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ON THE COMPUTATION OF THE CYCLE CLASS MAP FOR NULLHOMOLOGOUS CYCLES OVER THE ALGEBRAIC CLOSURE OF A FINITE FIELD

BY CHAD SCHOEN (1)

ABSTRACT. – This paper is concerned with a cycle class map $CH^r(W_k)_{hom} \rightarrow J_l^r(W_k)^{Gal(\bar{k}/k)}$, where the target space is constructed from the Galois cohomology of $H^{2r-1}(W_{\bar{k}}, \mathbb{Z}_l(r))$. We focus on the case in which there is a morphism to a curve $p: W \rightarrow X$ and on the problem of evaluating the cycle class map mod l on nullhomologous cycles supported in the fibers of p. When k is a finite field and p is a self-fiber-product of non-isotrivial, semi-stable, elliptic surfaces with section we find that $CH^2(W_{\bar{k}})_{hom}$ is not finitely generated. In very special situations the l-primary part of the Griffiths group is computed.

0. Introduction

Let W be a smooth, projective variety over a field k. Let $\bar{k}$ denote a separable closure of k and let G_k denote the Galois group $Gal(\bar{k}/k)$. Write

$$Z^r(W_{\bar{k}})_{rat} \subset Z^r(W_{\bar{k}})_{alg} \subset Z^r(W_{\bar{k}})_{hom}$$

for the groups of codimension r algebraic cycles on $W_{\bar{k}}$ which are rationally (respectively algebraically, respectively homologically) equivalent to zero. The Griffiths group $Gr^r(W_{\bar{k}}) := Z^r(W_{\bar{k}})_{hom}/Z^r(W_{\bar{k}})_{alg}$, and more generally the Chow group of nullhomologous cycles $CH^r(W_{\bar{k}})_{hom} := Z^r(W_{\bar{k}})_{hom}/Z^r(W_{\bar{k}})_{rat}$, are important but poorly understood invariants of $W_{\bar{k}}$. In fact the structure of $Gr^r(W_{\bar{k}})$ is unknown in every instance in which it is not the zero group. Even the structure of its torsion subgroup is very mysterious [Sch-T]. The main result of this paper is that the Griffiths group for a variety over the algebraic closure of a finite field can in certain cases contain a non-zero divisible subgroup. Previously it had not been known that these groups could be infinite. In §14 we shall prove the following more precise result :

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THEOREM 0.1. – *Suppose that k is a finite field whose characteristic is congruent to* $1 \bmod 3$. *Let E be the Fermat cubic curve. If l is an odd prime and $l = -1 \bmod 3$, then* $Gr^2(E^3_{\bar{k}}) \otimes \mathbf{Z}_l \simeq (\mathbf{Q}_l/\mathbf{Z}_l)^2$.

Here the hypothesis on the characteristic is essential, but the hypothesis on l appears to be a consequence of the method of proof and might conceivably be weakened if somewhat more elaborate arguments are applied.

We now outline the organization of the paper and the contents of the individual sections. The first section is devoted to preliminaries on arithmetic cycle class maps over a base field k which is finitely generated over the prime field. We begin by recalling an observation of Bloch [B] that the cycle class map

$$\mathbf{cl}^n_W : Z^r(W) \to H^{2r}(W, \mathbf{Z}/l^n(r))$$

defined in [Gr-D] when combined with the Hochschild-Serre spectral sequence gives rise to a cycle class map

$$cl^n_{W,0} : Z^r(W)_{hom} \to H^1(G_k, H^{2r-1}(W_{\bar{k}}, \mathbf{Z}/l^n(r))) \tag{0.2}$$

which annihilates $Z^r(W)_{rat}$. Given an explicit nullhomologous cycle, $z \in Z^r(W)_{hom}$, it is generally difficult to compute $cl^n_{W,0}(z)$ directly from the definition. The search for a more computationally feasible approach led to the observation (made independently and in much greater generality by Jannsen [Ja, §9]) that in certain circumstances (0.2) could be computed in terms of the first coboundary map attached to a short exact sequence of G_k-modules. This short exact sequence is in effect a piece of a long exact sequence for relative cohomology. By applying the inverse limit over n and the direct limit over finite extensions of k to (0.2), we obtain an "arithmetic Abel-Jacobi map"

$$\alpha^r : Z^r(W_{\bar{k}})_{hom} \to J^r_l(W), \tag{0.3}$$

where $J^r_l(W) \simeq H^{2r-1}(W_{\bar{k}}, \mathbf{Z}_l(r)) \otimes \mathbf{Q}_l/\mathbf{Z}_l$ when k is a finite field. The subgroup $Z^r(W_{\bar{k}})_{rat}$ will always be annihilated by α^r. In favorable instances $cl^1_{W,0}(z) \neq 0$ will already imply $\alpha^r(z) \neq 0$ and hence $z \notin Z^r(W_{\bar{k}})_{rat}$.

In §2 we describe circumstances under which $\alpha^r(Z^r(W_{\bar{k}})_{alg}) = 0$. In such situations α^r provides a tool for analyzing the Griffiths group.

The third section introduces specific cycles z for which the computation of $cl^n_{W,0}(z)$ might be less difficult than in general. The focus is on cycles which are supported in the fibers of a dominant morphism $p : W \to X$, where X is a smooth curve. Of particular interest is the case when k is a finite field, W has dimension 3, and the Picard number of every non-singular closed fiber is greater than the Picard number of the geometric generic fiber. The Leray spectral sequence for p is applied in §4 to reduce the computation of $cl^n_{W,0}(z)$ to a problem involving only constructible sheaves on the curve X. This problem can be quite difficult to analyze when the monodromy representations associated with these sheaves are complicated. There is however a geometric situation which gives rise to $SL(2, \mathbf{Z}/l^n)/\pm 1$ as the monodromy group. In the special case $n = 1$ the information from representation theory which we need has been worked out in [Sch-MR]. To describe this geometric situation begin with a non-isotrivial, semi-stable elliptic surface $\pi : Y \to X$

which has a section. Take W to be the blow up of the fiber product $Y \times_X Y$ along the singular locus. The resulting map $p : W \to X$ is the subject of §5. In particular we will discuss complex multiplication cycles which are certain nullhomologous cycles on W supported in fibers of p.

The sixth section contains the statements of three technical theorems culminating in

THEOREM 0.4. – *Let k be a finite field and let $p : W \to X$ be as above. Let $CH^2_{CM}(W_{\bar{k}})$ be the subgroup of $CH^2(W_{\bar{k}})_{hom}$ generated by complex multiplication cycles. Then for almost all primes l, there is a surjective map $CH^2_{CM}(W_{\bar{k}}) \to \mathbf{Q}_l/\mathbf{Z}_l$.*

After some preliminaries in §7, each of the sections 8, 9, and 10 is devoted to the proof of one theorem from §6. In particular, §10 contains a proof of (0.4). An important step in the argument involves showing that the image of the cycle class map on complex multiplication cycles contains an infinite group. Our method relies on a strong form of the Tchebotarev density theorem. Tchebotarev's theorem applies here, because we only consider the case of finite base field and because almost every fiber of the morphism $p : W \to X$ contains a complex multiplication cycle.

There is a different approach to showing that the image of the cycle class map on complex multiplication cycles can be infinite which works in the case of self-fiber-products of elliptic *modular* surfaces over finite fields. These varieties come equipped with Hecke correspondences. The action of Hecke correspondences on complex multiplication cycles is discussed in §11. Then in §12 the results are used to prove that under suitable hypotheses the image of α^2 applied to the complex multiplication cycles is a divisible group.

Although the image of (0.3) is a torsion group when the base field k is finite, it is not *a priori* clear which elements of $CH^r(W_{\bar{k}})_{hom}$ have finite order. We will apply a theorem of Soulé [So, Thm 3] to this problem. First, we must dominate W by a product of curves. This is done for one particular self-fiber-product of elliptic modular surfaces in §13 using the threefold product of an elliptic curve which is isogenous to the Fermat cubic curve. The consequences for the Griffiths group, including a proof of (0.1), are drawn in the final section.

0.5 NOTATIONS

k = a field, which after (1.2) will be assumed to be finitely generated over the prime field.

$\bar{k}$ = a separable closure of k.

$G_k = Gal(\bar{k}/k)$ endowed with the usual Krull topology.

$G_{k'/k} = G_k/G_{k'}$ for k'/k a Galois extension.

l = a prime distinct from the characteristic of k.

$H^1(G_k, T)$ = continuous crossed homomorphisms modulo coboundaries, where T is a $\mathbf{Z}_l$-module and G_k acts continuously with respect to the l-adic topology on T.

$X = X_k$ = is a smooth, projective, geometrically integral curve over k.

η = the generic point of X. $\bar{\eta}$ = a geometric generic point.

$\dot{X} \subset X$ = a non-empty open subset. $\Sigma_X = X - \dot{X}$.

$\pi_1^t(\dot{X}_k, \bar{\eta})$ = the tame fundamental group of $\dot{X}$. More precisely, the composition of all intermediate fields in the extension $\bar{\eta}/\eta$ which are finite over η, unramified over $\dot{X}$, and tamely ramified over Σ_X is Galois over η. The corresponding Galois group is denoted $\pi_1^t(\dot{X}_k, \bar{\eta})$.

$Z^r(W)$ = the group of codimension r cycles on the variety W.

$Z^r(W)_{rat} \subset Z^r(W)$ = the subgroup of cycles rationally equivalent to zero.

$CH^r(W) = Z^r(W)/Z^r(W)_{rat}$

$N^r(W) = Z^r(W)/Z^r(W)_{hom}$ = codimension r cycles modulo homological equivalence on a non-singular variety W. (The precise definition of $Z^r(W)_{hom}$ is given in (1.2) below.)

$Z^r_V(W)_{hom}$ = nullhomologous, codimension r cycles on the non-singular variety W whose support is contained in the subvariety V.

$N^r_V(W) = Z^r_V(W)_{hom}/Z^s(V)_{hom}$, where $V \subset W$ is non-singular of pure codimension $s - r$.

1. Preliminaries on the cycle class map

Let W be a smooth variety of dimension d over a field k. For each prime l distinct from the characteristic and each integer $n > 0$ there is a cycle class map [Mi, VI.9]

$$cl^n_W : Z^r(W) \to H^{2r}(W_{\bar{k}}, \mathbf{Z}/l^n(r)).$$

Passing to the inverse limit yields a map

$$(1.1) \qquad cl_W : Z^r(W) \to H^{2r}(W_{\bar{k}}, \mathbf{Z}_l(r)) := \varprojlim H^{2r}(W_{\bar{k}}, \mathbf{Z}/l^n(r)).$$

Using (1.1) for varying l we define

$$(1.2) \qquad Z^r(W)_{hom} = \text{Ker}\ [Z^r(W) \to \prod_{l \neq char(k)} H^{2r}(W_{\bar{k}}, \mathbf{Z}_l(r))].$$

Assume now and for the rest of the paper that k is finitely generated over the prime field and that W is geometrically integral and proper over k. The purpose of this section is to discuss the definition and properties of a cycle class map

$$(1.3) \qquad \alpha^r : Z^r(W_{\bar{k}})_{hom} \to J^r_l(W) := lim_{k'/k} H^1(G_{k'}, H^{2r-1}(W_{\bar{k}}, \mathbf{Z}_l(r))/\text{ tors}),$$

where the limit is over intermediate Galois extensions $k \subset k' \subset \bar{k}$ of finite degree over k. Galois cohomology is taken in the sense of [Ta1, §2]; the cocycles are continuous with respect to the Krull topology on the Galois group and the l-adic topology on the Galois module.

LEMMA 1.4. – (1) *The natural map* $H^1(G_{k'}, H^{2r-1}(W_{\bar{k}}, \mathbf{Z}_l(r))/\text{ tors}) \to J^r_l(W)$ *is injective.*

(2) *The torsion subgroup of* $J^r_l(W)$ *is divisible of finite corank.*

(3) *If k is a finite field,* $J^r_l(W)$ *is a torsion group.*

Proof. – Set $H = H^{2r-1}(W_{\bar{k}}, \mathbf{Z}_l(r))/\text{ tors}$. Let k''/k' be a finite Galois extension.

(1) It suffices to show that the first term in the exact sequence

$$0 \to H^1(G_{k''/k'}, H^{G_{k''}}) \to H^1(G_{k'}, H) \to H^1(G_{k''}, H)$$

is zero. In fact $H^{G_{k''}}$ is zero. This may be deduced from the Weil conjectures *via* a specialization argument [B, §1] or [Co-Ra, Theorem 1.5].

(2) Form the short exact sequence

$$0 \to H \to H \otimes \mathbf{Q}_l \to H \otimes \mathbf{Q}_l/\mathbf{Z}_l \to 0.$$

Again the Weil conjectures imply that $(H \otimes \mathbf{Q}_l)^{G_{k'}} = 0$. Thus the torsion in $H^1(G_{k'}, H)$ is isomorphic to $(H \otimes \mathbf{Q}_l/\mathbf{Z}_l)^{G_{k'}}$.

(3) If k is finite and $\phi \in G_k$ is the Frobenius, then

$$H^1(G_k, H \otimes \mathbf{Q}_l) \simeq H/(\phi - 1)H \ \otimes \mathbf{Q}_l.$$

This is zero by the Weil conjectures.

A starting point for the construction of α^r is the cycle class map of Grothendieck [Gr-D]

$$\mathbf{cl}^n_{W_{k'}} : Z^r(W_{k'}) \to H^{2r}(W_{k'}, \mathbf{Z}/l^n(r)). \tag{1.5}$$

The Hochschild-Serre spectral sequence gives a decreasing filtration $L^\bullet$ on $H^{2r}(W_{k'}, \mathbf{Z}/l^n(r))$ with graded pieces

$$L^0/L^1 \subset H^{2r}(W_{\bar{k}}, \mathbf{Z}/l^n(r))^{G_{k'}} \qquad \text{and} \qquad L^1/L^2 \subset H^1(G_{k'}, H^{2r-1}(W_{\bar{k}}, \mathbf{Z}/l^n(r))).$$

Since $\mathbf{cl}^n_{W_{k'}}(Z^r(W_{k'})_{hom}) \subset L^1$, (1.5) gives a map

$$cl^n_{W_{k'},0} : Z^r(W_{k'})_{hom} \to H^1(G_{k'}, H^{2r-1}(W_{\bar{k}}, \mathbf{Z}/l^n(r))).$$

Taking the inverse limit gives a map [Ta1, §2]

$$cl_{W_{k'},0} : Z^r(W_{k'})_{hom} \to H^1(G_{k'}, H^{2r-1}(W_{\bar{k}}, \mathbf{Z}_l(r))). \tag{1.6}$$

Now (1.5) and hence (1.6) are functorial with respect to finite separable extensions of the base field [Gr-D]. Thus (1.3) may be defined by composing (1.6) with the tautological map

$$H^1(G_{k'}, H^{2r-1}(W_{\bar{k}}, \mathbf{Z}_l(r))) \to H^1(G_{k'}, H^{2r-1}(W_{\bar{k}}, \mathbf{Z}_l(r))/ \text{ tors})$$

and taking direct limits with respect to finite Galois extensions k'/k.

There is a second natural way to associate to an element $z \in Z^r(W_{k'})_{hom}$ an element in $H^1(G_{k'}, H^{2r-1}(W_{\bar{k}}, \mathbf{Z}_l(r)))$. To discribe this write $|z|$ for the support of z and define

$$H^{2r}_{|z|}(W_{\bar{k}}, \mathbf{Z}/l^n(r))_0 = \text{Ker } [H^{2r}_{|z|}(W_{\bar{k}}, \mathbf{Z}/l^n(r)) \to H^{2r}(W_{\bar{k}}, \mathbf{Z}/l^n(r))]. \tag{1.7}$$

The fundamental class $[z]$ is a Galois invariant element of this group [Gr-D]. By purity [Mi,VI.9.1],

$$H^{2r-1}_{|z|}(W_{\bar{k}}, \mathbf{Z}/l^n(r)) = 0.$$

There results a short exact sequence of $G_{k'}$-modules,

(1.8)

$$0 \to H^{2r-1}(W_{\bar{k}}, \mathbf{Z}/l^n(r)) \to H^{2r-1}((W - |z|)_{\bar{k}}, \mathbf{Z}/l^n(r)) \to H^{2r}_{|z|}(W_{\bar{k}}, \mathbf{Z}/l^n(r))_0 \to 0.$$

Write
$$\beta^n_{k'} : H^{2r}_{|z|}(W_{\bar{k}}, \mathbf{Z}/l^n(r))_0^{G_{k'}} \to H^1(G_{k'}, H^{2r-1}(W_{\bar{k}}, \mathbf{Z}/l^n(r)))$$
for the first coboundary map in the long exact sequence of continuous $G_{k'}$-cohomology associated to (1.8).

PROPOSITION 1.9. – $\beta^n_{k'}([z]) = cl^n_{W_{k'},0}(z)$.

Proof. – See [Ja, 9.4].

The functoriality properties of the cycle class map are gathered together in

PROPOSITION 1.10. – *Abbreviate* β^n_k, $cl^n_{W_k,0}$, $cl^n_{W_{\bar{k}}}$, *respectively* $cl_{W_k,0}$ *with* β^n, cl^n_0, $cl^n_{\bar{k}}$, *respectively* cl_0.

(1) β^n *is functorial with respect to smooth pullback in the category of smooth, proper k-varieties.*

(2) β^n *is functorial with respect to direct image in the same category.*

(3) *Given* $z' \in Z^s(W)$ *define* $Z^r(z')(W)_{hom}$ *to be the subgroup of nullhomologous cycles all of whose components meet* z' *properly. The following diagram commutes*

$$\begin{array}{ccc} Z^r(z')(W)_{hom} & \xrightarrow{\cdot z'} & Z^{r+s}(W)_{hom} \\ \beta^n \downarrow & & \beta^n \downarrow \\ H^1(G_k, H^{2r-1}(W_{\bar{k}}, \mathbf{Z}/l^n(r))) & \xrightarrow{H^1(\ ,\cup cl_{\bar{k}}(z'))} & H^1(G_k, H^{2r+2s-1}(W_{\bar{k}}, \mathbf{Z}/l^n(r+s))). \end{array}$$

(4) β^n *annihilates cycles rationally equivalent to* 0.

(5) β^n *is functorial with respect to extension of the base field.*

(6) $\beta^n_{k'}$ *is* $G_{k'/k}$*-equivariant.*

(7) *Let* W' *be smooth and proper over k. Then a correspondence* $\Gamma \in Z^{d+s-r}(W \times W')$ *gives rise to a commutative diagram*

$$\begin{array}{ccc} CH^r(W)_{hom} & \xrightarrow{\Gamma_*} & CH^s(W')_{hom} \\ cl_0 \downarrow & & cl_0 \downarrow \\ H^1(G_k, H^{2r-1}(W_{\bar{k}}, \mathbf{Z}_l(r))) & \xrightarrow{H^1(\ ,cl_{\bar{k}}(\Gamma)_*)} & H^1(G_k, H^{2s-1}(W'_{\bar{k}}, \mathbf{Z}_l(s))). \end{array}$$

(8) *When* $r = 1$ *the cohomology class* $\beta^n(z)$ *may be represented by the crossed homomorphism* $G_k \to \mathrm{Pic}(W)(\bar{k})[l^n]$, $\sigma \to D - \sigma D$, *where* $l^n D \sim_{rat} z$.

Proof. – (1) This follows easily from the extension definition and the functoriality of the cycle class map to local cohomology with respect to pullback by smooth morphisms [Mi, VI.6.1c, 9.2].

(2) By Poincaré duality (1.7) may be rewritten in terms of homology [Ja, 9.0.1]. The assertion follows from the functoriality of the fundamental homology class with respect to proper direct image.

(3) *See* [Ja, 10.6].

(4) A cycle on W which is rationally equivalent to zero may be written as $pr_{W*}(\Gamma \cdot pr^*_{\mathbf{P}^1}(z))$, where $\Gamma \in Z(\mathbf{P}^1 \times W)$, $z \in Z^1(\mathbf{P}^1)_{hom}$, and Γ meets $pr^*_{\mathbf{P}^1}(z)$ properly. Since $H^1(\mathbf{P}^1_{\bar{k}}, \mathbf{Z}/l^n(1)) \simeq 0$, (4) follows from (1), (2), and (3).

(5) The cycle class map to local cohomology is functorial with respect to extension of the base field.

(6) Let $y = \cup_{\sigma \in G_{k'/k}} z^{\sigma}$. Replace $|z|$ by $|y|$ in (1.8). The first coboundary map associated to the resulting short exact sequence applied to $[z] \in H^{2r}_{|y|}(W_{\bar{k}}, \mathbf{Z}/l^n(r))_0^{G_{k'}}$ gives $\beta^n_{k'}(z)$. The coboundary commutes with the $G_{k'/k}$-action and thus sends z^{σ} to $\beta^n_{k'}(z^{\sigma}) = (\beta^n_{k'}(z))^{\sigma}$.

(7) $\Gamma_*(z) := pr_{W'*}(\Gamma \cdot pr_W^*(z))$. When $pr_W^*(z)$ meets Γ properly, the assertion follows from (1), (2), (3). In general one may replace $pr^*(z)$ by z' in the same rational equivalence class such that $\Gamma \cdot z'$ is well defined. Furthermore the rational equivalence class of the intersection is independent of the choice of z' [Ro]. The assertion follows from (1), (2), (3), and (4).

(8) Define

$$\text{(1.11)} \qquad \mathcal{K}_W := \text{Ker}\,[\bar{k}(W)^*/\bar{k}(W)^{*l^n} \xrightarrow{div} Div(W)_{\bar{k}}/l^n Div(W)_{\bar{k}}].$$

Kummer theory identifies $\mathcal{K}_W$ with the cyclic order l^n étale covers of W, that is with μ_{l^n}-torsors, hence with $H^1(W_{\bar{k}}, \mathbf{Z}/l^n(1))$. Write $(Div_{|z|}(W_{\bar{k}})/l^n)_0$ for the group of divisors on $W_{\bar{k}}$ with $\mathbf{Z}/l^n$-coefficients, whose class in $H^2(W_{\bar{k}}, \mathbf{Z}/l^n(1))$ vanishes and whose support is contained in $|z|$. When $r = 1$ (1.8) may be rewritten as

$$\text{(1.12)} \qquad 1 \to \mathcal{K}_W \to \mathcal{K}_{W-|z|} \to (Div_{|z|}(W_{\bar{k}})/l^n)_0 \to 0.$$

Now $\beta^n(z)$, the image of z under the first coboundary map associated with (1.12), is represented by the crossed homomorphism, $G_k \to \mathcal{K}_W$, $\sigma \to \sigma f/f$, where $f \in \bar{k}(W)$ and $div(f) = z - l^n D$. The assertion follows from the isomorphism

$$\mathcal{K}_W \simeq \text{Pic}(W_{\bar{k}})(\bar{k})[l^n], \qquad f \to \frac{1}{l^n} div(f).$$

The following proposition is a consequence of (1.10) :

PROPOSITION 1.13. – *The cycle class map $\alpha^r : Z^r(W_{\bar{k}})_{hom} \to J^r_l(W)$ satisfies*

(1) *functoriality with respect to smooth pullback, proper direct image and correspondences.*

(2) $\alpha^r(Z^r(W_{\bar{k}})_{rat}) = 0$.

(3) *α^r is G_k-equivariant.*

It will be useful later to make explicit a special case of (1.10(8)). For this we assume that k is a finite field of characteristic prime to l. W/k will be a smooth, projective, geometrically integral curve of positive genus and w, w_0 will be k-rational points of W. Define

$$T = (1 - \phi)\text{Pic}^0(W)[l^n], \qquad A = \text{Pic}^0(W)/T, \qquad \text{and} \qquad \varphi : A \to \text{Pic}^0(W),$$

where φ is the isogeny induced by multiplication by l^n on $\text{Pic}^0(W)$. Embed W in $\text{Pic}^0(W)$ using the base point w_0. Define

$$\tilde{W} = W \times_{\text{Pic}^0(W)} A.$$

Now $\tilde{W}$ is a geometrically integral curve. Moreover, $k(\tilde{W})$ is Galois over $k(W)$. The Galois automorphisms are given by translating by elements of the subgroup Ker φ, which consists entirely of k-rational points.

LEMMA 1.14. – *The canonical isomorphism,*

$$H^1(G_k, H^1(W_{\bar{k}}, \mu_{l^n})) \simeq \mathrm{Pic}(W)(\bar{k})[l^n]/T(\bar{k}) \simeq (\mathrm{Ker}\ \varphi)(k) \simeq Gal(k(\tilde{W})/k(W)),$$

sends $\beta^n(w - w_0)$ *to* $Frob_w^{-1}$.

Proof. – The first of the three isomorphisms sends a continuous crossed homomorphism to its value at the Frobenius element, $\phi \in G_k$. Choose $\hat{w} \in \mathrm{Pic}^0(W)(\bar{k})$ such that $l^n\hat{w} = w - w_0$. By (1.10(8))

$$\beta^n(w - w_0)(\phi) = \hat{w} - \phi\hat{w} \in \mathrm{Pic}(W)(\bar{k})[l^n]/T(\bar{k}).$$

Since $\hat{w}$ maps to the point w on the curve W embedded in $\mathrm{Pic}^0(W)$, the image w' of $\hat{w}$ in $A(\bar{k})$ lies on $\tilde{W}(\bar{k})$. By definition $Frob_w(w') = \phi w'$. Clearly translation by $-\beta^n(w - w_0)(\phi)$ in $A(\bar{k})$ sends w' to $\phi w'$. The lemma follows since an element of $\mathrm{Ker}\varphi$ is completely determined by its action on any point in $A(\bar{k})$.

In practice we shall not attempt to compute either $cl^n_{W,0}$ or β^n directly from the definition. Instead we will focus attention on a closely related map to be described presently. Let $i : V \to W$ denote the inclusion of a closed, reduced, non-singular k-subscheme of pure codimension, $r - s$. Denote by $Z^r_V(W)_{hom}$ (respectively $Z^r_V(W)_{rat}$) the group $i_*Z^s(V) \cap Z^r(W)_{hom}$ (respectively $i_*Z^s(V) \cap Z^r(W)_{rat}$). We will sometimes abuse notation and view $Z^r_V(W)_{hom}$ and $Z^r_V(W)_{rat}$ as subgroups of $Z^s(V)$. Thus

(1.15) $\qquad N^r_V(W) = Z^r_V(W)_{hom}/Z^s(V)_{hom} \qquad$ and $\qquad N^r_V(W)/Z^r_V(W)_{rat}$

have meaning. Write

$$\epsilon_n : H^{2r}_V(W_{\bar{k}}, \mathbf{Z}/l^n(r)) \to H^{2r}(W_{\bar{k}}, \mathbf{Z}/l^n(r))$$

for the obvious map and ϵ'_n for the corresponding map in degree $2r-1$. For $z \in Z^r_V(W)_{hom}$ there is an obvious commutative diagram with exact rows

(1.16)

$$\begin{array}{ccccccccc}
0 & \to & H^{2r-1}(W_{\bar{k}}, \mathbf{Z}/l^n(r)) & \to & H^{2r-1}((W - |z|)_{\bar{k}}, \mathbf{Z}/l^n(r)) & \to & H^{2r}_{|z|}(W_{\bar{k}}, \mathbf{Z}/l^n(r))_0 & \to & 0 \\
 & & p_n \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Coker}\ \epsilon'_n & \to & H^{2r-1}((W - |V|)_{\bar{k}}, \mathbf{Z}/l^n(r)) & \to & \ker \epsilon_n & \to & 0
\end{array}$$

in which p_n is the tautological map. Write δ^n_V for the first coboundary map in the long exact G_k-cohomology sequence associated to the bottom row of (1.16). Consider the composition

$$\theta^n_{V/W} := \delta^n_V \circ cl^n_{V_{\bar{k}}} : Z^r_V(W)_{hom} \to (\mathrm{Ker}\epsilon_n)^{G_k} \to H^1(G_k, \mathrm{Coker}\ \epsilon'_n)$$

and $\theta_{V/W} = \varprojlim \theta^n_{V/W}$.

LEMMA 1.17. – (1) $\theta^n_{V/W}(z) = p_n \circ \beta^n(z)$ *and* $\theta_{V/W}(z) = \varprojlim p_n \circ \beta^n(z)$.
(2) $\theta^n_{V/W}$ *and* $\theta_{V/W}$ *factor through* $N^r_V(W)/Z^r_V(W)_{rat}$.

Proof. – (1) This follows from (1.16) and the commutative diagram

$$\begin{array}{ccc} H^{2s}_{|z|}(V_{\bar{k}}, \mathbf{Z}/l^n(s)) & \to & H^{2r}_{|z|}(W_{\bar{k}}, \mathbf{Z}/l^n(r)) \\ \downarrow & & \downarrow \\ H^{2s}(V_{\bar{k}}, \mathbf{Z}/l^n(s)) & \to & H^{2r}_{|V|}(W_{\bar{k}}, \mathbf{Z}/l^n(r)) \end{array}$$

(2) $\beta^n(Z^r_V(W)_{rat}) = 0$ and $cl^n_{V_{\bar{k}}}(Z^s(V)_{hom}) = 0$.

Given an explicit cycle on an explicit subvariety $V \subset W$ it seems impractical to completely describe the cycle class $\theta_{V/W}(z) \in \varprojlim H^1(G_k, \mathrm{Coker}\, \epsilon'_n)$. However it may be possible to calculate $\theta^1_{V/W}(z) \in H^1(G_k, \mathrm{Coker}\ \epsilon'_1)$. The next lemma shows that the calculation of $\theta^1_{V/W}(z)$ gives important information about $\alpha^r(z) \in J^r_l(W)$.

LEMMA 1.18. – *Suppose that $z \in Z^r_V(W)_{hom}$ satisfies $\theta^1_{V/W}(z) \neq 0$. If $H^{2r-1}(W_{\bar{k}}, \mathbf{Z}_l)$ is torsion free, then $\alpha^r(z) \in J^r_l(W)$ is not zero.*

Proof. – By (1.17) $\beta^1(z) \neq 0$. By (1.9) $cl^1_{W,0}(z) \neq 0$, whence $cl_{W,0}(z) = \varprojlim cl^n_{W,0}(z) \neq 0$. Now (1.4.(1)) implies $\alpha^r(z) \neq 0$.

We will need a slight variant of (1.18) which involves correspondences which act on $N^r_V(W)$. To this end, let $P \in Z^d(W \times W) \otimes \mathbf{Z}_l$ meet $V \times W$ properly. Assume that $P \cap (V \times W) \subset V \times V$ so that P_* induces an endomorphism of $N^r_V(W)$. Assume also that P is a projector on cohomology, *i.e.* $P^2 = P$ in $N^d(W \times W) \otimes \mathbf{Z}_l$.

LEMMA 1.19. – *Suppose that $z \in N^r_V(W)$ satisfies $\theta^1_{V/W}(P_*z) \neq 0$. If $P_*H^{2r-1}(W_{\bar{k}}, \mathbf{Z}_l)$ is torsion free, then $P_*\alpha^r(z) \in J^r_l(W)$ is not zero.*

Proof. – Evidently $\beta^1(P_*z) \neq 0$, whence $cl_{W,0}(P_*z) = P_*cl_{W,0}(z) \neq 0$. Since $P_*H^{2r-1}(W_{\bar{k}}, \mathbf{Z}_l)$ is a direct summand of $H^{2r-1}(W_{\bar{k}}, \mathbf{Z}_l)/tors$, the obvious map

$$H^1(G_k, P_*H^{2r-1}(W_{\bar{k}}, \mathbf{Z}_l(r))) \to J^r_l(W)$$

is injective (1.4(1)). The lemma follows.

2. Preliminaries on the Griffiths group

We keep the notation of the previous section. In particular W/k is a smooth, projective, geometrically integral variety of dimension d. Let $C/\bar{k}$ be a smooth projective curve and suppose that the components of $\Gamma \in Z^r((C \times W)_{\bar{k}})$ are flat over $C_{\bar{k}}$. There is a map $\Gamma_* : Z^1(C_{\bar{k}}) \to Z^r(W_{\bar{k}})$ defined by $\Gamma_* z = pr_{W*}(\Gamma \cdot pr^*_C(z))$. Since the usual cycle class map (1.1) is compatible with correspondences [Lau, §6, §7], we get a map

$$\Gamma_* : Z^1(C_{\bar{k}})_{hom} \to Z^r(W_{\bar{k}})_{hom}.$$

Define the group of cycles algebraically equivalent to zero

$$Z^r(W_{\bar{k}})_{alg} = \sum \Gamma_*(Z^1(C_{\bar{k}})_{hom}) \subset Z^r(W_{\bar{k}})_{hom}, \tag{2.1}$$

where the sum is taken over all pairs $(C_{\bar{k}}, \Gamma)$ as above. The Griffiths group is defined by

$$Gr^r(W_{\bar{k}}) = Z^r(W_{\bar{k}})_{hom}/Z^r(W_{\bar{k}})_{alg}. \tag{2.2}$$

We also define

$$CH^r(W_{\bar{k}})_{alg} = Z^r(W_{\bar{k}})_{alg}/Z^r(W_{\bar{k}})_{rat}. \tag{2.3}$$

This is a divisible group since it is the image of the sum of the divisible groups, $CH^1(C_{\bar{k}})_{hom}$, as C ranges over all curves over $\bar{k}$.

The cycle class map α^r may sometimes be used to detect non-trivial elements in the Griffiths group, $Gr^r(W_{\bar{k}})$. To explain this we consider finite dimensional $\mathbf{Q}_\ell$-vector spaces U on which G_k acts continuously and which satisfy

$$\mathrm{Hom}_{G_{k'}}(H^1(C_{\bar{k}}, \mathbf{Q}_\ell(1)), U) = 0 \tag{2.4}$$

for every finite separable extension k'/k and every smooth projective curve C/k'.

Proposition 2.5. – *Let W/k be as above and let W'/k be a smooth variety. Let $Q \in Z^{d+s-r}(W \times W')$. If $U = Q_* H^{2r-1}(W_{\bar{k}}, \mathbf{Q}_\ell(r))$ has property (2.4) then*

$$Q_* \circ \alpha^r : CH^r(W_{\bar{k}})_{hom} \to J^s_\ell(W')$$

factors through $Gr^r(W_{\bar{k}})$.

Proof. – Fix a finite separable extension k'/k, a smooth projective curve C/k', and $\Gamma \in Z^r((C \times W)_{k'})$ whose components are flat over $C_{k'}$. By property (2.4) the map $Q_*\Gamma_* : H^1(C_{\bar{k}}, \mathbf{Z}_\ell(1)) \to H^{2s-1}(W'_{\bar{k}}, \mathbf{Z}_\ell(r))/tors$ is 0. From (1.10(7)) the map

$$cl_{W,0} \circ Q_* \circ \Gamma_* : Z^1(C_{k'})_{hom} \to H^1(G_{k'}, H^{2s-1}(W'_{\bar{k}}, \mathbf{Z}_\ell(r))/tors)$$

is 0. Thus $\alpha^s \circ Q_* Z^r(W_{\bar{k}})_{alg} = 0$. Since $\alpha^s \circ Q_* = Q_* \circ \alpha^r$ (1.13), the proposition follows.

Lemma 2.6. – *Let k be a finite field. Let $\phi \in G_k$ be the Frobenius. Suppose that no eigenvalue, ν, of ϕ^{-1} acting on $U(-1)$ is an algebraic integer. Then U has property (2.4).*

Proof. – Let $n = [k' : k]$. The eigenvalues of the inverse Frobenius element $\phi^{-n} \in G_{k'}$ acting on $H^1(C_{\bar{k}}, \mathbf{Q}_\ell)$ are algebraic integers. The eigenvalues of ϕ^{-n} acting on $U(-1)$ are of the form ν^n. Since ν is not an algebraic integer, neither is ν^n. Thus $\mathrm{Hom}_{G_{k'}}(H^1(C_{\bar{k}}, \mathbf{Q}_l), U(-1)) = 0$.

Proposition 2.7. – *Let k be a finite field, $\phi \in G_k$ the Frobenius element, W/k and W'/k smooth projective varieties, and l a prime distinct from the characteristic of k. Suppose that $Q \in Z^{d+2-r}(W \times W')$ is such that no eigenvalue of ϕ^{-1} acting on $Q_* H^{2r-1}(W_{\bar{k}}, \mathbf{Q}_l(r))(-1)$ is an algebraic integer. Then $Q_* CH^r(W_{\bar{k}})_{alg} \otimes \mathbf{Z}_l \simeq 0$.*

For the proof we make use of a cycle class map for smooth projective varieties

$$\lambda^r : CH^r(W_{\bar{k}})_{tors} \to H^{2r-1}(W_{\bar{k}}, \mathbf{Q}_l/\mathbf{Z}_l(r)) \tag{2.8}$$

defined by Bloch [B1].

PROPOSITION 2.9. – (1) λ^1 *agrees with the usual map from Kummer theory.*
(2) λ^2 *is injective on the l-torsion subgroup of* $CH^2(W_{\bar{k}})_{tors}$.
(3) $\lambda^{\cdot}$ *is functorial with respect to correspondences.*
(4) *Given* $z \in Z^s(W)$ *the following diagram commutes*

$$\begin{array}{ccc} CH^r(W)_{tors} & \xrightarrow{\cdot z} & CH^{r+s}(W)_{tors} \\ \lambda^r \downarrow & & \lambda^{r+s} \downarrow \\ H^{2r-1}(W_{\bar{k}}, \mathbf{Q}_l/\mathbf{Z}_l(r)) & \xrightarrow{\cup cl_{W_{\bar{k}}}(z)} & H^{2(r+s)-1}(W_{\bar{k}}, \mathbf{Q}_l/\mathbf{Z}_l(r+s)) \end{array}$$

Proof. – (1) [B1,3.6]. (2) [M-S,18.4]. (3) [B1,3.5]. (4) [B1,3.4].

Proof of 2.7. – For each finite extension k'/k and each smooth, projective curve C/k', $CH^1(C_{k'})_{hom}$ is a finite group. The images under correspondences of all such groups for all finite extensions of k generates $CH^r(W_{\bar{k}})_{alg}$, which is thus a subgroup of $CH^r(W_{\bar{k}})_{tors}$. For any $\Gamma \in Z^r((C \times W)_{k'})$, $Q_* \circ \Gamma_* H^1(C_{\bar{k}}, \mathbf{Q}_l(1)) \simeq 0$ by (2.6) and (2.5). To show that

$$\lambda^2(Q_*(CH^r(W_{\bar{k}})_{alg})) = Q_*(\lambda^r(CH^r(W_{\bar{k}})_{alg}) \simeq 0$$

it suffices to show that the induced map on Tate modules is zero [Su, 1.3]. This is indeed the case since $Q_* \circ \Gamma_* H^1(C_{\bar{k}}, \mathbf{Q}_l(1)) \simeq 0$ [Su, 3.1]. The proposition follows from (2.9(2)).

Remark 2.10. – (1) It would be interesting to know if (2.9(2)) generalizes to higher codimension.

(2) Both cycle class maps α^r and λ^r play a role in the sequel. However, we will not consider what relationship may exist between these two maps, because it will not be necessary to do so.

3. Cycles supported in fibers

Recall from §1 that a cycle $z \in Z^r(W_{\bar{k}})_{hom}$ gives rise to a non-trivial class in $CH^r(W_{\bar{k}})_{hom}$ if $\alpha^r(z) \in J_l^r(W)$ is not zero. The difficult task of showing $\alpha^r(z) \neq 0$ is easier if the cycle z is chosen carefully. With this in mind, this paper focuses on cycles supported in the fibers of a dominant morphism, $p : W \to X$, from a smooth, projective, d-dimensional variety to a smooth, geometrically integral curve. Our intention is to apply (1.17) to generators of $N^r_{p^{-1}(x)}(W)$. One problem with this approach is that there are many morphisms, $p : W \to X$, for which $N^r_{p^{-1}(x)}(W) = 0$ for most x in X. We devote this section to examples of morphisms for which $N^r_{p^{-1}(x)}(W) \otimes \mathbf{Q} \neq 0$ for many points x in X.

Fix $p : W \to X$ as above. Assume that p is generically smooth and that the geometric fibers are integral. Write $j : \dot{X} \to X$ for the inclusion of the largest open subset for which the base change $\dot{p} : \dot{W} \to \dot{X}$ of p is a smooth morphism. Also write $g : \eta \to X$ (respectively $g_{\bar{k}} : \eta_{\bar{k}} \to X_{\bar{k}}$) for the inclusion of the generic point.

LEMMA 3.1. – *Let* $x \in \dot{X}(k)$. *Suppose that the Tate conjecture holds for codimension* $d - r$ *cycles on* $p^{-1}(\eta)$. *Then*

$$\text{rank } N^r_{p^{-1}(x)}(W) = \text{rank } N^{r-1}(p^{-1}(x)) - \text{rank } N^{d-r}(p^{-1}(\eta))$$

Proof. – Since k is finitely generated over the prime field, so is η. Thus the Tate conjecture has meaning. There is a well defined specialization map for cycles modulo homological equivalence which is injective [Fu, 20.3.5] :

$$sp : N^{d-r}(p^{-1}(\eta)) \to N^{d-r}(p^{-1}(x)).$$

The lemma will follow immediately if we verify that the first of the two inclusions,

$$N^r_{p^{-1}(x)}(W) \otimes \mathbf{Q}_l \subset (sp(N^{d-r}(p^{-1}(\eta))))^{\perp} \otimes \mathbf{Q}_l \subset N^{r-1}(p^{-1}(x)) \otimes \mathbf{Q}_l$$

is an isomorphism. It is clear that a cycle class in the middle vector space is numerically equivalent to zero on W. We need to show that it is homologous to zero. For this we consider the following commutative diagram derived from the Leray spectral sequence (*see* 3.5) :

(3.2)

$$\begin{array}{ccccc} N^{r-1}(p^{-1}(x)) \otimes \mathbf{Q}_l & \to & H^{2r}_{p^{-1}(x)}(W_{\bar{k}}, \mathbf{Q}_l(r)) & \to & H^{2r}(W_{\bar{k}}, \mathbf{Q}_l(r)) \\ & & \simeq\uparrow & & \uparrow \\ & & H^2_x(X_{\bar{k}}, R^{2r-2}p_*\mathbf{Q}_l(r)) & \to & H^2(X_{\bar{k}}, R^{2r-2}p_*\mathbf{Q}_l(r)). \end{array}$$

We may substitute $j_*R^{2r-2}\dot{p}_*\mathbf{Q}_l(r)$ for $R^{2r-2}p_*\mathbf{Q}_l(r)$ in the bottom row. Poincaré duality gives a non-degenerate pairing [Mi, V.2.2(c), VI.11]

$$H^2(X_{\bar{k}}, j_*R^{2r-2}\dot{p}_*\mathbf{Q}_l(r)) \otimes H^0(X_{\bar{k}}, j_*R^{2d-2r}\dot{p}_*\mathbf{Q}_l(d-r)) \to \mathbf{Q}_l.$$

An element $z \in (sp(N^{d-r}(p^{-1}(\eta))))^{\perp} \otimes \mathbf{Q}_l$ gives rise to an element of $H^2(X_{\bar{k}}, j_*R^{2r-2}\dot{p}_*\mathbf{Q}_l(r))^{G_k}$ which is orthogonal to the cohomology classes of algebraic cycles in $H^0(X_{\bar{k}}, j_*R^{2d-2r}\dot{p}_*\mathbf{Q}_l(d-r))^{G_k}$. To show that z is homologous to zero on W, we need only show that this last cohomology vector space is generated by algebraic cycles. This follows from the Tate conjecture for codimension $d-r$ cycles on $p^{-1}(\eta)$:

$$\begin{aligned} N^{d-r}(p^{-1}(\eta)) \otimes \mathbf{Q}_l &\simeq (\varprojlim H^0(p^{-1}(\bar{\eta}), g^*j_*R^{2d-2r}\dot{p}_*\mathbf{Z}/l^n(d-r)))^{G_\eta} \otimes \mathbf{Q}_l \\ &\simeq \varprojlim H^0(p^{-1}(\eta), g^*j_*R^{2d-2r}\dot{p}_*\mathbf{Z}/l^n(d-r)) \otimes \mathbf{Q}_l \\ &\simeq \varprojlim H^0(X, g_*g^*j_*R^{2d-2r}\dot{p}_*\mathbf{Z}/l^n(d-r)) \otimes \mathbf{Q}_l \\ &\simeq H^0(X, j_*R^{2d-2r}\dot{p}_*\mathbf{Q}_l(d-r)) \simeq H^0(X_{\bar{k}}, j_*R^{2d-2r}\dot{p}_*\mathbf{Q}_l(d-r))^{G_k}. \end{aligned}$$

This completes the proof of (3.1).

Let $\pi : Y \to X$ be a generically smooth, dominant morphism from a smooth, projective, geometrically integral surface to a smooth curve. The base change of π with respect to $j : \dot{X} \to X$ will be denoted $\dot{\pi}$. Assume further that one can resolve the singularities of the fiber product $\bar{p} : \bar{W} := Y \times_X Y \to X$. Write $\sigma : W \to \bar{W}$ for the resolution of singularities. Set $p = \bar{p} \circ \sigma$.

Example 3.3. – If π is a semi-stable family, then $\bar{W}$ has only ordinary double point singularities. A single blow up centered at the singular locus resolves all singularities.

LEMMA 3.4. – *If the base field k is finite and if* $\mathrm{Pic}^0_{\pi^{-1}(\bar\eta)}$ *is a simple Abelian variety which is not isogenous to the base change of an Abelian variety defined over* $\bar{k}$*, then for all rational points* $x \in \dot{X}(k)$, rank $N^2_{p^{-1}(x)}(W) > 0$.

Proof. – We will make use of the Picard scheme for which we refer to [Grot, §3]. Consider the commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \to & N^1(\pi^{-1}(\eta))^{\oplus 2}\otimes \mathbf{Q} & \to & N^1(p^{-1}(\eta))\otimes \mathbf{Q} & \to & End\,(Pic^0(\pi^{-1}(\eta)))\otimes \mathbf{Q} & \to & 0 \\ & & \simeq\downarrow & & sp\downarrow & & \downarrow sp_{End} & & \\ 0 & \to & N^1(\pi^{-1}(x))^{\oplus 2}\otimes \mathbf{Q} & \to & N^1(p^{-1}(x))\otimes \mathbf{Q} & \to & End\,(Pic^0(\pi^{-1}(x)))\otimes \mathbf{Q} & \to & 0. \end{array}$$

The horizontal sequences are standard [Ta3, proof of Theorem 3] or [So, 2.3.1]. The vertical maps between Néron-Severi groups result from the fact that specialization of cycles respects homological equivalence [Fu, 20.3.5]. The induced specialization of endomorphisms corresponds to specializing endomorphisms of the relative Picard scheme $Pic^0_{\dot{Y}/\dot{X}}$ to the fiber over x. Thus sp_{End} is a ring homomorphism. Now the hypothesis of the lemma implies that $End\,(Pic^0_{\pi^{-1}(\bar\eta)})\otimes\mathbf{Q}$ does not contain a CM-field whose dimension over $\mathbf{Q}$ is twice the dimension of the Picard variety [Mu, p. 220]. As $End\,(Pic^0_{(\pi^{-1}(x))})\otimes\mathbf{Q}$ always contains such a field [Ta2, Thm. 1], neither sp_{End} nor sp is surjective. The Tate conjecture for codimension 1 cycles on the generic fiber was proved by Zarhin [Za, 4.1]. Now apply (3.1) with $d = 3$ and $r = 2$.

We end this section with a derivation of the Leray spectral sequence with supports for étale cohomology. This spectral sequence was used in (3.2) and will be encountered later as well.

LEMMA 3.5. – *Let* $f : A \to B$ *be a morphism of varieties,* $\mathcal{F}$ *an étale sheaf on* A *and* $i : Z \to B$ *a closed immersion. Then there is a spectral sequence*

$$H^p_Z(B, R^q f_*\mathcal{F}) \Rightarrow H^{p+q}_{f^{-1}(Z)}(A,\mathcal{F}).$$

Proof. – The argument parallels [Mi, III.1.18, 1.20]. Recall that $H^p_Z(B,*)$ is the p^{th} right derived functor of the left exact functor [Mi, p.91]

$$\mathcal{G} \to H^0_Z(B,\mathcal{G}) := \mathrm{Ker}(\mathcal{G}(B) \to \mathcal{G}(B - Z)).$$

Since $H^{p+q}_{f^{-1}(Z)}(A,*)$ is defined analogously, it is clear that

$$H^0_{f^{-1}(Z)}(A,\mathcal{F}) \simeq H^0_Z(B, f_*\mathcal{F})$$

for all sheaves $\mathcal{F}$ on A. Since f_* sends injectives to injectives [Mi, p. 68 and II.2.6(a)], the usual construction of the spectral sequence for the composition of two functors gives rise to the desired spectral sequence [Mi, Appendix B, Theorem 1].

4. Reduction via the Leray spectral sequence

We return to the morphism $p : W \to X$ described in the first paragraph of §3. Set $V = p^{-1}(x)$ for some $x \in \dot{X}(k)$. The purpose of this section is to describe the cycle class map $\theta^n_{V/W} : N^r_V(W) \to H^1(G_k, \mathrm{Coker}\,\epsilon'_n)$ of (1.17) in terms of the cohomology of constructible sheaves on the curve X. This reformulation will make it more feasible to try to calculate $\theta^n_{V/W}$. We introduce the notations $\check{X} = X - x$, $\check{W} = W - p^{-1}(x)$, $\mathcal{F}_n = \mathbf{Z}/l^n(r)$, and $\bar{\epsilon}_n$ for the standard map

$$H^2_x(X_{\bar{k}}, R^{2r-2}p_*\mathcal{F}_n) \to H^2(X_{\bar{k}}, R^{2r-2}p_*\mathcal{F}_n).$$

LEMMA 4.1. – *Suppose that*
(1) $H^0(X_{\bar{k}}, R^{2r-1}p_*\mathcal{F}_n) = 0$,
(2) $H^2(X_{\bar{k}}, R^{2r-3}p_*\mathcal{F}_n) = 0$, *then there is a commutative diagram,*

(4.2)

$$\begin{array}{ccccccccc} H^{2r-1}_V(W_{\bar{k}}, \mathcal{F}_n) & \xrightarrow{\epsilon'_n} & H^{2r-1}(W_{\bar{k}}, \mathcal{F}_n) & \xrightarrow{\rho} & H^{2r-1}(\check{W}_{\bar{k}}, \mathcal{F}_n) & \to & Ker\ \epsilon_n & \to & 0 \\ & & \omega_n \uparrow \simeq & & \uparrow \simeq & & \kappa_n \uparrow \simeq & & \\ 0 & \to & H^1(X_{\bar{k}}, R^{2r-2}p_*\mathcal{F}_n) & \to & H^1(\check{X}_{\bar{k}}, R^{2r-2}p_*\mathcal{F}_n) & \to & Ker\ \bar{\epsilon}_n & \to & 0. \end{array}$$

Here κ_n is obtained from the composition

$$H^2_x(X_{\bar{k}}, R^{2r-2}p_*\mathcal{F}_n) \simeq R^{2r-2}p_*\mathcal{F}_n(-1)_x \simeq H^{2r-2}(p^{-1}(x)_{\bar{k}}, \mathbf{Z}/l^n(r-1)) \simeq H^{2r}_V(W_{\bar{k}}, \mathcal{F}_n)$$

by restriction. Furthermore, $\epsilon'_n = 0$.

Proof. – The vertical isomorphism on the left follows from (1) and (2) and the Leray spectral sequence. Note also that

$$H^1_x(X_{\bar{k}}, R^{2r-1}p_*\mathcal{F}_n) \simeq 0 \simeq H^3_x(X_{\bar{k}}, R^{2r-3}p_*\mathcal{F}_n)$$

so that (1) and (2) remain true when $X_{\bar{k}}$ is replaced by $\check{X}_{\bar{k}}$. Thus the Leray spectral sequence yields the vertical isomorphism in the middle. It follows that ρ is injective, whence $\epsilon'_n = 0$. The map κ_n makes (4.2) commute and is clearly an isomorphism.

Remark 4.3. – Write $\delta_{(4.2)}$ for the first coboundary map in the long exact sequence of G_k-modules associated to the bottom row of (4.2). The top row in (4.2) coincides with the bottom row in (1.16). Consequently $\theta^n_{V/W}$ may be identified with $\delta_{(4.2)} \circ cl^n_V$.

LEMMA 4.4. – *There is a commutative diagram of G_k-modules with exact rows and surjective vertical maps*

(4.5)

$$\begin{array}{ccccccccc} 0 & \to & H^1(X_{\bar{k}}, R^{2r-2}p_*\mathcal{F}_n) & \to & H^1(\check{X}_{\bar{k}}, R^{2r-2}p_*\mathcal{F}_n) & \to & Ker\ \bar{\epsilon}_n & \to & 0 \\ & & \omega'_n \downarrow & & \downarrow & & \| & & \\ 0 & \to & H^1(X_{\bar{k}}, j_*R^{2r-2}\dot{p}_*\mathcal{F}_n) & \to & H^1(\check{X}_{\bar{k}}, j_*R^{2r-2}\dot{p}_*\mathcal{F}_n) & \to & Ker\ \bar{\epsilon}_n & \to & 0. \end{array}$$

Proof. – The kernel and cokernel of the restriction map $\xi : R^{2r-2}p_*\mathcal{F}_n \to j_*R^{2r-2}\dot{p}_*\mathcal{F}_n$ are skyscraper sheaves supported on $X - \dot{X}$. The surjectivity of the vertical arrows follows easily. To see that the bottom row is exact, we need only check the surjectivity on the right. This follows, since ξ gives rise to isomorphisms $H^2_x(X_{\bar{k}}, \xi)$ and $H^2(X_{\bar{k}}, \xi)$.

Remark 4.6. – Write $\delta_{(4.5)}$ for the first coboundary map in the G_k-cohomology sequence associated with the bottom row in (4.5). If $\delta_{(4.5)} \circ cl_V^n(z) \neq 0$ then $\theta^n_{V/W}(z) \neq 0$.

Remark 4.7. – If the map ξ is surjective, then the vertical maps in (4.5) are all isomorphisms. An example where this occurs is given in (5.15). In fact one may deduce from the local invariant cycle theorem [St, 5.12] that the surjectivity of ξ is a common phenomenon, at least in characteristic 0 or when there is a lifting to characteristic 0.

In many concrete situations it is possible to replace the coboundary map $\delta_{(4.5)}$ with the coboundary associated to a more manageable sequence than the bottom row of (4.5). This is done by using a projector $P \in CH^d(W^2) \otimes \mathbf{Z}_l$ to decompose the terms in the bottom row of (4.5) into simpler pieces. To this end suppose given $P \in Z^d(W^2) \otimes \mathbf{Z}_l$ with support in the subvariety $W \times_X W \subset W^2$ satisfying $P^2 = P \in CH^d(W^2) \otimes \mathbf{Z}_l$. The action of such a correspondence on cycles and cohomology induces endomorphisms of $N_V^r(W)$ and of the Leray spectral sequences for p and $\dot{p}$. Set $\mathcal{M}_n = P_* R^{2r-2}\dot{p}_*\mathcal{F}_n$. Applying P_* to the bottom line of (4.5) yields the short exact sequence of G_k-modules,

$$(4.8) \qquad 0 \to H^1(X_{\bar{k}}, j_*\mathcal{M}_n) \to H^1(\check{X}_{\bar{k}}, j_*\mathcal{M}_n) \to P_* \operatorname{Ker} \bar{\epsilon}_n \to 0,$$

for which the associated first coboundary map is denoted $\delta_{(4.8)}$.

LEMMA 4.9. – *Assume that the hypotheses of (4.1) hold and that $P_*H^{2r-1}(W_{\bar{k}}, \mathbf{Z}_l(r))$ is torsion free. Suppose $z \in Z_V^r(W)$. If $\delta_{(4.8)} \circ cl_V^n(P_*z) \neq 0$ for some n, then $P_*\alpha^r(z) \in J_l^r(W)$ is not zero.*

Proof. – By (1.19) it suffices to show $\theta^n_{V/W} \circ cl_V^n(P_*z) \neq 0$. By (4.6) this will follow if $\delta_{(4.5)} \circ cl_V^n(P_*z) \neq 0$. Since (4.8) is a direct summand of the bottom row of (4.5) the lemma follows.

5. Self-fiber-products of semi-stable elliptic surfaces with section

In this section we apply the considerations of the previous two sections to certain cycles on self-fiber-products of elliptic surfaces. Let $\pi : Y \to X$ be a semi-stable elliptic surface with section, whose j-invariant is transcendental over the base field k. As in the previous section we write $\bar{p} : \bar{W} := Y \times_X Y \to X$ for the tautological map. The blow up of $\bar{W}$ along the singular locus will be denoted $\sigma : W \to \bar{W}$. This is a resolution of singularities. Set $p = \bar{p} \circ \sigma : W \to X$. Define $j : \dot{X} \to X$ and $\dot{p} : \dot{W} \to \dot{X}$ as in the previous section and let $\dot{\pi} : \dot{Y} \to \dot{X}$ denote the base change of π with respect to j.

Definition. – An elliptic curve E defined over a field k is said to be CM if
(1) $\mathrm{E}nd(E_k) \simeq \mathrm{E}nd(E_{\bar{k}})$ and
(2) $\mathrm{E}nd(E_{\bar{k}})$ isomorphic to an order in an imaginary quadratic field.

According to this definition, no elliptic curve over $\mathbf{Q}$ is CM. If k is a finite field, then E is CM if and only if $E_{\bar{k}}$ is not supersingular.

If the fiber $\pi^{-1}(x)$ over the point $x \in \dot{X}(k)$ is CM, then the fiber $p^{-1}(x)$ carries so called complex multiplication cycles. To define these, let D be the discriminant of $\operatorname{End}(\pi^{-1}(x))$. Set $\varepsilon(D) = 2$ (respectively 1) when D is odd (respectively even). Given

$\omega \in \mathrm{End}(\pi^{-1}(x))$, write $\Gamma_\omega \subset p^{-1}(x)$ for the graph of ω. In particular Γ_1 denotes the diagonal. Let $\omega \to \bar{\omega}$ denote the non-trivial ring automorphism of $\mathrm{End}(\pi^{-1}(x))$. Fix a $\mathbf{Z}$-algebra generator ν of $\mathrm{End}(\pi^{-1}(x))$ and consider the divisor class

$$z_x = (\varepsilon(D)/2)(\Gamma_\nu - \Gamma_{\bar{\nu}}) \in N^1(p^{-1}(x)). \tag{5.0}$$

In fact up to a possible change of sign z_x is independent of the choice of generator ν. It is not difficult to check that z_x generates the free rank one $\mathbf{Z}$-module

$$N^1_{CM}(p^{-1}(x)) := \mathrm{Span}\{\pi^{-1}(x) \times e, e \times \pi^{-1}(x), \Gamma_1\}^\perp \subset N^1(p^{-1}(x)), \tag{5.1}$$

where e denotes the neutral element in the group law on the fiber $\pi^{-1}(x)$. In the case that the base field has characteristic zero an equivalent statement is proved in [Sch-CM, §2]. When the base field has positive characteristic one can reduce to the case of characteristic zero *via* the canonical lifting [D2, §3]. The cohomology class $cl^n_{p^{-1}(x)}(z_x) \in H^2(p^{-1}(x)_{\bar{k}}, \mathbf{Z}/l^n(1))$ in fact lies in the subgroup $[\mathrm{Sym}^2 H^1(\pi^{-1}(x)_{\bar{k}}, \mathbf{Z}/l^n)](1)$.

LEMMA 5.2. – *The cyclic subgroup generated by $cl^n_{p^{-1}(x)}(z_x)$ is isomorphic to $\mathbf{Z}/l^n$.*

Proof. – This follows because the cokernel of the natural map

$$N^1(p^{-1}(x)) \otimes \mathbf{Z}_l \to H^2(p^{-1}(x)_{\bar{k}}, \mathbf{Z}_l(1))$$

is torsion free [Ta3, proof of Lemma 1].

LEMMA 5.3. – *If $\pi^{-1}(x)$ is CM, then $N^2_{p^{-1}(x)}(W) \simeq \mathbf{Z}$. This group is generated by a multiple of z_x.*

Proof. – It is well known and easy to see that $H^2(p^{-1}(\bar{\eta}), \mathbf{Q}_l(1))^{G_\eta}$ has dimension 3 and is generated by algebraic cycles. Also $\mathrm{rank}\, N^1(p^{-1}(x)) = 4$. In the notation of the proof of (3.1) z_x is a generator of the free rank one $\mathbf{Z}$-module $(sp\,(N^1(p^{-1}(\eta))))^\perp$. Thus z_x generates $N^2_{p^{-1}(x)}(W) \otimes \mathbf{Q}$. In fact $N^2_{p^{-1}(x)}(W)$ is the free $\mathbf{Z}$-module generated by the least positive multiple of z_x which is homologous to zero on $W_{\bar{k}}$.

The next lemma strengthens (5.3).

LEMMA 5.4. – *$z_x \in Z^2(W)_{hom}$. In particular, $N^2_{p^{-1}(x)}(W) = \mathbf{Z}z_x$.*

Proof. – From the diagram (3.2) with $\mathbf{Z}_l$ replacing $\mathbf{Q}_l$ it is apparent that the map $i_* : N^1(p^{-1}(x)) \to H^4(W_{\bar{k}}, \mathbf{Z}_l(2))$ factors through

$$H^2(X_{\bar{k}}, R^2 p_* \mathbf{Z}_l(2)) \simeq H^2(X_{\bar{k}}, j_* R^2 \dot{p}_* \mathbf{Z}_l(2)).$$

The restriction of i_* to the subgroup $\mathbf{Z}z_x$ factors through $H^2(X_{\bar{k}}, j_* \mathrm{Sym}^2 R^1 \dot{\pi}_* \mathbf{Z}_l(2))$. To show that this group is zero, observe that there is a fixed integer n_0 such that multiplication by l^{n_0} annihilates $H^0(X_{\bar{k}}, j_* \mathrm{Sym}^2 R^1 \dot{\pi}_* \mathbf{Z}/l^n)$ for all n. Indeed, if this were not the case, then $X_{\bar{k}}$ would dominate an infinite tower of modular curves associated to various l-power level structures on elliptic curves. This is clearly impossible. Since the Poincaré duality pairing,

$$H^0(X_{\bar{k}}, j_* \mathrm{Sym}^2 R^1 \dot{\pi}_* \mathbf{Z}/l^n(-1)) \otimes H^2(X_{\bar{k}}, j_* \mathrm{Sym}^2 R^1 \dot{\pi}_* \mathbf{Z}/l^n(2)) \to \mathbf{Z}/l^n,$$

is perfect [Mi, V.2.2b], multiplication by l^{n_0} annihilates $H^2(X_{\bar{k}}, j_*\mathrm{Sym}^2 R^1\dot{\pi}_*\mathbf{Z}/l^n(2))$ for all n. Since the map in the inverse system

$$H^2(X_{\bar{k}}, j_*\mathrm{Sym}^2 R^1\dot{\pi}_*\mathbf{Z}/l^n(2)) \xrightarrow{l^{n_0}} H^2(X_{\bar{k}}, j_*\mathrm{Sym}^2 R^1\dot{\pi}_*\mathbf{Z}/l^{n-n_0}(2))$$

is zero, it follows that

$$H^2(X_{\bar{k}}, j_*\mathrm{Sym}^2 R^1\dot{\pi}_*\mathbf{Z}_l(2)) = \varprojlim\ H^2(X_{\bar{k}}, j_*\mathrm{Sym}^2 R^1\dot{\pi}_*\mathbf{Z}/l^n(2))$$

is zero. Thus $cl_W(z_x) \in H^4(W_{\bar{k}}, \mathbf{Z}_l(2))$ is zero as desired.

In order to apply the results of §4 to the cycles z_x on self-fiber-products of non-isotrivial, semi-stable elliptic surfaces with section, it is necessary to analyze the cohomology of the sheaf $R^1\dot{\pi}_*\mathbf{Z}/l^n$ and related sheaves in considerable detail. We proceed with this now. The goal is to be able to apply (4.9) to the present situation.

Definition 5.5. – *Denote by m_π the least common multiple of all integers m with the property that the semi-stable elliptic surface $\pi : Y \to X$ has a singular fiber of Kodaira type I_m.*

Since $\pi : Y \to X$ is semi-stable, the sheaf $R^1\dot{\pi}_*\mathbf{Z}/l^n$ is tamely ramified [Ogg, §II]. In other words, the sheaf is associated to a representation of the tame fundamental group, $\kappa : \pi_1^t(\dot{X}_{\bar{k}}, \bar{\eta}) \to GL(2, \mathbf{Z}/l^n)$. The representation respects the Weil-pairing on l^n-torsion points, so the image is actually contained in $SL(2, \mathbf{Z}/l^n)$.

Lemma 5.6. – *Let l denote, as usual, a prime distinct from the characteristic of the base field, k. If $\gcd(l, m_\pi) = 1$, then the monodromy representation $\kappa : \pi_1^t(\dot{X}_{\bar{k}}, \bar{\eta}) \to SL(2, \mathbf{Z}/l^n)$ is surjective.*

Proof. – We use standard facts about modular curves for which the reader may refer to [Gro, §1] and references therein. Let $\pi_0 : Y_0 \to \mathbf{P}^1$ denote an elliptic surface with section whose j-invariant is the identity map on $\mathbf{P}^1$ [Sil, p. 52]. The j-invariant for $\pi : Y \to X$ is a non-constant map $\mathbf{J} : X \to \mathbf{P}^1$ which determines the elliptic surface π up to a quadratic twist. Thus π is a twist of the pullback of π_0 via $\mathbf{J} : X \to \mathbf{P}^1$. Let $\hat{\rho}_0 : \hat{X}(l^n) \to \mathbf{P}^1$ be the complete non-singular curve obtained from the l^n-torsion in Y_0 by discarding the l^{n-1}-torsion and taking the Galois closure. The quotient of $\hat{X}(l^n)$ by the involution induced by inversion in Y_0, is the familiar modular curve for symplectic level-l^n-structure $\rho_0 : X(l^n) \to \mathbf{P}^1$. This construction is unaffected by quadratic twists and is thus independent of the choice of Y_0. We shall now deduce from the ramification of ρ_0 and $\mathbf{J}_{sep}$ over ∞ that the smooth curve C which normalizes the fiber product $X \times_{\mathbf{P}^1} X(l^n)$ is irreducible. By the hypothesis $\gcd(l, m_\pi) = 1$ the ramification of $\mathbf{J}_{sep}$ and hence of its Galois closure, is prime to l [Sil, Appendix C 14.1(b) or Table 15.1]. From the classical theory of modular curves $Gal(X(l^n)/\mathbf{P}^1) \simeq SL(2, \mathbf{Z}/l^n)/\pm 1$. The two elementary matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

generate the stabilizers of the cusps at 0 and $i\infty$. Each cusp has ramification index l^n. Since these two matrices generate $SL(2, \mathbf{Z})$ and hence also $SL(2, \mathbf{Z}/l^n)/\pm 1$, it follows that every intermediate extension in $X(l^n)/\mathbf{P}^1$ contains a cusp whose ramification index over $\infty \in \mathbf{P}^1$ is a positive power of l. This shows that $\mathbf{J}$ and ρ_0 are linearly disjoint and establishes the irreducibility of C. Projection to the first factor in the fiber product induces a Galois morphism $\rho : C \to X$ with group $SL(2, \mathbf{Z}/l^n)/\pm 1$. The monodromy group of the geometric generic fiber of π is a subgroup of $SL(2, \mathbf{Z}/l^n)$ which maps surjectively to $Gal(C/X)$. Thus it must be all of $SL(2, \mathbf{Z}/l^n)$.

COROLLARY 5.7. – *If* $\gcd(l, m_\pi) = 1$, *then* $H^0(X_{\bar{k}}, j_* R^1 \dot{\pi}_* \mathbf{Z}/l^n) \simeq 0$.
Write $\mathcal{M}_n$ for the sheaf $\mathrm{Sym}^2 R^1 \dot{\pi}_* \mathbf{Z}/l^n(2)$ on $\dot{X}$.

LEMMA 5.8. – *Let l be an odd prime. Then there is a projector $P \in CH^3(W \times W) \otimes \mathbf{Z}[\frac{1}{2}]$ with the property that for all n there is an isomorphism*

$$P_* R^2 \dot{p}_* \mathbf{Z}/l^n(2) = \mathcal{M}_n.$$

Furthermore P_* acts as the identity on complex multiplication cycles.

Proof. – The projector P will be constructed as the composition of two commuting projectors : $P = Q' \circ Q$. In order to define Q' consider the two involutions $\bar{\tau}, \bar{\tau}_1 \in \mathrm{Aut}(\bar{W})$, where $\bar{\tau}$ interchanges the factors in the fiber product and $\bar{\tau}_1$ acts by inversion on the first factor and the identity on the second. The corresponding involutions $\tau, \tau_1 \in \mathrm{Aut}(W)$ generate a subgroup, D_4, isomorphic to the dihedral group with 8 elements. Now Q' is defined as the idempotent in the group ring $\mathbf{Z}[\frac{1}{2}][D_4]$ corresponding to the representation $\varsigma : D_4 \to \{\pm 1\}$ with kernel $\langle \tau\tau_1 \rangle$.

In order to describe the projector Q introduce the notation $s : X \to Y$ for the distinguished section of the elliptic surface. The composition

$$Y \times_X Y \xrightarrow{pr_1} Y \xrightarrow{(id,s)} Y \times_X Y,$$

in which the first map is projection on the first factor gives rise to a morphism $q_1 : W \to W$ whose image is isomorphic to Y and which satisfies $q_1^2 = q_1$. Define $q_2 = \tau q_1 \tau$. Write $\Gamma_{q_i} \subset W^2$ for the graph of q_i and Δ for the diagonal. Define

$$Q = (\Delta - \Gamma_{q_1}) \circ (\Delta - \Gamma_{q_2}) \in CH^3(W^2).$$

Evidently Q commutes with D_4 and $P = Q' \circ Q$ is a projector. Since Q and Q' are supported on the subvariety $W \times_X W \subset W^2$, these correspondences act on the sheaves $R^j \dot{p}_* \mathbf{Z}/l^n$ and respect the Leray spectral sequence for the map $\dot{p}$. The first assertion of the lemma is now immediate from the actions of Q and Q' on the various components of the Künneth decomposition

$$R^2 \dot{p}_* \mathbf{Z}/l^n(2) \simeq \mathcal{M}_n \oplus (\Lambda^2 R^1 \dot{\pi}_* \mathbf{Z}/l^n)(2) \oplus (R^2 \dot{\pi}_* \mathbf{Z}/l^n \otimes \mathbf{Z}/l^n(2))^{\oplus 2}. \tag{5.9}$$

For the second assertion observe that the correspondence P preserves fibers of $\dot{p}$ and induces the identity map on $\mathrm{Sym}^2 H^1(\pi^{-1}(x)_{\bar{k}}, \mathbf{Z}_l)$ and hence on $N^1_{CM}(p^{-1}(x))$. We are now ready to bring (4.9) to bear on complex multiplication cycles. For this consider the short exact sequences of G_k-modules

$$0 \to H^1(X_{\bar{k}}, j_* \mathcal{M}_n) \to H^1(\check{X}_{\bar{k}}, j_* \mathcal{M}_n) \to H^2_x(X_{\bar{k}}, j_* \mathcal{M}_n) \to 0, \tag{5.10}$$

$$0 \to H^1(\dot{X}_{\bar{k}}, \mathcal{M}_n) \to H^1((\dot{X} - x)_{\bar{k}}, \mathcal{M}_n) \to H^2_x(\dot{X}_{\bar{k}}, \mathcal{M}_n) \to 0, \tag{5.11}$$

for which the associated first coboundary maps are denoted $\delta_{(5.10)}$ and $\delta_{(5.11)}$.

PROPOSITION 5.12. – *Fix a positive integer n. Suppose that l does not divide $2m_\pi$. If $\delta_{(5.10)} \circ cl^n_V(z_x) \neq 0$ or if $\delta_{(5.11)} \circ cl^n_V(z_x) \neq 0$, then $\alpha^2(z_x) \in J^2_l(W)$ is not zero.*

Proof. – Restricting cohomology classes from $X_{\bar{k}}$ to $\dot{X}_{\bar{k}}$ maps the short exact sequence (5.10) to (5.11) and is the identity on the right most term. Thus $\delta_{(5.11)} \circ cl^n_V(z_x) \neq 0$ implies $\delta_{(5.10)} \circ cl^n_V(z_x) \neq 0$. The proposition will follow from (4.9) once it has been established that the hypotheses of (4.9) hold. Specifically, we must check :

(1) $H^0(X_{\bar{k}}, R^3p_*\mathbf{Z}/l^n(2)) \simeq 0$.

(2) $H^2(X_{\bar{k}}, R^1p_*\mathbf{Z}/l^n(2)) \simeq 0$.

(3) $P_*H^3(W_{\bar{k}}, \mathbf{Z}_l(2))$ is torsion free.

We deduce (2) easily from Poincaré duality, the Künneth formula and (5.7) :

$$\begin{aligned} H^2(X_{\bar{k}}, R^1p_*\mathbf{Z}/l^n(2)) &\simeq H^2(X_{\bar{k}}, j_*R^1\dot{p}_*\mathbf{Z}/l^n(2)) \simeq H^0(X_{\bar{k}}, j_*R^1\dot{p}_*\mathbf{Z}/l^n(-1))^\vee \\ &\simeq (H^0(X_{\bar{k}}, j_*R^1\dot{\pi}_*\mathbf{Z}/l^n(-1))^\vee)^2 \simeq 0. \end{aligned}$$

In (5.16) below we show $R^3p_*\mathbf{Z}/l^n(2) \simeq j_*R^3\dot{p}_*\mathbf{Z}/l^n(2)$. Now

$$\begin{aligned} H^0(X_{\bar{k}}, j_*R^3\dot{p}_*\mathbf{Z}/l^n(2)) &\simeq H^0(X_{\bar{k}}, j_*(R^2\dot{\pi}_*\mathbf{Z}/l^n \otimes R^1\dot{\pi}_*\mathbf{Z}/l^n)(2))^2 \\ &\simeq H^0(X_{\bar{k}}, j_*R^1\dot{\pi}_*\mathbf{Z}/l^n(1))^2 \simeq 0. \end{aligned}$$

This establishes (1) and also implies that $H^3(W_{\bar{k}}, \mathbf{Z}/l^n(2)) \simeq H^1(X_{\bar{k}}, R^2p_*\mathbf{Z}/l^n(2))$.

The hypothesis $\gcd(l, 2m_\pi) = 1$ implies that $R^2p_*\mathbf{Z}/l^n(2) \to j_*R^2\dot{p}_*\mathbf{Z}/l^n(2)$ is surjective (*see* (5.15) below). Thus $H^1(X_{\bar{k}}, R^2p_*\mathbf{Z}/l^n(2)) \simeq H^1(X_{\bar{k}}, j_*R^2\dot{p}_*\mathbf{Z}/l^n(2))$. Apply the projector P and take inverse limits to get

$$P_*H^3(W_{\bar{k}}, \mathbf{Z}_l(2)) \simeq H^1(X_{\bar{k}}, j_*\mathrm{Sym}^2R^1\dot{\pi}_*\mathbf{Z}_l(2)). \tag{5.13}$$

Now assertion (3) follows from

LEMMA 5.14. – *If $\gcd(l, 2m_\pi) = 1$, then $H^1(X_{\bar{k}}, j_*\mathrm{Sym}^2R^1\dot{\pi}_*\mathbf{Z}_l(2))$ is torsion free.*

Proof. – (cf. [Ne, 2.2].) Multiplication by l on $\mathcal{M}_n$ gives rise to a short exact sequence

$$0 \to \mathcal{M}_1 \to \mathcal{M}_n \xrightarrow{l} \mathcal{M}_{n-1} \to 0.$$

By (5.6), $H^0(\dot{X}_{\bar{k}}, \mathcal{M}_n) \simeq 0$. It follows that

$$0 \to H^1(\dot{X}_{\bar{k}}, \mathcal{M}_1) \to H^1(\dot{X}_{\bar{k}}, \mathcal{M}_n) \xrightarrow{l} H^1(\dot{X}_{\bar{k}}, \mathcal{M}_{n-1}) \to 0$$

is exact. Induction on n implies that the inverse system $\{H^1(\dot{X}_{\bar{k}}, \mathcal{M}_n)\}_n$ is isomorphic to $\{(\mathbf{Z}/l^n)^h\}_n$, where $h = \dim_{\mathbf{Z}/l} H^1(\dot{X}_{\bar{k}}, \mathcal{M}_1)$. Thus $H^1(\dot{X}_{\bar{k}}, \mathrm{Sym}^2R^1\dot{\pi}_*\mathbf{Z}_l(2)) \simeq \mathbf{Z}_l^h$. Finally the natural map

$$H^1(X_{\bar{k}}, j_*\mathrm{Sym}^2R^1\dot{\pi}_*\mathbf{Z}_l(2)) \xrightarrow{j^*} H^1(\dot{X}_{\bar{k}}, \mathrm{Sym}^2R^1\dot{\pi}_*\mathbf{Z}_l(2))$$

is injective by the Leray spectral sequence for j and the left exactness of the inverse limit.

LEMMA 5.15. – *If* $\gcd(l, 2m_\pi) = 1$, *then* $R^2p_*\mathbf{Z}/l^n(2) \to j_*R^2\dot{p}_*\mathbf{Z}/l^n(2)$ *is surjective.*

Proof. – Recall from the first paragraph of this section the factorization $p = \bar{p}\circ\sigma$. Clearly $\sigma_*\mathbf{Z}/l^n(2) \simeq \mathbf{Z}/l^n(2)$. It suffices to show that the map $R^2\bar{p}_*\mathbf{Z}/l^n(2) \to j_*R^2\dot{p}_*\mathbf{Z}/l^n(2)$ is surjective. Since both sides may be decomposed using the Künneth theorem, one is reduced to showing that $R^i\pi_*\mathbf{Z}/l^n \to j_*R^i\dot{\pi}_*\mathbf{Z}/l^n$ is surjective for each i. This is well known. When $i = 1$ it follows from the theory of the Tate curve [Sil, Appendix C §14]. When $i \neq 1$ it is straightforward.

LEMMA 5.16. – *Suppose that* l *does not divide* $2m_\pi$. *Then the restriction map* r : $R^3p_*\mathbf{Z}/l^n \to j_*R^3\dot{p}_*\mathbf{Z}/l^n$ *is an isomorphism.*

Proof. – The Leray spectral sequence associated to the composition $p = \bar{p}\circ\sigma$ gives rise to the exact row in the diagram

$$
\begin{array}{ccccccc}
\bar{p}_*R^2\sigma_*\mathbf{Z}/l^n & \xrightarrow{b} & R^3\bar{p}_*\sigma_*\mathbf{Z}/l^n & \xrightarrow{h} & R^3p_*\mathbf{Z}/l^n & \to & 0 \\
 & & \bar{r}\downarrow & & & & \\
 & & j_*R^3\dot{p}_*\sigma_*\mathbf{Z}/l^n. & & & &
\end{array}
\tag{5.17}
$$

Recall that $\sigma_*\mathbf{Z}/l^n \simeq \mathbf{Z}/l^n$. Clearly $\bar{r} = r\circ h$. The surjectivity of $\bar{r}$ is easily deduced from the Künneth formula. As r is clearly an isomomorphism except possibly for the stalks above Σ_X, we fix a point $0 \in \Sigma_{X\bar{k}}$ where the fiber has Kodaira type I_m and study (5.17) restricted to Henselization of $X_{\bar{k}}$ at 0. The action of the inertia group on the l^n torsion in the fiber of π over the generic point in the Henselization depends only on the integer m. Thus (5.17) restricted to the Henselization depends only on m. Consequently, we may replace π with a complex elliptic surface with an I_m fiber. In this case an explicit description of the dual map $b^\vee$ is given in [Sch-CM, 1.10], from which it is apparent that the kernel of $b^\vee$ equals the image of $\bar{r}^\vee$. Since this is equal to the image of $h^\vee$, the lemma follows. (One may ignore that the reference computes with $\mathbf{Q}$-coefficients. Everything works with $\mathbf{Z}/l^n$-coefficients provided only that l is odd if $m = 2$.)

We end this section with a lemma which will simplify notation in later sections.

LEMMA 5.18. – *Let* $\pi : Y \to X$ *be a semi-stable, non-isotrivial elliptic surface with section over a finite field k. With W as defined above, complex multiplication cycles give well defined elements of* $CH^2(W_{\bar{k}})_{hom}$.

Proof. – By definition a complex multiplication cycle is a generator $z_x \in N^1_{CM}(p^{-1}(x))$, where $\pi^{-1}(x)$ is a CM elliptic curve. To show that z_x gives rise to a well defined element of $CH^2(W_{\bar{k}})_{hom}$ it is necessary to show that the obvious map

$$i_{W*} : CH^1(p^{-1}(x)_{\bar{k}})_{hom} \to CH^2(W_{\bar{k}})_{hom}$$

is zero. To see that this is the case use the isomorphism

$$CH^1(\pi^{-1}(x)_{\bar{k}})^2_{hom} \xrightarrow{(pr_1^*, pr_2^*)} CH^1(p^{-1}(x)_{\bar{k}})_{hom}$$

and the commutative diagram

$$\begin{array}{ccc} CH^1(\pi^{-1}(x)_{\bar{k}})_{hom} & \xrightarrow{i_{Y*}} & CH^2(Y_{\bar{k}})_{hom} \\ pr_i^* \downarrow & & (pr_i \circ \sigma)^* \downarrow \\ CH^1(p^{-1}(x)_{\bar{k}})_{hom} & \xrightarrow{i_{W*}} & CH^2(W_{\bar{k}})_{hom}. \end{array}$$

Since k is a finite field, Roitman's theorem [Mi2] and [Roi] implies that the natural map $CH^2(Y_{\bar{k}})_{hom} \to Alb_Y(\bar{k})$ is an isomorphism. For l prime to $2m_\pi$ and $x \in \dot{X}$ the restriction map

$$H^1(Y_{\bar{k}}, \mathbf{Z}/l^n) \to H^1(\pi^{-1}(x)_{\bar{k}}, \mathbf{Z}/l^n)$$

is zero (5.6). Thus the Albanese map $Y \to Alb_Y$ collapses the fiber, $\pi^{-1}(x)_{\bar{k}}$ to a point. It follows that i_{Y*} is the zero map. The lemma follows.

6. Locally constant sheaves on curves

The problem of showing that the cycle class map $\alpha^2 : CH^2(W_{\bar{k}})_{hom} \to J_l^2(W)$ takes a non-zero value when applied to an appropriate CM-cycle, z_x with $x \in \dot{X}(k)$, has been reduced in (5.12) to the computation of the first coboundary map for an appropriate short exact sequence of G_k-modules. This section begins by formulating the problem of computing this coboundary map in an abstract setting. Then some theorems are stated which give conditions under which the coboundary map is not zero.

The notations $k \subset \bar{k}$, G_k, $\eta \in \dot{X} \subset X$, l, $\pi_1^t(\dot{X}_k, \bar{\eta})$, etc. continue to have the meanings set out in (0.5). There is a short exact sequence

(6.1) $$1 \to \pi_1^t(\dot{X}_{\bar{k}}, \bar{\eta}) \to \pi_1^t(\dot{X}_k, \bar{\eta}) \to G_k \to 1.$$

Let M be an $\mathbf{F}_l$-vector space of finite dimension on which $\pi_1^t(\dot{X}_k, \bar{\eta})$ acts continuously. Write $\mathcal{M}$ for the associated locally constant sheaf on $\dot{X}$. For a point $x \in \dot{X}(k)$ the inclusion $j' : X' := \dot{X} - x \to \dot{X}$ induces a surjection

(6.2) $$j'_* \ : \ \pi_1^t(X'_{\bar{k}}, \bar{\eta}) \to \pi_1^t(\dot{X}_{\bar{k}}, \bar{\eta})$$

and thus an injection on the level of Galois cohomology :

(6.3) $$H^1(\pi_1^t(\dot{X}_{\bar{k}}, \bar{\eta}), M) \to H^1(\pi_1^t(X'_{\bar{k}}, \bar{\eta}), M).$$

Now (6.3) is canonically isomorphic to the first map in the exact sequence of G_k-modules,

(6.4) $$0 \to H^1(\dot{X}_{\bar{k}}, \mathcal{M}) \xrightarrow{j'^*} H^1(X'_{\bar{k}}, \mathcal{M}) \to H_x^2(\dot{X}_{\bar{k}}, \mathcal{M}).$$

We now assume that $\dot{X}$ is affine. This implies $H^2(\dot{X}_{\bar{k}}, \mathcal{M}) = 0$ [Mi, V.2.4a], so that the right hand arrow in (6.4) is surjective. Motivated by (5.12) we shall study the first coboundary map, $\delta_{(6.4)}$, in the long exact sequence of Galois cohomology associated with (6.4). Actually in (5.12) $\mathcal{M}$ was allowed to be a locally constant sheaf of $\mathbf{Z}/l^n$-modules for any n. Here we have restricted to the case $n = 1$ in order to minimize technical difficulties. We shall make some further simplifying assumptions. The first of these is that k be algebraically closed in the fixed field of the kernel of the representation

$$\pi_1^t(\dot{X}_k, \bar{\eta}) \to \mathrm{Aut}(M). \tag{6.5}$$

View this fixed field as the function field of a smooth projective curve C/k. The assumption implies that C/k is geometrically integral. Write Γ for the image of (6.5). We get a finite morphism of curves $\rho : C_k \to X_k$ which is Galois with Galois group Γ. Write $\dot{\rho} : \dot{C} \to \dot{X}$ for the base change of ρ with respect to the inclusion $\dot{X} \subset X$. Then $\dot{\rho}$ is étale and $\mathcal{M}|_{\dot{C}}$ is a constant sheaf. Define $\Sigma_C = C - \dot{C}$, $\Sigma'_C = \Sigma_C \cup \rho^{-1}(x)$, $C' = C - \Sigma'_C$, and

$$H^2_{\Sigma'_C}(C_{\bar{k}}, \mu_l)_0 = \mathrm{Ker} : H^2_{\Sigma'_C}(C_{\bar{k}}, \mu_l) \to H^2(C_{\bar{k}}, \mu_l).$$

Finally denote by $\delta_{(6.6)}$ the first coboundary map in the long exact sequence of G_k-cohomology associated to the short exact sequence of G_k-modules

$$0 \to H^1(C_{\bar{k}}, \mu_l) \to H^1(C'_{\bar{k}}, \mu_l) \to H^2_{\Sigma'_C}(C_{\bar{k}}, \mu_l)_0 \to 0. \tag{6.6}$$

A technical but important result is now

THEOREM 6.7. – *Let X, C, and M be as above. If the hypotheses (1)-(4) below are satisfied, then the coboundary map $\delta_{(6.4)}$ is not zero.*

(1) $\mu_l \subset k^*$.

(2) *G_k acts trivially on $H^1(\dot{C}_{\bar{k}}, \mu_l)$.*

(3) *All points of Σ_C are rational over k.*

(4) *There is an indecomposible, projective, $\mathbf{F}_l[\Gamma]$-submodule $P \subset H^1(C_{\bar{k}}, \mu_l)$ such that $(P \otimes_{\mathbf{Z}} M)^\Gamma \neq 0$. Furthermore there is $\mathfrak{d} \in (H^2_{\Sigma'_C}(C_{\bar{k}}, \mu_l)_0)^{G_k}$ such that the image of the homomorphism $\delta_{(6.6)}(\mathfrak{d}) \in \mathrm{Hom}(G_k, H^1(C_{\bar{k}}, \mu_l))$ is contained in P but not in* $\mathrm{rad}(P)$.

The main consequence of (6.7) which interests us is

THEOREM 6.8. – *Let X and C be as above. Let $S \subset X$ be a closed subscheme of dimension zero. If the hypotheses (1)-(4) below are satisfied, then, after replacing k be a finite extension if necessary, there exists $x \in \dot{X}(k), x \notin S(k)$, such that the coboundary $\delta_{(6.4)}$ is injective.*

(1) *k is a finite field,*

(2) *$\Gamma = SL(2, \mathbf{F}_l)/\pm 1$, with l an odd prime,*

(3) *the $\mathbf{F}_l[\Gamma]$-module M is isomorphic to the second symmetric power of the tautological representation of $SL(2, \mathbf{F}_l)$ on $\mathbf{F}_l^2$.*

(4) *Let $B \subset \Gamma$ be a Borel subgroup. If X has genus 0, then the (reduced) branching divisor of the cover $B\backslash C \to X$ has degree at least four.*

Using (6.8) we will show

THEOREM 6.9. – *Let k be a finite field. Let $\pi : Y \to X$ be a semi-stable elliptic surface with section. Let W and m_π have the same meaning as in §5. Then, if l does not divide $2m_\pi$, the image of the cycle class map $\alpha^2 : CH^2(W_{\bar{k}})_{hom} \to J_l^2(W)$ is an infinite group.*

The sections §8, §9, and §10 are devoted to the proofs of these theorems. The next section recasts the discussion of the exact sequence (6.4) in more elementary terms.

7. Reduction to the cohomology of finite groups

The coboundary map $\delta_{(6.4)}$ may be interpreted in terms of the cohomology of finite groups. This reformulation will be carried through in detail in this section. An advantage of this viewpoint is that the analysis of (6.4) takes on a more down to earth character. The operation of the Galois group becomes more transparent. Presumably this viewpoint is also better suited for explicit computations. As explicit computations will not be necessary to prove the theorems of §6, we shall return to the sheaf cohomology viewpoint in later sections.

The symbols X, C, C', $\dot{X}$, $\dot{C}$, M, etc. continue to have the same meaning as in §6. In particular $\dot{X}$ is affine. To this we add some additional notation :

$\dot{N} = \pi_1^t(\dot{C}_{\bar{k}}, \bar{\eta})_{ab} \ / \ l \cdot \pi_1^t(\dot{C}_{\bar{k}}, \bar{\eta})_{ab} \simeq \mathrm{Hom}(H^1(\dot{C}_{\bar{k}}, \mathbf{Z}/l), \mathbf{Z}/l)$,

$N' = \pi_1^t(C'_{\bar{k}}, \bar{\eta})_{ab} \ / \ l \cdot \pi_1^t(C'_{\bar{k}}, \bar{\eta})_{ab} \simeq \mathrm{Hom}(H^1(C'_{\bar{k}}, \mathbf{Z}/l), \mathbf{Z}/l)$,

$\dot{\mathcal{H}} =$ kernel of the tautological map $\pi_1^t(\dot{C}_{\bar{k}}, \bar{\eta}) \to \dot{N}$,

$\mathcal{H}' =$ kernel of the tautological map $\pi_1^t(C'_{\bar{k}}, \bar{\eta}) \to N'$.

The subgroups $\dot{\mathcal{H}} \subset \pi_1^t(\dot{X}_k, \bar{\eta})$ and $\mathcal{H}' \subset \pi_1^t(\mathcal{X}'_k, \bar{\eta})$ are closed and normal. Modding out by these in the commutative diagram with surjective vertical maps,

$$\begin{array}{ccccccccc} 1 & \to & \pi_1^t(C'_{\bar{k}}, \bar{\eta}) & \to & \pi_1^t(X'_k, \bar{\eta}) & \to & \Gamma \times G_k & \to & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \to & \pi_1^t(\dot{C}_{\bar{k}}, \bar{\eta}) & \to & \pi_1^t(\dot{X}_k, \bar{\eta}) & \to & \Gamma \times G_k & \to & 1, \end{array}$$

gives the following commutative diagram also with surjective vertical maps :

$$\begin{array}{ccccccccc} 0 & \to & N' & \to & \vartheta' & \to & \Gamma \times G_k & \to & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & \dot{N} & \to & \dot{\vartheta} & \to & \Gamma \times G_k & \to & 1. \end{array} \tag{7.1}$$

Pull back (7.1) with respect to the inclusion of Γ as the first factor in the product $\Gamma \times G_k$ to get another commutative diagram with surjective vertical maps :

$$\begin{array}{ccccccccc} 0 & \to & N' & \to & \theta' & \to & \Gamma & \to & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & \dot{N} & \to & \dot{\theta} & \to & \Gamma & \to & 1. \end{array} \tag{7.2}$$

LEMMA 7.3. – *In the commutative diagram,*

$$\begin{array}{ccc} H^1(\dot{\theta}, M) & \to & H^1(\theta', M) \\ \downarrow & & \downarrow \\ H^1(\pi_1^t(\dot{X}_{\bar{k}}, \bar{\eta}), M) & \to & H^1(\pi_1^t(X'_{\bar{k}}, \bar{\eta}), M), \end{array}$$

the vertical maps are isomorphisms.

Proof. – There is a commutative diagram of exact sequences,

$$\begin{array}{ccccccccc} 1 & \to & \pi_1^t(\dot{C}_{\bar{k}},\bar{\eta}) & \to & \pi_1^t(\dot{X}_{\bar{k}},\bar{\eta}) & \to & \Gamma & \to & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \to & \dot{N} & \to & \dot{\theta} & \to & \Gamma & \to & 1. \end{array}$$

Apply the Hochschild-Serre spectral sequence to obtain a commutative diagram :

$$\begin{array}{ccccccccc} 0 & \to & H^1(\Gamma,M) & \to & H^1(\dot{\theta},M) & \to & \mathrm{Hom}(\dot{N},M)^\Gamma & \to & H^2(\Gamma,M) \\ & & \| & & \downarrow & & \simeq\downarrow & & \| \\ 0 & \to & H^1(\Gamma,M) & \to & H^1(\pi_1^t(\dot{X}_{\bar{k}},\bar{\eta}),M) & \to & \mathrm{Hom}(\pi_1^t(\dot{C}_{\bar{k}},\bar{\eta}),M)^\Gamma & \to & H^2(\Gamma,M). \end{array}$$

The assertion of the lemma for the vertical arrow on the left now follows from the five lemma. The vertical arrow on the right is treated similarly.

The lemma together with the fact that $H^2(\dot{X}_{\bar{k}},\mathcal{M}) \simeq 0$ shows that (6.4) is canonically isomorphic to the short exact sequence,

$$0 \to H^1(\dot{\theta},M) \xrightarrow{j'^*} H^1(\theta',M) \to \mathrm{Coker}\, j'^* \to 0. \tag{7.4}$$

Let v be a place over $x \in X_{\bar{k}}$ in the cover of $X_{\bar{k}}$ with Galois group θ'. Write $D \subset \theta'$ for the decomposition (= inertia) group at v and $\bar{N}$ for the kernel of the left hand vertical map in (7.2). Then $D \subset \bar{N}$ and $D \simeq \mathbf{Z}/l$. The following lemma interprets $\mathrm{Coker}\, j'^*$ as a cohomology group.

LEMMA 7.5. – $\mathrm{Coker}\, j'^*$ *is canonically isomorphic to* $\mathrm{Hom}(D,M)$.

Proof. – The discrete valuation ring at v is naturally a subring of a Henselization A of $\mathcal{O}_{X_{\bar{k}},x}$. Write K for the fraction field of A and $\mathbf{D}$ for the absolute Galois group of K. Let a denote the closed point of Spec A. By excision [Mi, p. 92] the right hand arrow in (6.4) factors as

$$H^1(X'_{\bar{k}},\mathcal{M}) \to H^1(\mathrm{Spec}\ K,\mathcal{M}) \simeq H^2_a(\mathrm{Spec}\ A,\mathcal{M}) \simeq H^2_x(\dot{X}_{\bar{k}},\mathcal{M}).$$

The second term is isomorphic to

$$\mathrm{Hom}(\mathbf{D},M) \simeq \mathrm{Hom}(\mathbf{D}^t,M) \simeq \mathrm{Hom}(\mathbf{D}_{ab}/l,M),$$

where $\mathbf{D}^t$ (respectively $\mathbf{D}_{ab}$) is the maximal tame (respectively abelian) quotient of $\mathbf{D}$. Since the l-primary part of $\mathbf{D}_{ab}$ is procyclic [Mi3, p. 160], the natural map $\mathbf{D}_{ab}/l \to D$ is an isomorphism. Thus there is a commutative diagram :

$$\begin{array}{ccc} H^1(\theta',M) & \to & \mathrm{Hom}(D,M) \\ \simeq\downarrow & & \downarrow\simeq \\ H^1(\pi_1^t(X'_{\bar{k}},\bar{\eta}),M) & \to & \mathrm{Hom}(\mathbf{D}^t,M) \\ \simeq\downarrow & & \downarrow\simeq \\ H^1(X'_{\bar{k}},\mathcal{M}) & \to & H^2_x(\dot{X}_{\bar{k}},\mathcal{M}) \end{array} \tag{7.6}$$

Finally we note that the map $H^1(\theta',M) \to H^2_x(\dot{X}_{\bar{k}},M)$ factors through $\mathrm{Coker}\, j'^*$ by (7.3). The lemma follows.

The following short exact sequence translates (6.4) into the language of group cohomology :

$$0 \to H^1(\dot\theta, M) \xrightarrow{j'^*} H^1(\theta', M) \to \mathrm{Hom}(D, M) \to 0. \tag{7.7}$$

The action of the Galois group on (7.4) is determined by its action on the middle term. This is described with the help of the exact sequence

$$1 \to \theta' \to \vartheta' \xrightarrow{p} G_k \to 1. \tag{7.8}$$

Given $\sigma \in G_k$, $\tilde\sigma \in p^{-1}(\sigma)$, and a crossed homomorphism $F \in Z^1(\theta', M)$, then $t \to \tilde\sigma F(\tilde\sigma^{-1} t \tilde\sigma)$ is also a crossed homomorphism. Its class in $H^1(\theta', M)$ depends only on the class of F and the choice of $\sigma \in G_k$. This defines the action of G_k on $H^1(\theta', M)$. Since the subgroup $\dot\vartheta \subset \vartheta'$ surjects to G_k with kernel $\dot\theta$, the same perscription gives an action of G_k on $H^1(\dot\theta, M)$. This makes (7.4) a short exact sequence of G_k-modules.

The extension (7.8) gives rise to a homomorphism $\varsigma : G_k \to \mathrm{Aut}(\theta')/\mathrm{Inn}(\theta')$ with $G := G_k/\mathrm{Ker}(\varsigma)$ a finite group. The action of G_k on (7.4) or (7.7) factors through G. Since the map

$$H^1(G, H^1(\dot\theta, M)) \to H^1(G_k, H^1(\dot\theta, M))$$

is injective, the computation of $\delta_{(6.4)}$ may be treated entirely in the context of the cohomology of finite groups.

To analyze (7.4) and (7.7) it is helpful to apply the Hochschild-Serre spectral sequence to (7.2). This yields the following commutative diagram :

$$\begin{array}{ccc} H^1(\dot\theta, M) & \xrightarrow{j'^*} & H^1(\theta', M) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(\dot N, M)^\Gamma & \to & \mathrm{Hom}(N', M)^\Gamma. \end{array} \tag{7.9}$$

LEMMA 7.10. – *There is a commutative diagram of $\Gamma \times G_k$-modules in which the vertical maps are isomorphisms :*

$$\begin{array}{ccc} \mathrm{Hom}(\dot N, \mathbf{Z}/l) \otimes_{\mathbf{Z}} M & \to & \mathrm{Hom}(N', \mathbf{Z}/l) \otimes_{\mathbf{Z}} M \\ \dot\psi \downarrow & & \psi' \downarrow \\ \mathrm{Hom}(\dot N, M) & \to & \mathrm{Hom}(N', M). \end{array}$$

Proof. – The action of $\Gamma \times G_k$ on M is *via* projection on the first factor. The action of $\Gamma \times G_k$ on the Hom-groups in the bottom row is defined in analogy with the action of G_k on $H^1(\theta', M)$ except that (7.1) is used in place of (7.8). The action on the Hom-groups in the top row is defined similarly with trivial $\Gamma \times G_k$-module $\mathbf{Z}/l$ used in place of M. Now

$$\psi'(f \otimes m)(n) = f(n) \cdot m.$$

To check that ψ' is $\Gamma \times G_k$-linear take $\kappa \in \Gamma \times G_k$, a lifting $\tilde\kappa \in \vartheta'$, $F \in \mathrm{Hom}(N', \mathbf{Z}/l)$ and $m \in M$. Then

$$\psi'(\kappa F \otimes \kappa m)(n) = (\kappa F)(n) \cdot \kappa m = F(\tilde\kappa^{-1} n \tilde\kappa) \cdot \kappa m = \tilde\kappa \psi'(F \otimes m)(\tilde\kappa^{-1} n \tilde\kappa) = (\kappa \psi'(F \otimes m))(n).$$

DEFINITION 7.11. – *A functor,* Λ, *from finite type* $\mathbf{F}_l[\Gamma]$*-modules to* $\mathbf{F}_l$*-vector spaces is defined by* $\Lambda(T) = (T \otimes_{\mathbf{Z}} M)^\Gamma$.

One checks easily that Λ is left exact.

LEMMA 7.12. – *There is a commutative diagram of* G_k*-modules with exact rows :*

$$\begin{array}{ccccccccc} 0 & \to & H^1(\dot\theta, M) & \overset{j'^*}{\to} & H^1(\theta', M) & \to & \mathrm{Hom}(D, M) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow \simeq & & \\ 0 & \to & \Lambda(\mathrm{Hom}(\dot N, \mathbf{Z}/l)) & \to & \Lambda(\mathrm{Hom}(N', \mathbf{Z}/l)) & \to & \Lambda(\mathrm{Hom}(\bar N, \mathbf{Z}/l)) & \to & 0. \end{array}$$

Proof. – The left hand square comes from combining (7.9) and (7.10).

We now show the surjectivity of the horizontal arrow on the bottom right. First,

$$\mathrm{Hom}(N', \mathbf{Z}/l) \to \mathrm{Hom}(\bar N, \mathbf{Z}/l) \tag{7.13}$$

is a surjective homomorphism of $\mathbf{F}_l[\Gamma]$-modules. It suffices to check that (7.13) splits. This will occur if the right hand side is a free $\mathbf{F}_l[\Gamma]$-module. Write $c \in \rho^{-1}(x)(\bar k)$ for the place of $\dot C$ which lies below v. Now Γ acts simply transitively on $\rho^{-1}(x)(\bar k)$. It follows that the submodule $\mathbf{F}_l[\Gamma]D \subset \bar N$ is free of rank 1. This inclusion is in fact an isomorphism since the two vector spaces have the same dimension. Now the freeness of the right hand term in (7.13) follows from the $\mathbf{F}_l[\Gamma]$-isomorphism

$$\mathbf{F}_l[\Gamma] \to \mathrm{Hom}(\mathbf{F}_l[\Gamma], \mathbf{Z}/l), \qquad \gamma \to \delta_\gamma,$$

where δ_γ denotes the characteristic function of γ.

To explain the vertical arrow on the right, it suffices to show that the restriction map

$$(\mathrm{Hom}(\bar N, \mathbf{Z}/l) \otimes_{\mathbf{Z}} M)^\Gamma \to \mathrm{Hom}(D, \mathbf{Z}/l) \otimes_{\mathbf{Z}} M$$

is an isomorphism. For this, note that the tensor product representation $\mathbf{F}_l[\Gamma] \otimes_{\mathbf{Z}} M$ is a free $\mathbf{F}_l[\Gamma]$-module [Al, p. 53 Ex. 3]. In fact, it is easy to see that if $m_1, ..., m_r$ is an $\mathbf{F}_l$-basis for M, then $1 \otimes m_1, ..., 1 \otimes m_r$ is an $\mathbf{F}_l[\Gamma]$-basis for $\mathbf{F}_l[\Gamma] \otimes_{\mathbf{Z}} M$. Thus the map

$$\mathrm{Hom}(D, \mathbf{Z}/l) \otimes M \to (\mathrm{Hom}(\mathbf{F}_l[\Gamma] \otimes D, \mathbf{Z}/l) \otimes M)^\Gamma, \qquad \varphi \otimes m \to \sum_{\gamma \in \Gamma} \delta_\gamma \otimes \varphi \otimes \gamma m$$

is an isomorphism which is inverse to the restriction map. Now (7.12) follows.

To conclude this section, we describe the action of G_k on $\Lambda(\mathrm{Hom}(\bar N, \mathbf{Z}/l))$. This module becomes isomorphic to M if we assume that $\mu_l \subset k^*$ and fix an isomorphism $\mu_l \simeq \mathbf{Z}/l$. To describe this explicitly use the canonical identification of the following two exact sequences,

$$\begin{array}{cc} & 0 \to \mathrm{Hom}(\dot N, \mathbf{Z}/l) \to \mathrm{Hom}(N', \mathbf{Z}/l) \to \mathrm{Hom}(\bar N, \mathbf{Z}/l) \to 0 \\ \text{and} & 0 \to H^1(\dot C_{\bar k}, \mu_l) \to H^1(C'_{\bar k}, \mu_l) \to H^2_{\rho^{-1}(x)}(\dot C_{\bar k}, \mu_l) \to 0, \end{array} \tag{7.14}$$

which follows directly from the definitions of $\dot N$ and N' and the affineness of $\dot C$. The term on the right in the bottom sequence may be thought of as $\mathrm{mod}\, l$ divisors supported on $\rho^{-1}(x)$. The isomorphism of left $\mathbf{F}_l[\Gamma]$-modules given by,

$$\mathbf{F}_l[\Gamma] \to H^2_{\rho^{-1}(x)}(\dot C_{\bar k}, \mu_l), \qquad \sum_{\kappa \in \Gamma} a_\kappa \kappa \to \sum_{\kappa \in \Gamma} a_\kappa \kappa c,$$

together with the isomorphism

$$M \to (\mathbf{F}_l[\Gamma] \otimes_{\mathbf{Z}} M)^{\Gamma}, \qquad m \to \sum_{\kappa\in\Gamma} \kappa \otimes \kappa m,$$

gives rise to an identification

$$\chi : M \to \Lambda(H^2_{\rho^{-1}(x)}(\dot{C}_{\bar{k}}, \mu_l)), \qquad \chi(m) = \sum_{\kappa\in\Gamma} \kappa c \otimes \kappa m. \tag{7.15}$$

LEMMA 7.16. – *The G_k action on $\Lambda(H^2_{\rho^{-1}(x)}(\dot{C}_{\bar{k}}, \mu_l))$ is given with respect to the identification (7.15) by the group homomorphism $h : G_k \to \Gamma$ defined by :*

$$h(\sigma)^{-1} \circ c = c \circ \sigma^{-1} \qquad \forall \sigma \in G_k.$$

Proof. – Recall that $\Gamma \times G_k$ acts on $\rho^{-1}(x)(\bar{k}) = \mathrm{Morph}_k(\mathrm{Spec}\,\bar{k}, \rho^{-1}(x))$. In other words the actions of Γ and G_k commute. It follows that σ acts on the left on $\bmod l$ divisors supported on $\rho^{-1}(x)$ by

$$\sigma(\sum_{\kappa\in\Gamma} a_\kappa \kappa c) = \sum_{\kappa\in\Gamma} a_\kappa \kappa c \circ \sigma^{-1} = \sum_{\kappa\in\Gamma} a_\kappa \kappa h(\sigma)^{-1} c.$$

By (7.15)

$$\sigma\chi(m) = \sum_{\kappa\in\Gamma} \kappa c \circ \sigma^{-1} \otimes \kappa m = \sum_{\kappa\in\Gamma} \kappa h(\sigma)^{-1} c \otimes \kappa m = \sum_{\kappa'\in\Gamma} \kappa' c \otimes \kappa' h(\sigma) m = \chi(h(\sigma)m).$$

COROLLARY 7.17. – *Suppose that k is a finite field and that $\xi \in \Gamma$ generates the decomposition group of the scheme theoretic point of $\rho^{-1}(x)$ corresponding to c. Then the action of the Frobenius $\phi \in G_k$ on $H^2_x(\dot{X}_{\bar{k}}, \mathcal{M})$ may be identified with the action of a generator of $\langle\xi\rangle$ on M.*

8. A criterion for non-vanishing of the coboundary map

The purpose of this section is to prove (6.7). Recall from §6 that $\dot{X}$ and hence $\dot{C}$ are affine. Hypothesis (1) of (6.7) implies that $\mu_l \simeq \mathbf{Z}/l$. Thus the commutative diagram of G_k-modules (7.12) may be rewritten as

$$\begin{array}{ccccccccc} 0 & \to & H^1(\dot{X}_{\bar{k}}, \mathcal{M}) & \to & H^1(X'_{\bar{k}}, \mathcal{M}) & \to & H^2_x(\dot{X}_{\bar{k}}, \mathcal{M}) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow \simeq & & \\ 0 & \to & \Lambda(H^1(\dot{C}_{\bar{k}}, \mu_l)) & \to & \Lambda(H^1(C'_{\bar{k}}, \mu_l)) & \to & \Lambda(H^2_{\rho^{-1}(x)}(\dot{C}_{\bar{k}}, \mu_l)) & \to & 0 \end{array} \tag{8.1}$$

The coboundary associated to the top row in (8.1) is $\delta_{(6.4)}$. To show that $\delta_{(6.4)} \neq 0$ it will suffice to show that the first coboundary map, Δ, in the long exact G_k-cohomology sequence associated to the bottom row of (8.1) is not zero. To accomplish this, we would like to relate Δ to the first coboundary map, ∂, in the long exact G_k-cohomology sequence associated to

$$0 \to H^1(\dot{C}_{\bar{k}}, \mu_l) \to H^1(C'_{\bar{k}}, \mu_l) \to H^2_{\rho^{-1}(x)}(\dot{C}_{\bar{k}}, \mu_l) \to 0. \tag{8.2}$$

The group $\Gamma \times G_k$ acts on (8.2). Furthermore, the sequence (8.2) has an $\mathbf{F}_l[\Gamma]$-linear splitting, since $H^2_{\rho^{-1}(x)}(\dot{C}_{\bar{k}}, \mu_l)$ is a free rank one $\mathbf{F}_l[\Gamma]$-module.

As our problem is purely formal, we formulate it abstractly : Let

$$0 \to A \to B \to C \to 0 \tag{8.3}$$

be a short exact sequence of $\mathbf{F}_l[\Gamma \times G_k]$-modules. View $Z^1(G_k, B)$ (respectively $H^1(G_k, A)$) as $\mathbf{F}_l[\Gamma]$-modules *via* the action of Γ on B (respectively A). Now

$$\psi : B \to Z^1(G_k, B), \qquad \psi(b)(g) = gb - b$$

is $\mathbf{F}_l[\Gamma]$-linear. Suppose that there is an $\mathbf{F}_l[\Gamma]$-linear splitting, $s : C \to B$. Then the first coboundary in the long exact cohomology sequence associated to (8.3), $\partial : C^{G_k} \to H^1(G_k, A)$, is $\mathbf{F}_l[\Gamma]$-linear, since it is represented by $\psi \circ s$. The existence of the splitting s forces

$$0 \to \Lambda(A) \to \Lambda(B) \to \Lambda(C) \to 0$$

to be an exact sequence of G_k-modules. The associated coboundary will be written, $\Delta : \Lambda(C)^{G_k} \to H^1(G_k, \Lambda(A))$. There are canonical isomorphisms,

$$\kappa : \Lambda(C^{G_k}) = (C^{G_k} \otimes_{\mathbf{Z}} M)^\Gamma \simeq ((C \otimes_{\mathbf{Z}} M)^{G_k})^\Gamma \simeq ((C \otimes_{\mathbf{Z}} M)^\Gamma)^{G_k} = \Lambda(C)^{G_k},$$
$$\upsilon : \Lambda(H^1(G_k, A)) = (H^1(G_k, A) \otimes_{\mathbf{Z}} M)^\Gamma \simeq H^1(G_k, A \otimes_{\mathbf{Z}} M)^\Gamma \simeq H^1(G_k, \Lambda(A)).$$

LEMMA 8.4. – $\upsilon \circ \Lambda(\partial) = \Delta \circ \kappa$.

Proof. – Fix $\sum_j c_j \otimes m_j \in (C^{G_k} \otimes_{\mathbf{Z}} M)^\Gamma$. Now $s \otimes 1 : C \otimes_{\mathbf{Z}} M \to B \otimes_{\mathbf{Z}} M$ is $\mathbf{F}_l[\Gamma]$-linear, hence induces a map $(C \otimes_{\mathbf{Z}} M)^\Gamma \to (B \otimes_{\mathbf{Z}} M)^\Gamma$. Clearly $\sum_j s(c_j) \otimes m_j \in (B \otimes_{\mathbf{Z}} M)^\Gamma$ lifts $\sum_j c_j \otimes m_j$. As M is a trivial G_k-module, G_k acts on $(B \otimes_{\mathbf{Z}} M)^\Gamma$ *via* its action on B. It follows that $\Delta \circ \kappa(\sum_j c_j \otimes m_j)$ is represented by the crossed homomorphism

$$g \to \sum_j gs(c_j) \otimes m_j - \sum_j s(c_j) \otimes m_j.$$

On the other hand $\partial(c_j)$ is represented by $gs(c_j) - s(c_j)$. Thus the crossed homomorphism $g \to \sum_j (gs(c_j) - s(c_j)) \otimes m_j$ represents $\Lambda(\partial)(\sum_j c_j \otimes m_j)$. The lemma follows.

For the next lemma we introduce the following notation for the obvious restriction maps :

$$q : H^2_{\Sigma'_C}(C_{\bar{k}}, \mu_l)_0 \to H^2_{\rho^{-1}(x)}(\dot{C}_{\bar{k}}, \mu_l) \qquad \text{and} \qquad j_C^* : H^1(C_{\bar{k}}, \mu_l) \to H^1(\dot{C}_{\bar{k}}, \mu_l).$$

Also ∂ will once more refer to the coboundary map attached to (8.2).

LEMMA 8.5. – *Suppose given* $\mathfrak{d} \in H^2_{\Sigma'_C}(C_{\bar{k}}, \mu_l)_0^{G_k}$. *Then*

$$\partial(q(\mathfrak{d})) = H^1(G_k, j_C^*)(\delta_{(6.6)}(\mathfrak{d})).$$

Proof. – This is immediate from the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & H^1(C_{\bar{k}}, \mu_l) & \to & H^1(C'_{\bar{k}}, \mu_l) & \to & H^2_{\Sigma'_C}(C_{\bar{k}}, \mu_l)_0 & \to & 0 \\ & & j_C^* \downarrow & & \| & & q \downarrow & & \\ 0 & \to & H^1(\dot{C}_{\bar{k}}, \mu_l) & \to & H^1(C'_{\bar{k}}, \mu_l) & \to & H^2_{\rho^{-1}(x)}(\dot{C}_{\bar{k}}, \mu_l) & \to & 0. \end{array}$$

We are now ready to prove (6.7). By (8.1) it will suffice to show that the first coboundary map in the long exact sequence of Galois cohomology associated to

$$(8.6) \qquad 0 \to \Lambda(H^1(\dot{C}, \mu_l)) \to \Lambda(H^1(C'_{\bar{k}}, \mu_l)) \to \Lambda(H^2_{\rho^{-1}(x)}(\dot{C}_{\bar{k}}, \mu_l)) \to 0$$

is not zero. By (8.4) this reduces to checking that the map

$$\Lambda(\partial) : \Lambda(H^2_{\rho^{-1}(x)}(\dot{C}_{\bar{k}}, \mu_l)^{G_k}) \to \Lambda(H^1(G_k, H^1(\dot{C}_{\bar{k}}, \mu_l)))$$

is not zero. The hypothesis (2) of (6.7) allows us to write the Galois cohomology group in parenthesis on the right as $\mathrm{Hom}(G_k, H^1(\dot{C}_{\bar{k}}, \mu_l))$. Furthermore,

$$H^1(G_k, j_C^*) : \mathrm{Hom}(G_k, H^1(C_{\bar{k}}, \mu_l)) \to \mathrm{Hom}(G_k, H^1(\dot{C}_{\bar{k}}, \mu_l))$$

is an injective $\mathbf{F}_l[\Gamma]$-linear map. Choose $\mathfrak{d} \in H^2_{\Sigma'_C}(C_{\bar{k}}, \mu_l)_0^{G_k}$ as in (4) of (6.7). Then by (8.5)

$$\partial(q(\mathfrak{d})) = H^1(G_k, j_C^*)(\delta_{(6.6)}(\mathfrak{d})).$$

Furthermore the homomorphism $\delta_{(6.6)}(\mathfrak{d}) \in \mathrm{Hom}(G_k, H^1(C_{\bar{k}}, \mu_l))$ has image contained in the projective submodule $P \subset H^1(C_{\bar{k}}, \mu_l)$. This is a direct summand since projectivity and injectivity for $\mathbf{F}_l[\Gamma]$-modules coincide [Al,p. 41]. By hypothesis (4) of (6.7) there is an element $g \in G_k$ such that $\delta_{(6.6)}(\mathfrak{d})(g)$ generates P as an $\mathbf{F}_l[\Gamma]$-module. Now $\mathrm{Hom}(\langle g \rangle, P)$ is a direct summand of the projective $\mathbf{F}_l[\Gamma]$-module $\mathrm{Hom}(G_k, P)$ which *via* $H^1(G_k, j_C^*)$ is a direct summand of $\mathrm{Hom}(G_k, H^1(\dot{C}_{\bar{k}}, \mu_l))$. Let

$$pr : \mathrm{Hom}(G_k, H^1(\dot{C}_{\bar{k}}, \mu_l)) \to \mathrm{Hom}(\langle g \rangle, P)$$

be an $\mathbf{F}_l[\Gamma]$-linear projection. Now the $\mathbf{F}_l[\Gamma]$-linear map

$$pr \circ H^1(G_k, j_C^*) \circ \delta_{(6.6)} : H^2_{\Sigma'_C}(C_{\bar{k}}, \mu_l)_0^{G_k} \to \mathrm{Hom}(\langle g \rangle, P)$$

is surjective and, since $\mathrm{Hom}(\langle g \rangle, P) \simeq P$ is projective, it splits. Apply the functor Λ to this map. Since $\Lambda(P) \neq 0$ by hypothesis and

$$\Lambda(pr) \circ \Lambda(H^1(G_k, j_C^*) \circ \delta_{(6.6)}) = \Lambda(pr) \circ \Lambda(\partial) \circ \Lambda(q)$$

splits, we find that $\Lambda(\partial) \neq 0$. By (8.4), $\Delta \neq 0$. By (8.1), $\delta_{(6.4)} \neq 0$. This completes the proof of (6.7).

9. Injectivity of the coboundary map

The purpose of this section is to prove (6.8). The statement of the theorem permits replacing the base field k with an arbitrary finite extension field. Thus we may assume from the outset that (1), (2), and (3) of (6.7) hold. By assumption (3) in (6.8) M is an irreducible $\mathbf{F}_l[\Gamma]$-module [Al, p. 15]. Let P denote the projective envelope of M [Se1, 14.3]. Clearly P is indecomposible. It is also self-dual [Al, p. 52] so that

$$\Lambda(P) = (P \otimes_{\mathbf{Z}} M)^{\Gamma} \simeq (P^{\vee} \otimes_{\mathbf{Z}} M)^{\Gamma} \simeq \mathrm{Hom}_{\mathbf{F}_l[\Gamma]}(P, M) \neq 0.$$

PROPOSITION 9.1. – $P \subset H^1(C_{\bar{k}}, \mu_l)$.

Proof. – The $\mathbf{F}_l[\Gamma]$-module $H^1(C_{\bar{k}}, \mu_l)$ has a decomposition as a direct sum of indecomposible $\mathbf{F}_l[\Gamma]$-modules which is unique up to isomorphism. Write ν_P for the multiplicity of the summand P in this decomposition. When $l = 3$, $P = M$ is the $\bmod l$ Steinberg representation and the proof of the proposition is greatly simplified because it suffices to determine the multiplicity of the characteristic zero Steinberg representation in the Γ-module $H^1(C_{\bar{k}}, \mathbf{Q}_l)$ [Se1, III]. Write ψ (respectively τ) for the character of the Steinberg representation (respectively of the representation $H^1(C_{\bar{k}}, \mathbf{Q}_l)$). Write $B \subset \Gamma$ for a Borel subgroup and $<,>$ for the normalized inner-product of functions on Γ [Se1, §2]. Then

$$\begin{aligned}\nu_P &= <\tau, \psi> = <\tau, \mathrm{Ind}_B^{\Gamma} 1> - <\tau, 1> \\ &= <\mathrm{Res}\ \tau|_B, 1> - <\tau, 1> = 2g(B\backslash C) - 2g(X).\end{aligned}$$

When the genus of X is at least two, this number is positive by the Hurwitz formula. The same holds if $g(X) = 1$, since the cover $B\backslash C_{\bar{k}} \to X_{\bar{k}}$ is not Galois and therefore must ramify. To show that $\nu_P > 0$ when $g(X) = 0$ use that the degree 4 cover $B\backslash C_{\bar{k}} \to X_{\bar{k}}$ is branched in at least four points by hypothesis (4) of (6.8). The contribution of each branch point to the Hurwitz formula is 2. Thus $g(B\backslash C) - g(X) > 0$ as desired.

For $l > 3$ one has the formula [Sch-MR, 7.5,7.2] :

$$\nu_P = \xi - 3e(X_{\bar{k}}) + 2\mathrm{deg.}(\mathfrak{b}),$$

where $e(X_{\bar{k}})$ is the l-adic euler characteristic, $\mathfrak{b} \subset X$ is the (reduced) branch divisor of $\rho : C \to X$, and $\xi \geq -1$. When $g(X) \geq 2$ the positivity of ν_P is immediate. The same holds when $g(X) = 1$, since non-abelian covers are ramified. When $g(X) = 0$, $\nu_P > 0$ is a consequence of the assumption (4) in (6.8) which implies that $\mathrm{deg.}(\mathfrak{b}) \geq 4$.

The obvious observation that there are no non-trivial l^{th} roots of unity in $\bar{\mathbf{F}}_l$ implies that a semi-simple element of $SL(2, \mathbf{F}_l)$ has order prime to l. Let $\xi \in \Gamma = SL(2, \mathbf{F}_l)/\pm 1$ be an element of order $e > 1$, which is the image of a semi-simple element of $SL(2, \mathbf{F}_l)$. Let $\gamma : C \to \langle \xi \rangle \backslash C =: \bar{C}$ be the canonical quotient map. Fix a k-rational point $c_0 \in \Sigma_C$ and write $\bar{c}_0 = \gamma(c_0)$. Using the base points $\bar{c}_0 \in \bar{C}$ and $c_0 \in C$ we get embeddings, $\bar{i}_0 : \bar{C} \to \mathrm{Pic}^0(\bar{C})$ and $i_0 : C \to \mathrm{Pic}^0(C)$. Write $\bar{m}_l$ (respectively m_l) for multiplication by l in $\mathrm{Pic}^0(\bar{C})$ (respectively $\mathrm{Pic}^0(C)$). Define a fiber product over $\mathrm{Pic}^0(C)$, $\breve{C} = C \times_{(i_0, m_l)} \mathrm{Pic}^0(C)$ by means of the Cartesian square

$$\begin{array}{ccc} \breve{C} & \rightarrow & \mathrm{Pic}^0(C) \\ \downarrow & & \downarrow m_l \\ C & \xrightarrow{i_0} & \mathrm{Pic}^0(C). \end{array}$$

Similarly define two fiber products over $\mathrm{Pic}^0(\bar{C})$:

$$\tilde{C} = \bar{C} \times_{(\bar{i}_0, \bar{m}_l)} \mathrm{Pic}^0(\bar{C}), \qquad \hat{C} = C \times_{(\bar{i}_0 \gamma, \bar{m}_l)} \mathrm{Pic}^0(\bar{C}).$$

These curves are geometrically integral and fit into a commutative diagram with obvious maps :

$$\begin{array}{ccccc} \breve{C} & \rightarrow & \hat{C} & \rightarrow & \tilde{C} & & \\ & & \downarrow & & \downarrow & & \\ & & C & \rightarrow & \bar{C} & \xrightarrow{\bar{p}} & X. \end{array}$$

LEMMA 9.2. – *$\tilde{C}_k$ is Galois over $\bar{C}_k$ and $Gal(\tilde{C}_k/\bar{C}_k) \simeq (1+\xi+...+\xi^{e-1})_* \mathrm{Pic}(C)(k)[l]$.*

Proof. – Since $\gcd(e, l) = 1$, $\mathrm{Pic}^0(\bar{C})[l] \xrightarrow{\gamma^*} (1+\xi+...+\xi^{e-1})_* \mathrm{Pic}(C)[l]$ is an isomorphism. The assumption (2) of (6.7) implies that G_k acts trivially on $\mathrm{Pic}(C_{\bar{k}})[l]$. Thus the group scheme $\mathrm{Pic}^0(\bar{C})[l]$ consists entirely of k-rational points. An isomorphism $\mathrm{Pic}^0(\bar{C})(k)[l] \simeq Gal(\tilde{C}/\bar{C})$ is obtained by letting the torsion points act by translation on $\mathrm{Pic}^0(\bar{C})$.

LEMMA 9.3. – *$\bar{k}(\breve{C})$ is Galois over $\bar{k}(X)$.*

Proof. – Indeed $\bar{k}(\breve{C})$ is the maximal, unramified, abelian, exponent l extension of $\bar{k}(C)$. Since $\bar{k}(C)$ is Galois over $\bar{k}(X)$, so is $\bar{k}(\breve{C})$.

By the preceeding lemma we may replace the field k by a finite extension, again denoted by k, such that every element of $Gal(\breve{C}_{\bar{k}}/X_{\bar{k}})$ is defined over k. We make this extension, so $\breve{C}_k$ is now Galois over X_k.

LEMMA 9.4. – $(1+\xi+...+\xi^{e-1})_* P \not\subset \mathrm{rad}(P)$.

Proof. – Recall that P is the projective envelope of M. Thus $\mathrm{rad}(P)$ is the kernel of any essential map $P \rightarrow M$ [Se1, 14.3]. It suffices to show that $(1+\xi+...+\xi^{e-1})_* M \neq 0$. Let $\tilde{\xi} \in SL(2, \mathbf{F}_l)$ map to ξ. In the tautological two dimensional representation of $SL(2, \mathbf{F}_l)$, the transformation $\tilde{\xi}$ has two eigenvalues whose product is one. Clearly, in the second symmetric power representation, M, ξ will have 1 as an eigenvalue with multiplicity one. From $\gcd(e, l) = 1$, one deduces easily that $(1+\xi+...+\xi^{e-1})_* M \simeq M^{\langle \xi \rangle} \simeq \mathbf{F}_l$.

By (9.4) there is an element $p \in (1+\xi+...+\xi^{e-1})_* P$ which is not contained in $\mathrm{rad}(P)$. Use the inclusion,

$$(1+\xi+...+\xi^{e-1})_* P \subset (1+\xi+...+\xi^{e-1})_* H^1(C_{\bar{k}}, \mu_l) \underset{\gamma^*}{\xleftarrow{\sim}} Gal(\tilde{C}_k/\bar{C}_k),$$

to specify an element

$$(\xi, (\gamma^*)^{-1} p) \in Gal(C_k/\bar{C}_k) \times Gal(\tilde{C}_k/\bar{C}_k) \simeq Gal(\hat{C}_k/\bar{C}_k).$$

Choose $f \in Gal(\breve{C}_k/\bar{C}_k)$ which maps to $(\xi, (\gamma^*)^{-1}p)$. Make the following

ASSUMPTION 9.5. – There is $x \in \dot{X}(k)$ whose Frobenius conjugacy class in $Gal(\breve{C}_k/X_k)$ contains f.

Thus $\langle f \rangle$ is the decomposition group of a closed point of $\breve{C}$ in the fiber over x. Since $f \in Gal(\breve{C}_k/\bar{C}_k)$ this closed point lies over a k-rational point $\bar{c} \in \bar{C}$. Thus $\mathfrak{d} := \gamma^*(\bar{c} - \bar{c}_0) \in H^2_{\Sigma'_C}(C_{\bar{k}}, \mu_l)_0^{G_k}$.

LEMMA 9.6. – *The image of the homomorphism* $\delta_{(6.6)}(\mathfrak{d}) \in \mathrm{Hom}(G_k, H^1(C_{\bar{k}}, \mu_l))$ *is contained in* P*, but not in* $\mathrm{rad}(P)$.

Proof. – Consider the analog of (6.6) with $\bar{C}$ replacing C, $\bar{\rho}^{-1}(\{x\} \cup \Sigma_X)$ playing the role of Σ'_C, and $\bar{C} - \bar{\rho}^{-1}(\{x\} \cup \Sigma_X)$ replacing C'. Write $\bar{\delta}$ for the corresponding coboundary map. Since k is finite, G_k is a topologically cyclic group generated by the Frobenius, ϕ. There is an isomorphism,

$$\mathrm{Hom}(G_k, H^1(\bar{C}_{\bar{k}}, \mu_l)) \to H^1(\bar{C}_{\bar{k}}, \mu_l) \to Gal(k(\tilde{C})/k(\bar{C})), \qquad F \to F(\phi).$$

Under this isomorphism, $-\bar{\delta}(\bar{c} - \bar{c}_0)$ maps to $Frob_{\bar{c}} = (\gamma^*)^{-1}p$ (1.14). By functoriality of the cycle class map,

$$\delta_{(6.6)}(\gamma^*(\bar{c} - \bar{c}_0)) = -p \in \gamma^* H^1(\bar{C}_{\bar{k}}, \mu_l) = (1 + \xi + ... + \xi^{e-1})_* H^1(C_{\bar{k}}, \mu_l).$$

Now $-p \in (1 + \xi + ... + \xi^{e-1})_* P$, but $-p \notin \mathrm{rad}(P)$. This proves the lemma.

COROLLARY 9.7. – *Under the assumption (9.5) the coboundary map* $\delta_{(6.4)}$ *is injective.*

Proof. – By (9.6) and (6.7) the coboundary map $\delta_{(6.4)}$ is not zero. It remains only to show $H^2_x(\dot{X}_{\bar{k}}, \mathcal{M})^{G_k}$ is one dimensional. In fact there is a sequence of isomorphisms :

$$(9.8) \qquad H^2_x(\dot{X}_{\bar{k}}, \mathcal{M})^{G_k} \simeq (H^2_{\rho^{-1}(x)}(\dot{C}_{\bar{k}}, \mu_l) \otimes_{\mathbf{Z}} M)^{\Gamma \times G_k} \simeq M^{\langle \xi \rangle} \simeq \mathbf{F}_l.$$

The first isomorphism comes from (8.1), the second from (7.17), and the third from the proof of (9.4).

In order to prove (6.8) we need only show that if k is replaced by an appropriate extension field of finite degree, then there exists $x \in \dot{X}(k), x \notin S(k)$ such that assumption (9.5) holds. This is a consequence of the following strong form of the Tchebotarev density theorem.

PROPOSITION 9.9. – *Let* $\mathbf{F} \subset Gal(\breve{C}_k/X_k)$ *be a non-empty conjugacy class. Then given any integer* I *one has*

$$\#\{v \in X(k') : Frob_v = \mathbf{F}\} > I$$

for all finite extensions k'/k *of sufficiently large degree.*

Proof. – This may be derived from the Riemann hypothesis for Artin L-functions for function fields of transcendence degree 1 over a finite field as proved by Weil (*cf.* [La1]).

10. The cycle class map evaluated at certain complex multplication cycles

In this section we prove (6.9). Fix a finite field, k. Let $\pi : Y \to X$ denote a semi-stable elliptic surface with section defined over k and write W for the blow up of $Y \times_X Y$ along the singular locus. In (5.5) we associated to π an integer m_π. Let l be a prime, distinct from the characteristic of k, satisfying $\gcd(l, 2m_\pi) = 1$.

PROPOSITION 10.1. – *There exist a finite extension k'/k, a point $x \in \dot{X}(k')$, a complex multiplication cycle $z_x \in CH^2(W_{\bar{k}})_{hom}$ such that $\alpha^2(z_x) \in J_l^2(W)$ is not zero.*

Proof. – Let M denote the stalk of the sheaf $(Sym^2 R^1\dot{\pi}_*\mathbf{Z}/l)(2)$ at a geometric generic point $\bar{\eta}$ of $\dot{X}$. We shall check that the hypotheses of (6.8) are fulfilled in this situation. First observe that the image of the geometric monodromy representation

$$\pi_1^t(\dot{X}_{\bar{k}}, \bar{\eta}) \to \mathrm{Aut}(M)$$

is isomorphic to $SL(2, \mathbf{F}_l)/\pm 1$ by (5.6). Thus by replacing k by a finite extension field if necessary we may identify the image, Γ, of the arithmetic monodromy representation

$$\pi_1^t(\dot{X}_k, \bar{\eta}) \to \mathrm{Aut}(M) \tag{10.2}$$

with $SL(2, \mathbf{F}_l)/\pm 1$. Now hypotheses (1), (2), (3) of (6.8) are satisfied.

Prior to checking that hypothesis (4) in (6.8) holds in the present situation, we must recall that the homomorphism (10.2) gives rise to a finite Galois morphism of curves $\rho : C \to X$ over k with Galois group Γ. Again by replacing k by a finite extension field if necessary we may assume that $\mu_l \subset k^*$. In the proof of (5.6) C was identified with the normalization of $X \times_{\mathbf{P}^1} X(l)$, where $X(l)$ is the compactification of the modular curve which parametrizes elliptic curves with a symplectic level l-structrue and X maps to $\mathbf{P}^1$ via the j-invariant $\mathbf{J} : X \to \mathbf{P}^1$ associated to the family of elliptic curves π. We need to investigate the ramification of the cover $B\backslash C \to X$ where $B \subset \Gamma$ is a Borel subgroup. Now all Borel subgroups in Γ are conjugate. Using the classical theory of modular curves one checks easily that $B\backslash X(l)$ has a ramification point of index l over the cusp at ∞ on the j-line $\mathbf{P}^1$. Since $\gcd(l, 2m_\pi) = 1$, the ramification of $\mathbf{J}$ over ∞ has order prime to l. Thus $B\backslash C \to X$ will be ramified over each point in $\mathbf{J}^{-1}(\infty)(\bar{k})$.

To verify hypothesis (4) in (6.8) we need only check that $\mathbf{J}^{-1}(\infty)(\bar{k})$ consists of at least four points when X has genus zero. This is an immediate consequence of the following lemma (*cf.* [Be]).

LEMMA 10.3. – *A non-isotrivial, semi-stable elliptic surface with section, $\pi : Y_{\bar{k}} \to \mathbf{P}^1_{\bar{k}}$, has at least four singular fibers.*

Proof. – $H^1(Y_{\bar{k}}, \mathbf{Z}/l^n) = 0$ $\forall n$ by (5.6). To compute the l-adic euler characteristic of $Y_{\bar{k}}$ it suffices to compute

$$e(Y_{\bar{k}}) = \sum_{i \geq 0} (-1)^i \ dim_{\mathbf{F}_l} H^i(Y_{\bar{k}}, \mathbf{Z}/l).$$

Since the representation of $\pi_1(\dot{X}_{\bar{k}}, \bar{\eta})$ on the l-torsion subgroup of the geometric generic fiber is tamely ramified [Ogg, §II]

$$e(Y_{\bar{k}}) = \sum_{i=1}^{s} c_i$$

where s is the number of singular fibers and c_i is the number of irreducible components in the i^{th} singular fiber [Mi, V2.12]. It is easy to get a lower bound for the rank, r, of the Néron-Severi group in terms of the numbers c_i. Namely,

$$r \geq 2 + \sum_{i=1}^{s} (c_i - 1).$$

Since $e(Y_{\bar{k}}) - 2 = h^2(Y_{\bar{k}}) \geq r$, we conclude that $s \geq 4$ as desired.

Now let $S \subset \dot{X}$ denote the set of closed points, x, for which $\pi^{-1}(x)$ is not CM. This set is finite because $\mathbf{J}$ is non-constant and there are only finitely many supersingular j-invariants in $\bar{k}$ [Sil, V.3.1(iii)]. By (6.8) there is a finite extension field k' of k and a point $x \in \dot{X}(k'), x \notin S(k')$ such that

$$\delta_{(6.4)} : H^2_x(\dot{X}_{\bar{k}}, \mathcal{M})^{G_{k'}} \to H^1(G_{k'}, H^1(\dot{X}_{\bar{k}}, \mathcal{M}))$$

is injective. By (5.2) the class of the complex multiplication cycle

$$cl^1_{p^{-1}(x)}(z_x) \in (Sym^2 H^1(\pi^{-1}(x)_{\bar{k}}, \mathbf{Z}/l)(1))^{G_k} \simeq H^2_x(\dot{X}_{\bar{k}}, \mathcal{M})^{G_k}$$

is not zero. Now $\delta_{(6.4)}$ coincides with $\delta_{(5.11)}$ when $n = 1$. Thus $\delta_{(5.11)} \circ cl^1_{p^{-1}(x)}(z_x) \neq 0$. By (5.12) $\alpha^2(z_x) \in J^2_l(W)$ is not zero. This proves (10.1).

We may strengthen (10.1) as follows :

PROPOSITION 10.4. – *The image of* $\alpha^2 : CH^2(W_{\bar{k}})_{hom} \to J^2_l(W)$ *is isomorphic to a group of the form* $(\mathbf{Q}_l/\mathbf{Z}_l)^a \oplus \oplus_{i=1}^{b} (\mathbf{Z}/l^{n_i})$ *where* $b < \infty$ *and* $1 \leq a < \infty$.

Proof. – Write $H = H^3(W_{\bar{k}}, \mathbf{Z}_l(2))/tors$. By (1.4)

$$(10.5) \qquad J^2_l(W) \simeq lim_{k'/k} H^1(G_{k'}, H) \simeq lim_{k'/k}(H \otimes \mathbf{Q}_l/\mathbf{Z}_l)^{G_{k'}} \simeq H \otimes \mathbf{Q}_l/\mathbf{Z}_l.$$

The proposition will follow if the image of α^2 is infinite. To establish this it will suffice to construct for any finite extension field k_1/k a complex multiplication cycle whose image under α^2 is not contained in the subgroup $J^2_l(W)^{G_{k_1}}$.

Fix k_1. Since G_{k_1} is a topologically cyclic group, there is a finite extension field k_2/k_1 with the property that the natural map

$$H^1(G_{k'/k_1}, (H/l)^{G_{k'}}) \to H^1(G_{k_1}, H/l)$$

is surjective for any finite extension k'/k_2. Thus the inflation-restriction sequence implies that the map b in the commutative diagram

$$(10.6) \qquad \begin{array}{ccc} H^1(G_{k_1}, H/l) & \xrightarrow[b]{} & H^1(G_{k'}, H/l) \\ \uparrow & & \uparrow c \\ H^1(G_{k_1}, H) & \to & H^1(G_{k'}, H) \end{array}$$

is zero. Now apply (10.1) with k_2 playing the role of the base field. We find that there is a finite extension k'/k_2 and a point $x \in \dot{X}(k'), x \notin S(k')$ such that

$$(10.7) \quad \delta_{(5.10)} \circ cl^1_{p^{-1}(x)}(z_x) = \delta_{(5.10)} \circ cl^1_{p^{-1}(x)}(P_* z_x) \in H^1(G_{k'}, H^1(X_{\bar{k}}, j_* \mathcal{M}_1)) \simeq H^1(G_{k'}, P_* H/l)$$

is not zero. But the cycle class in (10.7) is the image of

$$\alpha^2(z_x) \in H^1(G_{k'}, H) \subset J^2_l(W)$$

under the map $c \circ P_*$. Suppose now that $\alpha^2(z_x)$ were contained in $J^2_l(W)^{G_{k_1}}$. By (10.5) $J^2_l(W)^{G_{k_1}} \simeq H^1(G_{k_1}, H)$. This forces $\alpha^2(z_x)$ to live in the lower left hand corner of (10.6). This means that $c \circ P_* \alpha^2(z_x)$ would be in the image of b which is not the case. Thus $\alpha^2(z_x) \notin J^2_l(W)^{G_{k_1}}$.

Let $Z_{CM}(W)$ denote the free abelian group on complex multiplication cycles.

COROLLARY 10.8. – *Suppose that $P_* H \otimes \mathbf{Q}_l$ is an irreducible representation of G_k. Then the arithmetic Abel-Jacobi map $\alpha^2 : CH^2(W_{\bar{k}})_{hom} \to J^2_l(W)$ is surjective.*

Proof. – The Tate module of $\alpha^2(Z_{CM}(W_{\bar{k}}))$ tensored with $\mathbf{Q}_l$ will be denoted V_l. It is a non-zero subrepresentation of $P_* H \otimes \mathbf{Q}_l$. Thus $V_l = P_* H \otimes \mathbf{Q}_l$. It follows formally that the inclusion of groups

$$\alpha^2(Z_{CM}(W_{\bar{k}})) \subset P_* J^2_l(W)$$

is an isomorphism [Su, 1.3].

To complete the proof of the corollary we need to produce a subgroup of $CH^2(W_{\bar{k}})_{hom}$ which α^2 maps surjectively to $(1-P_*)J^2_l(W)$. For this, consider the sum, $\mathcal{S}$, of the images of $CH^1(X_{\bar{k}})_{hom}$ in $CH^2(W_{\bar{k}})_{hom}$ under the correspondences $\sigma^* f_* \pi^*$:

$$\begin{array}{ccccc} Y & \xrightarrow{f} & Y \times_X Y & \xleftarrow{\sigma} & W \\ \downarrow \pi & & & & \\ X & & & & \end{array}$$

where f is the diagonal map (respectively $(id_Y, s \circ \pi)$) (respectively $(s \circ \pi, id_Y)$) and $s : X \to Y$ is a section of π. It is not difficult to deduce from the decomposition of $H^3(W_{\bar{k}}, \mathbf{Z}_l(2))/tors$ *via* the Leray spectral sequence as described in (5.9) and the proof of (5.12) that $\alpha^2(\mathcal{S}) = (1 - P_*)J^2_l(W)$. As this result will not be used in the sequel, details are left to the reader (compare also [Sch-CM, 1.7]).

COROLLARY 10.9. – *Write $\phi \in G_k$ for the Frobenius element. Suppose that no eigenvalue of ϕ^{-1} acting on $P_* H \otimes \mathbf{Q}_l(-1)$ is an algebraic integer. Then the Griffiths group $Gr^2(W_{\bar{k}})$ is not finitely generated.*

Proof. – By (10.4) there is a surjective map $P_* CH^2(W_{\bar{k}})_{hom} \xrightarrow{a} \mathbf{Q}_l/\mathbf{Z}_l$. By (2.7) $P_* CH^2(W_{\bar{k}})_{alg} \otimes \mathbf{Z}_l = 0$. Thus the torsion group $P_* CH^2(W_{\bar{k}})_{alg}$ is in Ker(a). The corollary follows.

It seems reasonable to guess that the hypothesis made in (10.9) concerning the eigenvalues of the ϕ^{-1} action holds rather frequently. Specific, although quite special examples are given in §12.

11. The action of Hecke correspondences on complex multiplication cycles

Presently it seems difficult to improve upon (10.4) without making further assumptions about the semi-stable elliptic surface $\pi : Y \longrightarrow X$. In this section we shall be concerned with certain special elliptic surfaces, so-called elliptic modular surfaces, which admit Hecke correspondences. After defining these notions we will study the action of the Hecke correspondences on complex multiplication cycles. These computations will be used in the next section to obtain further information about $\alpha^2(Z_{CM}(W_{\bar{k}}))$.

Let k be a field of characteristic $p \geq 0$. Fix a positive integer N, prime to the characteristic. Let F be a functor from affine k-varieties to sets, which associates to each affine variety the set of isomorphism classes of elliptic curves over the given affine variety with a certain prescribed sort of level N-structure. We require that F is representable by a smooth, geometrically integral, affine curve over k. Furthermore the projective, relatively minimal model of the universal elliptic curve is allowed only Kodaira type I_m singular fibers with $m|N$.

Examples. – (1) $N > 4$. $F(S) =$ isomorphism classes of elliptic curves over S with a section of exact order N.

(2) $N > 2$ and k contains the N^{th} roots of unity. $F(S) =$ isomorphism classes of elliptic curves over S with a symplectic level N structure.

(3) Further examples may be found in [Be2].

If F is taken from the above list, we refer to the non-singular, projective, relatively minimal model of the universal elliptic curve as an elliptic modular surface.

Fix an elliptic modular surface $\pi : Y \longrightarrow X$. The notations W, $\dot{X}$, etc. will continue to have the same meaning as in the previous section. For each prime l which does not divide pN, any iterated fiber product of Y with itself admits a Hecke correspondence, T_l [D] [Sch-CM, 1.13]. We describe this explicitly in the case of the threefold W. The normalization of the fiber product over the j-line, $\dot{X} \times_{\mathbf{P}^1} X_0(l)$, will be denoted $\ddot{X}$. Pulling back $\dot{Y}$ (respectively $\dot{W} = \dot{Y} \times_{\dot{X}} \dot{Y}$) by the obvious map, $h : \ddot{X} \rightarrow \dot{X}$ yields $\ddot{Y}$ (respectively $\ddot{W} = \ddot{Y} \times_{\ddot{X}} \ddot{Y}$). $\ddot{Y}$ contains a distinguished subgroup scheme Λ, of order l. Set $Y' = \ddot{Y}/\Lambda$ and write $f : \ddot{W} \rightarrow W'$ for the degree l^2 isogeny obtained by modding out by $\Lambda \times_{\ddot{X}} \Lambda$. Furthermore Y' inherits a level structure from $\ddot{Y}$ since l is prime to the level of Y. Because the functor with which we are dealing is representable the level structure on $\ddot{\pi} : \ddot{Y} \rightarrow \ddot{X}$ induces a unique fiber square

$$\begin{array}{ccc} W' & \xrightarrow{\tilde{g}} & \dot{W} \\ \downarrow & & \downarrow \\ \ddot{X} & \xrightarrow{g} & \dot{X} \end{array}$$

Both the correspondence $(T_l^X)_* := g_* \circ h^*$ and its transpose $(T_l^X)^* := h_* \circ g^*$ associate to each geometric point of $\dot{X}$ a cycle on $\dot{X}$ consisting of $l + 1$ geometric points. The effect of $(T_l)_* := (T_l^W)_* := \tilde{g}_* \circ f_* \circ \tilde{h}^*$ on a geometric fiber $\dot{p}^{-1}(x)$ is to mod out distinct copies of this fiber by each of the $l+1$ subgroups $\lambda \times \lambda$ where $\lambda \subset \pi^{-1}(x)$ is an order l subgroup. These correspondences are rational over the base field and extend to give correspondences on the compactifications X and W.

In order to describe the action of the Hecke correspondences on the complex multiplication cycles, we need a precise understanding of the behaviour of such cycles under isogeny. Let $f : E \to E'$ be an isogeny between CM elliptic curves. The induced maps

$$(f \times f)_* : N^1(E^2) \to N^1((E')^2) \qquad \text{and} \qquad (f \times f)^* : N^1((E')^2) \to N^1(E^2)$$

respect the subgroups generated by complex multiplication cycles. Fix generaters $z_E \in N^1_{CM}(E^2)$ and $z_{E'} \in N^1_{CM}((E')^2)$ so that $(f \times f)_* z_E = c z_{E'}$ with $c > 0$.

LEMMA 11.1. – *If* $g : E \to E'$ *is a second isogeny, then* $(g \times g)_* z_E = c' z_{E'}$, *with* $c' > 0$.

Proof. – The isogeny dual to f, $f' : E' \to E$ has the property that $f' \circ f \in End(E)$ is multiplication by the degree of f. Thus $(f' \times f')_* z_{E'}$ is a positive multiple of z_E. Set $h = f' \circ g \in End(E)$. We need only check that $(h \times h)_* z_E$ is a positive multiple of z_E. But $(h \times h)_*$ acts on $N^1(E^2)$ by scalar multplication by the degree of h.

The lemma permits us to choose a coherent system of complex multiplication cycles in each isogeny class of elliptic curves. In the remainder of this section and in the next section z_E will denote a generater of $N^1_{CM}(E^2)$. If $f : E \to E'$ is an isogeny, then the generater $z_{E'}$ of $N^1_{CM}((E')^2)$ will always be chosen so that $(f \times f)_* z_E$ is a positive multiple of $z_{E'}$. If $E = \pi^{-1}(x)$ we generally write z_x for $z_{\pi^{-1}(x)}$.

LEMMA 11.2. – *Let* $f : E \to E'$ *be an isogeny of CM elliptic curves. Define* $R = End(E)$, $R' = End(E')$, $D = disc(R)$, *and* $D' = disc(R')$. *Then,*

(1) *For any* $z \in N^1(E \times E)$, $(f \times f)_* z \cdot (f \times f)_* z = (\text{deg}.f)^2 z \cdot z$,

(2) $z_E \cdot z_E = D \cdot \epsilon(D)^2/2$,

(3) $(f \times f)_*(z_E) = c z_{E'}$ *with* $c = (\text{deg}.f)(\sqrt{D/D'})\epsilon(D)/\epsilon(D')$

Proof. – (1) It suffices to verify the identity when z is the pullback *via* $(f \times f)^*$ of an element in $N^1(E' \times E')$, in which case it is clear.

(2) One uses the explicit formula for z_E (5.0). Choose $\nu \in R$ such that $R = \mathbf{Z}[\nu]$, and compute $(\Gamma_\nu - \Gamma_{\bar{\nu}})^2 = -2\Gamma_\nu \cdot \Gamma_{\bar{\nu}}$. The isomorphism $pr_1 : \Gamma_\nu \to E$ identifies the intersection $\Gamma_\nu \cap \Gamma_{\bar{\nu}}$ with the kernel of $(\nu - \bar{\nu}) \in R$. The degree of this map is given by the determinant of the action on the Tate module. In terms of norms this is

$$N_{R/\mathbf{Z}}(\nu - \bar{\nu}) = 4N_{R/\mathbf{Z}}\nu - (tr_{R/\mathbf{Z}}\nu)^2 = -D.$$

This number is prime to p since E is not supersingular [Wa, Thm. 4.1].

(3) This is an immediate consequence of (1) and (2).

Fix a prime l distinct from the characteristic of the base field. The following lemma computes D' in terms of D when f is an order l isogeny. This gives a precise formula for the constant c.

LEMMA 11.3. – *If* $deg.f = l$ *then,*

(1) *If* $Ker(f)$ *is not an* R-*module, then* $D' = l^2 D$,

(2) *If Ker(f) is an R-module and l^2 does not divide D, then $D' = D$,*
(3) *If Ker(f) is an R-module and l^2 divides D, then $D' = l^{-2}D$.*

Proof. – Let $\hat{R} \subset R$ denote the order of relative conductor l. As $\hat{R}$ induces an endomorphism of the pair $(E, \text{Ker}f)$, there is an inclusion $\hat{R} \subset R'$. Furthermore one checks easily that the orders R and R' differ only at places over l. Recall that orders in quadratic fields are uniquely determined by their discriminant.

(1) In this case $R \not\subset R'$, whence $\hat{R} = R'$.

(2) In this case $R \subset R'$, whence D/D' is a non-negative even power of l, which is necessarily zero since l^2 does not divide D.

(3) Again $R \subset R'$ and D/D' is a non-negative even power of l. There is an algebraic integer γ such that $R \simeq \mathbf{Z}[l\gamma]$ and $\mathfrak{m} = (l, l\gamma)$ is the maximal ideal in R over l. We need to show that $R' \simeq \mathbf{Z}[\gamma]$. To establish that $\gamma \in R'$ it suffices to show that $l\gamma$ annihilates $E'[l]$. This will follow if $l\gamma \cdot f^{-1}(E'[l]) \subset \text{Ker}(f)$. We have

$$\text{Ker}(f) \subset f^{-1}(E'[l]) \subset E[l^2]$$

and the right hand term may be identified with R/l^2, since $\mu \in R$ annihilates $E[l^2]$ if and only if $l^2|\mu$. Now R/l^2 is a local ring and $f^{-1}(E'[l])$ is an index l, R/l^2-submodule. Thus $f^{-1}(E'[l])$ gets identified with the maximal ideal, $(l, l\gamma)$. It is easy to see that R/l^2 has a unique non-zero simple submodule. This leads to an identification of $\text{Ker}(f)$ with $(l^2\gamma)R/l^2$. It is now a simple computation using the fact that γ satisfies an equation of integral dependence, $\gamma^2 + b\gamma + d = 0$ with $b, d \in \mathbf{Z}$, to show that $l\gamma \cdot f^{-1}(E'[l]) = \text{Ker}(f)$. This shows $\mathbf{Z}[\gamma] \subset R'$. A consequence is that $f(E[l])$ is not a R' module. Indeed, if it were, then every element of R' would induce an endomorphism of E simply by modding E' out by $f(E[l])$. This is impossible since R' is not isomorphic to a subring of R. Now (3) follows by applying (1) to the isogeny $f' : E' \to E$ dual to f which has kernel $f(E[l])$. It also follows that $R' = \mathbf{Z}[\gamma]$.

For $x, x' \in \dot{X}(\bar{k})$ write $E = \pi^{-1}(x)$ and $E' = \pi^{-1}(x')$. If these elliptic curves are not supersingular, write D, D' for the discriminants of the respective endomorphism rings. Set $\mathcal{T}_i(x) = \{x' \in (T_l^X)_*x : D' = l^{2i}D\}$. As divisors on X, $(T_l^X)_*x = \sum_{-1\leq i\leq 1} \mathcal{T}_i(x)$. Similarly when considering the transposed correspondence, write $(T_l^X)^*x = \sum_{-1\leq i\leq 1} \mathcal{U}_i(x)$, where $\mathcal{U}_i(x) = \{x' \in (T_l^X)^*x : D' = l^{2i}D\}$.

COROLLARY 11.4. – *If l is an odd prime, or if $l = 2$ and either 2 ramifies in the fraction field of R or 4 divides the conductor of R, then*

$$(T_l)_*z_x = \sum_{-1\leq i\leq 1} \sum_{x' \in \mathcal{T}_i(x)} l^{1-i} z_{x'}.$$

Proof. – For arbitrary $x' \in (T_l^X)_*(x)$ the hypotheses on l imply that $\epsilon(D)/\epsilon(D') = 1$. The Corollary now follows immediately from the description of the action of $(T_l)_*$ on the fiber $p^{-1}(x)$ and the previous two lemmas. Furthermore it is evident how to write down the formula for $(T_2)_*z_x$ in complete generality. As this is not needed, it is left to the reader.

We now consider the effect of the transposed correspondence $(T_l)^* := \tilde{h}_* \circ f^* \circ \tilde{g}^*$.

COROLLARY 11.5. – *With hypotheses as in (11.4),*

$$(T_l)^* z_x = \sum_{-1\leq i\leq 1} \sum_{x'\in \mathcal{U}_i(x)} l^{1-i} z_{x'}.$$

Proof. – For $x' \in (T_l^X)^*(x)$ write $E = \pi^{-1}(x), E' = \pi^{-1}(x')$. The Hecke correspondence gives a degree l isogeny $f' : E' \to E$. When $x' \in \mathcal{U}_i(x)$, it must be shown that the positive integer c' which satisfies $c' z_{E'} = (f' \times f')^* z_E$ is l^{1-i}. By the projection formula $(f' \times f')_* z_{E'} = (l^2/c') z_E$. Let $f : E \to E'$ be such that $f \circ f'$ is multiplication by l. From (11.2) and (11.3), $(f \times f)_* z_E = l^{1-i} z_{E'}$. As multiplication by l acts by the scalar l^2 on $H_2(E' \times E', \mathbf{Z}_l)$,

$$l^2 z_{E'} = (f\circ f' \times f \circ f')_* z_{E'} = (l^2/c') l^{1-i} z_{E'},$$

whence $c' = l^{1-i}$.

Remark 11.6. – We intend to apply (11.5) in the following particular case. Consider $x \in \dot{X}(k)$, with $\pi^{-1}(x)$ a CM elliptic curve such that l^2 divides the discriminant D of $End(\pi^{-1}(x))$. As $E[l] \simeq R/l$, the action of $Gal(\bar{k}/k)$ on the order l subgroups is described by a homomorphism,

$$Gal(\bar{k}/k) \to (R/l)^*/(\mathbf{Z}/l)^* \simeq \mathbf{Z}/l.$$

Now the maximal ideal of R/l is identified with the distinguished order l subgroup which is R-stable. This is defined over k. If some order l subgroup is not k-rational, then at least it is defined over some $\mathbf{Z}/l$-extension k'/k. In this case $Gal(k'/k)$ cyclically permutes the non-distinguished subgroups and (11.5) yields

$$(T_l)^* z_x = \sum_{\sigma \in Gal(k'/k)} z_{\sigma x'} + l^2 z_{\hat{x}}.$$

where $x' \in \mathcal{U}_1(x)$ and $\{\hat{x}\} = \mathcal{U}_{-1}(x)$. Since T_l^W is defined over k, we may in fact write

(11.7) $$(T_l)^* z_x = \sum_{\sigma \in Gal(k'/k)} (z_{x'})^\sigma + l^2 z_{\hat{x}}.$$

An analogous formula holds for the transposed correspondence.

12. A divisibility result

Let p be a prime and k a finite field of characteristic p. As in the previous section W/k denotes an elliptic modular threefold of level N prime to p. Fix a prime number l, with $\gcd(l, Np) = 1$. Recall that $Z_{CM}(W_{\bar{k}})$ denotes the free abelian group on the complex multiplication cycles. Let $Q \in CH^3(W \times W) \otimes \mathbf{Z}_l$ be a correspondence which commutes with the Hecke operator T_l^*.

THEOREM 12.1 . – *Suppose that the l^{th}-Hecke operator T_l^* acts invertibly on $Q_* J_l^2(W_k)$. Then the cycle class map*

$$Q_* \circ \alpha^2 : Z_{CM}(W_{\bar{k}}) \to J_l^2(W)$$

has divisible image.

Proof. – Write $\mathcal{N}$ for the image of $Q_* \circ \alpha^2$. We need the following elementary facts about abelian groups.

LEMMA 12.2. – *$\mathcal{N}$ is a direct sum of a finite abelian group with a group of the form $(\mathbf{Q}_l/\mathbf{Z}_l)^r$. Furthermore T_l^* restricts to an automorphism of $\mathcal{N}$.*

Proof. – By (1.4) $J_l^2(W)$ is isomorphic to a finite direct sum of $(\mathbf{Q}_l/\mathbf{Z}_l)$'s. Now the two assertions of the lemma follow by applying Pontrjagin duality to the following well known facts about finitely generated $\mathbf{Z}_l$-modules :

(1) A homomorphic image of $\mathbf{Z}_l^n$ is isomorphic to a finite rank free module plus a finite group.

(2) A surjective endomorphism of a finitely generated $\mathbf{Z}_l$-module is an isomorphism.

Now T_l^* induces an automorphism of the finite dimensional $\mathbf{F}_l$-vector space $\mathcal{N}/l\mathcal{N}$. We need the explicit formulas for the action of T_l^* on complex multiplication cycles to argue that $\mathcal{N}/l\mathcal{N} = 0$.

Write $\bar{\alpha} : Z_{CM}(W_{\bar{k}}) \to \mathcal{N}/l\mathcal{N}$ for the map induces by $Q_* \circ \alpha^2$. Suppose that there is a CM cycle z_x for which $\bar{\alpha}(z_x) \neq 0$. By replacing the original base field with a finite extension (also denoted k), we may assume that $x \in X(k)$ and that the absolute Galois group G_k acts trivially on $\mathcal{N}/l\mathcal{N}$. Write $\mathbf{c}$ for the conductor of the order $End(\pi^{-1}(x))$ in the integers of its fraction field. Then by (11.5)

$$\gcd(l, \mathbf{c}) = 1 \text{ implies} \qquad T_l^* z_x = \sum_{y \in \mathcal{U}_1(x)} z_y + l \sum_{u \in \mathcal{U}_0(x)} z_u$$

$$\gcd(l, \mathbf{c}) = l \text{ implies} \qquad T_l^* z_x = \sum_{y \in \mathcal{U}_1(x)} z_y + l^2 \sum_{v \in \mathcal{U}_{-1}(x)} z_v.$$

Starting with $x_0 := x$ choose inductively $x_j \in \mathcal{U}_1(x_{j-1})$ so that $\bar{\alpha}(z_{x_j}) \neq 0$ for all $j \geq 0$. The points $\{x_0, x_1, x_2, ...\} \subset X(\bar{k})$ are distinct since the endomorphism rings of the elliptic curves $\pi^{-1}(x_j)$ have strictly increasing conductors. Choose i minimal so that $x_i \notin X(k)$. As $i \geq 1$ the second formula above applies. By (11.6) $Gal(k(x_i)/k) \simeq \mathbf{Z}/l$ and

$$T_l^* z_{x_{i-1}} = \sum_{\sigma \in Gal(k(x_i)/k)} (z_{x_i})^\sigma \quad + \ l^2 z_v.$$

Since G_k acts trivially on $\mathcal{N}/l\mathcal{N}$, $\bar{\alpha}$ sends the right hand side to zero. As T_l^* acts invertibly, this contradicts the assumption that $\bar{\alpha}(z_{x_{i-1}}) \neq 0$ and proves the theorem.

Remark 12.3. – Take for $Q \in Z^3(W \times W)$ the diagonal correspondence which induces the identity on Chow groups and cohomology. Assume $\gcd(l, 2Np) = 1$. Now (10.1) implies that $\alpha^2 : Z_{CM}(W_{\bar{k}}) \to J_l^2(W_k)$ is not zero. Thus (12.1) implies that the image is a non-zero divisible group.

Remark 12.4. – The simplest examples to which (12.1) and (12.3) apply are those for which $H^3(W_{\bar{k}}, \mathbf{Q}_l)$ is two dimensional. There are exactly six semi-stable elliptic modular surfaces which give rise to such threefolds [Be2] and [Sch-FP, §4 and §7]. These have levels 3, 4, 5, 6, 8, and 9. In all cases the varieties are defined over $\mathbf{Q}$ and have good reduction at primes which do not divide the level, N. Write $\Gamma \subset SL(2, \mathbf{Z})$ for the modular group associated to $W_{\mathbf{C}}$ [Be] or [Sch-FP, §4]. In each of the six cases the space of weight 4 cusp forms $S_4(\Gamma)$ is one dimensional and contains a unique element, f, whose Fourier expansion at the cusp $i\infty$, $f = \sum_{n\geq 1} b_n q^n$, has $b_1 = 1$.

LEMMA 12.5. – *Let W be one of the six elliptic modular threefolds mentioned above. Let N be the level of W. Let $k = \mathbf{F}_p$ with p prime to N. If* $\gcd(l, Np) = 1$, *T_l^* acts on $J_l^2(W) \simeq H^3(W_{\bar{k}}, \mathbf{Q}_l(2))/H^3(W_{\bar{k}}, \mathbf{Z}_l(2))$ by multiplication by the scalar b_l.*

Proof. – By [Sch-FP, 7.1(i)] and [Sh2, 2.6]

$$h^3(W(\mathbf{C}), \mathbf{C}) = 2h^{3,0}(W) = 2 = 2\ dim.(S_4(\Gamma)).$$

By [Zu, §12] and [Sch-CM, 1.11, 1.13] there is an isomorphism of modules for the algebra of Hecke operators, $H^3(W(\mathbf{C}), \mathbf{C}) \simeq S_4(\Gamma) \oplus \overline{S_4(\Gamma)}$. Now T_l^* acts on $S_4(\Gamma)$ by multiplication by b_l. Thus it acts on $H^3(W(\mathbf{C}), \mathbf{Z})/tors$ by scalar multiplication by b_l. Now standard specialization arguments such as used in the proof of (13.6) below show that T_l^* acts by the scalar b_l on $H^3(W_{\bar{\mathbf{F}}_p}, \mathbf{Z}_l)/tors$. The lemma follows.

PROPOSITION 12.6. – *Let W, k, p, N, l be as in the previous lemma. In addition assume that*
(1) $\gcd(l, 2Npb_l) = 1$.
(2) $\gcd(p, b_p) = 1$.
Then $CH^2(W_{\bar{k}})_{alg} \otimes \mathbf{Z}_l \simeq 0$ and $\alpha^2(Gr^2(W_{\bar{k}})) \subset J_l^2(W)$ contains a non-zero divisible group. If furthermore
(3) $T^2 - b_pT + p^3 \in \mathbf{Z}_l[T]$ *is irreducible,*
then $\alpha^2(Gr^2(W_{\bar{k}})) = J_l^2(W)$.

Proof. – Let $\phi \in G_{\mathbf{F}_p}$ denote the Frobenius. By [D], $T^2 - b_pT + p^3$ is the minimal polynomial of ϕ^{-1} acting on $H^3(W_{\bar{\mathbf{F}}_p}, \mathbf{Z}_l)/tors$. This polynomial is irreducible over $\mathbf{Z}$, since each root, ξ, has complex absolute value $p^{3/2}$ by Deligne's proof of the Riemann hypothesis. By (2), $tr(\xi/p) \notin \mathbf{Z}$, whence ξ/p is not an algebraic integer. The first assertion now follows from (2.7) by taking $W = W'$, $d = 3, r = 2$, and Q to be the diagonal correspondence. This shows that α^2 is well defined on $Gr^2(W_{\bar{k}})$. By (1), T_l^* acts invertibly on $J_l^2(W)$. By (12.1) and (12.3) $\alpha^2(Z_{CM}(W_{\bar{k}}))$ is a non-zero, divisible subgroup which is stable under the action of ϕ^{-1}. If (3) holds, the subgroup $\alpha^2(Z_{CM}(W_{\bar{k}})) \subset J_l^2(W)$ is all of $J_l^2(W)$ by (10.8).

Example 12.7. – We consider the case $N = 5$ in (12.4). Then $\Gamma = \Gamma_1(5)$, $S_4(\Gamma_1(5)) \simeq S_4(\Gamma_0(5))$. Let τ be the coordinate on $\mathbf{C}$ restricted to the upper half plane. Set $q = exp(2\pi i\tau)$ and $\eta(\tau) = q^{1/24}\prod_{n\geq 1}(1 - q^n)$. Then [Lig, p. 28]

$$f = \eta(\tau)^4\eta(5\tau)^4 = q - 4q^2 + 2q^3 + 8q^4 - 5q^5 - 8q^6 + 6q^7 - 23q^9 + 20q^{10} + 32q^{11} + 16q^{12} - 38q^{13} \ldots$$

is the normalized weight 4 cusp form on $\Gamma_0(5)$. When $l = 3, p = 7$ all hypotheses of (12.6) are satisfied. If l is prime and $l < 100$ one checks $\gcd(l, b_l) = 1$ for $l \neq 2$ or 5. This would seem to suggest that (12.6(1) and (2)) are usually satisfied when the normalized cusp form is not of CM-type.

Example 12.8. – We consider the cases $N = 3$ and $N = 9$ which are essentially identical since the corresponding elliptic modular threefolds are isogenous (*cf.* §13). The normalized weight 4 cusp form, f, lives in $S_4(\Gamma_0(9))$, which is one dimensional. It is most practical to describe its Fourier coefficients with the help of an appropriate Hecke character. Begin with the Hecke character, ψ, on $\mathbf{Q}(\mu_3)$ of conductor $3\mathbf{Z}[\mu_3]$ and infinity type $(1,0)$. There is a unique such. Explicitly, for each prime ideal $\mathbf{p} \neq \sqrt{-3}\mathbf{Z}[\mu_3]$, there is a unique generator $\pi = 1\text{mod}3$. Then $\psi(\mathbf{p}) := \pi$. Now ψ^3 is a Hecke character of conductor $\sqrt{-3}\mathbf{Z}[\mu_3]$ and infinity type $(3,0)$. By [Sh, Lemma 3] the inverse Mellin transform of the Hecke L-series $L(\psi^3, s)$ gives f. This allows us to describe f explicitly as follows : Set $q = exp(2\pi i\tau)$ and $f = \sum_{n\geq 1} b_n q^n$. Then

(1) $b_1 = 1$,
(2) $b_3 = 0$,
(3) $b_l = 0$ for $l = -1\text{mod}3$,
(4) When $l = 1$ mod, $b_l = \nu^3 + \bar{\nu}^3$ where $l = \nu\bar{\nu}$ and $\nu = 1\text{mod}\sqrt{-3}\mathbf{Z}[\mu_3]$.

In case (4) note that $b_l \neq 0 \bmod \nu$, whence $b_l \neq 0 \bmod l$. Thus the hypothesis (1) of (12.6) is satisfied exactly when $l = 1 \bmod 3$. Similarly hypothesis (2) is satisfied exactly when $p = 1 \bmod 3$. When both (1) and (2) hold, (3) fails since the roots of $T^2 - b_pT + p^3$ lie in $Z[\mu_3]$ and the prime l splits in this ring. In this case (12.1) implies $\alpha^2(Z_{CM}(W_{\bar{\mathbf{F}}_p})) \simeq (\mathbf{Q}_l/\mathbf{Z}_l)^r$ with $r \in \{1,2\}$. If $l = -1 \bmod 3$ then $T_l^* = 0$ and (12.1) is useless. Nonetheless we shall be able to get our most precise results in this case (*see* (14.2)).

Remark 12.9. – For elliptic modular threefolds of higher level the cohomology $H^1(X_{\bar{k}}, j_*\text{Sym}^2 R^1\dot{\pi}_*\mathbf{Q}_l)$ may be a direct sum of simple modules for the algebra of Hecke correspondences. In this case one might wish to choose the correspondence Q in (12.1) to be a projector to a simple summand. However, in the examples discussed in this paper, Q will always be simply the diagonal correspondence.

13. A map from an abelian 3-fold to an elliptic modular 3-fold

According to (1.4) the arithmetic intermediate Jacobian $J_l^2(W)$ is a torsion group when the base field k is finite. So far we have said nothing about whether or not complex multiplication cycles give classes of finite order in $CH^2(W_{\bar{k}})_{hom}$. There is (up to isogeny) only a single self-fiber product of non-isotrivial, semi-stable elliptic surfaces known to the author where this question has been resolved. In this particular case the solution depends on the purely geometric result (13.2) below. We state and prove (13.2) in this section and then apply it in the following section.

Let $E \subset \mathbf{P}^2$ be the elliptic curve defined by

$$t_0^3 - t_2 t_1^2 + t_2^3/4 = 0. \tag{13.1}$$

This curve is 3-isogenous over $\mathbf{Q}$ to the Fermat cubic curve [Ste, p.123]. Note that μ_3 acts on E by modifying t_0 by a root of unity and leaving t_1 and t_2 invariant. The substitution

$b = t_1 - t_2/2$ gives rise to the global Weierstrass equation

$$t_2(b^2 + bt_2) = t_0^3.$$

This is the curve 27A in the well known tables [B-Ku, p.83].

The functor which associates to each finite type $\mathbf{Q}$-scheme, S, the set of isomorphism classes of elliptic curves over S together with a point of order 3 and a disjoint subgroup scheme isomorphic to μ_3 is representable. Let $\pi : Y(3) \to X(3)$ denote the projective relatively minimal model for the the universal elliptic curve. Let $W(3)$ denote the variety obtained by blowing up the nodes of the fiber product $Y(3) \times_\pi Y(3)$. This variety has already appeared in (12.8) as the example with $N = 3$. The main result of this section is

THEOREM 13.2. – *There is a dominant rational map defined over* $\mathbf{Q}$, $f : E^3 \to W(3)$.

Proof. – We may identify π with the morphism associated to the rational map $\mathbf{P}^2 \to \mathbf{P}^1$, $(x_0 : x_1 : x_2) \to (x_0^3 + x_1^3 + x_2^3)/x_0x_1x_2$ [Be2]. It turns out to be more convenient to work with the isogenous elliptic surface $\pi' : Y_0(9) \to \mathbf{P}^1$, which is the minimal elliptic surface associated to the rational map $\mathbf{P}^2 \to \mathbf{P}^1$, $(r_0 : r_1 : r_2) \to (r_0^2r_2 + r_1^2r_0 + r_2^2r_1)/r_0r_1r_2$. As the notation indicates, this elliptic surface is associated to a congruence subgroup $\Gamma \subset SL(2, \mathbf{Z})$ with the property that $\pm Id \cdot \Gamma = \Gamma_0(9)$. A degree three isogeny $\kappa : Y_0(9) \to Y(3)$ is given by $r_i = x_i^2 x_{i+1}$. The fiber product $Y_0(9) \times_{\pi'} Y_0(9)$ with nodes blown up will be denoted $W_0(9)$. The isogeny dual to κ may be used to construct a dominant map $W_0(9) \to W(3)$.

The μ_3 action on E mentioned above gives rise to a product action of $(\mu_3)^3$ on E^3. Let $N \subset (\mu_3)^3$ denote the largest subgroup scheme which leaves the global regular 3-form on E^3 invariant. We shall in fact construct a map which factors through $N\backslash E^3$. We adopt the convention that a section $g \in \Gamma(E, \mathcal{O}_{\mathbf{P}^2}(1)|_E)$ when pulled back to the product E^3 by the projection pr_1 (respectively pr_2) (respectively pr_3) will be denoted g (respectively g') (respectively g''). Define sections $a, b \in \Gamma(E, \mathcal{O}_{\mathbf{P}^2}(1)|_E)$ by $a = t_1 + t_2/2$ and $b = t_1 - t_2/2$. Then $t_0^3 - t_2ab = 0$. With these notations consider the rational map $F : E^3 \to \mathbf{P}^2 \times \mathbf{P}^2$ defined by

$$\begin{aligned}(r_0 : r_1 : r_2) &= (-t_2a(t_0')^2t_0''b'' : -t_0at_2'b'(t'')_0^2 : t_0^2t_0't_2't_2''b'')\\ (s_0 : s_1 : s_2) &= (-t_0at_0't_2't_0''b'' : -t_0^2(t_0')^2(t_0'')^2 : t_2at_2'b't_2''b''),\end{aligned} \tag{13.4}$$

whose image is contained in the hypersurface $\bar{W}_0(9) \subset \mathbf{P}^2 \times \mathbf{P}^2$ defined by

$$r_0r_1r_2(\Sigma_{i\in\mathbf{Z}/3}s_i^2s_{i-1}) - s_0s_1s_2(\Sigma_{i\in\mathbf{Z}/3}r_i^2r_{i-1}) = 0.$$

This fact may be verified by a somewhat messy calculation in which one uses (13.4) to write

$$\Sigma_{i\in\mathbf{Z}/3}(s_i/s_{i+1} - r_i/r_{i+1}) = (t_0t_0't_0'')^{-1}[(at_2'b'' - t_2a'b'') + (-ba'a'' + ab'a'') + (-t_2b't_2'' + bt_2't_2'')].$$

Expanding the expression in square brackets as a linear combination of

$$\{t_1t_1't_1'', t_1t_1't_2'', t_1t_2't_1'', ..., t_2t_2't_2''\}$$

reveals that all coefficients are zero as required. Now $\bar{W}_0(9)$ is clearly birational to $W_0(9)$. As $N \subset Aut(E^3)$ is generated by $(t_0, t_0', t_0'') \to (\zeta_3t_0, \zeta_3^{-1}t_0', t_0'')$ and $(t_0, t_0', t_0'') \to (t_0, \zeta_3t_0', \zeta_3^{-1}t_0'')$, F clearly factors through $N\backslash E^3$.

It remains only to show that the image of F, which is irreducible, is three dimensional. Note that $p_0 := (0:1:2) \in E$ (13.1). Write $pr_1 : \mathbf{P}^2 \times \mathbf{P}^2 \to \mathbf{P}^2$ for projection on the first factor. Now $pr_1 \circ F|_{\Delta \times E}(p_0 \times p_0 \times E) = (0:1:0)$. Furthermore, the same map restricted to the small diagonal in E^3 is given by $(r_0 : r_1 : r_2) = (-a : -a : t_2)$. It follows that $F(\Delta \times E)$ is a surface. It is clear from (13.4) that this surface is contained in the hypersurface defined by $r_0 = s_0$. As $F(p_0 \times E \times E) = ((1:0:0),(0:0:1))$, which is not contained in the hypersurface $r_0 = s_0$, the image of F must be three dimensional. This completes the proof of (13.2).

Write $\Delta < \mu^3$ for the diagonal subgroup and $\Gamma_f \subset E^3 \times W(3)$ for the closed subvariety corresponding to the graph of f.

LEMMA 13.5. – *The map* $f^* : H^3(W(3)_{\mathbf{C}}, \mathbf{C}) \to H^3(E^3_{\mathbf{C}}, \mathbf{C})^{\Delta}$, *induced by* Γ_f, *is an isomorphism.*

Proof. – According to [Sch-CM, 1.7] or [Sch-FP, 7.1] $H^3(W(3)_{\mathbf{C}}, \mathbf{C}) \simeq H^{3,0} \oplus H^{0,3}$ and is two dimensional. Since f^* is injective on holomorphic 3-forms, the injectivity of f^* follows from the Hodge decomposition. The image is contained in $H^3(E^3_{\mathbf{C}}, \mathbf{C})^N \simeq H^3(E^3_{\mathbf{C}}, \mathbf{C})^{\Delta} \simeq H^{3,0}(E^3) \oplus H^{0,3}(E^3)$, which is also two dimensional.

LEMMA 13.6. – *For each prime* $p > 3$ *and* $l \neq p$, $f^* : H^3(W(3)_{\bar{\mathbf{F}}_p}, \mathbf{Q}_l) \simeq H^3(E^3_{\bar{\mathbf{F}}_p}, \mathbf{Q}_l)^{\Delta}$ *is an isomorphism.*

Proof. – Note that $\Gamma_f \subset E^3 \times W(3)$ may be spread out to a scheme which is flat over the base ring $\mathbf{Z}[1/6]$. It is clear how to specialize this cycle to any prime $p > 3$. The cycle class map to cohomology is compatible with specialization [Fu, 20.3.5]. Specialization on cohomology is compatible with the Künneth decomposition and Poincaré duality since both hold in the relative context for smooth proper morphisms. Thus there is a commutative diagram

$$\begin{array}{ccc} H^6((E^3 \times W(3))_{\bar{\mathbf{Q}}_p}, \mathbf{Q}_l(3)) & \to & H^6((E^3 \times W(3))_{\bar{\mathbf{F}}_p}, \mathbf{Q}_l(3)) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(H^3(W(3)_{\bar{\mathbf{Q}}_p}, \mathbf{Q}_l), H^3(E^3_{\bar{\mathbf{Q}}_p}, \mathbf{Q}_l)) & \to & \mathrm{Hom}(H^3(W(3)_{\bar{\mathbf{F}}_p}, \mathbf{Q}_l), H^3(E^3_{\bar{\mathbf{F}}_p}, \mathbf{Q}_l)) \end{array}$$

in which the horizontal arrows are isomorphisms. Now apply the Lefschetz principle and (13.5) to the Künneth component of $cl_{E^3 \times W(3)}(\Gamma_f)$ in $\mathrm{Hom}(H^3(W(3)_{\bar{\mathbf{Q}}_p}, \mathbf{Q}_l), H^3(E^3_{\bar{\mathbf{Q}}_p}, \mathbf{Q}_l))$.

14. Towards the structure of certain Griffiths groups

In this section we draw some conclusions about the Griffiths groups of the varieties $W(3)_{\bar{\mathbf{F}}_p}$ and $E^3_{\bar{\mathbf{F}}_p}$ introduced in the previous section.

The following result of Soulé will play an important role in the proof.

THEOREM 14.1. – *Let* $A_{\bar{\mathbf{F}}_p}$ *be a smooth projective threefold. Suppose that there is a dominant rational map from the product of three curves to* $A_{\bar{\mathbf{F}}_p}$. *Then for primes* $l \neq p$, $CH^2(A_{\bar{\mathbf{F}}_p})_{hom} \otimes \mathbf{Z}_l \subset CH^2(A_{\bar{\mathbf{F}}_p})_{tors} \otimes \mathbf{Z}_l$.

Proof. – [So, Thm. 3].

There is a natural map $H^3(W(3)_{\bar{\mathbf{F}}_p}, \mathbf{Q}_l/\mathbf{Z}_l(2)) \to H^4(W(3)_{\bar{\mathbf{F}}_p}, \mathbf{Z}_l(2))_{tors}$ [Co-S-S, p.774] whose kernel will be denoted by B_l. Now

$$B_l \simeq H^3(W(3)_{\bar{\mathbf{F}}_p}, \mathbf{Q}_l(2))/H^3(W(3)_{\bar{\mathbf{F}}_p}, \mathbf{Z}_l(2)) \simeq J_l^2(W(3)) \simeq (\mathbf{Q}_l/\mathbf{Z}_l)^2.$$

THEOREM 14.2. – *Let p and l be primes satisfying $p = 1 \bmod 3$, $l \neq p$, and $l > 3$.*

(1) *If $l = 1 \bmod 3$, then $Gr^2(W(3)_{\bar{\mathbf{F}}_p}) \otimes \mathbf{Z}_l$ is isomorphic to $(\mathbf{Q}_l/\mathbf{Z}_l)^2$ or to $\mathbf{Q}_l/\mathbf{Z}_l \oplus \mathbf{Z}/l^n$ for some non-negative integer n.*

(2) *If $l = -1 \bmod 3$, then $Gr^2(W(3)_{\bar{\mathbf{F}}_p}) \otimes \mathbf{Z}_l \simeq (\mathbf{Q}_l/\mathbf{Z}_l)^2$ and is generated by complex multiplication cycles.*

Proof. – Since $p = 1 \bmod 3$ (12.6) and (12.8) imply that $CH^2(W(3)_{\bar{\mathbf{F}}_p})_{alg} \otimes \mathbf{Z}_l = 0$. Thus $Gr^2(W(3)_{\bar{\mathbf{F}}_p}) \otimes \mathbf{Z}_l \simeq CH^2(W(3)_{\bar{\mathbf{F}}_p})_{hom} \otimes \mathbf{Z}_l$. Soulé's theorem implies that $Gr^2(W(3)_{\bar{\mathbf{F}}_p}) \otimes \mathbf{Z}_l \subset CH^2(W(3)_{\bar{\mathbf{F}}_p})_{tors}$. Thus there is an injective map (2.9(2))

$$\lambda^2 : Gr^2(W(3)_{\bar{\mathbf{F}}_p}) \otimes \mathbf{Z}_l \to B_l \simeq (\mathbf{Q}_l/\mathbf{Z}_l)^2.$$

This gives an upper bound on the size of $Gr^2(W(3)_{\bar{\mathbf{F}}_p}) \otimes \mathbf{Z}_l$. A lower bound on the size of $Gr^2(W(3)_{\bar{\mathbf{F}}_p}) \otimes \mathbf{Z}_l$ is obtained by computing its image under α^2. Under hypothesis (1) of the theorem, (12.6) or (10.4) implies that $Gr^2(W(3)_{\bar{\mathbf{F}}_p}) \otimes \mathbf{Z}_l$ admits a surjective map to $\mathbf{Q}_l/\mathbf{Z}_l$. The only subgroups of $(\mathbf{Q}_l/\mathbf{Z}_l)^2$ with this property are listed in (1).

For the second assertion we need a better lower bound on the size of $Gr^2(W(3)_{\bar{\mathbf{F}}_p}) \otimes \mathbf{Z}_l$. It would suffice to show that

$$\alpha^2(Z_{CM}(W(3)_{\bar{\mathbf{F}}_p})) \to J_l^2(W) \simeq (\mathbf{Q}_l/\mathbf{Z}_l)^2 \tag{14.3}$$

is surjective. To see that this is the case, we make use of certain natural automorphisms of $W(3)_{\mathbf{F}_p}$ which preserve complex multiplication cycles. These automorphisms are treated in detail in [Sch-BB, §1], so we need only sketch the argument here. Since $\mu_3 \subset \mathbf{F}_p^*$, $Y(3)_{\mathbf{F}_p}$ may be regarded as the universal elliptic curve for symplectic level 3 structure. Such structures form a principal homogeneous space for $SL(2, \mathbf{Z}/3)$. The action of $SL(2, \mathbf{Z}/3)$ on the moduli problem gives rise to actions on $Y(3)$ and on $X(3)$ so that $\pi : Y(3) \to X(3)$ is equivariant. There results a diagonal action on $W(3)$. The induced action on $H^3(W(3)_{\bar{\mathbf{F}}_p}, \mathbf{Q}_l) \simeq \mathbf{Q}_l^2$ factors through $\mathbf{Z}/3$ which is the quotient of $SL(2, \mathbf{Z}/3)$ by the 2-Sylow subgroup. This action is non-trivial. Since $l = -1 \bmod 3$ the representation is irreducible. By (10.4) the Tate module of $\alpha^2(Z_{CM}(W(3)_{\bar{\mathbf{F}}_p}))$ tensored with $\mathbf{Q}_l$ is a non-trivial subrepresentation, which must be all of $H^3(W(3)_{\bar{\mathbf{F}}_p}, \mathbf{Q}_l)$. The surjectivity of (14.3) follows formally [Su, 1.3].

THEOREM 14.4. – *Let p and l be primes satisfying $p \neq 3$, $l \neq p$, and $l > 3$.*

(1) *If $p = -1 \bmod 3$, then $Gr^2(E^3_{\bar{\mathbf{F}}_p}) \otimes \mathbf{Z}_l \simeq 0$.*

(2) *If $p = 1 \bmod 3$ and $l = 1 \bmod 3$, then $Gr^2(E^3_{\bar{\mathbf{F}}_p}) \otimes \mathbf{Z}_l$ is isomorphic to $(\mathbf{Q}_l/\mathbf{Z}_l)^2$ or to $\mathbf{Q}_l/\mathbf{Z}_l \oplus \mathbf{Z}/l^n$ for some non-negative integer n.*

(3) *If $p = 1 \bmod 3$ and $l = -1 \bmod 3$, then $Gr^2(E^3_{\bar{\mathbf{F}}_p}) \otimes \mathbf{Z}_l \simeq (\mathbf{Q}_l/\mathbf{Z}_l)^2$.*

In preparation for the proof of (14.4) consider the diagram

(14.5)

$$\begin{array}{ccc} CH^1(E^3_{\bar{\mathbf{F}}_p})_{tors} \otimes CH^1(E^3_{\bar{\mathbf{F}}_p}) & \xrightarrow{\lambda^1 \otimes cl_{E^3_{\bar{\mathbf{F}}_p}}} & H^1(E^3_{\bar{\mathbf{F}}_p}, \mathbf{Q}_l/\mathbf{Z}_l(1)) \otimes H^2(E^3_{\bar{\mathbf{F}}_p}, \mathbf{Z}_l(1)) \\ \downarrow & & \downarrow \cup \\ CH^2(E^3_{\bar{\mathbf{F}}_p})_{alg} & \xrightarrow{\lambda^2_{alg}} & H^3(E^3_{\bar{\mathbf{F}}_p}, \mathbf{Q}_l/\mathbf{Z}_l(2)), \end{array}$$

in which λ^2_{alg} denotes the restriction of λ^2 to $CH^2(E^3_{\bar{\mathbf{F}}_p})_{alg}$ and the left hand vertical arrow is the intersection product. By (2.9(4)) the diagram commutes. To see that the right hand vertical arrow is surjective apply the fact that the cohomology of an Abelian variety is an exterior algebra on H^1 to get that

$$H^1(E^3_{\bar{\mathbf{F}}_p}, \mathbf{Z}/l^n(1)) \otimes H^2(E^3_{\bar{\mathbf{F}}_p}, \mathbf{Z}_l(1)) \xrightarrow{\cup} H^3(E^3_{\bar{\mathbf{F}}_p}, \mathbf{Z}/l^n(2))$$

is surjective. Then pass to the direct limit. Note also that λ^1 is surjective.

The action of μ_3 on E described after (13.1) shows that E has complex multiplication by $\mathbf{Z}[\mu_3]$. Furthermore, E has good reduction at $p \neq 3$. We note a few consequences of these facts. First, when $p = -1 \bmod 3$, p is inert in $\mathbf{Z}[\mu_3]$, whence $E_{\mathbf{F}_p}$ is supersingular [La2, §10 Thm. 10]. The diagonal action of $\mathbf{Z}[\mu_3]$ on $E^3_{\bar{\mathbf{F}}_p}$ makes $H^1(E^3_{\bar{\mathbf{F}}_p}, \mathbf{Q}_l)$ a $\mathbf{Z}[\mu_3] \otimes_{\mathbf{Z}} \mathbf{Q}_l$-module, and thus gives rise to a decomposition

$$H^1(E^3_{\bar{\mathbf{F}}_p}, \mathbf{Q}_l) \otimes_{\mathbf{Q}_l} \mathbf{Q}_l(\mu_3) \simeq V \oplus \bar{V}$$

in which $\omega \in \mathbf{Z}[\mu_3]$ acts by $\omega Id_V \oplus \bar{\omega} Id_{\bar{V}}$. The inverse Frobenius $\phi^{-1} \in G_{\mathbf{F}_p}$ acts on $H^1(E^3_{\bar{\mathbf{F}}_p}, \mathbf{Q}_l)$ as the geometric Frobenius endomorphism $F_p \in End(E_{\mathbf{F}_p})$ [Mi, VI.13.5]. When $p = 1 \bmod 3$, $End(E_{\mathbf{F}_p}) \simeq \mathbf{Z}[\mu_3]$. Thus there is an algebraic integer $\pi \in \mathbf{Z}[\mu_3]$ with $\pi\bar{\pi} = p$ such that ϕ^{-1} acts on $H^1(E^3_{\bar{\mathbf{F}}_p}, \mathbf{Q}_l) \otimes \mathbf{Q}_l(\mu_3)$ by $\pi Id_V \oplus \bar{\pi} Id_{\bar{V}}$.

Proof of 14.4. – Suppose now that $p = -1 \bmod 3$. Since $E_{\mathbf{F}_p}$ is supersingular, the image of $cl_{E^3_{\bar{\mathbf{F}}_p}} : CH^1(E^3_{\bar{\mathbf{F}}_p}) \to H^2(E^3_{\bar{\mathbf{F}}_p}, \mathbf{Z}_l(1))$ tensored with $\mathbf{Z}_l$ has finite index. By the surjectivity of the right hand verticle arrow in (14.5) and the divisibility of $H^1(E^3_{\bar{\mathbf{F}}_p}, \mathbf{Q}_l/\mathbf{Z}_l(1))$, λ^2_{alg} is surjective. But λ^2 is injective on the l-power torsion subgroup of $CH^2(E^3_{\bar{\mathbf{F}}_p})_{tors}$ (2.9(2)). Thus $CH^2(E^3_{\bar{\mathbf{F}}_p})_{alg} \otimes \mathbf{Z}_l \simeq CH^2(E^3_{\bar{\mathbf{F}}_p})_{tors} \otimes \mathbf{Z}_l$. By (14.1) $CH^2(E^3_{\bar{\mathbf{F}}_p})_{hom} \otimes \mathbf{Z}_l \simeq CH^2(E^3_{\bar{\mathbf{F}}_p})_{alg} \otimes \mathbf{Z}_l$ and (14.4(1)) follows.

Write Γ_δ for the graph of $\delta \in \Delta \subset Aut(E^3_{\bar{\mathbf{F}}_p})$. To prove the remaining assertions of the theorem it is convenient to introduce the correspondences $Q = \sum_{\delta \in \Delta} \Gamma_\delta$ and $Q' = 3Id - Q \in Z^3(E^3_{\bar{\mathbf{F}}_p} \times E^3_{\bar{\mathbf{F}}_p})$. Note that

$$Q \circ Q = 3Q, \qquad Q' \circ Q' = 3Q', \qquad Q \circ Q' = 0 = Q' \circ Q,$$

so that up to a factor of 3, Q and Q' are orthogonal projectors. Q projects onto the Δ-invariants in cohomology or on Chow groups.

We assume for the remainder of the proof that $p = 1 \bmod 3$.

LEMMA 14.6. – *Assume* $\gcd(l, 3p) = 1$. *Then*
(1) $Q_* H^1(E^3_{\bar{\mathbf{F}}_p}, \mathbf{Q}_l/\mathbf{Z}_l) = 0$.
(2) $cl_{E^3_{\bar{\mathbf{F}}_p}}(CH^1(E^3_{\bar{\mathbf{F}}_p})) \otimes \mathbf{Z}_l$ *has finite index in* $Q_* H^2(E^3_{\bar{\mathbf{F}}_p}, \mathbf{Z}_l(1))$.
(3) $CH^2(E^3_{\bar{\mathbf{F}}_p})_{alg} \otimes \mathbf{Z}_l \simeq Q'_* H^3(E^3_{\bar{\mathbf{F}}_p}, \mathbf{Q}_l/\mathbf{Z}_l(2))$.
(4) $Q'_* CH^2(E^3_{\bar{\mathbf{F}}_p})_{hom} \otimes \mathbf{Z}_l \simeq CH^2(E^3_{\bar{\mathbf{F}}_p})_{alg} \otimes \mathbf{Z}_l$.

Proof. – The first assertion is clear so we begin with the second. Observe that

$$H^2(E^3_{\bar{\mathbf{F}}_p}, \mathbf{Q}_l(1)) \otimes \mathbf{Q}_l(\mu_3) \simeq \Lambda^2 V(1) \oplus V \otimes \bar{V}(1) \oplus \Lambda^2 \bar{V}(1).$$

The middle summand is both the subspace invariant under Frobenius and $Q_* H^2(E^3_{\bar{\mathbf{F}}_p}, \mathbf{Q}_l(1)) \otimes \mathbf{Q}_l(\mu_3)$. Thus (2) follows from the Tate conjecture, which follows from Tate's deep theorem [Ta3, Thm. 4] or may be easily checked directly in this instance.

For (3) note

$$Q_* H^3(E^3_{\bar{\mathbf{F}}_p}, \mathbf{Q}_l(2))(-1) \otimes \mathbf{Q}_l(\mu_3) \simeq \Lambda^3 V(1) \oplus \Lambda^3 \bar{V}(1).$$

Now ϕ^{-1} acts by $\frac{\pi^3}{p} Id_{\Lambda^3 V(1)} \oplus \frac{\bar{\pi}^3}{p} Id_{\Lambda^3 \bar{V}(1)}$. Since $\frac{\pi^3}{p}$ and $\frac{\bar{\pi}^3}{p}$ are not algebraic integers, $Q_* CH^2(E^3_{\bar{\mathbf{F}}_p})_{alg} \otimes \mathbf{Z}_l \simeq 0$ by (2.7). Thus

$$\text{(14.7)} \qquad CH^2(E^3_{\bar{\mathbf{F}}_p})_{alg} \otimes \mathbf{Z}_l = Q'_* CH^2(E^3_{\bar{\mathbf{F}}_p})_{alg} \otimes \mathbf{Z}_l \xrightarrow{\lambda^2_{alg}} Q'_* H^3(E^3_{\bar{\mathbf{F}}_p}, \mathbf{Q}_l/\mathbf{Z}_l(2)).$$

By (2) and (14.5) the cup product gives rise to an inclusion

$$\text{(14.8)} \qquad H^1(E^3_{\bar{\mathbf{F}}_p}, \mathbf{Q}_l/\mathbf{Z}_l(1)) \cup Q_* H^2(E^3_{\bar{\mathbf{F}}_p}, \mathbf{Z}_l(1)) \subset \lambda^2_{alg}(CH^2(E^3_{\bar{\mathbf{F}}_p})_{alg}).$$

By (1) the left hand side may be identified with $Q'_* H^3(E^3_{\bar{\mathbf{F}}_p}, \mathbf{Q}_l/\mathbf{Z}_l(2))$. Combining (14.7) and (14.8) yields

$$\lambda^2_{alg}(CH^2(E^3_{\bar{\mathbf{F}}_p})_{alg} \otimes \mathbf{Z}_l) \simeq Q'_* H^3(E^3_{\bar{\mathbf{F}}_p}, \mathbf{Q}_l/\mathbf{Z}_l(2)).$$

Since λ^2 is injective on $CH^2(E^3_{\bar{\mathbf{F}}_p})_{alg} \otimes \mathbf{Z}_l$, (3) follows. Finally, (4) follows from (3) *via* the injectivity of λ^2 on $CH^2(E^3_{\bar{\mathbf{F}}_p})_{hom} \otimes \mathbf{Z}_l$ and the fact that λ^2 commutes with the action of Q'_* (2.9(3)). This completes the proof of (14.6).

Returning to the proof of (14.4) we observe that (14.6(4)) implies

$$Q_* CH^2(E^3_{\bar{\mathbf{F}}_p})_{hom} \otimes \mathbf{Z}_l \simeq Gr^2(E^3_{\bar{\mathbf{F}}_p}) \otimes \mathbf{Z}_l.$$

Use the injectivity of λ^2 on $CH^2(E^3_{\bar{\mathbf{F}}_p})_{hom} \otimes \mathbf{Z}_l$ once more to view $Q_* CH^2(E^3_{\bar{\mathbf{F}}_p})_{hom} \otimes \mathbf{Z}_l$ as a subgroup of

$$Q_* H^3(E^3_{\bar{\mathbf{F}}_p}, \mathbf{Q}_l/\mathbf{Z}_l(2)) \simeq J^2_l(E^3_{\bar{\mathbf{F}}_p})^{\Delta} \simeq (\mathbf{Q}_l/\mathbf{Z}_l)^2.$$

This gives an upper bound on the size of the Griffiths group. To get the lower bound, apply (14.2) and the correspondence $Q \circ \Gamma_f$. It follows from (13.6) that the right hand vertical arrow in the commutative diagram

$$\begin{array}{ccc} Gr^2(W(3)_{\bar{\mathbf{F}}_p}) \otimes \mathbf{Z}_l & \xrightarrow{\alpha^2} & J_l^2(W(3)_{\mathbf{F}_p}) \\ {\scriptstyle Q_* \circ \Gamma_{f*}} \downarrow & & \downarrow {\scriptstyle Q_* \circ \Gamma_{f*}} \\ Gr^2(E^3_{\bar{\mathbf{F}}_p}) \otimes \mathbf{Z}_l & \xrightarrow{\alpha^2} & J_l^2(E^3_{\mathbf{F}_p})^{\Delta} \end{array}$$

may be identified with an isomorphism of $(\mathbf{Q}_l/\mathbf{Z}_l)^2$ with itself. Now (14.4) parts (2) and (3) follow from (14.2).

Remark 14.9. – In the statement of (14.4) we may replace the elliptic curve $E_{\bar{\mathbf{F}}_p}$ by any isogenous elliptic curve – for example, by the Fermat cubic curve.

COROLLARY 14.10. – *If* $p = -1 \bmod 3$ *and* $l \neq 3$, *then* $Gr^2(W(3)_{\bar{\mathbf{F}}_p}) \otimes \mathbf{Z}_l \simeq 0$.

Proof. – Soulé's theorem together with (2.9(2)) implies that $CH^2(W(3)_{\bar{\mathbf{F}}_p})_{hom} \otimes \mathbf{Z}_l \subset B_l \simeq (\mathbf{Q}_l/\mathbf{Z}_l)^2$. It would suffice to show that $CH^2(W(3)_{\bar{\mathbf{F}}_p})_{alg}$ contains a subgroup isomorphic to $(\mathbf{Q}_l/\mathbf{Z}_l)^2$. In fact $\Gamma_f^* CH^2(E^3_{\bar{\mathbf{F}}_p})_{alg} \otimes \mathbf{Z}_l$ is such a subgroup. This is because the lower line of (14.5) is surjective when $p = -1 \bmod 3$ and the map

$$\Gamma_f^* : H^3(E^3_{\bar{\mathbf{F}}_p}, \mathbf{Q}_l/\mathbf{Z}_l(2)) \to B_l$$

is surjective.

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