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OBSERVATIONS ON HARMONIC MAPS AND SINGULAR VARIETIES

BY BRENDON LASELL* AND MOHAN RAMACHANDRAN**

ABSTRACT. – Using facts about harmonic maps from Kähler manifolds to symmetric spaces, to buildings, and to Hilbert spaces, we prove results about the homomorphisms of fundamental groups induced by certain kinds of morphisms of Kähler manifolds. Our results are stated in terms of linear representations of these fundamental groups. Our main result, theorem 4.1, may be viewed as a non-abelian generalization of a fact following from the existence of functorial mixed Hodge structures on the complex cohomology of complex algebraic varieties. Our work is also related to Shafarevich’s question: “is perhaps the universal covering of a complete algebraic variety holomorphically convex?” [14].

1. Introduction

If X and Y are smooth complete varieties over the complex numbers, and ϕ is a morphism from Y to X with image Z , then the kernel of the map from $H^*(X; \mathbb{C})$ to $H^*(Y; \mathbb{C})$ induced by ϕ is the same as that of the map from $H^*(X; \mathbb{C})$ to $H^*(Z; \mathbb{C})$. This follows from the theory of weights, as is explained in proposition 8.2.7 of Deligne’s “Théorie de Hodge, III” [4]. Though the primary conclusion of the theory of weights over the complex numbers, that is the conclusion of the existence of functorial weight filtrations on the complex cohomology of separated schemes of finite type over the complex numbers, appears to be unrelated to analytic techniques, this appearance is illusory. In light of the arguments used to establish this primary conclusion in [4], the theory of weights appears as a partial adaptation of the view of geometry which sees its objects in terms of abstract algebra and category theory to the perspective and results of Hodge’s theory of harmonic integrals. One can make explicit the analytic arguments implicit in the proof by the theory of weights of the above assertion, insofar as this assertion relates to cohomology of the first degree, and one views this cohomology from a perspective suitable to analytic techniques. To be more precise, if X is connected, then $H^1(X; \mathbb{C})$ is the set of all group homomorphisms from $\pi_1(X)$ to \mathbb{C} . Each element of $H^1(X; \mathbb{C})$ corresponds to a flat line bundle on X with structural group contained in \mathbb{C} . Each such bundle has a “harmonic”

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section, unique up to addition of a constant, the differential of which is the harmonic one form on X representing the corresponding element of $H^1(X; \mathbb{C})$. If the element of $H^1(X; \mathbb{C})$ under consideration is in the kernel of the map to $H^1(Y; \mathbb{C})$, the harmonic section of the affine line bundle on X pulls back to give a harmonic function on Y . Any harmonic function on a Kähler manifold is harmonic in the usual sense when restricted to a complex analytic curve, and therefore by the maximum principle this harmonic function on Y is locally constant. Consequently the original global harmonic section of the bundle on X is locally constant on Z , the image of Y in X , and so provides a trivialization of the pullback of the bundle to Z . Therefore the element of $H^1(X; \mathbb{C})$, which by assumption is in the kernel of the map to $H^1(Y; \mathbb{C})$, is necessarily in the kernel of the map to $H^1(Z; \mathbb{C})$. Our purpose in this paper is to offer generalizations of this argument using more sophisticated harmonic map techniques discussed in Corlette's "Flat G -bundles with canonical metrics" [2] and in Gromov and Schoen's "Harmonic maps into singular spaces and p -adic superrigidity for lattices in groups of rank one" [5].

The above argument uses the existence of a harmonic section of a flat bundle on X corresponding to a group homomorphism from $\pi_1(X)$ to \mathbb{C} . Since \mathbb{C} is abelian, each homomorphism from $\pi_1(X)$ to \mathbb{C} forms a conjugacy class under the action of \mathbb{C} . We consider instead conjugacy classes of homomorphisms from $\pi_1(X)$ to arbitrary general linear groups. In fact, for each natural number n , we consider an affine scheme, contravariant in X , which we denote by $SS_n(\pi_1(X))$, and whose points over any field K correspond naturally to isomorphism classes of n -dimensional semi-simple representations of $\pi_1(X)$ over that field; *i.e.*, whose K -points correspond to conjugacy classes of homomorphisms from $\pi_1(X)$ to $GL_n(K)$ such that the Zariski closure of the image of $\pi_1(X)$ is reductive. Under the assumptions stated at the beginning, along with the assumption that X and Z are connected, we prove the following:

THEOREM 1.1. – *For each natural number n there is a finite quotient Δ_n of $\pi_1(Z)$ such that for any field K , those points of $SS_n(\pi_1(X))(K)$ that correspond to representations of $\pi_1(X)$ that are trivial when pulled back to the fundamental groups of the connected components of Y all pull back to points on $SS_n(\pi_1(Z))(K)$ corresponding to representations which factor through Δ_n .*

This theorem implies that to say a semi-simple representation of $\pi_1(X)$ is trivial when restricted to each of the fundamental groups of the connected components of Y is almost the same as to say it is trivial when restricted to $\pi_1(Z)$. The results we prove in the body of this paper are somewhat more general. *See* the beginning of the next section for our general assumptions and theorem 4.1 for our main results.

In addition to their relation with the theory of weights, our results have a connection with the so-called "Shafarevich conjecture" (based on a question of Shafarevich on p. 407 of his book *Basic Algebraic Geometry* [14]), which asserts that the universal cover of a complex projective variety is holomorphically convex, and with a question Nori asks in the paper "Zariski's conjecture and related problems" [12]. The theorem stated above is related to the Shafarevich conjecture in that if $\pi_1(Z)$ has infinite image in $\pi_1(X)$, but $\pi_1(Y)$ has finite image, then the inverse image of Z in the universal cover of X would be an analytic space whose connected components each have infinitely many compact irreducible components. The universal cover of X could not then be holomorphically convex. Our

theorem simply implies that it would be difficult to show such an example exists using linear representations of the fundamental group of X . In the forthcoming article “Some remarks on the Shafarevich conjecture for Kähler surfaces,” Katzarkov and the second author [7] use results from this paper, among others, to prove that if a smooth projective surface has a fundamental group which admits a faithful semi-simple representation on a complex vector space of finite dimension, then the universal cover of that surface is holomorphically convex. For the relation of our work, in particular corollary 4.4, to Nori’s question, as well as a related result, *see* the paper “Complex local systems and morphisms of varieties” [9] of the first author.

This paper is a modified version of our paper “Local systems on Kähler manifolds and harmonic maps,” which we have been distributing since March of 1994. The current version differs from the earlier version primarily in organization, though theorem 4.1 in this paper is more general than theorem 1.2 of the earlier version, in that the finite group Δ_n in theorem 4.1 is independent of the field K .

Finally, we would like to thank Professor Deligne for his suggestions regarding the introduction and for pointing out a more general and more elegant formulation of our main result, Professors Korevaar and Schoen for discussing their work on harmonic maps to non-positively curved spaces with us, and especially Professor Nori for listening to us and offering helpful criticism.

2. Two lemmas using harmonic maps

The natural setting for the harmonic map techniques which we apply in lemma 2.1 and lemma 2.2 is somewhat more general than that introduced above. We introduce here notation and assumptions which we conserve throughout the remainder of the paper:

Let M be a compact Kähler manifold, $\pi : \tilde{M} \rightarrow M$ be a Galois covering space with group of covering transformations Γ , N be a connected analytic subspace of \tilde{M} , all of the irreducible components of which are compact, and Γ' be a finitely generated subgroup of Γ which acts by automorphisms on N .

One can consider these assumptions as a more general version of those made at the beginning of the introduction if one replaces “ M ” by “ X ”, “ \tilde{M} ” by the cover of X corresponding to the normal subgroup of $\pi_1(X)$ generated by $\pi_1(Y)$, “ N ” by the inverse image of Z under the covering map, and “ Γ' ” by the image of $\pi_1(Z)$ in Γ . The condition that all the irreducible components of the inverse image of Z under the covering map under consideration are compact follows from the fact that the map from the compact space Y to X lifts to maps to the covering space of X , and the union of the images of these maps contains the inverse image of Z under the covering map.

Suppose now that $\sigma : \Gamma \rightarrow GL_n(k)$ is a semi-simple representation of Γ defined over a local field k . Assume that k is given an absolute value $|\cdot|$, which induces a locally compact topology on k . The topology on k induces a locally compact topology on $GL_n(k)$ via the embedding

$$\begin{aligned} GL_n(k) &\rightarrow M_n(k) \times M_n(k) \\ M &\mapsto (M, M^{-1}). \end{aligned}$$

LEMMA 2.1. – *The closure of $\sigma(\Gamma')$ in $GL_n(k)$ is compact.*

Proof. – Denote by G the Zariski closure of $\sigma(\Gamma)$ in $GL_n(k)$. By assumption, it is reductive over \bar{k} , the algebraic closure of k . Replacing Γ with a finite index subgroup if necessary, we may assume that G is connected. Since G is reductive over \bar{k} , over some finite extension k' of k it is isogenous to the product of the connected component of its center containing the identity (a torus defined over k') and its connected almost simple normal subgroups. (A group is almost simple if it is semi-simple and its quotient by its center is simple). Since a finite extension of a local field is a local field with an absolute value extending the given absolute value, we may assume that G is isogenous to such a direct product over k itself. By considering each projection onto an almost simple factor of G and the projection onto the center of G , we may assume that G is almost simple or a torus.

THE ARCHIMEDEAN CASE. – If k is Archimedean, we may assume that it is \mathbb{C} . In this case, let H be a maximal compact subgroup of G . The homogeneous space G/H , which we denote by X , is then a simply connected symmetric space with a non-positive curvature operator. X is homeomorphic to an Euclidean space, and so is contractible, and therefore there is a continuous Γ -equivariant map f from \tilde{M} to X . We can define such a map as follows. Suppose we are given a triangulation of M . This triangulation induces a triangulation of \tilde{M} . We define a continuous Γ -equivariant map f_n from the n -skeleton of this triangulation of \tilde{M} to X by induction on n . For f_0 we may take any equivariant map from the zero-skeleton to X , and given f_{n-1} , we choose one n -simplex in \tilde{M} over each n -simplex in M , extend f_{n-1} from the boundary of each of these n -simplexes across their interiors using the contractibility of X , and extend this map in the only possible way to a Γ -equivariant map from the n -skeleton to X .

By corollary 3.5 in [2], there is a Γ -equivariant homotopy of f with a harmonic map g from \tilde{M} to X . According to Corlette's generalization of the Hodge theorem, theorem 5.1 in [2], any harmonic map from a Kähler manifold to a Riemannian manifold with non-positive curvature operator is pluriharmonic; that is, the map is harmonic when restricted to any complex analytic curve. It follows that the composite of g with the square of the distance function from any given point on X is subharmonic when restricted to any curve in \tilde{M} . Therefore, by the maximum principle, g must be constant on any compact analytic subspace of \tilde{M} , and in particular $g(N)$ is a single point, which due to the equivariance of g must be a fixed point for the action of Γ' on X . The stabilizers of points of X are simply the maximal compact subgroups of G , and consequently the image of Γ' under the representation σ is contained in a compact subgroup of G .

THE NON-ARCHIMEDEAN CASE. – The results we use to prove the lemma in case k is not Archimedean are drawn entirely from the paper [5] of Gromov and Schoen.

Associated to the almost simple group G over the non-Archimedean local field k is a Bruhat-Tits building X . See Bruhat and Tits' article "Groupes réductifs sur un corps local. I. Données radicielles valuées" [1] for the construction of X . The building X is the geometric realization of a locally finite simplicial complex which is topologically contractible. Each simplex of X has a Riemannian metric which is the restriction of a metric defined on a neighborhood of a standard simplex with which it is identified, and

there is an embedding of X into a finite dimensional Euclidean space which is an isometry on each simplex. With this structure, X is a geodesic metric space with non-positive curvature, as explained in section one of [5]. G acts on X by isometries, and the stabilizer of any point in X is a compact subgroup of G .

As in the Archimedean case, since X is contractible there is a continuous Γ -equivariant map from \tilde{M} to X . By using a Lipschitz triangulation of M and using geodesic homotopies to extend a map from a boundary of an n -simplex to the entire n -simplex (*see* section four in [5]), the construction we use in the Archimedean case gives a Γ -equivariant Lipschitz map from \tilde{M} to X . This map has finite energy in the sense of [5], since M/Γ is compact.

If G is a torus, then X is simply a real Euclidean space and Γ acts by translations on X . Therefore the argument in the archimedean case applies to give a fixed point of Γ' acting on X .

If G is almost simple, then theorem 7.1 in [5] applies to give a harmonic Γ -equivariant map g from \tilde{M} to X . By theorem 7.3 in [5], g is pluriharmonic, so that as in the Archimedean case, the composite of g with the square of the distance function from any given point on X is subharmonic when restricted to any complex analytic curve in \tilde{M} . It follows that g is constant on any compact analytic subspace of \tilde{M} , and in particular $g(N)$ is a point, which due to the equivariance of g must be a fixed point of Γ' . As noted above, the image of Γ' under σ is consequently contained in a compact subgroup of G . \square

Note that nowhere in the proof of the lemma was the assumption that Γ' is finitely generated used.

COROLLARY 2.1. – *Under the assumptions of the lemma, if k is non-Archimedean, then the coefficients of the characteristic polynomials of the matrices in $\sigma(\Gamma')$ have absolute value at most one. If k is Archimedean, then the coefficients of the characteristic polynomials of the matrices in $\sigma(\Gamma')$ have absolute value at most n .*

Proof. – If k is a non-Archimedean local field, then the maximal compact subgroups of $GL_n(k)$ are all conjugate to the subgroup $GL_n(\mathcal{O}_k)$, where \mathcal{O}_k is the compact subring of k consisting of elements of k of absolute value at most one. So under the assumptions of the lemma, the coefficients of the characteristic polynomials of the matrices in $\sigma(\Gamma')$ are all in \mathcal{O}_k .

Similarly, if $k = \mathbb{C}$, then the maximal compact subgroups of $GL_n(k)$ are all conjugate to the unitary group $U_n(\mathbb{C})$ defined by the usual metric on \mathbb{C}^n . So under the assumptions of the lemma, the eigenvalues of all the matrices in $\sigma(\Gamma')$ all have absolute value one. Therefore the coefficients of the characteristic polynomials of the matrices in $\sigma(\Gamma')$ all have absolute value at most n . \square

COROLLARY 2.2. – *Suppose $\sigma : \Gamma \rightarrow GL_n(K)$ is a semi-simple representation of Γ defined over a field K which is either*

- (a) *a finite algebraic extension of either the field $\mathbb{F}_p(\mathbb{T})$ of rational expressions in one indeterminate with coefficients in a finite field \mathbb{F}_p , for some prime number p ; or*
- (b) *a finite algebraic extension of the field of rational numbers.*

Then $\sigma(\Gamma')$ is finite.

Proof. – According to lemma 2.1, $\sigma(\Gamma')$ is contained in a compact subset of $GL_n(\mathbb{A}_K)$, where \mathbb{A}_K is the group of Adèles associated to K ; *i.e.*, the restricted direct product of a set of representatives of all equivalence classes of completions K_ν of K with respect to a valuation ν . (See Weil's book *Basic Number Theory* [15] for a thorough discussion of Adèles). Since σ is by assumption defined over K , $\sigma(\Gamma')$ is also contained in the discrete subset $GL_n(K)$ of $GL_n(\mathbb{A}_K)$. Therefore $\sigma(\Gamma')$ is finite. \square

Let \mathcal{H} be a real Hilbert space and let

$$\sigma : \Gamma \rightarrow O(\mathcal{H})$$

be an orthogonal representation of Γ on \mathcal{H} .

LEMMA 2.2. – *If $\Gamma = \Gamma'$ then $H^1(\Gamma, \sigma) = 0$.*

Proof. – Suppose that

$$\phi : \Gamma \rightarrow \mathcal{H}$$

is a 1-cocycle for this representation, so that ϕ defines an affine action of Γ on \mathcal{H} by isometries, which has a fixed point if and only if ϕ is a 1-coboundary. The following result is a consequence of the work of Korevaar and Schoen in the paper “Global existence theorems for harmonic maps to non locally compact spaces” [8].

THEOREM 2.1. – *If the affine action of Γ on \mathcal{H} defined by ϕ has no fixed point, then there is a Hilbert space \mathcal{H}' on which Γ acts by isometries without fixed points and a pluriharmonic map*

$$g : \tilde{M} \rightarrow \mathcal{H}'$$

which is equivariant with respect to this action.

To assert that g is pluriharmonic in this situation is to assert that g is harmonic when restricted to any analytic curve in \tilde{M} .

If the action of Γ defined by ϕ has no fixed points, then, as in the proof of lemma 2.1, the map g given by the theorem of Korevaar and Schoen must be constant on the analytic subspace N . The point $g(N)$ is then a fixed point in \mathcal{H}' for the action of $\Gamma' = \Gamma$, which is a contradiction. Therefore $H^1(\Gamma, \sigma) = 0$ for any orthogonal representation σ of Γ . \square

Remark 2.1. – By a theorem of A. Guichardet (*see* [3]), it follows that Γ has D.A. Kazhdan's “property T.”

3. Moduli spaces of representations

In this section, making no claim to originality, we briefly define and denote certain affine schemes whose points over fields correspond to linear representations. Our understanding of these schemes, as well as some of our notation, is due primarily to the article “Varieties of representations of finitely generated groups” of Lubotzky and Magid [10].

Let Δ be a group generated by $\{\delta_1, \dots, \delta_d\}$ and defined by a set $\{r_q\}_{q \in Q}$ of relations. For any positive integer n , let $P_{n,d}$ be the polynomial ring with indeterminate set $\{x_{ij}^{(p)}, y^{(p)} \mid 1 \leq i, j \leq n, 1 \leq p \leq d\}$ and integral coefficients. We denote by $X^{(p)}$ the matrix in $M_n(P_{n,d})$ which has $x_{ij}^{(p)}$ in its i th row and j th column. Then we define $A_n(\Delta)$ to be the quotient of $P_{n,d}$ by the ideal generated by

$$\{\det(X^{(p)})y^{(p)} - 1, r_q(X^{(1)}, \dots, X^{(p)})_{ij} \mid 1 \leq p \leq d, q \in Q, 1 \leq i, j \leq n\}.$$

Note that there is a natural representation

$$\begin{aligned} \rho_n : \Delta &\rightarrow GL_n(A_n(\Delta)) \\ \delta_p &\mapsto X^{(p)}. \end{aligned}$$

The ring $A_n(\Delta)$, as is evident from its definition, has the following universal property: if S is a commutative ring and $\sigma : \Delta \rightarrow GL_n(S)$ is a group homomorphism, then there is a unique ring homomorphism $f : A_n(\Delta) \rightarrow S$ such that σ is the composite of the natural representation $\rho_n : \Delta \rightarrow GL_n(A_n(\Delta))$ with the group homomorphism from $GL_n(A_n(\Delta))$ to $GL_n(S)$ induced by f . It follows that the ring $A_n(\Delta)$ is well-defined up to a unique isomorphism, independently of the choice of presentation of Δ .

We denote by $R_n(\Delta)$ the scheme $\text{Spec}(A_n(\Delta))$. In the language of schemes, the universal property of $A_n(\Delta)$ implies that for any commutative ring S , the points of the space $R_n(\Delta)(S)$ correspond naturally to the representations of Δ on S^n .

Let T be an indeterminate, and let $M = \rho_n(\delta)$ be an element in the image of the natural homomorphism from Δ to $GL_n(A_n(\Delta))$. Define, for any s with $0 \leq s \leq n$, c_s^δ to be the element of $A_n(\Delta)$ such that

$$\det(M - T \cdot Id) = c_n^\delta T^n + \dots + c_0^\delta.$$

We define $B_n(\Delta)$ to be the subring of $A_n(\Delta)$ generated by

$$\{c_s^\delta \mid 0 \leq s \leq n, \delta \in \Delta\}.$$

If $A_n(\Delta)'$ is a ring defined as above for a different choice of generators and relations for Δ , and $B_n(\Delta)'$ is the subring of $A_n(\Delta)'$ defined as above, then the canonical isomorphism from $A_n(\Delta)'$ to $A_n(\Delta)$ induces an isomorphism from $B_n(\Delta)'$ to $B_n(\Delta)$.

By a basic result using the density theorem (*see* for example exercise 18.1 in Serre's book *Linear Representations of Finite Groups* [13]), two semi-simple representations of finite dimension over a field K are isomorphic if and only if for each δ in Δ , the characteristic polynomials of the images of δ under the two representations are the same. If σ is any representation of Δ in $GL_n(K)$, and $\bar{\sigma}$ denotes a semi-simplification of σ , then for any δ in Δ , the characteristic polynomial of $\sigma(\delta)$ is the same as that of $\bar{\sigma}(\delta)$. Therefore, two ring homomorphisms $f, g : A_n(\Delta) \rightarrow K$ have the same restriction to $B_n(\Delta)$ if and only if the semi-simplifications of the corresponding representations of Δ on K^n are isomorphic.

We denote by $SS_n(\Delta)$ the scheme $\text{Spec}(B_n(\Delta))$. In the language of schemes, the above property of $B_n(\Delta)$ implies that for any field K , the points of $SS_n(\Delta)(K)$ in the image of the map from $R_n(\Delta)(K)$ to $SS_n(\Delta)(K)$ correspond naturally to isomorphism

classes of semi-simple representations of Δ on K^n , while the fiber of this map over a point corresponding to a particular semi-simple representation corresponds to the set of all representations of Δ on K^n with semi-simplification isomorphic to this particular semi-simple representation.

If τ is a homomorphism from a finitely generated group E to Δ , then, given presentations of the groups E and Δ , τ induces a natural ring homomorphism τ_* from $A_n(E)$ to $A_n(\Delta)$ which takes $B_n(E)$ to $B_n(\Delta)$. For any ring S , the induced morphism of schemes τ^* from $R_n(\Delta)$ to $R_n(E)$ takes a point of $R_n(\Delta)(S)$ corresponding to a homomorphism σ from Δ to $GL_n(S)$ to the point of $R_n(E)(S)$ corresponding to the homomorphism $\sigma \circ \tau$ from E to $GL_n(S)$. Consequently, for any field K , it takes a point of $SS_n(\Delta)(K)$ corresponding to an isomorphism class of an n -dimensional semi-simple representation σ of Δ to the point of $SS_n(E)(K)$ corresponding to the isomorphism class of the *semi-simplification* of the representation given by $\sigma \circ \tau$.

Note that there is a natural action of the group scheme of finite type over the integers GL_n on $R_n(\Delta)$. This action is that which for any commutative ring S induces the action of $GL_n(S)$ on $R_n(\Delta)(S)$ given by the formula

$$GL_n(S) \times R_n(\Delta)(S) \rightarrow R_n(\Delta)(S) \\ (C, \sigma) \mapsto (\delta \mapsto C\sigma(\delta)C^{-1}),$$

if $R_n(\Delta)(S)$ is identified with the set of homomorphisms σ from Δ to $GL_n(S)$.

LEMMA 3.1. – *For an algebraic closure $\bar{\mathbb{Q}}$ of \mathbb{Q} , the ring $B_n(\Delta) \otimes \bar{\mathbb{Q}}$ is the ring of invariants of the $\bar{\mathbb{Q}}$ -algebra of finite type $A_n(\Delta) \otimes \bar{\mathbb{Q}}$ under the action of $GL_n(\bar{\mathbb{Q}})$ defined above.*

Proof. – If Δ is a finitely generated free group, this follows from the fact that the ring of polynomial invariants with $\bar{\mathbb{Q}}$ -coefficients of invertible matrices with entries in $\bar{\mathbb{Q}}$ is generated by the coefficients of characteristic polynomials. If Δ is simply a finitely presented group, with a presentation given by a morphism from a finitely generated free group E onto Δ , then the induced ring homomorphism from $A_n(E)$ onto $A_n(\Delta)$ is compatible with both the definitions of $B_n(E)$ and $B_n(\Delta)$ and the actions of GL_n , so that the identity of $B_n(\Delta) \otimes \bar{\mathbb{Q}}$ with the ring of invariants of the action of $GL_n(\bar{\mathbb{Q}})$ on $A_n(\Delta) \otimes \bar{\mathbb{Q}}$ follows from the corresponding identity for the finitely generated free group E . \square

COROLLARY 3.1. – *The $\bar{\mathbb{Q}}$ -algebra $B_n(\Delta) \otimes \bar{\mathbb{Q}}$ is of finite type.*

Proof. – This is special case of general fact which is prove in Mumford's *Geometric Invariant Theory* [11]: if a reductive group G over an algebraically closed field of characteristic zero acts on an affine scheme $\text{Spec}(A)$ of finite type over that field, then the ring of invariants of A is also of finite type over that field. \square

Since $B_n(\Delta) \otimes \bar{\mathbb{Q}}$ is of finite type over $\bar{\mathbb{Q}}$, so is $B_n(\Delta) \otimes \mathbb{Q}$, and therefore, there is a finite set of prime numbers Q_n so that if $Q_n(\Delta)^{-1}\mathbb{Z}$ denotes the localisation of \mathbb{Z} with respect to the multiplicative subset generated by $Q_n(\Delta)$, then $B_n(\Delta) \otimes Q_n(\Delta)^{-1}\mathbb{Z}$ is a $Q_n(\Delta)^{-1}\mathbb{Z}$ -algebra of finite type. (For a proof, see for example lemma 1.8.4.2 in Grothendieck and Dieudonné's *EGA IV* [6]).

4. Consequences of harmonic map lemmas in terms of moduli spaces

Our main result is the following.

THEOREM 4.1. – *For any natural number n there is a finite quotient Δ_n of Γ' and a finite set P_n of prime numbers such that for any field K :*

(a) *each point in $SS_n(\Gamma')(K)$ in the image of the natural map from $SS_n(\Gamma)(K)$ to $SS_n(\Gamma')(K)$ corresponds to an isomorphism class of semi-simple representations of Γ' which factor through Δ_n ; and*

(b) *if the characteristic of K is not in P_n , then the restriction of any semi-simple representation of Γ to Γ' is itself semi-simple.*

Given the definition of $B_n(\Gamma')$ and $SS_n(\Gamma')$ in section 3, corollary 2.1 implies that for any local field k , the pull-back of any regular function on the affine scheme $SS_n(\Gamma') \times \text{Spec}(k)$ to $R_n(\Gamma) \times \text{Spec}(k)$ is bounded in absolute value. If $\phi : X \rightarrow \mathbb{C}^m$ is a holomorphic map between analytic spaces such that every holomorphic function on \mathbb{C}^m pulls back to a bounded function on X , then by the maximum principle the image of ϕ is finite, with at most one point for every irreducible component of X . We model the following lemma and its proof on this fact about analytic spaces.

LEMMA 4.1. – *For any natural number n , any field K , and any minimal prime ideal \mathfrak{p} of $A_n(\Gamma) \otimes K$, the image of $B_n(\Gamma') \otimes K$ in $A_n(\Gamma) \otimes K/\mathfrak{p}$ is a finite algebraic extension of K .*

Proof. – We denote by i the inclusion of Γ' in Γ , and by i_* the induced map from $B_n(\Gamma')$ to $A_n(\Gamma)$.

The K -algebra $A_n(\Gamma) \otimes K/\mathfrak{p}$ is an integral domain of finite type over K , and therefore the algebraic closure of K in it is a finite algebraic extension of K . Since the ring $B_n(\Gamma')$ is generated by

$$\{c_s^\gamma \mid 0 \leq s \leq n, \gamma \in \Gamma'\}$$

(using the notation of section 3), to prove the lemma, it is sufficient to show that for any γ in Γ' and any s with $0 \leq s \leq n$, the K -algebra homomorphism

$$\begin{aligned} i_s^\gamma : K[T] &\rightarrow A_n(\Gamma) \otimes K \\ T &\mapsto i_*(c_s^\gamma) \end{aligned}$$

is not injective. If this map is not injective for a field K , it is not injective for any subfield or extension field of K , so we may assume that K is a non-archimedean local field.

Assume the map i_s^γ is injective for some γ in Γ' and some s with $0 \leq s \leq n$. Let x be an element of K of absolute value greater than one. Then $i_s^\gamma(T - x)$ is contained in a maximal ideal \mathfrak{m} of $A_n(\Gamma) \otimes K$. There is therefore a finite algebraic extension K' of K , and a homomorphism f from $A_n(\Gamma) \otimes K$ to K' with $i_*(c_s^\gamma) - x$ in its kernel. We denote by σ the homomorphism from Γ to $GL_n(K')$ corresponding to f . Since the image of $B_n(\Gamma') \otimes K$ under i_* in $A_n(\Gamma) \otimes K$ is in fact contained in $B_n(\Gamma) \otimes K$, there is an f so that σ is semi-simple. The image of $i_*(c_s^\gamma)$ in K' under f is x . But this means that the coefficient of T^s in the characteristic polynomial $\det(\sigma(\gamma) - T \cdot id)$ is x , an element of K' of absolute value greater than one, contradicting corollary 2.1. \square

COROLLARY 4.1. – *Any semi-simplification of the restriction to Γ' of a representation of Γ on a finite dimensional vector space over a field K is isomorphic to a representation defined over a finite extension of the prime field of K . (By the “prime field” we mean the minimal subfield of K ; i.e., \mathbb{Q} if the characteristic of K is zero, and \mathbb{F}_p if the characteristic of K is p).*

Proof. – We denote by F the prime field of K .

Clearly we may assume K is algebraically closed.

According to the universal properties of the rings $A_n(\Gamma)$ and $B_n(\Gamma')$ described in section 3, for any natural number n and any field K , the set of isomorphism classes of semi-simple representations of Γ' on K^n which contain a semi-simplification of the restriction to Γ' of a representation of Γ on K^n corresponds exactly to the set of F -algebra homomorphisms from $B_n(\Gamma') \otimes F$ to K which are pull-backs of F -algebra homomorphisms from $A_n(\Gamma) \otimes F$ to K . Lemma 4.1 implies that for any minimal prime \mathfrak{p} of $A_n(\Gamma) \otimes F$, the image of $B_n(\Gamma') \otimes F$ in $A_n(\Gamma) \otimes F/\mathfrak{p}$ is a finite algebraic extension F' of F , so that if g is an F -algebra homomorphism from $A_n(\Gamma) \otimes F/\mathfrak{p}$ to K , corresponding to a representation σ of Γ , then the restriction of g to F' is an isomorphism of F' onto a subfield K_0 of K . Since $A_n(\Gamma) \otimes F/\mathfrak{p}$ is an integral domain of finite type over F' , this isomorphism extends to an F -algebra homomorphism g' from $A_n(\Gamma) \otimes F/\mathfrak{p}$ to a finite algebraic extension \tilde{K}_0 of K_0 in K . As the restriction of g' to F' , the image of $B_n(\Gamma') \otimes F$, is the same as that of g to F' , g' corresponds to a representation σ' of Γ on \tilde{K}_0^n , such that any semi-simplification of the restriction of σ' to Γ' defines a representation over K isomorphic to that defined by g . Since any F -algebra homomorphism from $A_n(\Gamma) \otimes F$ to K must contain some minimal prime ideal of $A_n(\Gamma) \otimes F$, this completes the proof of the corollary. \square

COROLLARY 4.2. – *Any semi-simplification of the restriction to Γ' of a representation of Γ on a finite dimensional vector space over a field K has finite image when considered as a homomorphism from Γ' to $GL_n(K)$.*

Proof. – This is an immediate consequence of corollary 4.1 and, when the characteristic of K is zero, corollary 2.2. \square

COROLLARY 4.3. – *For any natural number n and any field K , there is a finite set of semi-simple representations of Γ' such that any semi-simplification of the restriction to Γ' of a representation of Γ on an n -dimensional vector space over K is isomorphic to some representation in this finite set.*

Proof. – Again we denote by F the prime field of K .

As in the proof of corollary 4.1, we identify the set of isomorphism classes of semi-simplifications of restrictions to Γ' of representations of Γ on n -dimensional vector spaces over K with the set of F -algebra homomorphisms from $B_n(\Gamma') \otimes F$ to K which are pull-backs by i_* of F -algebra homomorphisms from $A_n(\Gamma) \otimes F$ to K . We show this latter set is finite.

For any minimal prime \mathfrak{p} of $A_n(\Gamma) \otimes F$, since the image of $B_n(\Gamma') \otimes F$ in $A_n(\Gamma) \otimes F/\mathfrak{p}$ is a finite algebraic extension of F , there are only finitely many F -algebra homomorphisms from this image to K . Since $A_n(\Gamma) \otimes F$ is of finite type over F , it has only finitely many minimal primes, and since any F -algebra homomorphism from $A_n(\Gamma) \otimes F$ to K must

have a kernel which contains some minimal prime, it follows that the set of F -algebra homomorphisms from $B_n(\Gamma') \otimes F$ to K described above is finite. \square

The above lemma and its corollaries presuppose the choice of a field, but theorem 4.1 does not presuppose such a choice. We must therefore turn to the study of the rings $B_n(\Gamma')$ and $A_n(\Gamma)$ themselves.

LEMMA 4.2. – *For any natural number n there is*

(a) *a finite set of prime numbers S_n ,*

(b) *a finite integral extension Ω of the localisation $S_n^{-1}\mathbb{Z}$ of \mathbb{Z} away from the set of primes S_n , and*

(c) *a finite set H_n of homomorphisms from Γ' to $GL_n(\Omega)$,*

such that for any algebraically closed field K with characteristic not in S_n and any group homomorphism σ of Γ to $GL_n(K)$, there is a group homomorphism τ in H_n and an $S_n^{-1}\mathbb{Z}$ -algebra homomorphism f from Ω to K such that any semi-simplification of the restriction of σ to Γ' is isomorphic to the homomorphism to $GL_n(K)$ induced by τ and f .

Proof. – According to the definition of $Q_n(\Gamma')$, given near the end of section 3, $B_n(\Gamma') \otimes Q_n(\Gamma')^{-1}\mathbb{Z}$ is of finite type. According to lemma 4.1, applied to the field \mathbb{Q} , for any minimal prime \mathfrak{p} of $A_n(\Gamma) \otimes Q_n(\Gamma')^{-1}\mathbb{Z}$, the image of $B_n(\Gamma) \otimes Q_n(\Gamma')^{-1}\mathbb{Z}$ in $A_n(\Gamma) \otimes Q_n(\Gamma')^{-1}\mathbb{Z}/\mathfrak{p}$ is contained in a finite algebraic extension of \mathbb{Q} . Consequently, there is a finite set of primes, which we provisionally take for S_n , such that for each minimal prime \mathfrak{p} of $A_n(\Gamma) \otimes S_n^{-1}\mathbb{Z}$, the image of $B_n(\Gamma') \otimes S_n^{-1}\mathbb{Z}$ in $A_n(\Gamma) \otimes S_n^{-1}\mathbb{Z}/\mathfrak{p}$ is a finite integral extension of $S_n^{-1}\mathbb{Z}$. We choose embeddings of each of these finite extensions of $S_n^{-1}\mathbb{Z}$ in an algebraic closure of \mathbb{Q} , and denote by Ψ the compositum of the images of these embeddings.

If K is a field with characteristic not in S_n , then the set of group homomorphisms from Γ to $GL_n(K)$ corresponds exactly to the set of $S_n^{-1}\mathbb{Z}$ -algebra homomorphisms from $A_n(\Gamma) \otimes S_n^{-1}\mathbb{Z}$ to K . If σ is a homomorphism from Γ to $GL_n(K)$, corresponding to an $S_n^{-1}\mathbb{Z}$ -algebra homomorphism f , then the isomorphism class of a semi-simplification of the restriction of σ to Γ' corresponds to the pull-back of the $S_n^{-1}\mathbb{Z}$ -algebra homomorphism f to $B_n(\Gamma') \otimes S_n^{-1}\mathbb{Z}$. Such a pull-back is determined by the restriction of f to the image of $B_n(\Gamma') \otimes S_n^{-1}\mathbb{Z}$ in $A_n(\Gamma) \otimes S_n^{-1}\mathbb{Z}$. The kernel of f must contain one of the minimal primes of $A_n(\Gamma) \otimes S_n^{-1}\mathbb{Z}$, and given a choice of such a prime \mathfrak{p} , f defines a unique $S_n^{-1}\mathbb{Z}$ -algebra homomorphism from the image of $B_n(\Gamma') \otimes S_n^{-1}\mathbb{Z}$ in $A_n(\Gamma) \otimes S_n^{-1}\mathbb{Z}/\mathfrak{p}$ to K . Since Ψ is isomorphic to a finite integral extension of this image, if K is algebraically closed there is a $S_n^{-1}\mathbb{Z}$ -algebra homomorphism g from Ψ to K extending the restriction of f to the image of $B_n(\Gamma') \otimes S_n^{-1}\mathbb{Z}$. Therefore the pull-back $f \circ i_*$ of f to $B_n(\Gamma') \otimes S_n^{-1}\mathbb{Z}$ factors as the composite of a $S_n^{-1}\mathbb{Z}$ -algebra homomorphism, given by i_* and the choice of \mathfrak{p} , from $B_n(\Gamma') \otimes S_n^{-1}\mathbb{Z}$ to Ψ , and the $S_n^{-1}\mathbb{Z}$ -algebra homomorphism g from Ψ to K .

To define the set H_n , we choose for each minimal prime \mathfrak{p} of $A_n(\Gamma) \otimes S_n^{-1}\mathbb{Z}$ an $S_n^{-1}\mathbb{Z}$ -algebra homomorphism $f_{\mathfrak{p}}$ from $A_n(\Gamma) \otimes S_n^{-1}\mathbb{Z}$ to an algebraic closure $\bar{\mathbb{Q}}$ of \mathbb{Q} with kernel containing \mathfrak{p} . The restriction of each $f_{\mathfrak{p}}$ to $B_n(\Gamma') \otimes S_n^{-1}\mathbb{Z}$ has image in a finite extension of Ψ . For each such restriction, we choose a $S_n^{-1}\mathbb{Z}$ -algebra homomorphism $g_{\mathfrak{p}}$ from $A_n(\Gamma') \otimes S_n^{-1}\mathbb{Z}$ to $\bar{\mathbb{Q}}$ corresponding to a semi-simple representation of Γ' which is isomorphic to a semi-simplification of the restriction to Γ' of the representation of Γ

corresponding to f_p . The image of each g_p in $\bar{\mathbb{Q}}$ is then a finite extension of Ψ . After replacing S_n by a larger finite set of primes, we may assume this extension is integral. We take for Ω the compositum of all integral extensions of Ψ obtained in this way, and for H_n the set of homomorphisms from Γ' to $GL_n(\Omega)$ corresponding to the set of $S_n^{-1}\mathbb{Z}$ -algebra homomorphisms g_p from $A_n(\Gamma') \otimes S_n^{-1}\mathbb{Z}$ to Ω .

According to the above argument, for any algebraically closed field K , a semi-simplification of the restriction to Γ' of any homomorphism from Γ to $GL_n(K)$ is isomorphic to a *semi-simplification* of the composite of a homomorphism from Γ' to $GL_n(\Omega)$ in H_n and a $S_n^{-1}\mathbb{Z}$ -linear homomorphism from Ω to K . But all the representations of Γ' of $\bar{\mathbb{Q}}^n$ corresponding to elements of H_n are *themselves* semi-simple, so that the same is true for the corresponding representations on K^n ; this follows from the fact that a homomorphism from Γ' to $GL_n(\Omega)$ defines an irreducible representation on $\bar{\mathbb{Q}}^n$ if and only if its image is Zariski-dense in $GL_n(\bar{\mathbb{Q}})$. Indeed, for a subset of $GL_n(\Omega)$ to be dense in $GL_n(\bar{\mathbb{Q}})$ is the same as for it to be dense in $GL_n(\Omega)$ itself, and hence for its image in $GL_n(K)$ to be dense for every homomorphism induced by a ring homomorphism from Ω to a field K . \square

Proof of theorem 4.1. – We take for the group Δ_n in the statement of part (a) of the theorem the quotient of Γ' by the intersection of the following subgroups:

(a) the kernels of the homomorphisms in the finite set H_n defined above in the proof of lemma 4.2; and

(b) the kernels of the homomorphisms from Γ' to $GL_n(K)$ corresponding to the semi-simplifications of the restrictions to Γ' of representations of Γ defined on n -dimensional vector spaces over K , where K is any field with characteristic in the set S_n defined above in the proof of lemma 4.2.

According to corollary 4.3 and lemma 4.2, this is a finite set of subgroups of Γ' , and according to corollary 4.2, each of the subgroups in this set of Γ' is of finite index. Therefore the quotient Δ_n of Γ' so defined is a finite group. Finally, lemma 4.2 and corollary 4.1 together imply that if K is any field, and σ is a representation of Γ on K^n , then any semi-simplification of the restriction of σ to Γ' factors through Δ_n . When written in terms of the schemes $SS_n(\Gamma')$ and $SS_n(\Gamma)$, this is the statement we make in part (a) of the theorem.

As for part (b) of the theorem, lemma 2.1 implies, in the case $k = \mathbb{C}$, that, for an algebraic closure $\bar{\mathbb{Q}}$ of \mathbb{Q} , the image of the natural map from $R_n(\Gamma)(\bar{\mathbb{Q}})$ to $R_n(\Gamma')(\bar{\mathbb{Q}})$ contains only points corresponding to semi-simple representations of Γ' . According to part (a) of the theorem, this image is the union of a finite set of orbits of the natural action of $GL_n(\bar{\mathbb{Q}})$ on $R_n(\Gamma')(\bar{\mathbb{Q}})$. Each of these orbits is a closed subscheme over $\bar{\mathbb{Q}}$ (see [10] or [11]). Consequently there is a finite set of prime numbers, which we take for P_n , a finite integral extension Φ of the localisation $P_n^{-1}\mathbb{Z}$, and a quotient $C_n(\Gamma')$ of the ring $A_n(\Gamma') \otimes \Phi$, such that for any Φ -algebra Σ , the Φ -algebra homomorphisms from $C_n(\Gamma')$ to Σ correspond exactly to the representations of Γ' on Σ^n which are restrictions of representations of Γ . Since the Φ -algebra homomorphisms from $C_n(\Gamma')$ to $\bar{\mathbb{Q}}$ correspond only to semi-simple representations of Γ' on $\bar{\mathbb{Q}}^n$, for any field K the Φ -algebra homomorphisms from $C_n(\Gamma')$ to K correspond only to semi-simple representations of Γ' on K , for the reason involving Zariski-density given at the end of the proof of lemma 4.2. Any algebraically closed field

with characteristic not in the set P_n is a Φ -algebra, and therefore this completes the proof of part (b) of the theorem. \square

Our observations permit a simpler formulation in the case when $\Gamma = \Gamma'$. In the situation described in the introduction, this occurs for example when Z is a hyperplane section of X . The group Γ' is then the quotient of the fundamental group of X by the normal subgroup generated by the fundamental group (or groups) of Y . It has certain properties, which would be quite peculiar if it is not a finite group.

COROLLARY 4.4. – *If $\Gamma = \Gamma'$, then for each natural number n there is a finite quotient Δ_n of Γ and a finite set of prime numbers P_n such that for any field K :*

- (a) *any representation of Γ on K^n which is semi-simple factors through Δ_n ; and*
- (b) *if the characteristic of K is not in P_n , then any representation of Γ on K^n is semi-simple.*

Furthermore, Γ has D. A. Kazhdan's "property T."

Proof. – The first assertion is a special case of theorem 4.1, and the second is remark 2.1 at the end of section 2. \square

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