilde{\phi}$ obtained after the first simple inducing step.

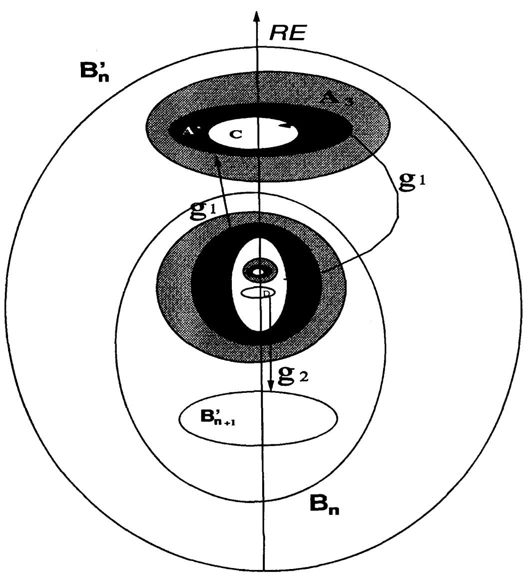

Fig. 3. – The mapping $\tilde{\phi}$ after one simple inducing step if the escape time is 2. Just one univalent domain of ϕ, called C and filled in white is shown. Similarly colored regions correspond by the dynamics indicated with arrows. A' and A_3 are separating annuli for B. g_1 denotes ψ. g_2 means $\phi_{|B} \circ \psi$. D is a maximal univalent domain of $\tilde{\phi}$. Another domain of $\tilde{\varphi}$ shown on the picture is the extremely small white disk a bit above D. Their common parent domain is $g_1^{-1}(B)$, *i.e.* the white-filled ellipse containing D.

Immediate preimages

If a type I box mapping Φ was constructed in a simple inducing step from another type I box map ϕ, then its *immediate* or *primary branches* are those univalent branches which are restrictions of the central branch of ϕ. Note that for real box mappings there may be none or two immediate preimages, depending on whether the real range of the central branch covers the critical point. In the case of a close return the immediate preimages are formed at the last stage of inducing on $\tilde{\phi}$. The well-known "Fibonacci case" is an example when all monotone branches occurring in the construction are immediate.

2.3. Geometry of box mappings

Rotation-like returns

Let ϕ be a type I box mapping obtained in a simple inducing step. Then its immediate branches are defined as in the previous paragraph.

DEFINITION 2.4. – *We say that ϕ exhibits a rotation-like return in the following situation. When the return is not close, it is rotation-like if and only if the critical value lands in an immediate branch of ϕ. When the return is close so that the k consecutive images of the critical point stay in the central domain, then consider the first exit $c = \phi^{k+1}(0)$. The return is rotation-like if c belongs to the domain of an immediate branch of ϕ and k consecutive images of $\phi(c)$ stay in the central domain.*

In other words, if the maximal number of consecutive images of 0 that stay in the central domain is k, then the return is rotation-like exactly when the central branch of the mapping obtained by a type I inducing step is $\zeta \circ \psi^{k+1}$ where ζ is an immediate branch of ϕ and ψ is the central branch.

Rotation-like sequences

We will say that a sequence of type I box mappings is *rotation-like* if each arises from the previous one by a standard inducing step, and each return, with the exception of the first one for which immediate preimages are not defined, is rotation-like.

Fact 2.1.

For a rotation-like sequence, there is an inductive formula relating consecutive central branches of the inducing procedure

$$f_{n+1} = f_{n-1} \circ f_n^{a_n},$$

where f_j denotes the central branch of ϕ_j, and a_n is a the smallest i such that the i-th iterate of the central branch of the n-th mapping maps the critical point outside of the central domain. E.g., $a_n = 1$ is equivalent to saying that the n-th map shows a non-close return. We define a_0 by a requirement that $f_2 = f^{a_0} \circ f_1^{a_1}$.

Following the analogy with the circle homeomorphisms we will introduce a concept of a rotation number for our class of maps.

DEFINITION 2.5. – *The rotation number $\rho(f)$ of ϕ is equal to*

$$\rho_n = \frac{1}{a_1 + \frac{1}{a_2 + \cdots}}$$

which can be written shortly as $[a_0, a_1, \cdots]$ using the formalism of continued fractions.

Theorem A about real box mappings

We state the theorem as follows.

THEOREM A. – *Let $\phi = \phi_0$ be a type I real box mapping and let ϕ_i be a rotation-like sequence derived from ϕ. Let B_i be the central domain of ϕ_i with $B'_i := B'$ as given by Definition 2.1 applies to $\phi := \phi_i$. Suppose that the initial ratio of lengths satisfies*

$$\frac{|B_0|}{|B'_0|} \leq 1 - \epsilon.$$

Specify a $\delta > 0$.

Then, for every $\epsilon > 0$ and $\delta > 0$ there is a number K depending on ϵ and δ only and independent of ϕ, with the property that for every $i \geq K$ we get

$$\frac{|B_i|}{|B'_i|} \leq \delta.$$

The proof of Theorem A uses purely real methods. It generalizes the result of [12]. A natural question is whether an analogous result can be demonstrated by real methods for an arbitrary induced sequence. In principle, that should be possible, but technical difficulties are daunting.

Sharp estimates for rotation-like mappings

THEOREM B. – *For any S-unimodal rotation-like map there exist positive constants K_1, K_2 and $\kappa_1, \kappa_2 < 1$ depending only on the initial geometry,* i.e. *the number ϵ of Theorem A, so that*

$$K_1 \kappa_1^{a_1 \cdot \ldots \cdot a_{n-1}} \leq \tau_n \leq K_2 \kappa_2^{a_1 \cdot \ldots \cdot a_{n-1}}$$

This is an improvement of Theorem A which also gives the lower bound on the rate of decay of box geometry. The proof is by purely real methods.

Growth of conformal moduli

THEOREM C. – *Let ϕ be a type II holomorphic box mapping, ϕ_0 be the type I mapping obtained from ϕ by filling-in and ϕ_i form a sequence, finite or not, of holomorphic box mappings set up so that ϕ_{i+1} is derived from ϕ_i by the type I inducing step for $i \geq 0$. Suppose that ϕ restricted to the real line is a real type II box mapping as given by Definition 2.1. Let B_i and B'_i denote the B and B', respectively, specified by Definition 2.2 applied to ϕ_i.*

Suppose that mod $(B'_0 \setminus B_0) \geq \beta_0$. For every $\beta_0 > 0$ there is a number $C > 0$ with the property that for every i

$$\text{mod } (B'_i \setminus B_i) \geq C \cdot i.$$

Theorem C claims a decay of the conformal geometry in a sequence of holomorphic box mappings derived by inducing. This phenomenon seems to be the basis of some recently obtained results, *see* [21].

Part II

Rotation-Like Sequences

3. Real induction

In this section we prove Theorem A. Suppose that in the situation of Theorem A a rotation-like sequence ϕ_n is given, $n = 0, 1, \cdots$. Thus, every ϕ_n makes a rotation-like return and ϕ_{n+1} is obtained from ϕ_n by a type I inducing step. Let a_n denote the number of iterations of the central branch after which the critical value escapes from the central domain. So, $a_n > 0$ and $a_n = 1$ exactly if ϕ_n makes a non-close return.

3.1. Rotation-like sequences

Notations

The central branch of ϕ_n will be called f_n. Then B_n is the central domain of ϕ_n and B'_n is the interval B' from Definition 2.1 applied to ϕ_n. Denote the endpoints of the box B'_n by s^-_{n-1} and s^+_{n-1}. We adopt the following convention of ascribing signs to points: '$+$' written as a superscript indicates the endpoint of B'_n which lies closer to the critical value of f_n. In other words $f_n(0) \in (0, s^+_{n-1})$. Also, denote by z^+_n and z^-_n the endpoints of B_n, defined so that z^+_n and s^+_n are on the same side of 0. These notations are explained on Figure 4. Each central branch can be represented as a composition of a diffeomorphism h_n and a quadratic map g. We know that the image of B_n is contained in B'_n. Generally, we will use (x, y) to denote the interval from x to y, regardless of the ordering of x and y.

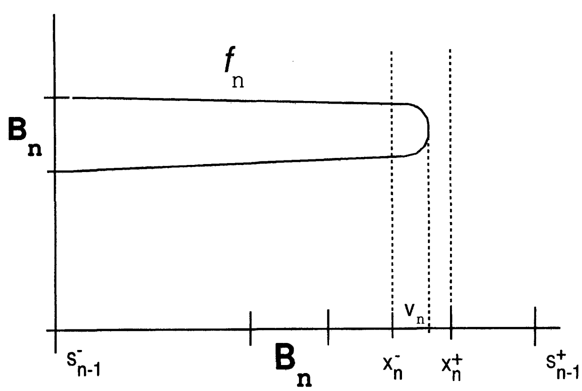

Fig. 4. – Notations explained.

Fact 3.1.

If $n \geq 3$, the diffeomorphism h_n extends on some neighborhood of $g(B_n)$ so that the image of the extension coincides with $(f_{n-2}(0), s_{n-3}^-)$.

Proof. – The proof follows by induction. We leave it to the reader as a good exercise in understanding the notations and the dynamics in the rotation-like case. □

In the "real part" of this paper we will extend diffeomorphisms h_n every time only over one side of their domains. We distinguish between two directions of one-sided extendibility of h_n. The key observation is that the points $\{f_{n-2}(0), f_n(z_n^+), f_n(0), s_{n-3}^-\}$ are always arranged according either to this or the reverse order of the real line. Observe, that f_n extends further "through the head" meaning in the direction of the critical value, where h_n can be extended up to s_{n-3}^-, than "through the legs" meaning in the direction of $f_n(\partial B_n)$ where the extension is only up to $f_{n-2}(0)$ which is closer to B_n' than s_{n-3}^+. The extendibility of central branches plays a crucial role in estimates of the distortion. By real Köbe's Lemma the distortion depends only on the relative scale of the images of domains with respect to the images of their extensions. As it happens (and will be proved), these scales will improve during inducing procedure. We will, however, need some initial extension to start with.

Ratios and cross-ratios

Suppose we have three points a, b, c arranged so that $a \notin [b, c]$. Let us define a few relative scales of the interval (b, c) with respect to (a, c).

Definition 3.1. – *The exclusive ratio of the interval (b, c) with respect to (a, c) is given by*

$$\mathbf{R}_e(b, c; a) = \frac{|b-c|}{\text{dist}((b, c), a)},$$

whereas their inclusive ratio by

$$\mathbf{R}_i(b, c; a) = \frac{|b-c|}{\max(|b-a|, |c-a|)}.$$

Set $\mathbf{R}(b, c; a)$ to be equal to the geometric mean of the inclusive and exclusive ratios.

$$\mathbf{R}(b, c; a) = \sqrt{\mathbf{R}_i(b, c; a)\mathbf{R}_e(b, c; a)}$$

Together with ratios we will often use cross-ratios. The types of cross-ratios we use are expanded by homeomorphisms with negative Schwarzian derivative.

Definition. – *Suppose we have a quadruple a, b, c, d ordered so that $a < b < c < d$ or reversely. Define their inclusive cross-ratio as*

$$\mathbf{Cr}_i(a, b, c, d) = \frac{|b-c||d-a|}{|c-a||d-b|},$$

and their exclusive cross-ratio as

$$\mathbf{Cr}_e(a,b,c,d) = \frac{|b-c||d-a|}{|b-a||d-c|}.$$

Finally set

$$\mathbf{Cr}(a,b,c,d) = \sqrt{\mathbf{Cr}_i(a,b,c,d)\mathbf{Cr}_e(a,b,c,d)}$$

Distortion

Suppose that we have an expression A which is defined in terms of distances between points (like ratios and cross-ratios.) Then we consider $f_*(A)$ obtained by replacing given points with their images by f. We will measure the distortion of this transformation by the ratio $f_*(A)/A$. For example, if we set $A = \mathbf{Cr}(a,b,c,d)$ then the distortion by f is equal to

$$\frac{\mathbf{Cr}(f(a),f(b),f(c),f(d))}{\mathbf{Cr}(f(a),f(b),f(c),f(d))}.$$

3.2. Induction parameters

In this subsection we will introduce quantities which will describe geometry of partitions given by our inducing procedure. Next we will compose a quasi-invariant which after a finite number of inducing steps will decrease at least exponentially fast. The real induction parameters formulated here will directly correspond to these in the complex part. The same concerns induction formulae. This suggests that estimates from the complex part of our work can somehow be translated into the corresponding ones in the real line. This would enable one to give a proof by purely real methods. However, the combinatorial complexity of such an approach seems formidable.

Recall the notations introduced at the beginning of this section. In addition, denote by (x_n^-, x_n^+) the domain of the *immediate* branch of ϕ_n which contains $f_n^{a_n}(0)$. Choose the point x_n^- so that $\phi_n(x_n^-) \in \partial B_n$ is in the range of f_{n+1}. Set $v_n = f_n^{a_n}(0)$.

DEFINITION 3.3. – *The center of* (x_n^-, x_n^+)*, denoted by* c_n*, is defined by the condition* $\phi_n(c_n) = 0$.

Observe that $v_n \in (x_n^+, c_n)$ and that for rotation-like maps x_n^- lies closer to zero than x_n^+.

Parameters of the induction measure sizes of domains of branches as well as and their separation from the critical point and the boundary of the relevant box. The distortion of these quantities will be controlled by bounds on τ and extendibility of branches. Here, we provide a full list of parameters.

- $\alpha_n = \mathbf{R}(z_n^+, z_n^-; s_{n-1}^+)$
- $\gamma_n = \mathbf{R}(x_n^+, x_n^-; 0)$,
- $\beta_n = \mathbf{Cr}(s_{n-1}^+, x_n^+, x_n^-, 0)$.

We will examine how these quantities change after a type I inducing step. Generally, none of these quantities decreases monotonically in the inducing procedure. Nevertheless, we can choose products of these parameters that show monotone decay. Consider the products $\alpha_n\gamma_n$ and $\alpha_n\beta_n$. We will see soon that immediate domains (x_n^-, x_n^+) stay always at the definite distance from the boundary of B'_n. This implies that in the case of rotation-like maps these products are equivalent and it is enough to consider only one of them.

PROPOSITION 1. – *Consider the quantity $\alpha_n\gamma_n$ for a rotation-like sequence*

$$\phi_0, \ldots, \phi_n, \ldots.$$

If $\tau_n \leq \tau = 0.37$ for every n, then there is an absolute constant $\Lambda < 1$ and a fixed integer N with the property that

$$\alpha_n\gamma_n \leq \Lambda^n \alpha_0\gamma_0$$

for all $n > N$.

Auxiliary quantities

Before we pass to the proof of Proposition 1 which will occupy the next three subsections we will introduce three auxiliary induction quantities $\overline{\gamma}_n$, ω_n and Ω_n.

$$\overline{\gamma}_n = \mathbf{R}(v_n, x_n^-; 0)$$

$$\omega_n = \frac{|f_n(0)|}{|s_{n-1}^+|} \qquad \text{and} \qquad \Omega_n = \frac{|v_n|}{|s_{n-1}^+|}$$

(we recall that $v_n = f_n^{a_n}(0)$.)

In addition, we have already defined τ_n and τ. The estimates in the next two subsections will be quite complicated. It may help the reader to think of central branches as quadratic polynomials, and of monotone branches as affine. In this model estimates are easier and actually give the right idea of the real situation. Then the distortion might be treated as a correction to formulae obtained in the "linear-quadratic" model.

3.3. A non-close return

Throughout this subsection we assume that ϕ_n makes a non-close return. This means that $s_n^+ = z_n^+$.

The distance of a point z to zero is denoted by $|z|$. Observe that in many cases this is independent of the superscript "+" or "-". We will often drop superscripts "+" and "-" if it does not matter which one should appear. We will start with the following simple observation.

$$\alpha_{n+1}^2 = 4 \cdot \mathbf{R}_e(g(z_{n+1}), g(0); g(z_n)).$$

The image of $\mathbf{R}_e(g(z_{n+1}), g(0); g(z_n))$ by $(h_n)_*$ is equal to

$$\overline{\gamma}_n \frac{\sqrt{|x_n^-||f_n(0)|}}{|x_n^-| + |s_{n-1}^-|}.$$

To find the distortion of $(h_n)_*$ on this ratio, we will complete the ratio to the cross-ratio $\mathbf{Cr}_e(g(z_n), g(z_{n+1}), g(0), h_n^{-1}(s_{n-3}^-))$. Since the cross-ratio is expanded, we get

$$\alpha_{n+1}^2 \leq 4 \cdot \overline{\gamma}_n \frac{\sqrt{|x_n||f_n(0)|}}{|x_n| + |s_{n-1}|} \frac{|s_{n-1}| + |s_{n-3}|}{|s_{n-3}| - |f_n(0)|}. \tag{1}$$

Fact 3.2

The distortion of $\overline{\gamma}_n$ and γ_n by a quadratic map is at least 2.

Proof. – This follows directly from the definition of $\overline{\gamma}_n$. □

We pass to estimating $\overline{\gamma}_{n+1}$. Take the image of $\overline{\gamma}_{n+1}$ by the quadratic map g. Fact 3.2 implies that

$$\overline{\gamma}_n \leq \frac{1}{2} \cdot \mathbf{R}(g(v_{n+1}(0)), g(x_n^-);\ g(0)).$$

Complete $g_*(\overline{\gamma}_{n+1})$ to the cross-ratio

$$\mathbf{Cr}\big(h_n^{-1}(f_{n-2}(0)), g(f_{n+1}(0)), g(x_n^-), g(0)\big)$$

and then push it forward by h_n. By the property of expanding cross-ratios we have that

$$\overline{\gamma}_{n+1} \leq \frac{1}{2} \cdot \frac{|z_{n+1}| + |f_{n+2}(0)|}{\sqrt{|f_n(0)|^2 - |z_{n+1}|^2}} \frac{|f_{n-2}(0)| + |f_n(0)|}{\sqrt{|f_{n-2}(0)|^2 - |z_{n+1}|^2}}. \tag{2}$$

Comment 1

Note that the estimate (2) remains true if we replace $f_{n+2}(0)$ by z_{n+1}^+.

Our next task is to combine estimates on $\overline{\gamma}_{n+1}$ and α_{n+1} and get the best possible upper bound of their product in terms of $\overline{\gamma}_n$ and α_n. To this end we prove

LEMMA 3.1. – *For arbitrary n the following inequality holds.*

$$\frac{\sqrt{|x_n||f_n(0)|}}{|s_{n-1}| + |x_n|} \frac{|f_{n+2}(0)| + |z_{n+1}|}{\sqrt{|f_n(0)|^2 - |z_{n+1}|^2}} \leq \frac{1}{4} \cdot \alpha_n \alpha_{n+1}(1 + \omega_{n+2}) \frac{|z_{n-1}|}{|f_n(0)| + |z_{n-1}|}.$$

Proof. – By the definition of α_n we have that

$$\frac{|z_{n+1}|}{\sqrt{|f_n(0)|^2 - |z_{n+1}|^2}} \leq \frac{1}{4} \cdot \alpha_{n+1}\alpha_n \frac{\sqrt{|z_{n-1}|^2 - |z_n|^2}}{|z_n|} \sqrt{\frac{|z_n|^2 - |z_{n+1}|^2}{|f_n(0)|^2 - |z_{n+1}|^2}}. \tag{3}$$

The last factor in the inequality (3) is decreasing with respect to $|z_{n+1}|$. Thus, the right-hand side of (3) is bounded by

$$\leq \frac{1}{4} \cdot \alpha_{n+1}\alpha_n \frac{|z_{n-1}|}{|f_n(0)|}.$$

To complete the reasoning we will need the following elementary fact:

For any three positive numbers $0 < x < y < z$ the inequality

$$\frac{\sqrt{xy}}{z+x} < \frac{y}{z+y}$$

holds.

which can be readily proved by calculus. From there,

$$\frac{\sqrt{|x_n||f_n(0)|}}{|s_{n-1}|+|x_n|} \leq \frac{|f_n(0)|}{|f_n(0)|+|s_{n-1}|}$$

which completes the proof.

Comment 2

Replace $|f_{n+2}(0)|$ by $|z_{n+1}|$ in the estimate of Lemma 3.1. By the same reasoning we obtain

$$\text{(4)} \qquad \frac{\sqrt{|x_n||f_n(0)|}}{|s_{n-1}|+|x_n|}\frac{|z_{n+1}|}{\sqrt{|f_n(0)|^2-|z_{n+1}|^2}} \leq \frac{1}{2} \cdot \alpha_n\alpha_{n-1}\frac{|z_{n-1}|}{|z_{n-1}|+|f_n(0)|}.$$

Multiply the inequalities (1) and (2) and then combine with the inequality in Lemma 3.1. As a result we get the following recursive formula

$$\alpha_{n+1}\overline{\gamma}_{n+1} \leq \lambda_n\alpha_n\overline{\gamma}_n,$$

where λ_n is less than

$$\text{(5)} \qquad \frac{1}{2} \cdot (1+\omega_{n+2})\frac{|s_{n-1}|}{|f_n(0)|+|s_{n-1}|}\frac{|f_{n-2}(0)|+|f_n(0)|}{\sqrt{|f_{n-2}(0)|^2-|z_{n+1}|^2}}\frac{|s_{n-1}|+|s_{n-3}|}{|s_{n-3}|-|f_n(0)|}.$$

We will bound from above λ_n by maximizing (5) with respect to a location of $f_n(0)$. To this end consider

$$\epsilon_n = \frac{|f_{n-2}(0)|+|f_n(0)|}{(|s_{n-3}|-|f_n(0)|)(|s_{n-1}|+|f_n(0)|)}$$

as a function of $|f_n(0)|$ on the interval $(0, |s_{n-1}|)$.

LEMMA 3.2. – *The function ϵ_n achieves a global maximum in* 0.

Proof. – The sign of the derivative of ϵ_n with respect to $|f_n(0)|$ is the same as the sign of

$$-(|f_{n-2}(0)| - |s_{n-1}|)|s_{n-3}| + 2|f_n(0)||f_{n-2}(0)| + |f_n(0)|^2.$$

The smaller root of the above quadratic polynomial is always less than zero. Thus, the function ϵ_n can have only a local minimum in the interval $(0, |s_{n-1}|]$. A direct computation shows that if $\tau^2 \leq 1/3$, then $\epsilon_n(0) \geq \epsilon_n(|s_{n-1}|)$ □

Finally, by Lemma 3.2 and the definition of $\tau \geq \tau_n$ for any n, λ_n is less than

$$\frac{1+\omega_{n+2}}{2}\frac{1+\tau^2}{\sqrt{1-\tau^6}}. \tag{6}$$

Comment 3

The same computation based on Comments (1) and 2 yields

$$\gamma_n\alpha_n \leq \frac{1+\tau^2}{\sqrt{1-\tau^6}}\gamma_{n-1}\alpha_{n-1}. \tag{7}$$

3.4. A close return

Throughout this subsection we assume that n-th return is close. This means that $a_n > 1$. The scheme of the proof is much the same as in the previous case. The only difference is that the reasoning is a bit more way around and requires several repetitions of the estimates similar to these found in the last subsection.

Let ζ_n be the ratio of lengths of B'_{n+1} to B_n. Put

$$\overline{\alpha_{n+1}} = \mathbf{R}(z^+_{n+1}, z^-_{n+1}, z^+_n).$$

Denote the preimage of B'_{n+1} in (x^-_n, x^+_n) by f^{-a_n} by $(x^-_{n+1/2}, x^+_{n+1/2})$ chosen so that x^+_n and $x^+_{n+1/2}$ are on the same side.

Recursion

We will write a recursion for the sequence $\alpha_n\gamma_n$. By definition,

$$\alpha_{n+1} \leq \frac{\overline{\alpha}_{n+1}}{\zeta_n\sqrt{1-\tau^2}}. \tag{8}$$

For i ranging from 1 to $a_n - 1$ let $x^-_{n+1/2,i}$ stand for $f_n^{-i}(x^-_{n+1/2})$ and $x^+_{n+1/2,i}$ for $f_n^{-i}(x^+_{n+1/2})$. To bound $f_{n*}(\gamma_{n+1}\overline{\alpha}_{n+1})$ from above we will use similar arguments as in the previous section. Push forward γ_{n+1} by the quadratic map g. Then

$$\gamma_{n+1} \leq \frac{1}{2}R(g(x^+_{n+1}), g(x^-_{n+1}); g(0)).$$

Complete $g_*(\gamma_{n+1})$ to the cross-ratio

$$\mathbf{Cr}(h_n^{-1}(f_{n-2}(0)), g(x_{n+1}^+), g(x_{n+1}^-), g(0))$$

and then push it forward by h_n. By the property of expanding cross-ratios we have that

$$\gamma_{n+1} \leq \frac{|z_{n+1}|}{\sqrt{|f_n(0)|^2 - |z_{n+1}|^2}} \frac{|f_{n-2}(0)| + |f_n(0)|}{\sqrt{|f_{n-2}(0)|^2 - |z_{n+1}|^2}}. \tag{9}$$

In the same way as in the proof of Lemma 3.1 (see the inequality (3)) we obtain

$$\frac{|z_{n+1}|}{\sqrt{|f_n(0)|^2 - |z_{n+1}|^2}} \leq \frac{1}{4} \cdot \overline{\alpha}_{n+1} \alpha_n \frac{|s_{n-1}|}{|f_n(0)|}.$$

By definition of $\overline{\alpha}_{n+1}$,

$$\overline{\alpha}_{n+1} = 4 \cdot \mathbf{R}_e(g(z_{n+1}), g(0);\, g(z_n)) \leq 4\mathbf{Cr}(g(z_n), g(z_{n+1}), g(0), h_n^{-1}(s_{n-3}^-)).$$

Again using the expanding property of cross-ratios we get

$$\overline{\alpha}_{n+1}^2 \leq 4 \cdot \frac{|x_{n+1/2,a_n-1}^- - f_n(0)|}{|x_{n+1/2}^-| + |s_{n-1}|} \frac{|s_{n-1}| + |s_{n-3}|}{|s_{n-3}| - |f_n(0)|}.$$

The estimates for γ_{n+1} and $\overline{\alpha}_{n+1}^2$ lead to the following formula

$$\overline{\alpha}_{n+1}\gamma_{n+1} \leq \alpha_n \eta_n \frac{|f_n(0)| - |x_{n+1/2,a_n-1}^-|}{\sqrt{|f_n(0)|^2 - |z_{n+1}|^2}},$$

where η_n is equal to

$$\frac{|s_{n-1}|}{|s_{n-1}| + |x_{n+1/2,a_n-1}^-|} \frac{|s_{n-1}| + |s_{n-3}|}{|s_{n-3}| - |f_n(0)|} \frac{|f_{n-2}(0)| + |f_n(0)|}{\sqrt{|f_{n-2}(0)|^2 - |z_{n+1}|^2}}.$$

Let $0 \leq i < a_n$. We shall write $\gamma_{n,i}$ for the ratio

$$\frac{|f_n^{a_n-i}(0)| - |x_{n+1/2,i}^-|}{\sqrt{|f_n^{a_n-i}(0)||x_{n+1/2,i}|}}. \tag{10}$$

Now, compute

$$\frac{\dfrac{|f_n(0) - x_{n+1/2,a_n-1}^-|}{\sqrt{|f_n(0)|^2 - |z_{n+1}|^2}}}{\mathbf{R}(g \circ f_n(0), g(x_{n+1/2,a_n-1}^-);\, g(z_{n+1}))} \leq \frac{|x_{n+1/2,a_n-1}^-|}{|x_{n+1/2,a_n-1}^-| + |f_n(0)|}.$$

Increase this factor to $1/2$ and decrease η_n replacing $|x^{-1}_{n+1/2,a_n-1}|$ with $|f_n(0)|$. As a result we obtain an upper bound of $\overline{\alpha}_{n+1}\gamma_{n+1}$ which can be written in the form

$$\frac{1}{2}\alpha_n\eta'_n\mathbf{R}(g\circ f_n(0), g(x^-_{n+1/2,a_n-1}); g(z_{n+1})),$$

where

$$\eta'_n = \frac{|s_{n-1}|}{|f_n(0)|+|s_{n-1}|}\frac{|f_{n-2}(0)|+|f_n(0)|}{\sqrt{|f_{n-2}(0)|^2-|z_{n+1}|^2}}\frac{|s_{n-1}|+|s_{n-3}|}{|s_{n-3}|-|f_n(0)|}.$$

By Lemma 3.2 we can only worsen estimates letting $|f_n(0)| = 0$ in the expression for η'_n. Therefore,

$$\overline{\alpha}_{n+1}\gamma_{n+1} \le \frac{1}{2}\alpha_n\frac{1+\tau^2}{\sqrt{1-\tau^6}}\mathbf{R}(g\circ f_n(0), g(x^-_{n+1/2,a_n-1}), g(z_{n+1})). \tag{11}$$

Complete the last ratio to an appropriate cross-ratio by adjoining a fourth point at $h_n^{-1}(f_{n-2}(0))$ and then push it forward by h_n. The points $f_n(z_{n+1})$ and $x^-_{n+1/2,a_n-2}$ lie on the opposite sides of zero. The resulting cross-ratio can be only increased if we move the point $f_n(z_{n+1})$ in the direction of zero. Hence, for $a_n > 2$

$$\mathbf{R}(g\circ f_n(0), g(x^-_{n+1/2,a_n-1}), g(z_{n+1})) \le \frac{\gamma_{n+1/2,a_n-2}}{1-\tau^2} \tag{12}$$

If $a_n = 2$ then put $1-\tau\Omega_n$ in the place of the denominator of (12) in order to have the correct estimate.

We can use the sequence of estimates starting from (10) again to prove

$$\gamma_{n,i} \le \frac{1}{2}\cdot\frac{\gamma_{n+1/2,i-1}}{1-\tau^2} \tag{13}$$

provided $i > 1$ and

$$\gamma_{n,1} \le \frac{1}{2}\cdot\frac{\gamma_{n+1/2,i-1}}{1-\tau\Omega_n} \tag{14}$$

when $i = 1$. From inequalities (11), (12), (13) and (14) we obtain

$$\overline{\alpha}_{n+1}\gamma_{n+1} \le \overline{\Lambda}_n\alpha_n\gamma_{n+1/2,0} \tag{15}$$

where $\overline{\Lambda}_n$ is bounded from above by

$$\frac{1}{2^{a_n-1}}\frac{1+\tau^2}{\sqrt{(1-\tau^6)}(1-\tau^2)^{a_n-2}(1-\tau\Omega_n)}. \tag{16}$$

In the last step of our reasoning we exploit the fact that $\gamma_{n+1/2,0}$ is substantially less than γ_n. We claim that

LEMMA 3.3.

$$\gamma_{n+1/2,0} \leq \zeta_n \gamma_n.$$

Proof. – We will actually prove

$$\mathbf{R}(x^+_{n+1/2}, x^-_{n+1/2}; 0) \leq \zeta_n \gamma_n$$

and use

$$\gamma_{n+1/2,0} < \mathbf{R}(x^+_{n+1/2}, x^-_{n+1/2}; 0)$$

which follows directly from the definition of $\gamma(n+1/2, 0)$. First observe that $|x^-_{n+1/2} - x^+_{n+1/2}| \leq \zeta_n |x^-_n - x^+_n|$. Indeed, the centers of B'_{n+1} and B_n coincide and the hyperbolic length of $B_{n+1/2}$ with respect to B_n is not increased by the pullback by a diffeomorphism with a non-positive Schwarzian. Since the element of the hyperbolic length is the smallest in the middle of an interval we conclude that pull-backs are nested with the ratio at most ζ_{n+1}. Denote by u_1 and u_2 the midpoints of $(x^-_{n+1/2}, x^+_{n+1/2})$ and (x^-_n, x^+_n). A straightforward calculation shows that if these midpoints coincide then the Lemma follows. Suppose that $|u_1|$ is less than $|u_2|$ since otherwise then we are done. Push x^+_n toward u_1 so far that the centers coincide again. This operation can only increase the ratio of γ_n to $\mathbf{R}(x^+_{n+1/2}, x^-_{n+1/2}; 0)$. The ratio of the lengths of the resulting, concentric intervals, is again at most ζ_n, which completes the proof. □

Finally, Lemma 3.3 and inequalities (8) and (15) imply that

$$\alpha_{n+1}\gamma_{n+1} \leq \Lambda_n \alpha_n \gamma_n, \tag{17}$$

where Λ_n is less than

$$\frac{1}{2^{a_n-1}} \frac{1+\tau^2}{\sqrt{(1-\tau^6)(1-\tau^2)}(1-\tau^2)^{a_n-2}(1-\tau\Omega_n)}.$$

Clearly, Λ_n is the largest for $a_n = 2$ since $\tau^2 < \frac{1}{2}$. For $a_n = 2$ we get

$$\frac{1}{2} \frac{(1+\tau^2)}{(1-\tau\Omega_n)\sqrt{(1-\tau^6)(1-\tau^2)}} \tag{18}$$

as an upper bound of Λ_n in the case of a close return.

Comment 4

In this subsection unlike for non-close returns we worked with the quantity γ_n *instead of* $\overline{\gamma}_n$. *The estimates become stronger if we decrease the left-hand side of (17) substituting* γ_n *by* $\overline{\gamma}_n$.

(19)
$$\alpha_{n+1}\overline{\gamma}_{n+1} \le \lambda_n \alpha_n \gamma_n,$$

where λ_n *is less than*

(20)
$$\frac{1+\omega_{n+2}}{4}\frac{(1+\tau^2)}{(1-\tau\Omega_n)\sqrt{(1-\tau^6)(1-\tau^2)}}.$$

The proof is the same, except that the estimate (9) ought to be replaced by (2).

3.5. Proof of Proposition 1

The sequence ω_n plays a crucial role in the inductive formulae derived in the last two subsections. The decay of geometry depends directly on the separation of this sequence from 1. We will begin by writing a recursion for the sequence Ω_n. Clearly, $\omega_n \le \Omega_n$. We will consider two cases.

A non-close return

In this case $\Omega_n = \omega_n$.

LEMMA 3.4. – *Assume that* ϕ_n *makes a non-close return. Then*

$$\Omega_{n+1}^2 \le (1+\tau^2)\frac{\Omega_n + \Omega_{n+2}\tau^2}{1+\Omega_n}.$$

Proof. – Take the image of Ω_n by the quadratic map g

$$\Omega_{n+1}^2 = \mathbf{R}_i(g(v_{n+1}), g(0); g(z_n))$$

and next push it forward by h_n. As a result we obtain

(21)
$$f_{n*}(\Omega_{n+1}) = \frac{|f_n(0)| + |f_{n+2}(0)|}{|f_n(0)| + |s_{n-1}|}.$$

To compute the distortion brought in by h_n complete $g_*(\Omega_n)$ to the cross-ratio

$$\mathbf{Cr}_i(g(v_{n+1}), g(x_{n+1}), g(0), h_n^{-1}(z_{n-3}^-)).$$

Note that the cross-ratio is expanded. We obtain

$$\frac{|z_{n-1}| + |z_{n-3}|}{|z_{n-3}| + |f_{n+2}(0)|}.$$

as a correction to (21). Observe that $\frac{|f_{n+2}(0)|}{|s_{n-1}|} \leq \Omega_{n+2}\tau^2$ which establishes the claim of Lemma 3.4. □

A close return

Assume that f_n shows a close return. Then $\Omega_{n+1} = |v_{n+1}|/|s_n|$. By definition

$$\Omega_{n+1}^2 = \mathbf{R}_i(g(v_{n+1}), g(0); g(z_{n+1/2})).$$

Complete the last ratio to an appropriate cross-ratio by adjoining a fourth point at $h_n^{-1}(z_{n-3}^-(0))$ and then push it forward by h_n. By the property of expanding cross-ratios we get

$$\Omega_{n+1}^2 \leq \frac{|f_n(0)| + |f_{n+2}(0)|}{|f_n(0)| + |f_n(s_n)|} \frac{|f_n(s_n)| + |z_{n-3}|}{|z_{n-3}| + |f_{n+2}(0)|}.$$

Let us denote the ratio $|f_n(s_n)|/|z_n|$ by σ. Replace $|f_n(0)|$ by $|f_n(s_n)|$ in the inequality above. We obtain a new bound of Ω_{n+1}, namely

$$\Omega_{n+1}^2 \leq \frac{1}{2} \cdot \left(1 + \tau_{n+1}\frac{|z_{n+1/2}|}{|z_n|\sigma}\Omega_{n+2}\right)(1 + \sigma\tau^3). \tag{22}$$

Clearly, $\frac{|z_{n+1/2}|}{|z_n|\sigma} < 1$. We will estimate σ from above under the assumption that the critical value of f_n remains in the central domain for at least two iterates. The point $f_n(s_n)$ is $f_n^{-a_n+1}(z_n^-)$. Thus, σ is the greatest when $f_n(s_n)$ coincides with a boundary point of B_n. The next inequality is obtained by completing the ratio $g_*(\sigma)$ to an "inclusive" cross-ratio with a fourth point at $h_n^{-1}(s_{n-3})$ and then pushing it forward by h_n.

$$\sigma^2 \leq \frac{|f_n(0)| + |z_n|}{|f_n(0)| + |s_{n-1}|} \frac{|s_{n-3}| + |s_{n-1}|}{|s_{n-3}| + |z_n|},$$

and finally,

$$\sigma \leq \sqrt{\frac{2\tau(1+\tau^2)}{1+\tau}} \tag{23}$$

So, we write (22) as

$$\Omega_{n+1}^2 \leq \frac{1}{2}(1 + \tau\Omega_{n+2})(1 + \sigma\tau^3), \tag{24}$$

where σ is bounded by (23).

A bound

We want to find an upper bound of Ω_n. To this end, observe that

- Ω_{n+1} is an increasing function of Ω_n and Ω_{n+2}.
- As long as the value of Ω_n is greater than 0.7, the estimate of Lemma 3.4 gives a lower value of Ω_{n+1} than inequality (24).

The last statement can be easily justified by direct computation. Indeed, the right-hand side of the estimate of Lemma 3.4 is smaller than $(1/2)(1+\tau^2)(1+\tau^2\Omega_{n+2})$ while that of (24) is larger than $(1/2)(1+\tau\Omega_{n+2})$. We will be done once we show that

$$(1+\tau^2)(1+0.7\cdot\tau^2) < 1+0.7\cdot\tau,$$

which clearly holds for $\tau < 0.4$.

So we consider the recursion given by assuming equality in (24). The function

$$y \to \sqrt{1+\frac{\tau y}{2}}$$

has exactly one attracting fixed point for $y \leq 0.823562$ in the positive domain. It follows that if Ω_{n+1} in (24) is greater than this fixed point, then Ω_{n+2} has to be less than Ω_{n+1}. Thus, we set $\Omega = 0.823562$ as an bound of Ω_n and ω_n. We note that this bound is attained in for all values of n sufficiently large depending only on ϵ stipulated by Proposition 1. Indeed, τ_n is smaller than $\tau = 0.37$, and then it is clear that Ω_n decreases at least by a uniform amount for each step of the recursion given by (24) as long as it is greater than Ω.

Final estimates

After these preparations we will prove Proposition 1. We will conduct estimates by splitting the sequence of box mappings into blocks. Each block save perhaps the first will have a mapping with a close return immediately followed by a maximal sequence of consecutive non-close returns. E.g., a single mapping with a close return is a block. The only exception from the above rule of constructing blocks occurs when ϕ_0 makes a non-close return. Then the first block consists of a maximal sequence of box mappings with non-close returns.

Below we list the rules which will give recursive estimates within a given block of box mappings.

1. Suppose ϕ_n exhibits no close return and is the last such mapping in a given block. Then we use formula (7) to estimate

$$\alpha_{n+1}\gamma_{n+1} \leq 1.13837\cdot\alpha_n\overline{\gamma}_n.$$

If ϕ_n is not last in its block, we use the inequality (6)

$$\alpha_{n+1}\overline{\gamma}_{n+1} \leq 0.56919 \cdot (1 + \omega_{n+2})\, \alpha_n \overline{\gamma}_n.$$

2. Let n-th box mapping exhibits close return.

If ϕ_n is not a block in its own right, then we apply formula (18)

$$\alpha_{n+1}\overline{\gamma}_n \leq 0.80402 \cdot \alpha_n \gamma_n.$$

If ϕ_n is a block by itself, then (20) implies that

$$\alpha_{n+1}\gamma_{n+1} \leq 0.881181 \cdot \alpha_n \gamma_n.$$

We will consider two cases:

Blocks with at least two box mappings with non-close returns

Suppose that a series of at least two box mappings with non-close returns begins at the moment n. We will show that the separation of the critical value $f_n(0)$ from the boundary of the box $B_{n'}$ improves with n growing. Indeed, by Lemma 3.4

$$\Omega_{n+1} \leq (1 + \tau^2)\sqrt{\frac{\Omega}{1 + \Omega}} \leq 0.76403$$

and next

$$\Omega_{n+2} \leq \sqrt{(1 + \tau^2)\frac{0.76403 + \Omega\tau^2}{1.76403}} \leq 0.751714.$$

By monotonicity of this formula with respect to Ω_n, all $\Omega_k \leq 0.751714$ for $k \geq n + 2$. Consequently, for $k \geq n$ we obtain that

$$\alpha_{k+1}\overline{\gamma}_{k+1} \leq 0.9971 \cdot \alpha_k \overline{\gamma}_k.$$

It happens that $0.80402 \cdot 1.13837 < 1$. Thus, if a block of length $k \geq 3$ starting with ϕ_n is taken as whole, then

$$\alpha_{n+k}\gamma_{n+k} \leq (0.9971)^{k/3} \alpha_n \gamma_n.$$

Shorter blocks

For a block of a single map with a close return, or one close and one non-close, it follows immediately from our rules that the product $\alpha_n \gamma_n$ decreases after passing through a block by a definite constant less than 1.

Conclusion

To see that $\alpha_n \gamma_n$ goes down to 0 at least exponentially fast, first wait N steps for bounds on Ω to be achieved (N is bounded in terms of τ). Then pick a $k > 2N$, and construct the blocks starting from ϕ_N. Cut off the last block at ϕ_k. The uniform exponential estimate follows at once from our considerations of the rate of decay within blocks. So, Proposition 1 follows.

3.6. Decay of box geometry

General picture

In this subsection we will estimate the rate of the decay of box geometry proving eventually that for all S-unimodal rotation-like maps the rate is always at least exponential. This will prove Theorems A and B. In the course of inducing a subtle interaction between α_n and γ_n takes place. Namely, after a long series of non-close returns, γ_n is approximately equal to the second power of α_n. The first close return will violate this simple relation between γ_n and α_n by decreasing α_n stronger than γ_n. If the close return is deep enough (*i.e.* the critical value needs a lot of iterates to escape from the central domain) then γ_n and α_n can even become comparable. At the moment when we are leaving box maps with close returns, γ_n and α_n will quickly regain (exponentially fast with the number of steps of non-close inducing) their square-law relation. However, the product $\alpha_n \gamma_n$ for rotation-like maps of bounded type decreases asymptotically with each step of inducing by a constant uniformly separated from 0 and 1. It means that while switching between patterns of inducing "oscillations" between α_n and γ_n destroy monotone (known from the Fibonacci example) fashion of the decay of boxes.

Theorem B expresses what we mean by an exponential decay of box geometry. From Theorem B it follows immediately that there is a whole class of S-unimodal maps with at most exponential decay of box geometry. The dynamics of maps from this class, purely characterized in terms of rotation number, is certainly different from the "Fibonacci pattern".

COROLLARY 1. – *For any S-unimodal rotation-like map with a rotation number of the constant type* [3] *there exist constants* $K > 0$ *and* $0 < \kappa < 1$ *so that*

$$\tau_n > K \kappa^n.$$

The constant κ *depends only on the upper bound of the coefficients of the continued fraction representation of the rotation number while the constant* K *depends solely on the initial geometry of* ϕ_0*, in particular is uniformly controlled by the parameter* ϵ *of Theorem A.*

Proof of Theorem B. – This proof will be based on Proposition 2 whose proof is postponed until the next section. We will start the proof with two Lemmas.

[3] Let us recall that a number ρ is of the constant type if and only if all coefficients in its continuous fraction representation are bounded.

LEMMA 3.5. – *Under the hypotheses of Theorem B, there is a constant K depending only on the initial estimate ϵ so that*

$$\gamma_n \leq K\alpha_n.$$

Proof. – The proof follows from the observation that the image of γ_n by f_n is comparable with γ_n and $f_{n*}(\gamma_n)$ can exceed α_n by no more than a uniform constant. □

LEMMA 3.6. – *The ratio τ_n of two consecutive boxes goes to zero at least exponentially fast.*

Proof. – Suppose that n is so large that $\tau_n < \tau = 0.37$. This n can be chosen based on Proposition 2. Let us recall that $\alpha_n\gamma_n$ goes to zero at least exponentially fast. We will actually prove that α_n decreases exponentially which is easily equivalent. Consider two cases:

- f_n shows a close return.

We push forward α_n by $f_{n*}^{a_n}$. Using (13) we obtain

$$\alpha_{n+1}^2 \leq K\lambda^{a_n}\gamma_n, \tag{25}$$

where $\lambda < 1$ depends only on τ. Finally, by Lemma 3.5

$$\alpha_{n+1}^4 \leq K\lambda^{2a_n}\alpha_{n-1}\gamma_{n-1}.$$

Proposition 1 concludes the proof.

- f_n shows a close return.

Similarly as before we get that

$$\alpha_{n+1}^2 \leq K\gamma_n.$$

Lemma 3.6 likewise follows. □

Lemma 3.6 together with the inequality 25 give the upper estimate of Theorem B.

To prove the opposite estimate make τ go to 0 arbitrary small in all distortion estimates. So, we can reverse the directions of the inequalities estimating $\alpha_n\gamma_n$ from below. Next, observe that λ_n and Λ_n appearing in the recursive scheme for for $\alpha_n\gamma_n$ are after a finite number of inducing step greater than $(\frac{1}{2} - \epsilon)^{a_n}$. This completes the proof of Theorem B.

Theorem A follows directly from Theorem B and Proposition 2.

4. Initial bounds

Our goal is to prove the following initial estimate for the box ratio in rotation-like sequences.

PROPOSITION 2. – *Let $\phi = \phi_0$ be a type I real box mapping and let ϕ_i be a rotation-like sequence derived from ϕ. Let B_i be the central domain of ϕ_i with $B'_i := B'$ as given by Definition 2.1 applied to $\phi := \phi_i$. Suppose that the initial ratio of lengths satisfies*

$$\frac{|B_0|}{|B'_0|} \leq 1 - \epsilon\,.$$

Then, for every $\epsilon > 0$ *there is* N *with the property that*

$$\frac{|B_i|}{|B'_i|} < 0.37$$

for every $i \geq N$.

The proof of Proposition 2 will occupy the rest of this section.

4.1. Geometrical setup

Notations

Fix the attention on some mapping ϕ_n from the rotation-like sequence mentioned in the statement of Proposition 2. The new domain B_{n+1} is formed as the preimage by f_n of an immediate domain of ϕ_n which is always filled-in so that it maps onto B'_{n+1}. Let us call this domain (u, v) and let us say that u is closer to the critical point. The branch defined on (u, v) extends at least onto the image B'_n. The domain of this extension will be called (U, V), and again say that U is closer to the critical point. Then, denote $B'_{n+1} := (-w, w)$ and say that w is on the side of the critical value. Note that B'_{n+1} equals B_n if ϕ_n makes a non-close return, but otherwise it is smaller. Also, let $-W := f_n(w)$. Notice that in the rotation-like return $-W$ and W are on the same side of 0. Finally, let $s := s^-_{n-3}$ in the notations of the previous section, or the endpoint of B_{n-2} which is on the same side of 0 as W. The reader may try to get familiar with this notation by trying to see that the ordering of points is

$$-W, -w, 0, w, U, u, v, V, W, s$$

or perhaps is completely reversed. These notations are applicable for both close and non-close returns.

Observe that

$$\mathbf{Cr}_i(U, u, v, V) \leq \mathbf{Cr}_i(W, -w, w, W).$$

We will do the estimates in two rounds. For the first one, we don't assume any estimate on how much smaller the interval (U, V) is compared in (w, W). In the second round, we use the results of the first round to get still better estimates.

An extremal problem

Let us recall that f_n can be represented as a composition of the quadratic map g and the diffeomorphism h_n. The extendibility properties of f_n are formulated in Fact 3.1. We will denote preimages of points by h_n by adding primes in the notation. To find preliminary bounds we solve the following problem:

Lemma 4.1. – *Suppose that* $\mathbf{Cr}_i(U, u, v, V) = C$ *and that*

$$L^{-1} = \mathbf{Cr}_e((-W)^1, w, W, s).$$

Then,

$$\frac{|u'-v'|}{|(-W)'+v'|} < \frac{C}{L(\sqrt{1-C}+\sqrt{1+L^{-1}})^2}.$$

Proof. – This is viewed as a conditonal extremum problem We assume that the interval (U', V') has the unit length. Then $|(-W)' - w'| > L$. Denote $|V' - u'|$ by ρ and $|v' - U'|$ by λ. Then

$$C = \frac{\rho+\lambda-1}{\rho\lambda}. \tag{26}$$

We want to maximize

$$\frac{\rho+\lambda-1}{L+\lambda} = C\frac{\rho\lambda}{L+\lambda},$$

which is equivalent to minimizing

$$M = \frac{1}{\rho}\left(1+\frac{L}{\lambda}\right).$$

In the extremal point the gradients of M and C are parallel. Hence

$$\frac{1-\lambda}{1-\rho} = 1+\frac{\lambda}{L}. \tag{27}$$

Calculate ρ from (26) and substitute into (27). The result can be seen as a quadratic equation for λ^{-1} at the extermal point. The result is

$$\lambda_0^{-1} = 1+\sqrt{(1-C)(1+L^{-1})}$$

where λ_0 is the value of λ at the extremal point. The value ρ_0 can then be obtained from (26) and afterwards the minimal value of M can be calculated. For this minimal value we get

$$\begin{aligned} M_0 &= \frac{1}{\rho_0}\left(1+\frac{L}{\lambda_0}\right) \\ &= \left(\frac{1-C}{\sqrt{(1-C)(1+L^{-1})}}+1\right)\left(1+L+L\sqrt{(1-C)(1+L^{-1})}\right) \\ &= L(2\sqrt{(1-C)(1+L^{-1})}+(1-C)+(1+L^{-1})) = L(\sqrt{1-C}+\sqrt{1+L^{-1}})^2. \end{aligned}$$

Lemma 4.1 follows directly. □

Initial estimates

We apply the results of Lemma 4.1 in the following fashion. First, observe that

$$\frac{|B_{n+1}|}{|B'_{n+1}|} \leq \frac{|u' - v'|}{|(-W)' - v'|}. \tag{28}$$

Next, let τ be chosen so that $|B'_j|/|B'_{j-1}| \leq \tau$ for $j = n$ and $j = n - 1$. If C and L are chosen as in the statement of Lemma 4.1, then intoduce $\gamma := \sqrt{1 - C}$ and observe that

$$L^{-1} = \mathbf{Cr}_e((-W)', w, W, s) = \gamma\beta^{-1}\frac{1 + \tau^2}{1 - \tau^2}$$

with some $\beta \geq 1$. The constant β is equal to 1 if the inequalities defining τ are both equalities, $(U, V) = (w, W)$, and the return of ϕ_n is non-close. In the second round of estimates, we will need to use the fact that β is strictly larger than 1 in most cases. Let us introduce $T := 1 - \tau^2/1 + \tau^2$. Then, by (28) and Lemma 4.1, we get

$$\frac{|B_{n+1}|}{|B'_{n+1}|} \leq \tau_{n+1} := \frac{\sqrt{1 - \gamma^2}}{\sqrt{T\beta\gamma} + \sqrt{1 + \frac{\beta T}{\gamma}}}. \tag{29}$$

4.2. Computation of initial bounds

Basic procedures

Set $T' = 1 - \tau_{n+1}^2/1 + \tau_{n+1}^2$ where τ_{n+1} is derived from (29). Therefore, T' should be thought of as a function of T, γ and β. We will study the condition $T' - T > 0$. To represent this difference algebraically, we bring $T' - T$ to the common denominator and examine the sign of the numerator. We see that $T' - T > 0$ is equivalent to

$$T\left((1 - T)\beta\left(\gamma + \frac{1}{\gamma}\right) - 2\right) + \gamma^2(1 + T) + 2\beta T(1 - T)\sqrt{1 + \frac{\gamma}{\beta T}} > 0. \tag{30}$$

The *first procedure* finds an answer to the following problem.

Given $\beta \geq 1$ *and* $0 \leq \gamma_l < \gamma_u \leq 1$ *find a value* T_1, *so that* $T'(T, \gamma, \beta) - T > 0$ *for every* $0 < T < T_1$ *and* $\gamma_l < \gamma < \gamma_u$.

The idea is to find T_1 as large as possible, but we make no attempt to determine the actual maximal value.

The *second procedure* is as follows.

Let $0 \leq \gamma_l < \gamma_u \leq 1$ *be given, together with* $\beta \geq 1$ *and* $0 < T < 1$. *Find* T'_1 *so that* $T'(T, \gamma, \beta) \geq T'_1$ *for every* $\gamma \in (\gamma_l, \gamma_u)$.

Finally, let give an explicit formula for the function T':

$$T'(t, \gamma) = \frac{t(\gamma + \frac{1}{\gamma} + 2\sqrt{1 + \frac{\gamma}{t}}) + \gamma^2}{2 - \gamma^2 + t(\frac{1}{\gamma} + \gamma + 2\sqrt{1 + \frac{\gamma}{t}})} \tag{31}$$

where $t = \beta T$.

A uniform bound

We also obtain as a corollary:

Fact 4.1.

Let ϕ be a type I box mapping of rank n, let $|B_n|/|B_{n'}| \leq 1 - \epsilon$ and assume that the central branch of ϕ is ϵ-extendible. If another type I box mapping of rank m is obtained from ϕ in a sequence of type I inducing steps, then

$$\frac{|B_m|}{|B'_m|} \leq 1 - K(\epsilon)$$

where K is a continuous function of ϵ only, positive when $\epsilon > 0$.

Proof. – Clearly, the ratio will remain bounded away from 1 for two first type I inducing steps. This gives some value of T bounded away from 0 in terms of ϵ. Then, formula (30) implies that T' will never fall below this value as the condition (30) is clearly satisfied for T close to 0. This means an upper bound for τ_{n+1}, as needed. □

Implementation of the first procedure

The first procedure will take five parameters. Of those, β, γ_u and γ_l have already been specified. The other two will have to be picked by trial-and-error to get the biggest possible T_1. For practical reasons, we can assume without a loss of generality that $\gamma_l \geq 0.1$. if $\gamma < 0.1$ then one easily checks that $T'(T, \gamma, \beta) - T > 0$ whenever $T \leq 0.9$ and $\beta \geq 1$. Hence, we can always increase γ_l to 0.1 if we agree to get no more than $T_1 = 0.9$ for the answer of the first procedure.

Another parameter σ gives the step size. We will cover the range $[\gamma_l, \gamma_u]$ with finitely many closed intervals of length σ. We will solve the first problem for each subinterval separately, and give the answer equal to the minimum of estimates on all subintervals and 0.9. Lastly, we have a parameter $0 < \nu < 1$ which must be no less than T_1 for the procedure to be valid. So the game here is a trial-and-error picking ν small, but then having to try again if T_1 turns out larger that ν.

To finish the description, we have to explain how the problem is solved on a subinterval $[\gamma_1, \gamma_2]$. The condition of [30] is clearly weaker than

$$T((1-T)\beta\left(\gamma_2 + \frac{1}{\gamma_2}\right) - 2) + \gamma_1^2(1+T) + 2\beta T(1-T)\sqrt{1 + \frac{\gamma_1}{\beta\nu}} > 0.$$

This gives us a quadratic inequality on T which is solved to give us T_1 on this subinterval.

Implementation of the second procedure

This procedure will take five parameters. Four are already determined in the statement of the second procedure. Again, we make $\gamma_l \geq 0.01$. If $\gamma < 0.01$, then from formula (31) it is evident that $T'(T, \beta, \gamma) \geq 0.9$ provided that $T \geq 0.2$ and $\beta \geq 1$. In all cases when we use the second procedure this makes no difference, since T is always more than 0.2 and the answer T'_1 is never more than 0.9.

We use the same procedure of dividing $[\gamma_l, \gamma_u]$ into subintervals, taking a lower bound for T' on each interval, and taking the minimum with respect to all subintervals for a final answer. The fifth parameter σ gives the length of subintervals.

On each subinterval $[\gamma_1, \gamma_2]$, formula (31) bounds T' from below by

$$T' = \frac{t\left(\gamma_2 + \frac{1}{\gamma_2} + 2\sqrt{1 + \frac{\gamma_1}{t}}\right) + \gamma_1^2}{2 - \gamma_1^2 + t\left(\frac{1}{\gamma_1} + \gamma_1 + 2\sqrt{1 + \frac{\gamma_2}{t}}\right)} .$$

The first estimate on τ

The first estimate on τ is obtained by calling the first procedure with parameters $\gamma_l = 0$, $\gamma_u = 1$, $\beta = 1$, $\mu = 0.7$, $\sigma = 10^{-5}$. The result is $T_1 > 0.69901$. Since

$$T = \frac{1 - \tau^2}{1 + \tau^2} ,$$

$\tau(1) = 0.4209$ will guarantee that $T' - T > 0$. We claim that from any initial box mapping the value of τ will eventually drop below $\tau(1)$ in a number of type I inducing steps bounded in terms of the ϵ from the statement of Proposition 2. Indeed, from (30) by a compactness argument it is clear that if $0 < \eta \leq T \leq 0.69901$, the increment $T' - T$ is bounded away from 0. However, η is bounded away from 0 in terms of ϵ by Fact 4.1. So we can proceed as follows. Start with $T = \eta$. After three type I inducing steps we can adjust $T := T'(T, \gamma, 1$ and this increases T be a definite amount. We proceed in the same way until T is decreased below 0.69901. But means that τ_{n+1} is always bounded from above by $\tau(1)$.

Better estimates on the box ratio

We assume that an estimate τ is chosen so that $|B_j|/|B_j'| \leq \tau$ for $j = n-2, n-1, n$. First, we take $\tau := \tau(1)$. We will get improved estimates for τ_{n+1} based on making β larger than 1. The procedure depends on the combinatorics of close and non-close returns.

A close return for ϕ_n

The opportunity to significantly improve β arises for the following reason. The immediate branch has an extension onto B_n', and so it follows that

$$\frac{|V - U|}{|W - w|} \leq \frac{|B_n|}{|B_n'|} \leq \tau .$$

Hence, one can take $\beta = \tau^{-1}$. Next, one uses procedure two with

$$t = \beta \frac{1 - \tau^2}{1 + \tau^2}$$

$\sigma = 10^{-5}$ and the full range of γ from 0 to 1. For $\tau = \tau(1)$ this gives $\tau(21) = 0.29728$.

A non-close return following a close return

In this case, we will use the second procedure and get an improvement from two sources. First, γ_l will be quite large as a result of the box ratio being very small in the case of a close return. Secondly, β is greater than 1 as well, since $|B'_n|/|B'_{n-1}|$ is at least τ times $|B'_n|/B_{n-1}$. This last ratio b is estimated from above by

$$b = \sqrt{\frac{2\tau}{1+\tau}(1+\tau^2)}\,.$$

Now β is at least

$$\beta \geq \frac{1-bd^2}{1+bd^2} \cdot \frac{1+d^2}{1-d^2}$$

where $d \leq \tau$. Thus, if we want to get a better estimate on the box ratio, d should be no more than this expected lower estimate. Using $d = 0.37$ and $\tau = \tau(1)$ we get $\beta \geq 1.04694$. Also, $\gamma_l \geq \frac{1-\tau(21)}{1+\tau(21)}$. With $\gamma_u = 1$, $\sigma = 10^{-5}$ and $t = 0.749$, procedure one gives $T = 0.7483$ corresponding to $\tau(22) = 0.3795$.

A non-close return followed by a non-close return

In this case we also use procedure two. The improvement is obtained from a better γ_l given by

$$\gamma_l = \frac{1-\tau(1)}{1+\tau(1)}$$

as well as better β. Here, β can easily be estimated

$$\beta \geq \sqrt{\frac{1+\tau}{2\tau(1+\tau^2)}}$$

Taking $\tau := \tau(1)$ with $\sigma = 10^{-5}$, $\gamma_u = 1$, this β and T determined by $\tau(1)$, we get $T = 0.7449$ which gives $\tau(23) = 0.38236$.

Now, the combination of estimates in all cases implies that the box ratio will eventually decrease below the maximum of $\tau(21)$, $\tau(22)$ and $\tau(23)$. In fact, this will happen after no more than four inducing steps since the box ratio has sunk below $\tau(1)$. The minimum happens to be $\tau(23)$. So, we can take $\tau(2) = \tau(23) = 0.38236$.

A second round of estimates

We now repeat the same sequence of estimates in all three cases using τ^2 as our original bound instead of τ^1. In the case of a close return this gives

$$\tau(31) = 0.27736.$$

In the second situation, we get $b \leq 0.79629$. With this, and $d = 0.369$, we obtain $\beta \geq 1.05793$. Also, $\gamma_l = 0.56572$. With $T = \frac{1-\tau^2(2)}{1+\tau^2(2)}$, this β and $\sigma = 10^{-5}$ procedure two yields $\tau(32) = 0.36983$. In the last case, we get $\gamma_l = 0.4468$ and $\beta = 1.25582$. We feed those into procedure one together with $\gamma_u = 1$ and the same T to get $T = 0.75983$ which corresponds to $\tau(33) = 0.36942$.

We see that indeed the box ratio eventually goes below 0.37 which proves Proposition 2.

Part III

Growth of Separating Moduli

5. Combinatorial Analysis

5.1. Plan of the work

Introduction

In this part of the work we will prove Theorem C. Consider a holomorphic type I box mapping ϕ which arose from another type I holomorphic box mapping ϕ_{-1} by the way of a simple inducing step showing a non-close return. This is always the case if ϕ was obtained from a type I inducing step, since that by definition is a sequence of simple inducing steps ending with one showing a non-close return. In this case *immediate branches* of ϕ are defined as the only univalent branches of ϕ which are restrictions of the central branch of ϕ_{-1}. If ϕ arose in a simple inducing step showing a close return, then ϕ has *pseudo-immediate* branches defined inductively as follows. Immediate branches defined above are also pseudo-immediate. After a simple inducing step in the situation of a close return, look at the parent branches in the form $b \circ \psi$ where b is pseudo-immediate for ϕ. The maximal branches belonging to the parent branches are *pseudo-immediate* for the mapping obtained from ϕ by a simple inducing step.

We then say that ϕ shows a *rotation-like return* if the critical value of ϕ lands in the domain of a pseudo-immediate branch. Otherwise, we say that ϕ exhibits a *non-rotational* or non- rotation-like return. This is in agreement with Definition 2.4.

The statement of Theorem C is that the moduli mod $(B_i' \setminus B_i)$ grow at a certain rate. The problem is that they do not grow monotonically. This is the reason why we will introduce another quantity, called the *separation index*. The separation index differs from the moduli mentioned in Theorem C only by a multiplicative constant, so it is sufficient to show that it grows with i at a linear rate. It is not difficult to see that the separation index does not decrease in a sequence of simple inducing steps. In order to prove Theorem C, we are still required to show that the separation index actually increases, though not necessarily by each step. First, we will see such increase in "typical cases" which occur in a sequence of a few type I inducing steps not all of which are rotation-like and none of which shows a close return with a large escaping time. Typical cases are handled by very simple-minded, yet quite involved combinatorially, considerations of the separating annuli.

More advanced tools are needed to solve the remaining special cases. Those include Teichmüller's Modulsatz as well as Grötzsch' extremal domain. The case of a close return with large escaping time is dealt with directly using complex-analytic tools only. It should be emphasized that the entire part of the proof which reduces the problem to rotation-like sequences only requires complex-analytic tools and therefore is valid for all holomorphic box mappings. The assumption that restricted to the real line they give real box mappings is only used in the analysis of rotation-like returns. Here Theorem A and other statements proved for real rotation-like sequences will be used.

Separation bounds

Let φ be a type I holomorphic box mapping. As usual, B is the domain of the central branch of φ and B' is the range of the central branch. Let D a univalent branch of φ. It is assumed that φ was obtained by filling-in from some type II holomorphic box map. In particular, every univalent branch of φ has an analytic continuation as a univalent map onto B'. This continuation will be denoted by $\mathcal{E}_D$. The domain of $\mathcal{E}_D$ is contained in $B' \setminus B$ provided that the domain of D is contained in $B' \setminus B$.

Separating annuli

The *separating annuli* for D are any five annuli $A_i(D)$, $i = 1, \cdots, 4$, and $A'(D)$, either open or degenerated to Jordan curves, which satisfy the conditions listed below.

- All annuli are contained in B'.
- The complement of $A_2(D)$ contains B in its bounded component and the domain of D in its unbounded component.
- The bounded component of the complement of $A_1(D)$ contains $A_2(D)$.
- The annulus $A'(D)$ is uniquely determined as the set-theoretical difference between the domains of $\mathcal{E}_D$ and D.
- The complement of $A_3(D)$ contains $A'(D)$ in its bounded component and B in its unbounded component.
- The complement of $A_4(D)$ contains $A_3(D)$ in its bounded component.

Figure 4 shows a choice of separating annuli for domain D, which is the same as domain D from Figure 2.

Separation symbols

Remain in the same set-up, *i.e.* assume that φ is a type I holomorphic box mapping derived by filling-in from a type II holomorphic box mapping and that D is the domain of a univalent branch of φ.

Definition 5.1. – *A* separation symbol $s(D)$ *for* D *is a choice of separating annuli as described above together with a quadruple of numbers* $s_i(D)$ *for* $i = 1, \cdots, 4$ *so that the*

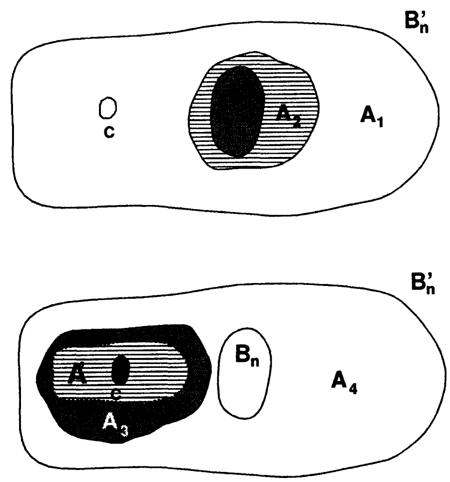

Fig. 5. – A choice of separating annuli for C. Note that the outermost annuli A_1 and A_4 are filled in white.

following inequalities hold:

$$s_2(D) \leq \text{mod } A_2(D) \textit{ and}$$
$$s_1(D) \leq \text{mod } A_2(D) + \text{mod } A_1(D) \textit{ and}$$
$$s_3(D) \leq \text{mod } A'(D) + \text{mod } A_3(D) \textit{ and}$$
$$s_4(D) \leq \text{mod } A'(D) + \text{mod } A_3(D) + \text{mod } A_4(D).$$

The dependence on D will often be suppressed in our subsequent notations, so we will just write s_1 or A'.

Normalized symbols

We will now impose certain algebraic relations among various components of a separation symbol. Choose a number β, and $\alpha := \beta/2$, together with $\lambda_1(D)$ and $\lambda_2(D)$. Assume that

$$\alpha \geq \lambda_1(D), \lambda_2(D) \geq -\alpha$$

and

$$\lambda_1 + \lambda_2 \geq 0.$$

If these quantities are connected with a separation symbol $s(D)$ as follows

$$s_1(D) = \alpha + \lambda_1(D),$$
$$s_2(D) = \alpha - \lambda_2(D),$$
$$s_3(D) = \beta - \lambda_1(D),$$
$$s_4(D) = \beta + \lambda_2(D).$$

we will say that $s(D)$ is normalized with norm β and corrections $\lambda_1(D)$ and $\lambda_2(D)$.

This leads to a definition:

DEFINITION 5.2. – *For a type I holomorphic box mapping, a positive number β is called its separation index provided that normalized separation symbols with norm β exist for all univalent branches.*

Nesting annuli

We introduce an "addition" of nesting annuli. Suppose that A^1 is contained in the bounded component cut off by A^2. Then we define $A^1 \oplus A^2 = A^2 \oplus A^1$ to be the annulus contained between the inner component of the boundary of A^1 and the outer component of the boundary of A^2. We have mod $(A^1 \oplus A^2) \geq$ mod $A_1 +$ mod A_2.

Statement of the result

Suppose now that a type I holomorphic box mapping ϕ is given which later undergoes k consecutive type I inducing steps. We denote by ϕ_i the complex box mappings obtained in the process so that $\phi_0 = \phi$ and ϕ_{i+1} arises from ϕ_i in a type I inducing step. Also, let us use notations ψ_i for central branches of ϕ_i as well as B_i and B'_i for the domain and range of ψ_i, respectively.

PROPOSITION 3. – *Choose $0 < j < k$. Let β be a separation index for ϕ. Assume in addition that $j \geq 3$ and ϕ_j does not show a rotation-like return. Assume that for each $1 \leq i \leq j$ if the escaping time of ϕ_i is E_i, then for some constant $L > 0$ independent of i,*

$$\text{mod } \psi^{1-E_i}(B'_i \setminus B_i) \geq L.$$

Then, for every β and L positive, there is a positive number K such that $\beta + K$ is a separation norm for ϕ_{j+1}.

As will be shown later, L can be determined in terms of β with the exception of hyperbolic close returns with very large escaping times in which cases the increase of the separation index can be seen by other methods.

The set-up

We assume that a type I holomorphic box mapping ϕ is given which arose in a simple inducing step. Let ψ denote the central branch of ϕ with domain B and range B'. Recall from Chapter I about parent branches, subordination of branches and maximal branches. Making assumptions about the normalized separation symbols for branches of ϕ, we will proceed to construct separation symbols for branches of the mapping $\tilde{\phi}$ obtained from ϕ by a simple inducing step. Our estimates will depend on the combinatorial situation as follows.

We fix a univalent branch b of $\tilde{\phi}$. Let $\tilde{p}$ be the parent branch of b so that $\tilde{p} = p \circ \psi$ where p is a univalent branch of ϕ. Another important branch of ϕ is P defined by the condition that the domain of P contains the critical value $\psi(0)$. The process of building the separation symbol for b will depend on the combinatorial situation.

Combinatorial analysis

We will distinguish the following combinatorial situations and handle them one by one.

- b is not a maximal branch,
- $P = \psi$, *i.e.* ϕ makes a close return,
- ϕ makes a non-close return which is subdivided further:

1. $\tilde{p}$ is immediate,
2. p and P are independent,
3. p is subordinate to P,
4. P is subordinate to p.

5.2. General returns

All lemmas we state in this section have implicit assumptions spelled out above under the heading of the "set-up".

Separation of domains

LEMMA 5.1. – *Let $\tilde{\phi}$ be a holomorphic type I box mapping derived from another holomorphic type I box mapping ϕ by a simple inducing step showing a non-close return. If β is a separation index of ϕ, then the domain of any branch of $\tilde{\phi}$ which is contained in B is separated from the boundary of B by an annulus of modulus at least $\beta/4$. Here, B denotes the central domain of ϕ which is the same as B' for $\tilde{\phi}$.*

Proof. – As usual, define P as the branch of ϕ whose domain contains the critical value of ϕ. Since the central domain of $\tilde{\phi}$, called $\tilde{B}$, is the preimage by ψ of the domain of P, the preimage by ψ of $A'(P) \oplus A_3(P)$ is an annulus separating $\tilde{B}$ from the boundary of B. The modulus of this annulus is at least $\frac{\beta-\lambda_1(P)}{2} \geq \beta/4$. The domain of any univalent branch of $\tilde{\phi}$ is surrounded by the preimage of this annulus using its canaonical extension. Thus, the domain of any branch of $\tilde{\phi}$ is indeed separated from the boundary of B by an annulus of modulus at least $\beta/4$. □

Separation of parent branches

LEMMA 5.2. – *Let ϕ be a type I holomorphic box mapping which arose in a type I inducing step from another map. Denote the central branch of ϕ by ψ and the escaping time by $E < \infty$. As usual, let B be the domain of the central branch and B' its range. Assume that*

$$\operatorname{mod} \psi^{1-E}(B' \setminus B) \geq L.$$

If $\tilde{\phi}$ is derived from ϕ by a type I inducing step, then the domains of all all parent branches of $\tilde{\phi}$ are separated from the complement of B' by ring domains with moduli at least $L/2$.

Proof. – Consider the type I holomorphic box mapping ϕ' chosen so that $\tilde{\phi}$ arises from ϕ' in a simple inducing step. This means that ϕ' is obtained from ϕ by a sequence of $E-1$ close returns. The central domain of ϕ' is $\psi^{1-E}(B)$, and the range of its central branch is $\psi^{1-E}(B')$. Now the domains of all branches of ϕ' are separated from the complement of the range of its central branch by annuli with modulus at least L. Since the domains of the parent branches are the preimages of those by ψ, the claim follows. □

Remarks on separation symbols

LEMMA 5.3. – *Consider a normalized separation symbol*

$$s = (s_1 = \beta/2 + \lambda_1, s_2 = \beta/2 - \lambda_2, s_3 = \beta - \lambda_1, s_4 = \beta + \lambda_2)$$

and suppose that $(s_1, s_2, s_3 + \epsilon, s_4 + \epsilon)$ *is another valid separation symbol where* $\epsilon > 0$. *Then there is a normalized separation symbol with norm* $\beta + \frac{\epsilon}{2}$.

Proof. – If we take $\beta = \beta + \frac{\epsilon}{2}$, $\lambda_1' = \lambda_1 - \frac{\epsilon}{4}$, $\lambda_2' = \lambda_2 + \frac{\epsilon}{4}$, then the normalized separation symbol with norm β and corrections λ_1' and λ_2' is

$$\left(s_1, s_2, s_3 + \frac{3}{4}\epsilon, s_4 + \frac{3}{4}\epsilon\right)$$

which is obviously valid. □

LEMMA 5.4. – *Suppose that for some univalent branch of a holomorphic type I box mapping one has a normalized separation symbol with norm* $\beta + \epsilon$, $\beta, \epsilon > 0$. *Then, for the same branch a normalized separation symbol* (s_1, s_2, s_3, s_4) *exists with norm* β *so that* mod $(A_4(b) \oplus A_3(b) \oplus A'(b)) - s_4 \geq \epsilon/2$.

Proof. – Consider the normalized symbol with norm β and corrections $\lambda_1' = \lambda_1 \frac{\beta}{\beta+\epsilon}$ and $\lambda_2' = \lambda_2 \frac{\beta}{\beta+\epsilon}$ where λ_i are corrections for the symbol with norm $\beta + \epsilon$. Observe that this will give an algebraically admissible separation symbol, moreover with weaker estimates, and that

$$s_4' = \beta + \lambda_2' \leq \beta + \lambda_2 - \frac{\epsilon}{2} = s_4 + \frac{\epsilon}{2}$$

and our claim follows from the superadditivity of moduli. □

b not maximal

The next lemma will allow us to consider separation symbols for maximal branches only.

LEMMA 5.5. – *Suppose that* b *is not a maximal branch of* $\tilde{\phi}$, *thus another branch* b' *exists so that* b *is subordinate to* b'. *Assume that the domains of all branches of* ϕ *are separated from the complement of* B' *by ring domains with modulus at least* $2L'$. *Also, assume that* $A_2(b')$ *separates the domain of* b' *from* $A'(b')$. *If a separation symbol for* b' *has bounds* (s_1, s_2, s_3, s_4) *and a separating annulus* $A_3(b')$, *then there is a separation symbol for* b *with bounds*

$$(s_1, s_2, s_3 + L', s_4 + L')$$

with $A_3(b)$ *disjoint from the unbounded component of the complement of* $A_3(b')$.

Proof. – Observe that as a consequence of our assumption, the domains of all parent branches of ϕ' are separated from the complement of $\tilde{B}$ by annuli with modulus L'. We can take $A_1(b) = A_1(b')$ and $A_2(b) = A_2(b')$. Likewise, we can certainly adopt $A_4(b) = A_4(b')$. The annulus $A'(b')$ is the preimage of the annulus $B' \setminus B$ by the parent branch of b. The annulus $A'(b)$ is the preimage of the same annulus by the canonical extension of b, so it has the same modulus. By assumption, there is an annulus A of modulus L surrounding b inside the domain of its canonical extension, hence inside the bounded component of the complement of $A_3(b')$. Thus, we can adopt $A_3(b) = A \oplus A_3(b')$ and the claim of the Lemma follows directly. □

We observe that the assumption about $A_2(b')$ separating the central domain from $A'(b')$ and not just the domain of b' is automatically satisfied for the separating annuli constructed in all lemmas of this section.

b immediate

LEMMA 5.6. – *Suppose that ϕ makes a non-close return and let P have a normalized separation symbol (s_1, s_2, s_3, s_4) with norm β and corrections $\lambda_1(P)$ and $\lambda_2(P)$. Then an immediate branch b of $\tilde{\phi}$ has a normalized separation symbol with norm β and corrections*

$$\lambda_1(b) = \frac{\lambda_2(B)}{2}, \qquad \lambda_2(b) = \frac{\lambda_1(B)}{2}.$$

If, in addition, D is the domain of P and

$$\text{mod}\,(B' \setminus P) \geq s_4 + \epsilon,$$

then a normalized strong separation symbol exists for b with norm $\beta + \frac{\epsilon}{4}$. In both cases, the complement of $A_3(b)$ contains the immediate branches of $\tilde{\phi}$ different from b itself in its unbounded component.

Proof. – The annulus $\overline{A}_2(b)$ will be the preimage of $A'(P) \oplus A_3(P)$ by the ψ. Then, $\overline{A}_1(b)$ is the preimage of $A_4(P)$ by ψ. It follows that we can take

$$\tilde{s}_1 = \frac{\beta + \lambda_2(P)}{2}$$

and

$$\tilde{s}_2 = \frac{\beta - \lambda_1(P)}{2}$$

However, if the additional assumption holds, we can add $\epsilon/2$ to both estimates.

The annulus $\overline{A}'(b)$ is determined and biholomorphically equivalent to $B \setminus \tilde{B}$. Hence, if the additional assumption is satisfied, its modulus is at least $\tilde{s}_1 + \frac{\epsilon}{2}$. Make $\overline{A}_3(b)$ the preimage of $A_2(P)$, and $\overline{A}_4(b)$ the preimage of $A_1(P)$ by ψ. We get

$$\tilde{s}_3 = \frac{\beta + \lambda_2(P)}{2} + \alpha - \lambda_2(P)$$

and

$$\tilde{s}_4 = \tilde{s}_3 + \frac{\lambda_1(P) + \lambda_2(P)}{2} = \frac{\beta}{2} + \alpha + \frac{\lambda_1(P)}{2}.$$

Thus, if we put

$$\lambda_1(b) = \frac{\lambda_2(P)}{2}, \qquad \lambda_2(b) = \frac{\lambda_1(P)}{2}$$

we get a normalized separation symbol.

If the additional assumption is satisfied, then $\tilde{s}_3$ and $\tilde{s}_4$ can be increased by $\epsilon/2$, hence by Lemma 5.3 a normalized symbol can be built with norm $\beta + \frac{\epsilon}{4}$. □

p subordinate to P

LEMMA 5.7. – *Suppose that ϕ makes a non-close return, b is a maximal branch of $\tilde{\phi}$, p and P are chosen as explained in the introduction to this section, and p is subordinate to P. Assume that β is a separation index for ϕ and that the domain of the canonical extension P_e of P is separated from the complement of B' by a ring domain with modulus at least L'. Then, a normalized separation symbol exists for b with norm $\beta + \frac{L'}{4}$. The complement of $A_3(b)$ contains the immediate branches of $\tilde{\phi}$ different from b itself in its unbounded component.*

Proof. – The canonical extension P_e maps the domain of P onto B and the domain of p goes onto the domain of some univalent branch p' of ϕ. Denote $G = P_e \circ \psi$ and let A be the annulus separating the domain of P_e from the complement of B'. Set

$$\begin{aligned} A_2(b) &= G^{-1}(A_2(p')) \\ A_1(b) &= G^{-1}(A_1(p')) \oplus \psi^{-1}(A) \\ A_3(b) &= G^{-1}(A_3(p')) \\ A_4(b) &= G^{-1}(A_4(p')). \end{aligned}$$

The separation symbol is like in the proof of Lemma 5.11 except for extra terms due to the existence of A, hence:

$$\begin{aligned} \tilde{s}_1 &= \frac{\alpha + \lambda_1(p')}{2} + \frac{L'}{2} \\ \tilde{s}_2 &= \frac{\alpha - \lambda_2(p')}{2} \\ \tilde{s}_3 &= \beta + \frac{\alpha - \lambda_1(p') + L'}{2} \\ \tilde{s}_4 &= \beta + \frac{\alpha + \lambda_2(p') + L'}{2}. \end{aligned}$$

As shown in the proof of Lemma 5.11, when $L' = 0$ these estimates yield a normalized separation symbol with norm β. By Lemma 5.3, the presence of these extra terms allows one to construct a normalized symbol with norm $\beta + \frac{L'}{4}$. □

P subordinate to p

LEMMA 5.8. – *Suppose that ϕ makes a non-close return, b is a maximal branch of $\tilde{\phi}$, p and P are chosen as explained in the introduction to this section, and P is subordinate to p. Assume that β is a separation index for ϕ and that the domain of the canonical extension p_e of p is separated from the complement of B' by a ring domain with modulus at least L'. Then, a normalized separation symbol exists for b with norm $\beta + \frac{L'}{4}$. The complement of $A_3(b)$ contains the immediate branches of $\tilde{\phi}$ different from b itself in its unbounded component.*

Proof. – The canonical extension p_e maps the domain of p onto B and the domain of P goes onto the domain of some univalent branch P' of ϕ. Denote $G = p_e \circ \psi$ and let A be the annulus separating the domain of p_e from the complement of B'.

This case becomes similar to the case of immediate branches covered by Lemma 5.6. If we construct the separating annuli as preimages of the appropriate separating annuli for P' by G, we get the same separation symbol as in the proof of Lemma 5.6. The existence of A gives us an extra contribution of $L'/2$ in $\tilde{s}_3$ and $\tilde{s}_4$. The proof is concluded by invoking Lemma 5.3. □

P and p are independent

LEMMA 5.9. – *Suppose that ϕ makes a non-close return and let β be a separation index for ϕ. Suppose that the domain of the canonical extension of P is separated from the complement of B' by a ring domain with modulus at least L'. We make two claims:*

- *If b is a maximal univalent branch of $\tilde{\phi}$, p and P are chosen as explained in the set-up for this section, and p and P turn out to be independent, then a normalized separation symbol exists for b with corrections*

$$\lambda_1(b) = \frac{\delta}{2},$$
$$\lambda_2 = \frac{\alpha - \lambda}{2}$$

where $\delta = \max(\lambda_2(P), \lambda - \alpha + L')$. If $\lambda_1(P) + \lambda_2(P) \geq \epsilon$, then b has a normalized separation symbol with norm $\beta + \frac{\epsilon}{4}$.

- *Let $s = (s_1, s_2, s_3, s_4)$ be a normalized separation symbol for P with norm β. If, in addition,*

$$\operatorname{mod}\,(A'(P) \oplus A_3(P) \oplus A_4(P)) \geq s_4 + \epsilon,$$

then a normalized separation symbol exists for b with norm $\beta + \frac{\epsilon}{4}$. In all cases, the complement of $A_3(b)$ contains the immediate branches of $\tilde{\phi}$ different from b itself in its unbounded component.

Proof. – Observe that Since p and P are independent, the domains of their canonical extensions are disjoint. To pick $A_2(b)$, we take $\psi^{-1}(A'(P))$. The ring domain $A'(P)$ is biholomorphically equivalent to $B' \setminus B$, and hence its modulus is at least $\alpha + \lambda$. Hence,

$$\tilde{s}_2 = \frac{\alpha + \lambda}{2}.$$

Set $A_1(b) = \psi^{-1}(A_3(P) \oplus A_4(P))$. There are two ways of estimating $\tilde{s}_1$. One estimate is

$$\tilde{s}_1 = \frac{\beta + \lambda_2(P)}{2}.$$

Another is based on the observation that mod $A_1 \geq L'$. So, we can also put $\tilde{s}_1 = \tilde{s}_2 + \frac{L}{2}$. Thus,

$$\tilde{s}_1 = \frac{\beta + \delta}{2}$$

is always a valid estimate. Observe that if the extra assumption is satisfied, we can pick $\delta := \min(\lambda - \alpha + L, \lambda_2(P) + \epsilon)$ and the estimate for s_1 is still valid.

As always, $\tilde{A}'(b)$ is determined with modulus at least $\tilde{s}_1$. The annulus $\tilde{A}_3(b)$ will be obtained as the preimage by ψ of $A'(p)$. This has modulus at least $\alpha + \lambda$ in all cases as argued above. The annulus $\tilde{A}_4(b)$ is the preimage of $A_3(p) \oplus A_4(p)$. By induction,

$$\tilde{s}_3 = \frac{\beta + \delta}{2} + \alpha + \lambda$$

and

$$\tilde{s}_4 = \tilde{s}_3 + \frac{\beta + \lambda_2(p) - \alpha - \lambda}{2} = \beta + \frac{\alpha + \delta + \lambda + \lambda_2(p)}{2}.$$

If the extra assumption is satisfied, then both $\tilde{s}_3$ and $\tilde{s}_4$ can be increased by $\epsilon/2$.

We put $\lambda_1(b) = \frac{\delta}{2}$ and $\lambda_2(b) = \frac{\alpha - \lambda}{2}$. We check that

$$\tilde{s}_3 + \lambda_1(b) = \frac{\beta}{2} + \alpha + \delta + \lambda \geq \beta + \delta + \lambda.$$

Note that

$$\delta + \lambda \geq \lambda + \lambda_2(P) \geq 0.$$

Moreover, if the extra assumption is satisfied, the $\delta + \lambda \geq \epsilon$.

In a similar way one verifies that

$$\tilde{s}_4 - \lambda_2(b) \geq \beta + \frac{\lambda + \delta}{2}.$$

Also, it is clear that $\lambda_1(b), \lambda_2(b) \geq 0$. Finally,

$$\lambda_1(b) + \lambda_2(b) = \frac{1}{2}(\alpha + \delta - \lambda) \geq \frac{\alpha - \lambda + \lambda - \alpha + L}{2} > 0.$$

Hence, by possibly decreasing $\tilde{s}_3$ and $\tilde{s}_4$ we get a normalized separation symbol with norm β. If either $\delta + \lambda \geq \lambda_1(P) + \lambda_2(P) \geq \epsilon$ or the extra assumption of the second claim holds, we actually see that s_3 and s_4 should both be decreased by at least $\epsilon/2$ to give a normalized symbol. In that case, Lemma 5.3 implies that the norm can be increased by $\epsilon/4$. □

Summary

The results of this section are summarized as follows.

LEMMA 5.10. – *Suppose that ϕ makes a non-close return. Assume also that the domains of its parent branches are all separated from the complement of B' by ring domains with modulus L'. Suppose that β is a separation index for ϕ, and that the post-critical branch P has a normalized separation symbol (s_1, s_2, s_3, s_4) with norm β and that mod $(A'(P) \oplus A_3(P) \oplus A_4(P)) - s_4 \geq \epsilon$. Then for every $\epsilon > 0$ and $L' > 0$ there is a $K > 0$ so that the mapping resulting from ϕ after a simple inducing step (here equivalent to a type I inducing step) has separation index $\beta + K$.*

Proof. – Let $\tilde{\phi}$ denote the mapping obtained from ϕ by a simple inducing step. Let us choose a univalent branch b of $\tilde{\phi}$, and find the corresponding branch p of ϕ. If b is not maximal, our claim follows from Lemma 5.5. So let us assume that b is maximal. If b is immediate, our claim follows from Lemma 5.6. If p and P are independent, or p is subordinate to P, or P is subordinate to p, the claim follows from Lemmas 5.9, 5.7 and 5.8, respectively. ☐

5.3. Close returns

ϕ makes a close return

LEMMA 5.11. – *Suppose that ϕ makes a close return. Assume that a normalized separation symbol exists for p with norm β. Then a normalized separation symbol*

$$\tilde{s} = (\tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4)$$

exists for b with norm β. Moreover, $\tilde{s}_3 \leq s_3 +$ mod $(B' \setminus B)$ while A_3 is conformally equivalent to $A'(p) \oplus A_3(p)$.

Proof. – We mostly repeat the analysis of the close return from Chapter I. Consider $A_2(p)$ and $A_1(p)$. Their preimages by the central branch give us $A_2(b)$ and $A_1(b)$, respectively. The estimates are

$$\tilde{s}_2 = \frac{\alpha - \lambda_2(p)}{2}$$

and

$$\tilde{s}_1 = \frac{\alpha + \lambda_1(p)}{2}.$$

The annulus $A'(b)$ is uniquely determined with modulus at least $\tilde{s}_1$, and $A_3(b)$ will be the preimage by ψ of $A_3(p) \oplus A'(p)$. Finally, $A_4(b)$ will be the preimage of $A_4(p)$ by ψ. The estimates are

$$\tilde{s}_3 = \frac{\alpha + \lambda_1(p)}{2} + \beta - \lambda_1(p) = \beta + \frac{\alpha - \lambda_1(p)}{2}$$

and

$$\tilde{s}_4 = \tilde{s}_3 + \frac{\lambda_1(p)+\lambda_2(p)}{2} = \beta + \frac{\alpha+\lambda_2(p)}{2}.$$

Set

$$\tilde{\lambda}_1 = \frac{-\alpha+\lambda_1(p)}{2}$$

and

$$\tilde{\lambda}_2 = \frac{\alpha+\lambda_2(p)}{2}.$$

The requirements of a normalized symbol are satisfied. □

Improved separation bounds

LEMMA 5.12. – *Under the hypotheses of Proposition 3, suppose that some ϕ_i for $i \geq 1$ makes a close return with escaping time $E > 1$. Let ϕ denote the mapping obtained from ϕ_i by $E-1$ simple inducing steps. Thus, ϕ shows a non-close return and ϕ_{i+1} results from ϕ in a simple inducing step, so we denote $\tilde{\phi} = \phi_{i+1}$.*

Assume that β is a separation index for ϕ_i. Then every univalent and non-immediate branch b of ϕ_{i+1} has a normalized separation symbol with norm $\beta + K$ where $K > 0$ depends only on L from the statement of Proposition 3. The complement of $A_3(b)$ contains the immediate branches of ϕ_{i+1} different from b itself in its unbounded component.

Proof. – To fix the notations, assume that ϕ arose from some ϕ_{-1} in a simple inducing step showing a close return. Let B_{-1} and B'_{-1} denote the domain and range of the central branch of ϕ_{-1}, respectively. Take some univalent non-immediate b branch of $\tilde{\phi}$ and find branches p and P of ϕ in the usual way. Unless b is maximal, we are done by Lemma 5.5. Observe that any parent branch of ϕ extends in a univalent fashion onto the image B'_1. By assumptions of Proposition 3, mod $(B'_1 \setminus B_1) \geq 2L$. Hence, if p and P are in the relation of subordination, we are also done by Lemmas 5.7 and 5.8. Hence we only need to consider the case when p and P are independent with b maximal.

Let us next look at the simple inducing step in which ϕ was created. Unless P is maximal, we are done. Indeed. by Lemma 5.5 P then has an improved separation symbol. By Lemma 5.4 this means that the extra assumption of Lemma 5.11 is satisfied and Lemma 5.12 follows. So let P' be the parent branch of P, and define P_{-1} (a univalent branch of ϕ_{-1}) by the requirement $P' = P_{-1} \circ \psi_{-1}$. Similarly, Let p' be the parent branch of p so that $p' = p_{-1} \circ \psi_{-1}$. There are two distinct cases we encounter depending of whether p_{-1} is subordinate to P_{-1} or they are independent.

If p_{-1} is subordinate to P_{-1}, then there is a mapping $P_{-1,e}$ (the canonical extension of P) which sends the domain of P_{-1} to B_{-1}, and the domain of p_{-1} to the domain of some branch p'_{-1} of ϕ_{-1}. Denote $G = \psi \circ P_{-1,e}$ and consider annuli defined as follows:

- $A^i(P) = G^{-1}(A_2(p'_{-1})) \oplus A'(P)$,
- $A^e(P) = G^{-1}(A_1(p'_{-1}))$,
- $A^i(p) = G^{-1}\left[A_3(p'_{-1}) \oplus A'(p'_{-1})\right] \oplus A'(p)$,
- $A^e(p) = G^{-1}(A_4(p'_{-1}))$.

To estimate their moduli, we observe that G maps univalently onto B'_{-1} and that $A'(p)$ and $A'(P)$ are conformally equivalent to $B' \setminus B$, hence they have modulus at least L. So,

$$\begin{aligned} \text{mod } A^i(P) &\geq L + \alpha - \lambda_2(p'_{-1}) \\ \text{mod } A^e(P) + \text{mod } A^i(P) &\geq L + \alpha + \lambda_1(p'_{-1}) \\ \text{mod } A^i(p) &\geq L + \beta - \lambda_1(p'_{-1}) \\ \text{mod } A^i(p) + \text{mod } A^e(p) &\geq L + \beta + \lambda_2(p'_{-1}) \end{aligned}$$

We now define $A_2(b) = \psi^{-1}(A^i(P))$ and

$$\tilde{s}_2 = \frac{\beta - \lambda_2(p'_{-1})}{2} + \frac{L}{2}.$$

Next, $A_1(b) = \psi^{-1}(A^e(P))$ and

$$\tilde{s}_1 = \frac{\alpha + \lambda_1(p'_{-1})}{2} + \frac{L}{2}.$$

Then $A'(b)$ is determined with modulus at least $\tilde{s}_1$, and we set $A_3(b) = \psi^{-1}(A^i(p))$, hence

$$\tilde{s}_3 = \beta + \frac{\alpha - \lambda_1(p'_{-1})}{2} + L.$$

Also, $A_4(b) = \psi^{-1}(A^e(p))$ and

$$\tilde{s}_4 = \beta + \frac{\alpha - \lambda_2(p'_{-1})}{2} + L.$$

With $L = 0$ these are the same estimates as in the analysis of the close return (see the proof of Lemma 5.11). So they yield a normalized symbol with norm β. The extra L term gives the possibility of constructing a better symbol with norm $\beta + \epsilon/2$ (*see* Lemma 5.3.) Hence in this case Lemma 5.12 follows.

We still need to analyze the situation with P_{-1} and p_{-1} independent. In tis situation, we get the annulus $A = \psi^{-1}(A'(p_{-1}))$ with modulus at least $2L$ which separates p from P inside $A_3(p)$. Moreover, from the analysis of the close return (see the proof of Lemma 5.11), we see that

$$\beta - \lambda_1(p) \geq \text{mod } A'(p) + \text{mod } A + \text{mod } \psi^{-1}(A_3(p_{-1})).$$

The annuli $A_1(b)$ and $A_2(b)$ are chosen exactly as in the proof of Lemma 5.9, hence

$$\tilde{s}_1 = \frac{\beta + \delta}{2}$$

and

$$\tilde{s}_2 = \frac{\alpha + \lambda}{2}$$

Next,

$$A_3(b) = \psi^{-1}(A) \oplus A'(p)$$

and

$$A_4(b) = \psi^{-1}(\psi^{-1}(A_3(p_{-1})) \oplus A_4(p))$$

to get (in the notations of Lemma 5.9)

$$\tilde{s}_3 = \frac{\beta + \delta}{2} + \alpha + \lambda + 2L$$

and

$$\tilde{s}_4 = \beta + \frac{\alpha + \delta + \lambda + \lambda_2(p)}{2} + L.$$

In the proof of Lemma 5.9 we show that for $L = 0$ this yields a normalized symbol with norm β, so by Lemma 5.3 our estimates give a norm $\beta + \frac{L}{2}$. This concludes the proof of Lemma 5.12 □

5.4. Derivation of Proposition 3

Reduction

First, note that Proposition 3 follows if we can prove the following

Statement

For all non-immediate univalent branches of ϕ_j normalized separation symbols exist with norm $\beta + K$ where $K > 0$ depends on L only. Indeed, let ϕ be chosen so that ϕ_{j+1} is derived from ϕ by a simple inducing step. In particular, ϕ shows a non-close return and ϕ is either equal to ϕ_j or derived from it by a sequence of simple inducing steps all showing close returns. As a consequence of our assumption, using Lemmas 5.5 and 5.11 we show that all branches of ϕ save the pseudo-immediate ones have normalized separation symbols with norm $\beta + K'$ where K' only depends on L. By Lemma 5.4, this also implies that normalized separation symbols with norm β can be built for all univalent and not pseudo-immediate branches so that mod $(A' \oplus A_3) - s_3 \geq \frac{K'}{2}$. Note here that the parameter L' which occurs in the statement of several lemmas can be taken to be $L/2$ by Lemma 5.2. Now our reduced statement implies Proposition 3 by Lemma 5.10.

So we will concentrate on proving the reduced statement. If ϕ_{j-1} shows a close return, our statement is a direct consequence of Lemma 5.12. We consider two distinct cases:

- ϕ_{j-2} shows a close return, or
- ϕ_{j-2} shows a non-close return.

Either way, this is followed by a non-close return for ϕ_{j-1}.

A close return for ϕ_{j-2}

Notice that we are done if ϕ_{j-1} shows a non-rotational return. Indeed, by Lemma 5.12 it means that the post-critical branch has an improved normalized separation symbol with norm $\beta + K$. Hence, in view of Lemmas 5.3 and 5.10, the separation index improves already for ϕ_j. So let us assume that ϕ_{j-1} makes a rotation-like return. Notice that in view of our constructions separation symbols for branches of ϕ_{j-1}, for every univalent branch

p' different from P, the annulus $A_3(p')$ separates the domain of p' from the domain of P. Let us use the established notations with $\phi := \phi_{j-1}$ and $\tilde{\phi} := \phi_j$. Choose a non-immediate univalent branch b of $\tilde{\phi}$ and try to pick a separation symbol for b with norm $\beta + K$ where $K > 0$ depends on L. If b is not maximal, or p and P are in a relation of subordination, this is possible as seen in the proof of Lemma 5.10. So the only case we need to concentrate on is when b is maximal and p is independent from P.

In this case, we construct the separating annuli $A_1(b)$ and $A_2(b)$ is in the proof of Lemma 5.9. However, $A_3(b)$ can be set equal to $\psi^{-1}(A_3(p))$, and then $A_4(b) = \psi^{-1}(A_4(p))$. Using notations λ and δ introduced in Lemma 5.9, this leads to estimates:

$$\begin{aligned}
\tilde{s}_1 &= \frac{\beta+\delta}{2}\\
\tilde{s}_2 &= \frac{\alpha+\lambda}{2}\\
\tilde{s}_3 &= \beta - \lambda_1(p) + \frac{\beta+\delta}{2}\\
\tilde{s}_4 &= \beta + \frac{\beta+\delta}{2} + \frac{\lambda_2(p)-\lambda_1(p)}{2}
\end{aligned}$$

If we set

$$\begin{aligned}
\lambda_1(b) &= \delta/2\\
\lambda_2(b) &= \frac{\alpha-\lambda}{2}
\end{aligned}$$

then we already checked in the proof of Lemma 5.9 that $\lambda_1(b) + \lambda_2(b) \geq 0$ and $|\lambda_1(b), \lambda_2(b)| \leq \alpha$. Next,

$$\tilde{s}_3 + \lambda_1(b) = \beta + \alpha + \delta - \lambda_1(p).$$

As $\delta \geq \lambda - \alpha + L$, we get

$$\beta + \alpha + \delta - \lambda_1(p) \geq \beta + L.$$

Then

$$\begin{aligned}
s_4 - \lambda_2(p) &= \beta + \frac{\alpha+\lambda+\delta+\lambda_2(p)-\lambda_1(p)}{2}\\
&\geq \beta + \frac{\alpha+\delta+\lambda_2(p)}{2}\\
&\geq \beta + \frac{\lambda+\lambda_2(p)+L}{2} \geq \beta + \frac{L}{2}
\end{aligned}$$

where we again used the estimate $\delta \geq \lambda - \alpha + L$. Hence, after decreasing s_3 and s_4 by at least $L/2$, we still get a normalized separation symbol with norm β, hence by Lemma 5.3, a normalized symbol with norm $\beta + \frac{L}{4}$ also exists.

This shows that Proposition 3 holds if ϕ_{j-2} shows a non-close return.

ϕ_{j-2} makes a non-close return

Again use our usual notations making $\phi := \phi_{j-1}$ and choose a univalent non-immediate branch b of $\tilde{\phi} := \phi_j$. Like in the previous case, we can restrict our attention to the case when b is maximal while p and P are independent. We first analyze the combinatorial situation of P. If P has a normalized separation symbol with norm $\beta + K$ where $K > 0$ depends only on L, we are done by Lemmas 5.4 and 5.10. Hence P must be maximal. Next, suppose that P is immediate. Then we are exactly in the situation considered above when ϕ_{j-2} made a close return. Again, we are done.

For further analysis, we need the post-critical branch P^* of ϕ_{j-2} and a univalent branch P' of ϕ_{j-2} defined so that $P' \circ \psi_{j-2}$ is the parent branch of P. Unless P^* and P' are independent, we are done by Lemmas 5.7 and 5.8. If they are independent, then by Lemma 5.9

$$\lambda_1(P) = \frac{\delta'}{2}$$
$$\lambda_2(P) = \frac{\alpha - \lambda'}{2}$$

where the improtant feature of δ' is that $\delta' \geq \lambda - \alpha + L$, thus

$$\lambda_1(P) + \lambda_2(P) \geq \frac{L}{2}.$$

Now Lemma 5.9 implies that b has a normalized separation symbol with norm $\beta + \frac{L}{8}$.

This concludes the proof of Proposition 3.

6. Consequences of the Modulsatz

6.1. Technical machinery

Plan of the work

In the preceding section we analyzed cases when an increase of the separation index followed by "combinatorial" reasons. We are left with two situations left unsolved by Proposition 3:

• when the assumption about L is violated, that is when a close return occurs with large escaping time,

• an unbroken sequence of rotation-like returns.

In both cases, the increase will follow from the fact that the equality in the superadditivity estimate for nesting ring domains

$$\text{mod } (A \oplus B) \geq \text{mod } A + \text{mod } B$$

occurs only in a special geometric configuration. Based on this, we will be able to solve the first difficulty completely and for all holomorphic type I mappings by showing that if it occurs, then an increase in the separation index follows for reasons related to the Modulsatz. For the second difficulty (rotation-like returns) we will furnish a solution based on assuming that the holomorphic box mappings are extensions of real box mappings. Here the estimates we got for real rotational-like sequences will be crucial.

The statement of Teichmüller's Modulsatz

Fact 6.1.

Let A_1 and A_2 be two disjoint open annuli situated so that A_1 separates 0 from A_2 while A_2 separates A_1 from ∞. Assume further that both are contained in the ring $A = \{z : r < |z| < R\}$ for some $0 < r < R$. By C denote the set (annulus) of all points from $A \setminus (A_1 \cup A_2)$ separated from 0 and ∞ by $A_1 \cup A_2$. Then, for every $\delta > 0$ there is a number $\epsilon > 0$ with the following property: if

$$\text{mod } A_1 + \text{mod } A_2 \geq \text{mod } A - \epsilon,$$

then a ρ exists for which the ring

$$\{z : \rho < |z| < (1+\delta)\rho\}$$

contains C.

Fact 6.1 follows directly from the "Modulsatz" of [22].

Conformal roughness

DEFINITION 6.1. – *Let w be a Jordan curve in the plane. We say that w is (ϵ, M)-rough if for every pair of open annuli C_1 and C_2 subject to the conditions*

- *C_1 is contained in the bounded component of the complement of w,*
- *w is contained in the bounded component of the complement of C_2,*
- *the moduli of both annuli are at least M,*

the inequality

$$\text{mod } (C_1 \oplus C_2) > \text{mod } C_1 + \text{mod } C_2 + \epsilon$$

holds.

For example, a consequence of Teichmüller's Modulsatz is that every non-analytic Jordan curve is $(0, M)$-rough for every positive M. The important idea is that the Modulsatz can imply roughness of some curves and that roughness can help us get stronger estimates for separating annuli.

Technical facts

We will first prove a few technical facts. The reader may choose to skip them now and return when they are referenced from the text.

LEMMA 6.1. – *Let w_i and w_o be Jordan curves in the complex plane so that w_o is contained in the unbounded component of the complement of w_i. Let G be the ring domain bounded by w_i and w_o. Let v_i be a Jordan curve in the unbounded component of the complement of w_i situated so that the Hausdorff distance from w_i to v_i is less than $\delta \cdot \operatorname{diam} w_i$. Let M denote the modulus of the ring domain bounded by v_i and w_o if v_o is contained in G, or 0 otherwise. Then for every $\epsilon > 0$ there is a $\delta_0 > 0$ so that if $\delta \leq \delta_0$, then mod $G - M \leq \epsilon$.*

Proof. – Fix an $\epsilon > 0$. If mod $G \leq \epsilon$, there is nothing to prove. Otherwise, consider the Jordan curve t which is the image of a circle centered at 0 by the canonical mapping of G, chosen so that the modulus of the ring domain bounded by w_i and t is ϵ. We will be done if v_i is enclosed between w_i and t. We need a fact which follows directly from Teichmüller's module theorem stated on page 56 in [13]:

Fact 6.2.

If the ring domain G separates the points 0 and z_1 from z_2 and ∞, then the bound

$$\operatorname{mod} G < \log \frac{|z_1| + |z_2|}{|z_1|} + C\left(\left|\frac{z_2}{z_1}\right|\right)$$

holds where the function C is bounded by $2 \log 4$ and goes to 0 as its argument decreases to 0.

Choose a point on w_i. By moving the frame we assume without loss of generality that this point is at 0. There is a point z_1 on w_i whose distance from 0 is at least half of the diameter of w_i. If z_2 is a point on t, then Fact 6.2 allows one to bound $|z_2|/|z_1|$ in terms of ϵ. Taking twice this bound as δ_0, we conclude the proof. □

If γ_1 and γ_2 are Jordan curves in the plane, let $M(\gamma_1, \gamma_2)$ denote the modulus of the ring domain bounded by γ_1 and γ_2 if the curves do not intersect, or 0 otherwise.

An application of the Modulsatz

LEMMA 6.2. – *Let w be a Jordan curve in the plane. For every M and δ positive there numbers K, which is independent of δ and $\Delta > 0$ so that if w is not (Δ, M)-rough, then there is a K-quasiconformal Jordan curve in the Hausdorff distance less than $\delta \cdot \operatorname{diam} w$ from w.*

Proof. – Choose annuli C_1 and C_2 as in the Definition 6.1 so the both have the same modulus $M > 0$. Let C be $C_1 \oplus C_2$ and H be the standard conformal mapping from a ring $\{z : r < |z| < R\}$. We can choose the scale so that $r = R^{-1}$. Let $t = H^{-1}(w)$ and choose the smallest closed ring with inner radius r_1 and outer radius R_1 which contains t. Then for every $M > 0$ there is $\lambda > 1$ so that $r_1 > \lambda r$ and $R_1 < R/\lambda$. Indeed, this follows from Teichmüller's module theorem, quoted earlier in this paper as Fact 6.2.

Assume that mod $C < 2M + \log \lambda$. Observe first that it implies $r_1 \leq \sqrt{\lambda} \exp(-M)$ and $R_1 \geq \exp(M)/\sqrt{\lambda}$. Change the frame so that the diameter of w becomes 1 and w separates

0 from ∞ and consider the family of mappings H from the ring $\{z : e^{-M} < |z| < e^{M}\}$ onto ring domains containing w but avoiding 0 and ∞. This is an equicontinuous family (which follows from Fact 6.5). Hence, for every $\delta > 0$ there is a $\delta_1 > 0$ so that if $\log R_1/r_1 \leq \delta_1$, then the Hausdorff distance from w to the image s of the circle centered at 0 with radius r_1 is less than δ. Note that s is a K-quasiconformal Jordan curve where K only depends on λ (from Lemma 6.3), thus ultimately on M. On the other hand, if $\log \frac{R_1}{r_1} > \delta_1$, then Teichmüller's Modulsatz (quoted here as Fact 6.1) implies that

$$\operatorname{mod} C \geq 2M + \Delta$$

where Δ depends on δ_1.

We see that for every M and δ either s can be chosen which follows w in the Hausdorff distance δ, or for every C_1 and C_2 chosen as above with modulus exactly M

$$\operatorname{mod} C \geq 2M + \Delta$$

with Δ depending on δ and M. This last situation implies (Δ, M)-roughness since when C_1 has modulus larger than M it can be split into the sum of two nesting annuli whose moduli add up exactly to mod C_1 and the one adjacent to w has modulus M. The same reduction applies when the modulus of C_2 is larger than M. □

Bounded turning

A very convenient critierion for checking whether a Jordan curve is K-quasiconformal, was given by Ahlfors, *see* [13] where it is called "bounded turning" (also known as the "three-point property"):

Fact 6.3.

Let U be a Jordan curve in the Riemann sphere. For every pair of points z_1 and z_2 consider u_1 and u_2 defined as diameters, in the spherical metric, of the two arcs cut from U by z_1 and z_2. Define

$$Z := \sup \left\{ \frac{\min(u_1, u_2)}{\operatorname{dist}(z_1, z_2)} \ : \ z_1, z_2 \in U \right\}.$$

For every finite Z there is a Q so that U is Q-quasiconformal. Conversely, if U is a Q-quasiconformal, then Z is bounded in terms of Q.

Distortion in ring domains

LEMMA 6.3. – *Let G be a ring domain. Let g be a canonical univalent mapping from a ring domain bounded by two circles concentric at 0 onto G. Let w be the image by g of some circle centred at 0 so that both components of the complement of w in G are annuli with moduli at least $\delta > 0$. For every $\delta > 0$ there is a Q so that g restricted to $g^{-1}(w)$ can be continued to a Q-quasiconformal homeomorphism of the plane. In particular, w is a Q-quasiconformal Jordan curve.*

Proof. – Lemma 6.3 is a direct corollary from the following.

Fact 6.4.

Let $w_0 : G \to G'$ *be a* K*-quasiconformal mapping and* F *a compact subset of the domain* G*. There exists a quasiconformal mapping of the whole plane which coincides with* w_0 *on* F *and whose maximal dilatation is bounded by a number depending only on* K*,* G *and* F*.*

Fact 6.4 is a verbatim quotation of Theorem 8.1, page 96, from [13].

Normalize our situation so that g^1 of w is the unit circle. Then w_0 is defined at least on the ring

$$\{z : e^{-\delta} < |z| < e^{\delta}\}$$

and we can take this ring as G in Fact 6.4. Then w_0 is g restricted to this ring, which is 1-quasiconformal, while F is the unit circle. The claim of Lemma 6.3 follows directly. □

6.2. Deeply nested close returns

PROPOSITION 4. – *Let* ϕ *be a type I holomorphic box mapping. Denote by* β *a separation index of* ϕ*. Also, adopt the usual notations* ψ *for the central branch of* ϕ*,* B *for the domain of* ψ *and* B' *for its range. Let* E *be the escaping time of* ϕ *with* $E \geq 2$*. Suppose that* $\beta \geq \beta_0$ *and* mod $(B' \setminus B) \geq \alpha_0$*.*

For every $\alpha_0 > 0$ *and* $\beta_0 > 0$ *there are positive numbers* δ_0 *and* δ_1 *so that one of the following holds true:*

- *after three type I inducing steps (assuming that they are feasible),* ϕ *yields a type I holomorphic box mapping with separation index* $\beta + \delta_0$*, or*
-

$$\text{mod } \psi^{1-E}(B' \setminus B) \geq \delta_1 .$$

The first step of the proof is as follows.

LEMMA 6.4. – *Let* ϕ *be a type I holomorphic box mapping. Denote by* β *a separation index of* ϕ*. Also, adopt the usual notations* ψ *for the central branch of* ϕ*,* B *for the domain of* ψ *and* B' *for its range. Let* E *be the escaping time of* ϕ *with* $E \geq 2$*. Suppose that* $\beta \geq \beta_0$ *and* mod $(B' \setminus B) \geq \alpha_0$*. Let* ϕ_1 *denote the holomorphic box mapping derived from* ϕ *is a type I inducing step.*

For every $\alpha_0 > 0$*,* $\beta_0 > 0$ *and* M *there are positive numbers* δ_0, δ_1 *and* η *so that one of the following holds true:*

- ϕ_1 *has separation index* $\beta + \delta_0$*, or*
-

$$\text{mod } \psi^{1-E}(B' \setminus B) \geq \delta_1, \textit{or}$$

- *the boundary of the central domain of* ϕ_1 *is the preimage by* $z \to z^2$ *of an* (η, M)*-rough curve.*

Proof. – We will split the proof of Lemma 6.4 into a number of steps.

Step I of the proof

Let us introduce the notation B^i for $0 \leq i \leq E$ to denote $\psi^{-i}(B')$. Consider the following statement:

Choose parameters $\delta > 0$ and $K \geq 1$. For every such choice, and for every $\alpha_0 > 0$ in the statement of Lemma 6.4, there are parameters $\Delta_0 > 0$, $\delta_1 > 0$ and $\eta_1 > 0$ for which at least one statement in the following alternative holds true:

a.

$$\text{mod}\,(B^{E-1} \setminus B^E) + \text{mod}\,(B' \setminus B^{E-1}) \leq \text{mod}\,(B' \setminus B^E) - \eta_1.$$

b.

There is no K-quasiconformal Jordan curve contained in the closure of B^E in Hausdorff distance less than $\Delta_0 \cdot \operatorname{diam} B^E$ from the border of B^E.

c.

The Hausdorff distance from B^{E-1} to B^E is less than $\delta \cdot \operatorname{diam} B^E$.

d.

The modulus of $B^{E-1} \setminus B^E$ is at least δ_1.

Let w be a K-quasiconformal Jordan curve contained in the closure of B^E. Consider the ring domain G delimited by w and the boundary of B'. Let H be the canonical (conformal) mapping from the round ring

$$\{z : 1 < |z| < R\}$$

onto G chosen so that w corresponds to the circle with radius 1 by the continuous continuation of H. Observe that H can be continued as a K^2-quasiconformal mapping defined on $\{z : R^{-1} < |z| < R\}$. Indeed, this is done by the quasiconformal reflection in w and the inner boundary of G'. Keep the notation H for this extended homeomorphism. As a consequence of $E \geq 2$, we have $R \geq R_0 := e^{\alpha_0/2}$. Adjust the frame so that 0 belongs to the bounded component of the complement of w and $\operatorname{diam} w = 1$. Consider the family $\mathcal{W}$ containing all eligible mappings H, that is the family of all K^2-quasiconformal homeomorphisms from the ring $G' = \{z : R_0^{-1} < |z| < R_0\}$ which avoid a point z_0 with $|z_0| < 1$ and the complement of the unit disk. We now quote Theorem 4.1 from page 69 in [13]:

Fact 6.5.

Let $\mathcal{W}$ be a family of Q-quasiconformal mappings from the domain G into the complex plane. If every mapping $w \in \mathcal{W}$ omits two values whose spherical distance is greater than a fixed positive number d (the omitted values need not to be fixed), then $\mathcal{W}$ is equicontinuous in G.

We see that our family $\mathcal{W}$ is equicontinuous. Let μ be the module of continuity, *i.e.* a function chosen so that if $|x-y| < \mu(\epsilon)$, then $|H(x) - H(y)| < \epsilon$ for every $\epsilon > 0$ and $x, y \in G'$. The module of continuity only depends of K and α_0. For every $\delta > 0$ specified in the claim of Step I, choose the number ϵ_1 to be the minimum of $R_0 - \sqrt{R_0}$ and $\mu(\frac{\epsilon_1}{2})$. Next, specify δ_1 in the claim of Step I so that e^{δ_1} is the minimum of ϵ_1 and R_0.

Let t be the preimage by H of the border of B^{E-1}. Suppose that case **d.** fails with this choice of δ_1. This means that t contains a point z_0 with $|z_0| < e^{\delta_1}$. Then ask whether t is contained in the annulus $\{z : 1 < r_1 < |z| < r_2\}$ with $\log \frac{r_2}{r_1} < \epsilon_1$. By the choice of δ_1 and ϵ_1 this implies that t is contained in G'. Moreover, the Hausdorff distance from t to the unit circle is less than $\mu(\delta)$. We see that case **c.** must occur. If t is not contained in this band, then by Teichmüller's Modulsatz (quoted as Fact 6.1), for some $\eta_2 > 0$ we get

$$\text{mod } (B' \setminus B^{E-1}) + \text{mod } U \leq \text{mod } V - \eta_2.$$

Here U is the annulus delimited by w and the border of B^{E-1}, and V is bounded by w and the boundary of B'. Observe that η_2 depends only on ϵ_1, thus ultimately on α_0, δ and K. From Lemma 6.1, for every η_2 we can choose $\Delta_0 > 0$ so that if the Hausdorff distance from w to the border of B^E is less than $\Delta_0 \text{diam}\, w$, then

$$\text{mod } U \leq \text{mod } (B^{E-1} \setminus B^E) + \frac{\eta_2}{10}$$
$$\text{mod } V \leq \text{mod } (B' \setminus B^E) + \frac{\eta_2}{10}.$$

With this choice of Δ_0 in the claim of Step I, and $\eta_1 := \eta_2/2$, we see that either case **a.** or **b.** occurs. This concludes the proof of the claim of Step I.

Step II

We show the following statement:

Given any α_0 and K in Step I, by choosing δ appropriately and making Δ_0 possibly smaller than in Step I, but depending only on α_0 and K, it is possible to get rid of case **c.**

Observe that B^E is a preimage of B^{E-1} by ψ, and hence is a topological disc bounded by a Jordan curve and symmetrical with respect to the transformation $z \to -z$. Let us first make $\delta \leq \frac{1}{10}$. If case **c.** holds, then there is a point on the border of B^{E_1} in the distance at least $\frac{1}{2}\text{diam}\, B^{E-1}$ from ψ^0, and another one in the distance less than $\delta \text{diam}\, B^{E-1}$. The branch ψ is a composition $H(z^2)$ where H is univalent and extends at least onto B', so by Köbe's distortion theorem its distortion on the preimage of B^{E-1} is bounded in terms of α_0. It follows that on the border of B^E there are points z_1 and z_2 so that

$$|z_1| < K_1 |z_2| \sqrt{\delta} \tag{32}$$

where K_1 depends on α_0 only. Now suppose that $\Delta_0 \leq \frac{1}{10}$. Choose points $z_1' \in w$ on the segment $(0, z_1]$, $z_1'' \in w$ on the segment $(0, -z_1]$, $z_2' \in w$ on the segment $(0, z_2]$ and $z_2'' \in w$ on the segment $(0, -z_2]$. Clearly, $|z_1'|, |z_1''| \leq |z_1|$ and $|z_2'|, |z_2''| \geq \frac{|z_2|}{2}$. Both arcs of w joining z_1' and z_1'' have diamters at least $\frac{|z_2|}{2}$, while the euclidean distance between these points is at most $2|z_1|$. For every K and by estimate (32), one can choose a small δ, depending on α_0 and K so that the corresponding three-point property (Fact 6.3) is violated for w, thus w is not K-quasiconformal. Then case **c.** in Step I can be eliminated by showing that it leads to case **b.** The constant Δ_0 should ultimately be chosen equal to $\min(\Delta_0, \frac{1}{10})$ where Δ_0 is obtained from Step I for α_0, K and δ picked out above.

Step III

We show the following:

For every M_0, it is possible to specify K in the claim of Step II, and to find another Δ_0, both depending only on M_0 and α_0, so that if case **b.** *occurs, then the boundary of B^E is (Δ, M)-rough in the meaning of Definition 6.1.*

Our tool here is Lemma 6.2 applied with $w := \partial B^E$ and $M := M_0$. Using that lemma, first choose K depending on M only. Then specify Denote by δ as chosen in Step II for α_0 and some value of K. Then the claim of Step III is implies directly by Lemma 6.2.

Step IV

Now we move towards proving Lemma 6.4. Given arbitrary parameters M_0, α_0 from the statement of Lemma 6.4, we specify K by Step III. Then we return the triple alternative left by Step II and assume that case **d.** occurred. This leads immediately to the second part of the alternative of Lemma 6.4 being satisfied if we choose δ_1 as in Step II.

It requires a little work to see that case **a.** leads to the first possibility in Lemma 6.4. Let us observe that case **a.** implies

$$\sum_{j=1}^{E} \mathrm{mod}\ (B^{j-1} \setminus B^j) \leq \mathrm{mod}\ (B^0 \setminus B^E) - \eta_1$$

If ϕ' is the mapping obtained from ϕ after $E-1$ simple inducing steps, then a normalized separation symbol with norm β is obtained for ϕ' by iterating the construction of Lemma 5.11. According to the extra claim of the lemma in an inductive fashion, for every univalent branch b of ϕ',

$$s_3 = \mathrm{mod}\ A + \sum_{j=1}^{E} \mathrm{mod}\ (B^{j-1} \setminus B^j)$$

while $A_3(b)$ is $A \oplus A'$ where A' is conformally equivalent to the $\oplus$ sum of $B^{j-1} \setminus B^j$ where j ranges from 1 to $E-1$. Here A is some annulus nested around the preimage of B^0 by the univalent continuation of b. Hence, our assumption means that mod $(A_3(b) \oplus A'(b)) \geq s_3 + \eta_1$. Also, $A_3(b) \geq \alpha_0$. Hence we can use Lemma 5.10 with $L' = \alpha_0$ and $\epsilon := \eta_1$ to in order to get the first part of the alternative of Lemma 6.4.

Step V

In view of Steps III and IV, all we still need to prove is that for every choice of parameters α_0 and M in Lemma 6.4, one can choose M_0 and $\Delta > 0$ depending only on α_0 and M so that if B^E is (Δ, M_0)-rough, then the third possibility occurs in Lemma 6.4.

Recall the mapping ϕ' introduced in Step IV. A simple inducing step performed on ϕ' will result in a mapping ϕ_1 which is the same as the outcome of the type I inducing step carried out on ϕ. Let B_1 denote the central domain of ϕ_1. Then B_1 is the preimage of B^E by the central branch of ϕ_1. The central branch of ϕ_1 is the mapping $z \rightarrow z^2$ composed with a map F that is univalent onto its image B'. We need to demonstrate to that the Jordan curve $w := F^{-1}(\partial B^E)$ is (η, M)-rough with η chosen depending on the parameters of Lemma 6.4. Let D denote the domain of F. Then let us choose annuli C_1 and C_2 as in Definition 6.1. If C_2 is in D, then certainly

$$\text{mod } C_1 + \text{mod } C_2 + \Delta < \text{mod } (C_1 \oplus C_2)$$

with Δ chosen in Step III. In the general situation, without loss of generality we can assume that the inner component of the boundary of C_2 is w. Suppose that an annulus U is contained $D \cap C_2$ and that w is its inner boundary while $N > 0$ is the modulus. Consider the canonical map H from the ring domain $\{z : 1 < |z| < R\}$ onto C_2. Notice that $H^{-1}(U)$ fills the ring bounded outside by $C(0, \gamma)$ where $\gamma > 1$ depends only on N. Indeed, then γ is determined from Teichmuüller's module theorem (Fact 6.2).

Next, we bound show how to find U so that N is bounded from below in terms of M and α_0. If $\partial D \in C_2$, then just take $U = F^{-1}(B' \setminus B_E)$ and the bound by $\alpha_0/2$ is evident. Otherwise, let V be the unbounded component of the complement of w together with the point at infinity. Now take the Riemann mapping H' from the unit disc to V taking 0 to the point at infinity. By Teichmüller's module theorem (Fact 6.2) both $(H')^{-1}(t)$ and the preimage by H' of the outer component of the boundary of A_o are inside $D(0, \rho)$ where $\rho < 1$ depends only on the minimum of moduli of the two annuli involved. Hence we get $N = -\log \rho$.

Now let us look at the image C of $\{z : 1 < |z| < \gamma\}$ by H. It is an annulus contained in $C_2 \cap D$. Moreover, mod $C_2 =$ mod $C +$ mod $(C_2 \setminus C)$. Setting $M_0 := \gamma$ and choosing Δ from Step III, we see that w is (Δ, M)-rough. □

A property of separation symbols

LEMMA 6.5. – *Let ϕ be a type I holomorphic box mapping with separation index $\beta \geq \beta_0$. Also, assume that mod $(B' \setminus B) \geq \alpha_0$. Suppose that ϕ_1 is derived from ϕ by a type I inducing step. For every positive α_0 and β_0 there is a positive η with which the following is fulfilled.*

For every univalent branch b of ϕ_1, if D denotes the domain of b, a normalized separation symbol with norm β exists for which at least one of the two extra estimates holds:

- *$A_2(b)$ is mapped exactly onto itself by the symmetry $z \rightarrow -z$ and mod $A_2(b) \geq \frac{\alpha_0}{4}$, or*
- *$s_4(b) \leq$ mod $(B'_1 \setminus B_1) - \eta$.*

Here, B'_1 and B_1 denote the range and domain of the central branch of ϕ_1.

Proof. – Assume that the escaping time of ϕ is $E \geq 1$ and adopt the notations used in the proof of the previous lemma. Let ϕ^i be the mapping obtained from ϕ upon i simple inducing steps. Thus. $\phi^0 = \phi$ and $\phi' = \phi^{E-1}$. Let b_1 and b_2 be two univalent branches of ϕ^{E-2} with domains D_1 and D_2, respectively. Consider annuli U_1 and U_2, both contained in $B^{E-1} \setminus B^E$ so that U_1 contains D_1 in the bounded component of its complement and D_2 is the unbounded component, while U_2 contains D_2 in the bounded component of its complement, and D_1 in the unbounded component. The claim is that U_1 and U_2 like this exist so that mod U_1 + mod $U_2 \geq \alpha_0$.

The proof goes by induction. For ϕ^0 the analogous claim is obvious. One has to consider the cases when b_1 and b_2 are independent or one is subordinate to the other and the claim follows right away since $A'(b_1)$ and $A'(b_2)$ both have modulus at least α_0. For the induction, consider first the case when b_1 and b_2 are independent. This means that their domains can be mapped forward by a univalent map until they are found in different parent domains. But parent domains are already separated as needed. If b_1 is subordinate to b_2, then a common neighborhood of their domains can be mapped by a univalent trasformation until the domain of b_2 goes onto the central domain, while the domain of b_1 goes onto a univalent domain. But the branch from this univalent domain extends to the range B_0, and so its domain is separated from the central domain by an annulus with modulus at least α_0.

As a corollary we get that the analogous separation property is satisfied when D_1 and D_2 are two parent domains of ϕ'.

Let b be a univalent branch of ϕ_1 with domain D. The following alternative holds:

• a normalized separation symbol with norm β exists for b corresponding to a choice of separating annuli so that $A_2(b)$ is symmetric with respect to the rotation by π and satisfies mod $A_2(b) \geq \alpha_0/4$, or

• a normalized separation symbol with norm β exists for b, and M can be chosen in Step III depending only on α_0 so that $s_4(b) \leq$ mod $(B'_1 \setminus D) - \eta_3$ with η_3 depending on β_0, α_0 and M.

For the proof, we assume first that b has an immediate parent domain. In this case, the construction of Lemma 5.6 provides for a normalized separation symbol with norm β and $s_2 \geq \alpha_0/2$. So let us suppose that b is not immediate. Choose p to the branch of ϕ' so that $p \circ \psi$ is a parent branch of b, and choose P so that the domain of P contains $\psi(0)$. If p and P are in the relation of subordination, or p fails to be maximal, then by Lemmas 5.5, 5.7, 5.8 and 5.4, the second part of the alternative holds. So let us assume that p and P are independent. Since p is maximal, then P and p are in different parent domains. Call D_1 the parent domain of P and D_2 the parent domain of p. Suppose that D_1 is surrounded inside $B^{E-1} \setminus B^E$ by an annulus with modulus at least $\alpha_0/2$ separating it from D_2. One can choose $\gamma_0 > 0$ depending on α_0 so that this annulus has modulus γ_0 and is bounded by the outer boundary of $A'(P)$ and an image of a circle centered at 0 by the canonical mapping from a round ring domain onto the annulus delimited by the border of B^{E-1} and the outer boundary of $A'(P)$. This reasoning is analogous to the bound on γ is Step III. Then the construction of Lemma 5.9 can be amended so that $A_2(b)$ has modulus $\gamma_0/2$ while the estimates are unchanged. So the first part of the alternative holds.

The other possiblity is that an annulus with modulus $\alpha_0/2$ surrounds D_2 separating it from D_1. Again, one can choose $\gamma_0 > 0$ depending on α_0 so that the outer boundary of this annulus is the image of a circle by the canonical mapping. In this case, estimates $\tilde{s}_3$ and $\tilde{s}_4$ obtained in Lemma 5.9 can be increased by $\alpha_0/2$ leading to the first part of the alternative being satisfied. □

6.3. Rough central domains

LEMMA 6.6. – *Let ϕ be a type I holomorphic box mapping derived in a type I inducing step and with separation index $\beta \geq \beta_0$ and mod $(B' \setminus B) \geq \alpha_0$. Assume that the boundary of B is either $(2\epsilon, M)$-rough or is a preimage by $z \to z^2$ of an (ϵ, M) rough Jordan curve. Suppose that the critical value of ϕ is in the domain of a univalent branch b. Assume that for P a normalized separation symbol exists with norm β and at least of the following holds:*

- *$A_2(b) \geq \alpha_1$ and A_2 is the preimage by $z \to z^2$ of an annulus,*
- *$s_4(b) + \Delta \leq$ mod $(B' \setminus D)$ where D is the domain of P.*

Then for every choice of positive β_0, α_0, α_1, Δ, and ϵ, there is are positive numbers δ and M, which is inpependent of ϵ, so that for immediate branches of the map ϕ_1 arising from ϕ after a type I inducing step there is a normalized separation symbol with norm β satisfying $s_3(b_1) \leq$ mod $(A'(b_1) \oplus A_3(b_1)) - \delta$ where b_1 is the immediate branch of ϕ_1.

Proof. – If $s_4(b) + \Delta \leq$ mod $(B' \setminus D)$, then Lemma 5.6 directly implies that the separation norm for immediate branches grows by $\Delta/4$. If we choose δ to be no more than $\Delta/8$, then the claim of Lemma 6.6 follows by Lemma 5.4. So we now assume that the first part of the alternative holds.

Let ϕ_1 denote the mapping obtained from ϕ by a type I inducing step. Let B_1 and B_1' denote the domain and range, respectively, of the central branch of ϕ_1. Also, call b_1 the immediate branch of ϕ_1. Consider the annulus $A = B_1' \setminus B_1$. We have mod $A \geq \frac{\beta_0}{4}$. If M is chosen equal to the minimum of $\beta_0/4$ and α_1, then

$$\text{mod}\,(A \oplus A_2) \geq \text{mod}\,A + \text{mod}\,A_2(b) + \epsilon \geq \frac{\beta + \lambda_2(b)}{2} + \alpha - \lambda_2(b) + \epsilon. \tag{33}$$

This follows directly if the border of B is $(2\epsilon, M)$-rough. If it is a preiamge by $z \to z^2$ of a $(2\epsilon, M)$-rough Jordan curve, then map A and $A_2(b)$ forward by $z \to z^2$ to get the estimate. This is possible since both annuli are symmetric with respect to $z \to -z$.

Then we just repeat the reasoning of Lemma 5.6 to construct the separation symbol for b_1. Since $A_3(b_1) = \psi^{-1}(A_2(b))$ and $A'(b_1) = \psi^{-1}(A)$, then (33) implies that mod $(A_3(b_1) \oplus A'(b_1) \geq s_3(b_1) + \epsilon$ and the claim follows if we choose δ to be no more than ϵ. □

LEMMA 6.7. – *Let ϕ be a type I holomorphic box mapping derived in a type I inducing step and with separation index $\beta \geq \beta_0$ and mod $(B' \setminus B) \geq \alpha_0$. Assume that the boundary of B is either (ϵ, M)-rough or is a preimage by $z \to z^2$ of an (ϵ, M) rough Jordan curve. Suppose that ϕ makes a close return. Then for every choice of positive β_0, α_0 and ϵ, there are a positive numbers δ and M, which is independent of ϵ, so that for $\beta + \delta$ is a separation index for the map arising from ϕ after one type I inducing step.*

Proof. – We can apply the first condition with $C = \psi^{-1}(B' \setminus B)$ and $C' = B' \setminus B$. Hence, $M = \frac{\alpha_0}{2}$. If the first condition fails, then by the argument of Step IV in the proof of Lemma 6.4, after a single type I inducing step we get the desired growth of the separation index. □

Proof of Proposition 4. – Now we conclude the proof of Proposition 4.

By Lemma 6.4, it is enough to show that if the third alternative occurs, then the separation index grows by the following two type I inducing steps. We have to adjust M from Lemma 6.4 to β_0 and α_0 and show that after a few type I inducing steps the separation index grows. If a close return occurs, we are done with both tasks by Lemma 6.7. So let us assume a non-close return. Adopt the notations whereby $\tilde{\phi}$ is the map resulting from ϕ upon a simple inducing step. We will also use notations $\tilde{B}$, $\tilde{B}' := B$ and $\tilde{\psi}$ adhering to our usual convention. Let b be a univalent branch of $\tilde{\phi}$. We also use notations p where p is the branch of ϕ chosen so that $\phi \circ \psi$ is the parent branch of b and P for the branch of ϕ whose domain contains $\psi(0)$. We will construct a normalized separation symbol for b and prove that mod $(A'(b) \oplus A_3(b)) \geq s_3(b) + \epsilon$ where $\epsilon > 0$ only depends on β_0 ands α_0. Then, for the mapping derived from $\tilde{\phi}$ by another type I inducing step an increase occurs by Lemma 5.10 (if $\tilde{\phi}$ makes a close return, note that the condition $s_3(b) + \epsilon \leq$ mod $(A'(b) \oplus A_3(b))$ for every branch persists under simple inducing steps with close returns) and so Proposition 4 will follow.

To prove this statement, we consider the typical combinatorial case analysis.

• b is not maximal. Here, by Lemma 5.5 the norm of the separation index exceeds β. Hence our claim follows by Lemma 5.4. So in the future analysis we assume that b is maximal.

• b is immediate. By Step VII, either we are done using Lemma 6.6 where $M' = \alpha_0/4$, or the additional assumption of Lemma 5.6 is satisfied with $\epsilon = \eta_3$. In both cases, we are done. So in the future analysis, assume that b is not immediate.

• P is not subordinate to p, *i.e.* either p and P are independent, or p is subordinate to P. In those cases the domain of the canonical extension p_e of p does not contain $\psi(0)$. By inspection of the construction of the separating symbol carried out in the proof of Lemma 5.1, we see that $A_3(b)$ has the structure $\psi^{-1}(p_e^{-1}(B' \setminus B) \oplus A)$ where A may be $A_3(p)$ or may be degenerate. The estimate $\tilde{s}_3$, at the same time, is a lower bound for

$$\text{mod } A'(b) + \text{mod } \psi^{-1}(A'(p)) + \text{mod } \psi^{-1}(A).$$

So, it suffices if we show that

$$\text{mod } (A'(b) \oplus \psi^{-1}(A'(p)) \geq \text{mod } A'(b) + \text{mod } \psi^{-1}A'(p) + \epsilon$$

where ϵ only depends on β_0. However, $A'(b) = (p_e \circ \psi)^{-1}(B \setminus \tilde{B})$ while $A'(p) = p_e^{-1}(B' \setminus B)$. Applying the assumption of the Lemma with $A = B \setminus \tilde{B}$ and $M = \beta_0/8$, we are done.

• P is subordinate to p. Notice that Lemma 5.8 is pretty useless here since we have no bound for the parameter L' used there. However, the argument of Lemma 5.8 about the analogy with the immediate case still applies. More specifically, P is then in the form $P' \circ p_e$ where P' is another univalent branch of ϕ. We define the separating annuli

$$A'(b) = \psi^{-1}(p_e^{-1}(B \setminus \tilde{B}))$$
$$A_3(b) = \psi_1 \circ (p_e^{-1}(A_2(P'))$$

The proof of Proposition 4 is finished.

7. Rotation-Like Returns

7.1. Outline of the work

In order to finish the proof of Theorem C, we need the following:

PROPOSITION 5. – *Let $\phi := \phi_0$ be holomorphic type I box mapping. Suppose that ϕ restricted to the real line is a type I box mapping. Let β be a separation index of ϕ. Let ϕ_i for $i = 0, 1, \cdots$ be a sequence (finite or not) of type I holomorphic box mappings set up so that ϕ_i is derived from ϕ_{i-1} by a type I inducing step for $i > 0$. Assume also that all ϕ_i show rotation-like returns.*

Then, for every $\beta_0 > 0$, if $\beta \geq \beta_0$, there is an integer j_0 and a number $\epsilon > 0$ so that $\beta + \epsilon$ is a separation index for ϕ_{j_0}.

Note that Proposition 5, unlike our past work in the direction of Theorem C, requires the mapping to be real. The proof uses already familiar tools, such as the Modulsatz, and some real-variable work. The absolute version will be derived in the following chapter using the technique of pull-back.

Plan of the proof

The proof will be presented as a sequence of steps. We will pick some j and will try showing that if j is large enough, than $j + 10$ can be used as j_0 in Proposition 5. We begin by reducing the problem to the situation when central domains have suitable geometric shape (no pinching). Next, we introduce a real measure of separation and show that in the quasi-roundness situation it is equivalent to a separation index. Finally, we prove that the real measure of separation increases.

Estimates in rotation-like sequences

We begin the proof of Proposition 5. Consider the rotation-like sequence ϕ_i introduced in the hypothesis of the Proposition. This is a subsequence of the sequence of type I holomorphic box mappings φ_k which is derived from ϕ_0 by *simple* inducing steps. We have $\phi_i = \varphi_{k_i}$ for every i. For every k we define $u(k)$ to be the greatest i for which k_i is not greater than k.

LEMMA 7.1. – *For every β_0, there are numbers $\eta, \theta > 0$ with the following property. Choose any $k \geq k_1$ and assume that $\beta + \eta$ is not a separation index for $\phi_{u(k)+5}$. Then a* *pseudo-immediate branch b of φ_k has a normalized separation symbol with norm β and selected so that $s_2(b) \geq \theta$.*

Proof. – The claim of our lemma is obvious when $k = k_i$ for some i. From the way $A_2(b)$ is selected in the construction (*see* Lemma 5.6), s_2 is at least $\beta/4$. In the same situation, we can also assume that

$$\beta/2 \geq 2s_1(b) \geq \text{mod }(B_i' \setminus B_i). \tag{34}$$

Indeed, the subsequent estimates are based on the presumption of equality, so if this were violated, we could get an improved norm of the separation symbol for immediate branches right away. That would lead to the increase of the separation index after another type I inducing step (*see* Lemma 5.10.)

Passing to the case of an arbitrary k, consider $m = k - u(k)$. From the construction of separating symbols in central returns, we see that $s_2(b) \geq \beta/2^{m+2}$. Now apply Proposition 4 and specify η equal to δ_0 from that Proposition. We see that either we get growth of the separation index, or mod $(B' \setminus B) \geq \delta_1$ for φ_k. From estimate (34), we get $\beta \geq 2^{m+2}\delta_1$, hence $s_2(b) \geq \frac{\delta_1}{16}$. □

LEMMA 7.2. – *There are numbers Q and $\eta > 0$ depending only on β_0 with the following property. Choose some $k \geq k_1$ and consider φ_k. Suppose that $\beta + \eta$ is not a separation index for $u(k) + 5$. Let x and y be two points of the boundary of the central domain B of φ_k. Then $|x|/|y| < Q$.*

Proof. – First we will show that the boundary B is not (ϵ, M)-rough for some positive ϵ and M. Assume without loss of generality that for the normalized separation symbol of the immediate branch with norm β, we can assume $s_2 \geq \theta$ where θ comes from Lemma 7.1 and depends only on β_0. Indeed, otherwise Lemma 7.1 implies that the separation index grows by a definite amount η for $\phi_{u(k)+5}$. If we choose the same η in Lemma 7.2, this will be ruled out.

Next, we have two cases depending on whether φ_k shows a close return or not. In the case of a non-close return we refer to Lemma 6.6. Use that Lemma with parameters β_0 taken from the hypothesis of Proposition 5, α_0 and α_1 equal to θ from Lemma 7.1, Δ equal to 0. The last parameter ϵ will be specified later. By Lemma 6.6 we get that for every positive ϵ there are M, independent of ϵ, and $\delta(\epsilon)$ so that if the central domain is (ϵ, M)-rough, then the separation index for $\phi_{u(k)+3}$ grows by $\delta(\epsilon)$. In the case of a close return for φ_k, we get to the same conclusion by Lemma 6.7.

Next, we specify ϵ depending ultimately only on β_0. If we specify η in Lemma 7.2 to be less or equal to $\delta(\epsilon)$, the property derived in the preceding paragraph means that the boundary of B is not (ϵ, M)-rough. The parameter ϵ will be chosen from Lemma 7.2. The parameters of that Lemma are set as follows. M is the M chosen above which ultimately depends only on β_0, and δ is free. This gives us a K which only depends on β_0 and $\Delta(\delta)$. We pick ϵ equal to $\Delta(\delta)$.

We need only to prove that making δ small enough, depending on K, implies the desired property of the border of B. Observe that if diam $B/|y| = C^{-1}$, y on the border of B, then there are points y' and y'' on w so that $|y' - y''| \leq (C + 2\delta)\text{diam}\, B$ while the diameters of both arcs of w joining y' and y'' are at least $(1-2\delta)\text{diam}\, B$. This is true since $-y$ is also on the border of B, and y'' can be chosen close to $-y$. But $(C + 2\delta)/(1 - 2\delta) \geq Q_1(K) > 0$ from the three-point property (*see* Fact 6.3). So choosing $\delta < 1/4$ and $C, \delta \leq \frac{1}{10}Q_1(K)$ will imply the desired property. □

7.2. Real separation

The real separation index

Let $\phi = \phi_j$ for $j \geq 2$. Denote by D_0 the domain of the central branch on the real line and by D_1 the domain of some univalent branch also intersected with the real line. Let $2x_1$ be the length of $B \cap \mathbf{R}$ and x_2 denote $\text{dist}(0, D_1) + |D_1|$. Then, the quantity

$$\gamma(\phi) = \log \frac{x_1 x_2}{|D_0| \cdot |D_1|}$$

is the *real separation index.*

Real and complex separation

LEMMA 7.3. – *Choose $k \geq k_2$ with separation index β so that φ_k makes a non-close return. Assume that γ is the real separation index of φ_k.*

Then, there are numbers K and $\eta > 0$, both only depending on the parameter β_0 from Proposition 5 so that the following alternative holds:

- *$\beta + \eta$ is a separation index for $\phi_{u(k)+6}$,*
- *the immediate branches of φ_{k+1} have normalized separation symbols with norm $\gamma - K$.*

Proof. – Choose η the same as obtained in Lemma 7.2 and suppose that the first part of the alternative does not hold . Then the hypotheses of Lemma 7.2 are satisfied for φ_{k-1}, φ_k and φ_{k+1}, and hence if B is the central domain of any of these maps, then $|x|/|y| < Q$ for all x, y in the border of B and with Q dependent only on β_0. Notice that for φ_k and φ_{k+1} the same estimate holds for x, y in the border of B', since B' is the same as B one simple inducing step before.

Let ϕ be either φ_k are φ_{k+1} with real parameters x_1 and x_2 as in the definition of the real separation index. Use notations D_0 and D_1 for the intersections of domains with the real line. We first build $A_2(b)$ as the round ring domain bounded by the circle centered at 0 with radius $Q \cdot |D_0|$ and another concentric circle with radius $x_2/2$. Let us assume that $x_2 > 2Q|D_0|$. Otherwise, make $A_2(b)$ degenerate. Then $A_2(b)$ contains B in the bounded component. The domain of b is the preimage of B by the canonical extension of b. This canonical extension has distortion bounded on the domain of b depending on β_0, from Köbe's distortion lemma. Hence, this domain is contained in the disc of radius $Q_1|D_1|$ centered in the endpoint of D_1 further from B. The constant Q_1 is determined by Q and the distortion of the canonical extension of b, hence ultimately by β_0 only. Thus, unless

$$x_2 < 2Q_1|D_1|$$

the domain of b is in the unbounded component of the complement of $A_2(b)$. On the other hand, if

$$\frac{x_2}{|D_1|} < 2Q_1$$

then also $\frac{x_1}{|D_0|} < 2Q_1$ this time using the canonical extension on the real line which has negative Schwarzian derivative. In that case, γ is bounded by a constant and we take the whole separation symbol to be degenerate. Then we built $A_1(b)$ as the ring domain

$$\left\{z : \frac{x_2}{2} < |z| < \frac{x_1}{Q}\right\}.$$

Since $|x|/|y| < Q$ for x, y in the border of B', this is a valid choice. Hence we can pick

$$s_1 = \log \frac{x_1}{|D_0|} - K_1$$
$$s_2 = \log \frac{x_2}{|D_0|} - K_1$$

where K_1 is a constant depending only on β_0.

Next, construct the separating annuli for an immediate branch $\tilde{b}$ of φ_{k+1}. The construction of $A_2(b)$ follows the same reasoning as above. By elementary calculation,

$$\frac{\tilde{x}_1}{|\tilde{D}_0|} \geq K_2 \sqrt{\frac{|x_1|}{|D_1|}}$$

where K_2 is a positive constant depedning on the distortion, and thus on β_0. Similarly,

$$\frac{\tilde{x}_2}{|D_0|} \geq K_2 \sqrt{\frac{x_2}{|D_0|}}.$$

Hence, we can take

$$\tilde{s}_1 = \frac{1}{2} \log \frac{x_1}{|D_1|} - K_3$$
$$\tilde{s}_2 = \frac{1}{2} \log \frac{x_2}{|D_1|} - K_3$$

Then $A'(\tilde{b})$ also has modulus at least $\tilde{s}_1$. The annulus $A_3(\tilde{b})$ is obtained by taking the preimage of $A_2(b)$ constructed for ϕ by the central branch of ϕ, and to get $A_4(\tilde{b})$ we take the preimage of $A_1(b)$ by the central branch. We get

$$\tilde{s}_3 = \gamma - \frac{1}{2} \log \frac{x_1}{|D_1|} - 2K_3$$
$$\tilde{s}_4 = \gamma - \frac{1}{2} \log \frac{x_1}{|D_1|} + \frac{1}{2} \log \frac{x_1}{x_2} - 2K_3$$

where K_3 is a constant determined by β_0. Set

$$\begin{aligned}\beta' &= \frac{2}{3}\gamma - 2K_3\\ \lambda_1 &= \frac{1}{2}\log x_1 - \frac{1}{2}\log|D_1| - \frac{\gamma}{3}\\ \lambda_2 &= -\frac{1}{2}\log x_2 + \frac{1}{2}\log|D_1| + \frac{\gamma}{3}\end{aligned}$$

It is clear that $\lambda_1 + \lambda_2 \geq 0$ and s_i are determined by β' and the corrections as demanded by the definition of the normalized separation symbol. It remains to see that $|\lambda_i| \leq \frac{\gamma}{3}$. To this end, it is sufficient to demonstrate $\lambda_1, \lambda_2 \leq \beta'/2 = \gamma/3 - K_3$. We estimate

$$\lambda_1 = \frac{1}{2}\log x_1 - \frac{1}{2}\log|D_1| - \frac{\gamma}{3} = \frac{\gamma}{2} - \frac{1}{2}\log x_2 + \frac{1}{2}\log|D_0| - \frac{\gamma}{3} \leq \frac{\gamma}{6}.$$

We indeed have $\gamma/6 \leq \frac{\gamma}{3} - K_3$ unless $\gamma < 6K_3$, which means that γ is bounded by a constant depending only on β_0 and we can take the complex separation index to be 0. To bound λ_2, observe again that

$$\frac{x_1}{|D_0|} \geq \frac{x_2}{|D_1|}.$$

Thus,

$$\lambda_2 = -\frac{1}{2}(\log x_2 - \log|D_1|) + \frac{\gamma}{3} \leq -\frac{\gamma}{4} + \frac{\gamma}{3} = \frac{\gamma}{12}.$$

Again, either $\gamma/12 \leq \beta'/2$ or γ is bounded by a constant. □

Proof of Proposition 5. – We are assuming that we are in the situation of Proposition 5.

Step I

We prove that ϕ_0 has a real separation index at least $\beta - K$ where K depends only on β_0. Also, $|B_0|/|B_0'| \leq 1 - K'$ where $K' > 0$ also depends on β_0 only.

We will use the notations D_i and x_i as introduced by the definition of the real separation index. Shift the frame by a translation so that $z_1 > 0$ is an endpoint of D_0, 0 is the other endpoint of D_0 and $z_2 > 0$ is an endpoint of D_1 closer to D_0. From Fact 6.2 we get

$$\frac{2x_1}{|D_0|} \geq s_1 - 2\log 4.$$

Hence, $\log x_1/|D_0| \geq s_1 - K_1$ where K_1 is a constant. In a similar fashion we estimate that

$$\log\frac{x_2}{|D_1|} \geq s_3 - K_1.$$

The claim of Step I follows.

Step II

Next, we prove that the real separation indexes $\gamma(\phi_i)$ increase. More precisely, for every Q there is N depending only on Q and β_0 so that $\gamma(\phi_i) - \gamma(\phi_0) > Q$ for all $i \geq N$.

By Theorem A, there is a number N_1 depending only on K' from the claim of Step I so that $|B_i|/|B_i'| < 0.37$ for all $i \geq N_1$. Next, we need to look into Proposition 1. Subsitutting $\phi_0 := \phi_{N_1}$ in the statement of that Proposition, we discover the following connection between that numbers α_n and γ_n used there and the parameters used to define the real separation index:

$$\alpha_n^{-1} = \frac{x_1(\phi_{n+N_1})}{|D_0(\phi_{n+N_1})|}$$
$$\gamma_n^{-1} = \frac{x_2(\phi_{n+N_1})}{|D_1(\phi_{n+N_1})|} - 1.$$

Hence,

$$\gamma(\phi_{n+N_1}) \geq \log \frac{1}{\alpha_n \gamma_n}.$$

In this situation, the claim of Step II follows directly from Proposition 1.

Step III - conclusion

To avoid a clash of notations, introduce K_1 and make it equal to K obtained in Lemma 7.3. In the claim of Step II, choose $Q = 2(K + K_1)$ with K from Step I. Take N from Step II. We know that for any $i \geq N$, the real separation index $\gamma(\phi_i)$ exceeds β by more that $2K_1$. Next, pick any i at least equal to $N+1$ and 3, and pick k so that φ_k yields ϕ_i in a simple inducing step. We apply Lemma 7.3 in this situation. One thing we leave to the reader to check is that $\gamma(\varphi_k) \geq \gamma(\phi_{u(k)})$ (*i.e.* the real separation index does not decrease in close returns.) Lemma 7.3 leaves us with two alternatives. The first one means that the claim of Proposition 5 is satisfied with $j_0 = N + 6$ and $\epsilon := \eta$ where η is taken from Lemma 7.3. But the second alternative also leads to the Proposition holding true, this time with $j_0 = N + 2$ and ϵ depending and K_1. When the second part of the alternative holds, we get an improvement in the norm of the separaton symbol for the immediate branches of ϕ_{i+1}, and then we need one more inducing step to translate that to the increase of the separation index by Lemma 5.10. Now Proposition 5 has been demonstrated.

Proof of Theorem C. – The first step is to estimate the separation index for the mapping ϕ_0 in the statement of Theorem C. As a consequence of the fact that ϕ_0 is derived from ϕ by filling-in, every univalent branch of ϕ_0 can be assigned a separation symbol with bounds $(\beta_0, 0, \beta_0, \beta_0)$. It is easy to see that by decreasing some of these bounds we get a normalized separation symbol with norm $\beta_0/2$.

Next, we prove that the separation symbol grows. That is, we demonstrate the following **Claim**:

if ϕ_0 is a holomorphic type I box mapping whose restriction to the real line is a real box mapping and whose separation index is $\beta \geq \beta_1$, then there are numbers N and $\delta > 0$ depending only on β_1 so that $\beta + \delta$ is the separation index for ϕ_N.

Let us first look at Proposition 4. Its parameters β_0 and α_0 can be made $\beta_1/4$, and then we can apply this Proposition to any $\phi := \phi_j$ for $j \geq 0$. In any case, we get parameters δ_0 and δ_1. Next, try to apply Proposition 3 with $j \geq 3$ and $L := \delta_1$. From Proposition 4, we see that if the assumption of Proposition 3 is violated, then our Claim holds with $N := j+3$ and $\delta := \delta_1$. But, if the these assumptions are satisfied and if ϕ_j makes a non-rotational return, then the Claim also holds with $N := j + 1$ and δ equal to K from Proposition 3.

Now take the parameter j_0 from Proposition 5. The reasoning conducted above can be applied with any $3 \leq j \leq j_0 + 3$ and shows that the Claim is valid unless for all these j the mapping ϕ_j shows a rotation-like return. Then, however, we apply Proposition 5 to the rotation-like sequence beginning with ϕ_3 and we see that the Claim holds again. So we proved the Claim.

Now, the separation index grows at a linear rate as is evident from using the Claim to ϕ_0, ϕ_N, etc. Theorem C follows by Lemma 5.1.

Part IV

Construction of Box Mapping

Preliminaries

The construction of holomorphic box structure is based on two crucial concepts: the method of inducing and a combination of real induction with hyperbolic geometry. Our approach to constructing domains in the complex plain avoids complicated initial estimates, which are usually necessary to make the real induction work. Instead we are looking for a certain weak topological condition of "expansion" in the complex domain, *i.e.* small sets being mapped to larger sets. Expansion measured in terms of moduli, once established and uniform, will persist (even grow) during the inducing procedure due to our complex induction until a terminal map is obtained. As we will see, the lack of this weak "expansion" in the complex plane must be compensated by the rapidly growing expansion on the real line around the critical point. This will lead to the direct construction of the uniform complex box structure after a uniform number (large) of inducing steps. Finally, by compactness argument, we will show that our procedure fails to produce the uniform complex box structure in renormalizable situation only if a unimodal map exhibits a renormalization of bounded combinatorics or has an almost parabolic point of large depth.

8. Real box mappings

We review the method of inducing and define the *real inducing* of real box mappings. We emphasis that the real inducing, though conceptually very similar, is a different procedure from type I inducing introduced in the first part of the paper.

Type III of real box mappings

With a box mapping ϕ we can always associate two domains B, and B'. Suppose that ϕ is not terminal. Then the orbit $\phi^i(0)$ eventually escapes from B. Take the first point of this orbit not in B and assume that it belongs to the domain Δ_p of a monotone branch χ_p. We will call Δ_p a *postcritical* domain and χ_p a postritical branch of ϕ, respectively.

According to the definition we can have many non-equivalent states of box mappings. For our current needs we introduce type III.

DEFINITION 8.1. – *A box mapping ϕ is of* type *III if the range of the postritical branch, called B'', is an interval symmetric with respect to* 0 *and satisfies*

$$\overline{B} \subset B'' \subset B'$$

and the critical value $\phi(0)$ belongs to B''.

We see that type III box mapping has one marked monotone branch, the postritical one, whose image is larger than images of all others non-marked monotone branches.

8.1. Real inducing

Real inducing describes the dynamics of real box mappings.

Step A

Suppose that a box mapping ϕ is given. Choose an interval $U = (-a, a)$ to be contained in the range of any monotone branch of ϕ and so that a and $-a$ are not in the domain of ϕ. We will now describe an algorithm which transforms ϕ into a new box mapping ϕ_{new} of *type* III with $B'' = U$. The operation will be called *inducing step A*. Note that the outcome of Step A is determined by ϕ as well as U.

Filling-in

For any $x \in \Delta$, where Δ is a connected component of the domain of ϕ different from the central domain, we define its *landing time* to be

$$l(x) := \min\{i = 0, 1, \cdots : \phi^i(x) \in B\}.$$

Note that the orbit $\phi^i(x)$ may be finite. By convention, $l(x)$ is infinite if the orbit never hits B. The set of points with infinite landing time is forward invariant and stays away from B. The outcome of filling-in is the mapping $\phi_{filled-in}$ defined on the set $W := \{x \in \Delta \ : \ l(x) < \infty\}$ by

$$\phi_{filled-in}(x) := \phi^{l(x)}(x).$$

Dynamical extendibility of branches after filling-in

After filling-in every monotone branch of $\phi_{filled-in}$ has a dynamical extension with range U. Observe that for every branch of $\phi_{filled-in}$, the domain of its extension onto U is either contained in or disjoint with boxes $B' \supset B'' = U \supset B$.

Definition of ϕ_{new}. – Consider now the mapping $\phi_{filled-in}$ on the postcritical domain Δ_p of ϕ. Assume that the critical value $\psi(0)$ lands in the interior of the domain V of a branch of $\phi_{filled-in}$. Let ζ be the extension of the branch defined on V with range U. Then outside of the domain of ζ, set $\phi_{new} = \phi_{filled-in}$. On the domain of ζ set $\phi_{new} = \zeta$.

It is immediately seen that ϕ_{new} is a box mapping of *type* III with the same box structure as ϕ. By definition, the *central* branches of ϕ and ϕ_{new} coincide.

Step B - critical filling

Let ϕ be a box mapping. A box mapping ϕ_{new} will arise as a result of *critical filling*. By definition, ϕ_{new} coincides with ϕ except on the central domain B of ϕ. On the central domain B, we set ϕ_{new} to be equal to:

- $\phi \circ \psi$ on $B \setminus \psi^{-1}(B \cup \Delta)$.
- ψ on $\psi^{-1}(B)$.

It can be easily verified that *critical filling* gives a box mapping.

Complete inducing step

We will join step A and step B into one *inducing step* which is, by definition, a composition of a certain number of steps B followed by step A. The number of steps B in the procedure is determined by so called *escaping time* of the critical point. The complete inducing step will transform a type III box mapping into another type III box mapping.

Close and non-close returns

A box mapping ϕ exhibits a *close return* if $\psi(0) \in B$. A non-close return is the complementary case.

DEFINITION 8.2. – *The* escaping time *of ϕ is defined to be*

$$E := \min\{i = 1, 2, 3, \cdots : \psi^i(0) \notin B\}.$$

By convention, $E = \infty$ if the above set is empty.

If ϕ is induced by a unimodal mapping f, then $E = \infty$ means that f has a restrictive interval. For now, we rule this case out. We have very simple characterization of returns in terms of the escaping time. Namely, $E = 1$ if and only if ϕ shows a non-close return.

DEFINITION 8.3. – *Given a type III box mapping ϕ with escaping time E, the* complete inducing step *is defined as follows.*

$$\phi_{new} = \text{stepA} \circ \underbrace{\text{stepB} \circ \cdots \circ \text{stepB}}_{E \text{ iterates}}(\phi),$$

Here Step A is carried out with U equal to the central domain of the box mapping $\tilde{\phi}$ which results from ϕ after $E - 1$ Steps B.

By construction, ϕ_{new} is a type III box mapping. Observe that $B'(\phi_{new})$ equals the range of the postcritical branch of ϕ while $B''(\phi_{new})$ is contained in the central domain B of ϕ, and equal to B exactly when ϕ makes a non-close return.

Hyperbolic and parabolic returns

We distinguish between two different types of returns: a *hyperbolic return* defined by condition that 0 is covered by the range of the central branch, and a *parabolic return* which is the complementary case.

There is an important analytic difference between parabolic and hyperbolic close returns. A deep hyperbolic close return (E is large) means that the critical value ψ_0 lands in a great proximity of a repelling periodic point. This implies some expansion and makes the case relatively easy to deal with. A parabolic close return with long escaping time is the hard case since no expansion is implied. This is the reason we had to make an exception for almost parabolic points in the statement of Theorem 2.

9. From real to holomorphic box mapping

Standardized picture

Consider a type III box mapping φ. Let $B' = (-a, a)$ denote the smallest interval symmetric with respect to 0 and containing the range of the central branch, $B'' = (-b, b)$ the range of the postcritical branch, B the domain of the central branch, and ψ the central branch itself. The important measures of the real box geometry of φ are its *characteristic ratio* $\alpha(\varphi) = \frac{b}{a}$ and the *characteristic cross-ratio* $\tau(\varphi) = 2\mathbf{Poin}(-a, c, b, a)$. If φ is not terminal and after a complete inducing step another type III box mapping φ_1 arises, analogous notations will be used: $B'_1 = B''$, $B''_1 = (-c, c)$ and B_1.

PROPOSITION 6. – *Let φ be a type III box mapping and consider its standardized picture. Choose the orientation so that $(\psi(0), a)$ is the range of the central branch. Let the escapng time of φ be E. For some $\varepsilon > 0$, suppose that $\alpha(\varphi) \leq 1 - \varepsilon$ and a point x exists in $(-a, \psi(0))$ but outside of the domain of the postcritical branch of the map obtained from φ after $E - 1$ Steps B, so that*

$$2\mathbf{Poin}(-a, x, \psi(0), a) \leq 1 - \varepsilon\,.$$

Then, a type I box mapping ϕ exists which can be obtained from φ by a combination of Steps A and B. This ϕ has an analytic continuation Φ as a type I holomorphic box mapping (see Definition 2.2) with separation index $K > 0$ where K depends only on ε.

Further on, we will refer to the construction of Φ from φ as the *general geometric construction*. The proof of the Proposition will occupy the rest of this section. We start the proof with the definition of a well-known object in holomorphic dynamics.

Quadratic-like maps

The concept of quadratic-like maps due to Douady and Hubbard turned out to be especially fruitful in the study of iterates of rational maps. We will generalize their construction.

DEFINITION 9.1. – *A quadratic-like map* $g : U \overset{\text{onto}}{\longrightarrow} V$ *is a proper holomorphic map of degree* 2 *between two open topological disks* U *and* V, U *is compactly contained in* V. *We call the modulus mod* $(V \setminus U)$ *a quadratic-like norm of* g.

Geodesic neighborhoods

The following convenient tool was introduced by Sullivan [20]. We use several technical properties and refer to literature for proofs [18]. Consider an interval $[x-y, x+y]$, $y > 0$, on the real line.

DEFINITION 9.2. – *Look at two circles passing through* $x-y$ *and* $x+y$, *one centered at* $x+it$, *and the other at* $x-it$ *where* $t \in \mathbf{R}$. *Suppose that the circles intersect the real line at angle* $\alpha \leq \pi/2$. *Consider the two open discs delimited by these circles. The union of these discs will be called the geodesic neighborhood of* $[x-y, x+y]$ *with angle* $\pi - \alpha$, *and denoted* $\mathcal{D}(\pi-\alpha, [x-y, x+y])$. *Likewise, the intersection of these discs will be called the geodesic neighborhood of* $[x-y, x+y]$ *with angle* α *and denoted* $\mathcal{D}(\alpha, [x-y, x+y])$.

The following property makes the geodesic neighborhoods convenient.

Fact 9.1.

Let h *be a real diffeomorphism with range* U *and domain* V. *Suppose that* h^{-1} *from* V *has an analytic continuation as a univalent mapping of the interior of the upper half-plane into itself.*

Then, for any $\alpha \in (0, \pi)$ *the preimage of* $\mathcal{D}(\alpha, U)$ *by the analytic continuation of* h *is contained in* $\mathcal{D}(\alpha, V)$.

9.1. General geometric construction

Consider the standardized picture for a type III box mapping ϕ. Choose the orientation so that the central branch ψ takes its minimum at 0. If the escape time for φ is E, look at the box mapping ϕ obtained from φ by E Steps B. The box mapping ϕ has the postritical branch χ_p mapping over B''. Let Δ_p be the postcritical domain, then Δ_p is contained in one connected component of $B'' \setminus B''_1$.

LEMMA 9.1. – *Suppose that for some* $x \in (-a, \psi(0))$ $2\mathbf{Poin}(-a, x, \psi(0), a) < 1$. *Then there exists a disk* G *so that*

$$\psi^{-1}(G) \subset \mathcal{D}(\psi^{-1}(G) \cap \mathbf{R}).$$

Proof. – Let G denote $\mathcal{D}(\frac{\pi}{2}, (x, a))$ in the sense of Definition 9.2, or simply speaking just the geometric disk with diameter (x, a). The key observation is that $\psi^{-1}(G) \subset \mathcal{D}(\frac{\pi}{2}, B)$ if only $\mathbf{Poin}(-a, x, \psi(0), a) \leq \frac{1}{2}$. Indeed, representing $\psi(x) = h(x^2)$ where h is a diffeomorphism with non-positive Schwarzian derivative, this assumption implies that

$|h^{-1}(x, \psi(0))| \le |h^{-1}(\psi(0), a)|$ while by Fact 9.1 the preimage of G by the analytic continuation of h is contained in the geometric disc with diameter $(h^{-1}(x), h^{-1}(a))$. Now it follows immediately that indeed $\psi^{-1}(G) \subset \mathcal{D}(\frac{\pi}{2}, B)$. Observe that if a return is hyperbolic, $\psi(0) \in (-b, 0)$, and $x < -b$ then $\psi^{-1}(G) \subset G$. □

Construction of a quadratic-like map

In the hyperbolic case the construction is immediate, if only $x < -b$, since by Lemma 9.1, $\psi^{-1}(G) \subset G$. (see *Fig.* 6). In the parabolic case of inducing φ cannot be terminal and thus $\psi(0) \in (c, b)$ (recall that $(-c, c)$ stands for B_1' in the standardized picture). Assume that $-a < x < c$. We proceed to note that

(35) $$\chi_p^{-1} \circ \psi^{-1}(G) \subset \chi_p^{-1}(\mathcal{D}(B)) \subset \mathcal{D}(\Delta_p \cap \mathbf{R}) \subset G\,.$$

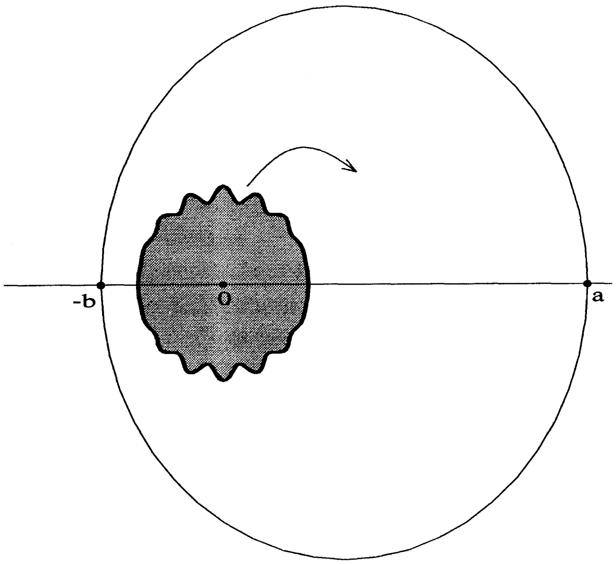

Fig. 6. – Construction of a holomorphic box mapping in the case of a hyperbolic return. The central branch ψ maps the shaded region onto the ball based on the interval $(-b, a)$. We obtain a uniform lower bound on the modulus by moving $x = -b$ to the left.

So, the map $\chi_p \circ \psi$ is polynomial-like of degree 2

(36) $$\chi_p \circ \psi : \psi^{-1} \circ \chi_p \circ \psi^{-1}(G) \longrightarrow \psi^{-1}(G)\,.$$

Figure 7 shows the construction of a holomorphic box mapping in the case of a parabolic return.

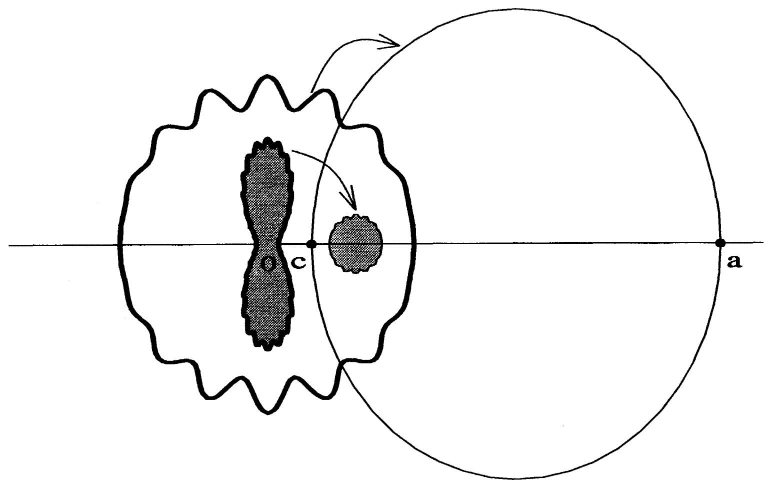

Fig. 7. – Construction of a holomorphic box mapping in the case of a parabolic return. The shaded regions represent respectively the central domain of the holomorphic box mapping and the preimage by the postcritical branch χ_p of the quasidisk with the bold boundary. By our assumptions, the quasidisk is the preimage of the ball based on the interval (c, a) by the central branch ψ.

Norm of quadratic-like maps

We will estimate the norm of the quadratic-like maps: (36) and $\psi : \psi^{-1}(G) \longrightarrow G$ in the parabolic and the hyperbolic cases respectively.

Two estimates

We continue to work in the standardized picture. The configuration in the hyperbolic setting is defined by: $\psi(0) \in (-b, 0)$ and $b \leq x < \psi(0)$.

Fact 9.2.

Let $\gamma := 2\mathbf{Poin}(-a, x, \psi(0), a) < 1 - \varepsilon$, $\varepsilon > 0$. *There is a positive constant* K *so that for every* φ *in the hyperbolic setting one can choose a point* $-a < y < x$ *so that*

- $\mathbf{Poin}(-a, y, \psi(0), a) \leq \frac{1}{2}$,
- $\frac{|(y,x)|\cdot|(-x,a)|}{|(y,-x)|\cdot|(x,a)|} \geq K\varepsilon$,

Proof. – Let $\frac{|x|}{|a|} = 1 - \rho$, $0 < \rho < 1$. With $y = x$, the $\mathbf{Poin}$ cross-ratio named is less than $\frac{1}{2}(1-\varepsilon)$. One can move y by $\delta := \frac{1}{4}(1-\rho)\varepsilon|x|$ to the left and the first estimate is still satisfied. Indeed, $\frac{2|a|}{|a-\psi(0)|} < \frac{2}{\rho}$ and $\frac{\delta}{|y|+|a|} < \frac{1}{4}(1-\rho)\rho\varepsilon$. Therefore, the first cross-ratio is bounded from above by $\frac{1}{2}(1-\rho\varepsilon)$. Clearly, after this move $y > -a$. This will make $\frac{|(y,x)|}{|(y,-x)|}$ at least $\frac{1}{12}(1-\rho)\varepsilon$ while the complementary ratio in the second condition is bounded from below by $1 - \frac{1}{2}\gamma \geq \frac{1}{2}$ by simple algebra. For $\rho < \frac{9}{10}$ the Fact follows. Hence, assume that $\rho \geq \frac{9}{10}$. Then $\mathbf{Poin}(-a, x, \psi(0), a)$ is bounded from above by $\frac{2(1-\rho)}{2-\rho} < \frac{1}{2}$ and we can move $y = x$ by $\frac{1}{9}|x|$ to the left preserving the first estimate. Evidently, $y > -a$. The second cross-ratio is larger than $\frac{1}{100}$.

In the parabolic setting assume that $\psi(0) \in (c, b)$ and $x = c$. We have the following estimate.

Fact 9.3.

Let $\gamma := 2\mathbf{Poin}(-a, b, x, a)$ *be less than* $1 - \varepsilon$, $\varepsilon > 0$. *There is a positive constant* K *so that for every* φ *in the parabolic setting one can choose a point* $-a < z < c$ *so that*

- $\mathbf{Poin}(-a, z, \psi(0), a) \leq \frac{1}{2}$,
- $\frac{|(z,c)| \cdot |(b,a)|}{|(z,b)| \cdot |(c,a)|} \geq K\varepsilon$,

Proof. – With $z = c$ the cross-ratios $\mathbf{Poin}$ is less than $\frac{1}{2}(1 - \varepsilon)$. One can move z by $\varepsilon|b - c|$ to the left and the first estimate is still satisfied. Clearly, after this move $z > -b$. This will make $\frac{|(z,c)|}{|(z,b)|}$ at least $\varepsilon/2$ while the complementary ratio is bounded from below by $1 - \frac{1}{2}\gamma \geq \frac{1}{2}$. □

Cross-ratio versus modulus

We choose points y and z from Fact 9.2 and Fact 9.3. As a consequence of Proposition 10.3 and Facts 9.2 and 9.3, we have $\psi^{-1}(G) \subset \mathcal{D}(\frac{\pi}{2}, B)$ and

(37)
$$\frac{|(y,x)| \cdot |(-x,a)|}{|(y,-x)| \cdot |(x,a)|} \geq K\varepsilon$$

(38)
$$\frac{|(z,c)| \cdot |(b,a)|}{|(z,b)| \cdot |(c,a)|} \geq K\varepsilon$$

where K is a positive constant.

In the hyperbolic setting $G \subset \psi^{-1}(G)$ while in the parabolic the preimage of the diameter $G \cap \mathbf{R}$ of G by $\chi \circ \psi$ is contained in $(-b, -c)$. In both cases, the cross-ratios which measure the nesting of $(\chi_p^{-1} \circ \psi^{-1})(G)$ and $\psi^{-1}(G)$ respectively inside G are at least $K\varepsilon$. Since both cross-ratio and modulus are linear-fraction invariants, the moduli of

(39)
$$G \setminus \left[(\chi_p^{-1} \circ \psi^{-1})(G)\right] \qquad \text{and} \qquad G \setminus \psi^{-1}(G)$$

are larger than the modulus of the annulus $\mathcal{D}(-1, 1) \setminus \mathcal{D}(-r, r)$, where r satisfies

$$\frac{(1-r)^2}{(1+r)^2} = K\varepsilon\,.$$

By algebra, $|r| < 1 - K'\varepsilon$ for a different constant K' and finally we get $K''\sqrt{\varepsilon}$ as a lower bound of the moduli (39). This completes the estimates in the hyperbolic case. In the parabolic case we use a simple estimate of the modulus of the preimage of the annulus A by a holomorphic mapping ψ of the degree 2

(40)
$$\text{mod}\,(\psi^{-1}(A)) \geq \frac{1}{2}\,\text{mod}\,(A)\,.$$

Therefore, by (40), the modulus of

$$(\psi^{-1} \circ \chi_p \circ \psi^{-1})(G) \setminus \psi^{-1}(G)$$

is larger than $(1/2)K''\sqrt{\varepsilon}$. The estimate depends only on ε.

Complex domains of monotone branches

Concentrate on the mapping φ' derived from ϕ by $E-1$ Steps B, but let ψ still be the "old" central branch of φ.

The range of every monotone branch χ of φ' contains B and therefore, the branch can be analytically extended on a neighborhood of its real domain Δ_χ so that the image of this extension contains $\mathcal{D}(B) \subset \psi^{-1}(G)$. For a hyperbolic return it is sufficient to construct a holomorphic type II box mapping Φ. Although the border of G may intersect the domain of some univalent branch, the preimage of this domain will have no intersection with the real line and hence can be dropped.

In the parabolic case φ cannot be terminal and the analytic extension of χ maps

$$\chi^{-1}(\psi^{-1}(G)) \subset \mathcal{D}(\Delta_\chi)$$

univalently onto $\psi^{-1}(G)$. Now, do the critical pullback of all univalent branches χ. By the Markov property and Fact 9.1, again we obtain a holomorphic box mapping of type II.

In both cases, after holomorphic filling-in, we get a type I box mapping Φ with the box structure $B := \psi^{-1} \circ \chi_p \circ \psi^{-1}(G)$ and $B' = \psi^{-1}(G)$. We already know that the number $m := \frac{1}{3}\text{mod}\ (B' \setminus B)$ is larger than zero in terms of ε only.

Separation index

We are left with the task of estimating the separation index β of the box mapping Φ. By the construction, the domain of every univalent branch of Φ together with its natural univalent extension (mapping over B') is either contained in B' or disjoint with B'. Thus, the separation symbol $(s_i), i = 1,2,3,4$ of Φ might be bounded from below by $(m, 0, m, m)$. Elementary algebra shows that a separation index $\beta = m/2$ will work for Φ.

10. From unimodal to box mapping

10.1. The initial box mapping

The fundamental inducing domain

DEFINITION 10.1. – *The interval* $(-q, q)$ *will be called* the fundamental inducing domain.

By the assumption that all periodic orbits are repelling, every chaotic f has a fixed point $q > 0$.

Induced mappings

If f is a unimodal polynomial then a piecewise map ϕ is said to be *induced by* f if every branch of ϕ has the form f^i for some $i \geq 0$.

Construction

Given a mapping $f \in \mathcal{F}_\eta$ there is a standard way of obtaining a box mapping $\tilde{\phi}_1$ induced by f. Let $B^0 = J$ be the fundamental inducing domain of f. Then $\tilde{\phi}_1$ is defined as the first return mapping from the interior of the fundamental inducing domain into itself. One easily verifies that so defined $\tilde{\phi}_1$ is a box mapping. Let B^1 be the central domain of $\tilde{\phi}_1$. The box structure of $\tilde{\phi}_1$ consists only of two boxes $B^0 \supset B^1$.

Real *a priori* bounds

LEMMA 10.1. – *There exists a constant ε_0 dependent only on η, $f \in \mathcal{F}_\eta$, so that* $|B^1|/|B^0| < 1 - \varepsilon_0$.

Proof. – Define J_k for $k > 0$ as the set of points in $1 > x > 0$ whose first entry time into the fundamental inducing domain is k. Then $J_{k+1} = f_r^{-1}(-J_k)$ where f_r denotes the right "lap" of f. The derivative of f_r on $(q, 1)$ is greater than 1, since it is so at the endpoints q and 1 and Schwarzian derivative is negative. So $|J_k|$ form a decreasing sequence. Now B^1 is the preimage by $f = h(x^2)$ of this J_{k_0} which contains the critical value of f. At the same time, $B^0 \setminus B^1$ is the preimage of the union of J_k for $k < k_0$. The Poincare length of J_{k_0} with respect to $(q, 1+\eta)$ is less than a uniform constant depending solely on η. Since h^{-1} contracts the Poincare distance and x^2 distorts the ratio by at most squaring, the ratio $|B^1|/|B^0|$ is indeed bounded away from 1 in terms of η only. □

Every monotone branch of $\tilde{\phi}_1$ ranges over B^0. We subject $\tilde{\phi}_1$ to Step A with U chosen to be B^0. That results in a box mapping ϕ_1 of type III with $B'(\phi_1) = B''(\phi_1) = B^0$ and the same central domain as $\tilde{\phi}$. As we proceed by another complete inducing step applied to ϕ_1, we get a mapping ϕ_2 with boxes $\tilde{B}$, $\tilde{B''} \subset B$ and $\tilde{B'} = B'' = B'$. As the result of the estimate of $|B^1|/|B^0|$, for the characteristic ratio of ϕ_2 we get $\alpha(\phi_2) < 1 - \varepsilon_0$.

The induced sequence

DEFINITION 10.2. – *The sequence ϕ_i for $i \geq 2$ constructed so that ϕ_i is obtained from ϕ_2 by $i - 2$ complete inducing steps is called* the induced sequence.

10.2. Conditional construction property of the real inducing

Now, we are in position to formulate the main geometric property of the real inducing procedure. Loosely speaking, we have always two possibilities: either the induced sequence terminates ($< N(\eta)$) almost immediately or after $n \leq N(\eta)$ complete inducing steps we can build an induced holomorphic box mapping Φ with the separation index $\beta(\eta) > 0$.

PROPOSITION 7. – *Let $f \in \mathcal{F}_\eta$ be a polynomial unimodal map. Then there are numbers K and N dependent on η only, so that for every $f \in \mathcal{F}_\eta$ an induced sequence of the box* *mappings ϕ_i, $0 \leq i \leq n$ exists, which satisfies the conditions:*

1. $n \leq N$,

2. *there is either a holomorphic box mapping Φ obtained from ϕ_n by the* general geometric construction *with the separation index larger than K or ϕ_n is terminal.*

The following observation stated as Lemma 10.2 plays the crutial role in the proof of Proposition 7. The characteristic ratio $\alpha(\phi_1)$ of the output of the real inducing procudure is either smaller by the definite factor then that of an input or the characteristic cross-ratio $\tau(\phi_1)$ is smaller than 1. In the latter case, one will always be able to apply Proposition 6 with $x = c$ for parabolic returns and $x = b$ for hyperbolic ones and thus prove the existence of an induced holomorphic box mapping Φ.

LEMMA 10.2. – *Consider a type III box mapping φ and choose δ so that $\alpha(\varphi) \leq \delta$. Then for every $\delta < 1$ there is a number $\lambda < 1$ so that whenever in the standardized picture*

$$\frac{c}{b} \geq \lambda\alpha(\varphi),$$

then $\tau(\varphi) \leq 1 - \frac{(1-\delta)^2}{5}$.

Proof. – Denote $\alpha = \alpha(\varphi)$ and $\alpha_1 = \frac{c}{b}$.

$$\tau(\varphi) = 2\mathbf{Poin}\left(-\frac{1}{\alpha}, \alpha_1, 1, \frac{1}{\alpha}\right) = \frac{4\alpha(1-\alpha_1)}{(1-\alpha_1\alpha)(1+\alpha)}. \tag{41}$$

Notice that this expression decreases with α_1 increasing as α is kept fixed and increases as α increases and α_1 is fixed. Hence, assuming $\alpha_1 \geq \lambda\alpha$ for some positive λ, we get

$$1 - \tau(\varphi) \geq 1 - \frac{4\alpha_0(1-\lambda\alpha_0)}{(1-\lambda\alpha_0^2)(1+\alpha_0)}.$$

The difference on the right-hand for $\lambda = 1$ expresses nicely as

$$\left(\frac{1-\alpha_0}{1+\alpha_0}\right)^2 \geq \frac{(1-\alpha_0)^2}{4}.$$

The value of $\lambda < 1$ can be picked for every δ to satisfy the claim of the lemma by continuity. □

Estimates of characteristic cross-ratios

LEMMA 10.3. – *There are constants ε and N depending only on the a priori bound $1 - \varepsilon_0$ (ε_0 is a function of η only), and an index $2 \leq j < N$ so that either*

- *ϕ_j is terminal*
- *both $\alpha(\phi_j)$ and $\tau(\phi_j)$ are less than $1 - \varepsilon$.*

Proof. – Assume that for $j < N$ none of the box mappings ϕ_j is terminal. We will show that there is a universal constant K and an index $j < N$ so that $\alpha(\phi_j) < 1 - \varepsilon_0$ and $\tau(\phi_j) < 1 - K\epsilon_0^2$, where ϵ_0 is a constant from Lemma 10.1. We know that $\alpha(\phi_2) < 1 - \varepsilon_0$.

Substitute $\delta := \alpha(\phi_2)$ into Lemma 10.2 to get the λ. Observe that λ is a universal constant. Next, choose j as the smallest integer at least equal to 2 so that either $\alpha(\phi_{j+1}) \geq \lambda\alpha(\phi_j)$ or $\alpha(\phi_j) \leq \frac{1}{4}$. Proposition 10.3 follows when we show:

- that j is bounded from above depending only ε_0.
- that the estimates claimed in Proposition 10.3 hold for ϕ_j.

The first statement is immediate since until j is reached $\alpha(\phi_j)$ have to decrease with ratio λ. For the second statement, observe first that $\alpha(\phi_j) \leq 1 - \varepsilon_0$. Indeed, this estimate held $j := 2$ and the characteristic ratios keep decreasing until j is reached. If $\alpha_{j+1} \geq \lambda\alpha_j$, then the needed estimate for $\tau(\phi_j)$ follows directly from Lemma 10.2 with $K = \frac{1}{5}$. All we are left to do is to prove that $\tau(\phi_j)$ is bounded as needed when $\alpha(\phi_j) \leq \frac{1}{4}$.

The case of a small nest

Returning to the proof of Lemma 10.2, we recall the estimate (41) which is applied here for $\varphi := \phi_j$, $\alpha := \alpha(\phi_j)$ and $\alpha_1 := \alpha(\phi_{j+1})$. If $\alpha \leq \frac{1}{4}$, the maximal value of the right-hand side is obtained by setting $\alpha_1 = 0$, and this value is $\frac{4}{5}$. In this case, $K = \frac{1}{2}$ will do. □

Now, Proposition 7 follows from Lemma 10.3 and Proposition 6 applied with $x = c$ for a parabolic return and $x = -b$ for a hyperbolic one.

11. Long return time

For a map $f \in \mathcal{F}_\eta$, let k denote the maximum depth of almost parabolic points with periods less than the return time of the restrictive interval if f is renormalizable, or ∞ if not.

PROPOSITION 8. – *Suppose that the map ϕ_n in the induced sequence $(\phi_i)_{i=2}^n$ is terminal and $n < N$, where N is a constant of Proposition 7. If the return time of the restrictive interval is larger than $M(k, \eta)$, then there exist a number $K(\eta)$ and an induced holomorphic box mapping Φ obtained from ϕ_i, $i \leq n$, by the general geometric construction so that the separation index of Φ is larger than $K(\eta)$.*

We will start with the following observation. Even though a box mapping in an induced sequence is not derived from its predecessor as the first return map to the central domain, recall close returns, it shares one important property with the latter.

LEMMA 11.1. – *Every branch of the box mapping from the induced sequence $(\phi_j)_{i=1}^n$ has the property that no intermediate images of the domain of the branch enter the central domain.*

Proof. – The statement is verified by induction. By the definition, ϕ_0 is the first return time to the fundamental inducing domain while ϕ_1 is derived from ϕ_0 through filling-in, *i.e.* by taking the first landing map to the central domain. The central domain of ϕ_i is defined as the preimage of the 1-box extension of the domain of a monotone branch of ϕ_{i-1} by the $\phi_{i-1}^{E_{i-1}}$ and thus by the induction hypothesis has no intermediate images entering itself. The same is true for all newly created long branches. They will be next subjected to the filling-in procedure which is defined as the first landing map to the new central domain. This completes the proof. □

The next lemma essentially expresses the compactness of finite induced sequences.

LEMMA 11.2. – *Let ϕ_n be a terminal real box mapping in the induced sequence $(\phi_i)_{i=2}^n$, $n < N$. For any number $\delta > 0$, there exist integers $M(k, \delta, \eta)$ and $i \leq n$ so that if the return time of the restrictive interval of f is larger than $M(k, \delta, \eta)$ then either*

- $\frac{|B(\varphi_n)|}{|B'(\varphi_n)|} < \delta$ *or*
- *there exists $2 \leq i \leq n$ so that $\alpha(\phi_i) < \delta$.*

Proof. – Let $\mathcal{F}_\eta(k)$ be the subclass of $\mathcal{F}_\eta$ such that the depths of almost parabolic points of period less than the return time of the restrictive interval are less than k. We begin by observing that $\mathcal{F}_\eta(k)$ is a bounded in the $C^{2,1}$ norm, and thus normal in the C^2 topology. Indeed, all members of this family are in the form $h_f(z - 1/2)^2$. Diffeomorphisms h_f are of negative Schwarzian derivative and uniformly η-extendible. It is a well known fact the Schwarzian derivative of an ϵ-extendible iterate of a one-dimensional map with finitely many polynomial-type singularities is bounded from below uniformly in terms of ϵ (see a proof of a very similar estimate in [5].) Thus the normality follows.

Suppose the Lemma is false. Then the lengths of the central boxes B_i,$1 < i \leq n$, would be larger than $L(f)\delta^i$, where $L(f)$ stands for the length of the fundamental inducing interval. It is not a hard fact (*see* [7]) that L is bounded away from zero in terms of η only. Consider a limit g of maps from $\mathcal{F}_\eta(k)$ which have increasing return times of the restrictive intervals while the lengths of central boxes remain bounded away from 0. Now one can easily see that g has a homterval, *i.e.* an interval on which all iterations of g are monotone. We conclude that g must have a non-repelling, thus neutral cycle. We also notice that g continues to expand cross-ratios, thus by [19] this neutral orbit is unique and the critical point is in the immediate basin of one point, say p. Now carry out the inducing process for g. The critical point and p will always stay together in the central branch, since branches in the inducing construction are separated by preimages of the fixed point. Next, Fact 11.1 says that for any branch, no intermediate images enter the central domain. From this observation it follows that return times on the central branch in the inducing process for g cannot jump over the period of p. Thus, after finitely many steps an induced map is obtained which exhibits a close return (which must be parabolic, *i.e.* the image of the real central branch does not cover the critical point). Now, if we take a map f from the sequence which allegedly contradicts the claim of the lemma which is very close to g in the C^2 topology, the construction is conducted in the same way for f, since the course of the construction only depends on where the critical value falls with the respect to the mesh built up by finitely many preimages of the fixed point. The map f will show a parabolic return, but will recover from it after a large number of inducing steps B. Since it takes a long time for the critical value to escape the central domain, and this time can be made arbitrarily large by choosing f close enough to g, we can obtain a map f with an arbitrary long escaping time, contradiction. □

An induced holomorphic box mapping

Without loss of generality, see Proposition 8 and Proposition 6, we may assume that all characteristic ratios $\alpha(\phi_i)$ are less than $1 - \varepsilon_0$, $\varepsilon_0 > 0$ is function of η only. Let $\delta = \frac{1}{4}$. Then, by Lemma 11.2, there exists $1 < i \leq n$ so that the characteristic cross-ratio $\tau(\phi_i)$ is less than $4/5$. Proposition 8 follows by Proposition 6 with the usual substitution of $x = -b$ for a hyperbolic return and $x = c$ for the parabolic one.

12. Analysis of hyperbolic close returns

In this section we prove that if ϕ_i, $i < N$, exhibits a hyperbolic close return (the escape time $E > 1$) and has no a holomorphic box structure then after a bounded number of steps A of the real inducing procedure we reach a uniform holomorphic box structure.

Assumptions and formulation of results

Consider the standarized picture φ for a type III box mapping ϕ_i obtained from $f \in \mathcal{F}_\eta$ by the inducing construction. By Lemma 10.1 and Proposition 10.3, the characteristic ratio $\alpha(\varphi)$ is less than $1 - \varepsilon_0$ and $\varepsilon_0 > 0$ depends on η only.

We define a collection of intermediate domains that arise while iterating step A of the real inducing procedure.

DEFINITION 12.1. – *Let $B^0 =: B$. Define inductively domains B^i, $0 < i$, by*

$$B^{i+1} =: \psi^{-1}(B^i)$$

Fact 12.1.

The intersection of all intervals $\bigcap_{i=1}^{\infty} B^i$ is equal to the interval $[r, r']$, where r is a repelling fixed point of ψ and r' is its preimage. In particular, both r, $r' \in B^{E-1}$.

We **normalize** B^1, by affine change of coordinates to $(-1, 1)$. Denote by φ_i, $1 < i < E$, an induced real box mapping obtained from φ by i inducing steps A.

PROPOSITION 9. – *There exist constants E_0 and K and an induced holomorphic box mapping Φ so that if the escape time $E > E_0$ then*

- *$i \leq E_0$ and Φ is obtained from φ_i by the* general geometric construction.
- *Φ has a holomorphic box structure with the separation index larger then K.*

The constants E_0 and K depend only on the characteristic ratio $\alpha(\varphi)$.

Without loss of generality we may assume that $|B^1|/|B'| \geq 1/4$. Indeed, since if not then the characterictic cross-ratio $\tau(\varphi_1)$ is less than $4/5$ and by Proposition 6 applied with $x = -b$ we have an induced holomorphic box mapping Φ with the separation index bounded by a positive number independent from f.

Real geometry of hyperbolic close returns

The central branch $\psi = h(x^2)$. By the real Koebe Lemma, h has bounded distortion on $h^{-1}(B') \supset [0,1]$. The bound depends solely on $\alpha(\varphi)$. Observe that the derivative of h restricted to $[0,1]$ is less than a constant that depends again only on $\alpha(\varphi)$. To see this apply the Mean Value Theorem, *i.e.* find a point $x \in B^1$ so that the derivative $h'(x) \leq 2|B^0|/|B^1|$. We spread out this estimate obtained at x, by the bounded distortion property of h, over $[0,1]$.

The next Fact results from the following simple computation

$$1 < \frac{d}{dz}\psi(r) = \frac{d}{dz}h(r^2) \cdot 2|r|\,.$$

Fact 12.2.

The repelling fixed point r of ψ is at a definite distance from the critical point 0.

Without loss of generality we may assume that both the preimage q' and the critical value $\psi(0)$ are positive. Of course, $|r| = |r'|$. Denote the endpoint of B^i contained in $\mathbf{R}_-$ by a_i. We will show that the sequence a_i approaches q exponentially fast with the uniform rate. In particular it will imply that the eigenvalue of r is uniformly greater than 1.

LEMMA 12.1. – *For any a_i, $i > 2$, the following inequality holds*

$$\frac{|r - a_i|}{|r - a_1|} \leq \mathbf{Poin}(a_1, a_{i-1}, 0, r')\,\frac{|r - a_{i-1}|}{|r - a_1|}\,.$$

Proof. – Observe that

$$\frac{|r - a_i|}{|r - a_1|} = \frac{|a_i|}{|a_1|}\mathbf{Poin}(a_1, a_i, r, 0) \tag{42}$$

which, by expanding cross-ratio property, is smaller than

$$\frac{|a_i|}{|a_1|}\mathbf{Poin}(a_0, a_{i-1}, r, \psi(0)).$$

The assumption that the map ϕ is not terminal, $\psi(0) > r'$, and algebra implies that

$$\mathbf{Poin}(a_0, a_{i-1}, r, \psi(0)) \leq \frac{|r - a_{i-1}|}{|r - a_1|}\,\frac{|r' - a_1|}{|r' - a_{i-1}|}. \tag{43}$$

Clearly, $|a_i| < |a_{i-1}|$. Combine (42) and (43) together and replace in the resulting inequality $|a_i|$ by $|a_{i-1}|$. The Lemma follows. □

LEMMA 12.2. – *The ratio $|a_1|/|a_2|$ is smaller than $K < 1$ and the bound K depends on $\alpha(\varphi)$ only.*

Proof. – The diffeomorphism h identifies the triples

$$(|a_1|^2, |a_2|^2, 0) \xrightarrow{h} (|a_0|, |a_1|, \psi(0))$$

and distortes distances only by a bounded amount. In our situation, $|\psi(0)| < |a_1|$ which yields the claim of the Lemma. □

Construction of Φ

From Lemmas 12.1 and 12.2 we infer that the cross-ratio $\mathbf{Poin}(a_1, a_{i-1}, 0, r')$ is smaller than $1 - \delta$ and $\delta > 0$ depends only on $\alpha(\varphi)$. Lemma 12.1 asserts that the sequence a_i tends to q uniformly fast. Therefore, after a bounded number of inducing Steps A the characteristic cross-ratio $\tau(\varphi_i) < 2\mathbf{Poin}(a_0, a_i, r, -a_i)$ will become smaller than 1. If we choose i carefully, *i.e.* i should not be too large, then $\tau(\varphi_i)$ is bounded away from 1 in terms of $\alpha(\varphi)$ only.

The ranges of all monotone branches of a real box mapping φ_i contain the interval $(a_i, -a_i)$. We fill-in all these monotone branches. As an outcome we obtain a type III real box mapping with the real box structure $B = (a_i, -a_i)$, $B'' = (a_{i-1}, -a_{i-1})$ and B' unchanged from the initial one. The proof of the existence of Φ is completed by invoking Proposition 6 with x chosen to be the endpoint of B'' closer to the critical value.

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