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Uniformization of surface laminations

UNIFORMIZATION OF SURFACE LAMINATIONS

By Alberto CANDEL

ABSTRACT. − A surface lamination is a metric space that carries a foliation with leaves of dimension two. Given a riemannian metric along the leaves we study the problem of finding another such metric, in the same conformal class, for which all leaves have the same constant curvature. As for surfaces, the existence of such metric is determined by the Euler characteristics of the lamination. These numbers are obtained by evaluating the invariant transverse measures on the curvature form of the given metric. We prove that there is a metric of curvature −1 (resp. 1) if and only if all Euler characteristics are negative (resp. positive). Using harmonic measures we prove a similar statement holds for flat metrics.

Introduction

The classical Uniformization Theorem of Koebe-Poincaré-Klein asserts that the universal covering of any Riemann surface is conformally equivalent to exactly one of the following: the plane $\mathbf{C}$, the sphere $\mathbf{S}^2$, or the unit disc Δ, and so there are three types of Riemann surfaces: euclidean, spherical and hyperbolic. It is remarkable that the conformal type of a compact Riemann surface is completely determined by a topological invariant, its Euler characteristic, which can be computed from the curvature of a riemannian metric by means of the Gauss-Bonnet formula. Furthermore, according to the type of the surface, a metric exists of curvature 0, 1 or −1.

In this work we analyze the problem of constructing metrics of constant curvature on surface laminations. A surface lamination is a topological space locally homeomorphic to the product of a disc in the plane and a piece of metric space, with the overlap homeomorphisms preserving the disc factor of this product structure. The discs glue together to form surfaces, the leaves of the lamination, whose global behaviour is usually very complicated. If the overlap homeomorphisms are holomorphic functions of the disc coordinate, then we have a Riemann surface lamination.

The metric uniformization of surfaces relies on the fact that a riemannian metric on an oriented surface is the same as a complex structure. Analytically, this is the existence of solutions to the Beltrami equation. To have the same relation between Riemann surface laminations and oriented surface laminations with riemannian metric we then need a regularity theorem for the Beltrami equation depending on parameters. This is precisely what Ahlfors and Bers proved in their classical paper [1], [6]. This fact established, we ask: given a compact Riemann surface lamination M, is there a riemannian

metric on M inducing the complex structure (a conformal metric) for which all the leaves have the same constant curvature?

The problem studied here has also a motivation within the framework of foliation theory. In [4], Cantwell and Conlon studied the existence of constant curvature metrics on the leaves of a three manifold smoothly foliated by surfaces, and gave a construction of such a metric for proper foliations without toroidal leaves. Other results in this direction are due to Cairns and Ghys [3] for totally geodesic foliations, and to Verjovsky [18] for hyperbolic foliations of complex hyperbolic manifolds.

Here is a brief description of the contents of this paper. By using the local product structure of the surface lamination M, we can speak of differential forms, namely, differential forms on the leaves locally parametrized by the transverse coordinates. There is a natural differential operator along the leaves and, just as for manifolds, one obtains a leafwise de Rham complex with cohomology groups $H^p(H, \mathbf{R}_l)$, which are trivial if $p > $ leaf dimension. We associate to M a topological invariant $e_l(M)$ which is the Euler class of the tangent bundle to the leaves TM. This is a cohomology class in $H^2(M, \mathbf{R}_l)$ which may be represented by the curvature form of a riemannian metric g on M.

To obtain a numerical invariant we need to integrate the Euler class against the "fundamental classes" of the lamination, in a way analogous to the Gauss-Bonnet formula for surfaces. These fundamental classes are invariant transverse measures: Radon measures on transversals to the leaves, invariant by the local transformations defining the lamination.

The dual de Rham complex has homology groups

$$H_p(M, \mathbf{R}_l) \cong \mathrm{Hom}(H^p(M, \mathbf{R}_l), \mathbf{R}).$$

The Ruelle-Sullivan map gives an isomorphism between $H_2(M, \mathbf{R}_l)$ and the space of invariant transverse measures for M. Therefore these objects became truly fundamental classes, and we can define the Euler characteristic of an invariant measure μ as $\chi(M, \mu) = \mu(e_l(M))$ by that isomorphism.

In Section 4 we prove the following uniformization theorem.

THEOREM. — *Let M be a compact oriented surface lamination with a riemannian metric g. Then $\chi(M, \mu) < 0$ for every positive invariant transverse measure if and only if g is conformal to a metric of curvature -1. In particular, this holds true if M has no invariant measure.*

Together with Reeb's Stability theorem this gives:

THEOREM. — *Let M be a compact, connected, oriented surface lamination with a nontrivial invariant transverse measure and let g be a riemannian metric on M. Then $\chi(M, \mu) > 0$ for every positive invariant transverse measure if and only if the metric g is conformal to a metric of curvature 1. Furthermore, M is a two-sphere bundle over a metric space and the fibers are the leaves of M.*

While these two theorems completely characterize those surface laminations for which all leaves are either hyperbolic or spherical, simple examples show the analogous statement for $\chi(M, \mu) = 0$ is not true. There is, however, a cohomological condition on the Euler

class which is equivalent to the existence of a flat metric. To any Riemann surface lamination we associate a cohomology group $H^1(M, \mathscr{H})$ whose dual is isomorphic to the space of harmonic measures of [8] and where the Euler class of M naturally lives. We say that the Euler class $e_h(M)$ of M in $H^1(M, \mathscr{H})$ is >0 (resp., <0) if $m(e_h(M)) > 0$ (resp., $m(e_h(M)) < 0$) for every positive harmonic measure m on M. As invariant transverse measures, when combined with a volume form on the leaves, are also harmonic measures, the uniformization theorem for surface laminations may be stated as follows.

THEOREM. — *Let* M *be a compact Riemann surface lamination. Then there is a conformal metric for which all leaves*

(a) *have curvature* -1 *if and only if* $e_h(M) < 0$,

(b) *are flat if and only if* $e_h(M) = 0$,

(c) *have curvature* 1 *if and only if* $e_h(M) > 0$.

A corollary of this work is that Riemann surface laminations and Riemann surfaces, while being different in many respects, share many geometric properties. The differences usually came from the dynamical aspect of the laminations. This makes it interesting to study how the objects one usually attaches to Riemann surfaces (divisors, line bundles, Teichmüller spaces, etc.) behave for laminations. For instance, Teichmüller spaces of surface laminations are used in [17] in the study of one-dimensional dynamics.

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1. Fundamentals of laminations

1.1. LAMINATIONS. — Let M be a separable, locally compact, metrizable space. We say that M is a p-dimensional lamination if there is a cover of M by open sets U_i (called flow boxes or charts) and homeomorphisms

$$\varphi_i : U_i \to D_i \times T_i$$

with D_i open in $\mathbf{R}^p$, and such that the overlap maps $\varphi_j \varphi_i^{-1}$ are of the form

$$\varphi_j \varphi_i^{-1}(z, t) = (\lambda_{ji}(z, t), \tau_{ji}(t)),$$

where each map

$$\lambda_{ji} : \varphi_i(U_i \cap U_j) \subset D_i \times T_i \to D_j$$

is of class C_l^∞, that is, smooth in the first variable and all its partial derivatives with respect to the first variable are continuous functions of all the variables. We call $\varphi_i^{-1}(D_i \times \{t\})$ a plaque. The plaques smoothly glue together to form maximal connected sets called leaves, which are p-dimensional manifolds. If $p=2$, M is called a surface lamination.

By [7], we may assume the cover $\{U_i, \varphi_i\}$ is regular: the U_i's have connected plaques, each plaque in U_i meets at most one plaque in U_j, and there is another cover $\{V_i, \phi_i\}$ such that the U_i's have compact closure $\bar{U}_i \subset V_i$ and the homeomorphism ϕ_i extends φ_i.

A transversal is a Borel subset of M which intersects each leaf in a countable sub-sets. The standard ones are those of the form $x \times T$ for some flow box $D \times T$. Regular transversals are those contained in some standard transversal. A regular trans-versal can be slid along the plaques into one of the standard transversals $x \times T$. Maps between parts of regular transversals produced by iterations of this operation and its inverse are local homeomorphisms called holonomy transformations.

The main examples of laminations are foliated manifolds. But even in foliation theory one often considers objects that are not manifolds, for instance, minimal sets of foliations. Laminations by surfaces of three-manifolds are objects of much current interest in low dimensional topology. Other type of examples appear in [17].

1.2. FUNCTION SPACES AND METRICS. — Let M, N be laminations. With $\mathscr{C}(M, N)$ we denote the space of continuous maps $f : M \to N$ with the compact open topology. The closed subspace of $\mathscr{C}(M, N)$ consisting of those continuous maps which take leaves of M to leaves of N is denoted by $\mathscr{C}_l(M, N)$ and its elements are called leafwise continuous maps.

A map $f : M \to N$ between laminations is said to be of class C_l^∞, or smooth map of laminations, (resp. of class C_l^r) if it is continuous, takes leaves to leaves, and for all flow boxes $\varphi : U \to D \times T$ in M and $\psi : V \to D' \times T'$ in N the functions

$$D \to D \times T \xrightarrow{\psi f \varphi^{-1}} D' \times T' \to D'$$

$$z \mapsto (z, t) \mapsto (z', t') \mapsto z'$$

are smooth maps (resp. C^r maps) for all $t \in T$, and the partial derivatives of all orders (resp. up to order r) of $\psi f \varphi^{-1}$ with respect to the leaf variables are continuous functions of all the variables. The space of smooth maps between laminations M, N is denoted by $\mathscr{C}_l^\infty(M, N)$, and with $\mathscr{C}_l^r(M, N)$ we denote the space of C_l^r-maps. We dote these spaces with the weak C_l^r-topology ($0 \le r \le \infty$), that is, the topology of uniform convergence of all derivatives up to order r with respect to the leaf variables. When $r = 0$ this is the compact-open topology.

Laminations are objects of a category, morphisms being C_l^∞ maps (or smooth maps of laminations). This category contains smooth manifolds and smooth maps, so that all terminology and constructions on manifolds extend to laminations. The smoothness assumption in the definition of lamination is no restriction: A lamination of class C_l^1 always has a subordinate C_l^∞-structure.

A vector bundle $\pi : E \to M$ of rank n over a p-dimensional lamination M is smooth if E has the structure of a lamination of dimension $p + n$ which is compatible with the local product structure of the bundle and if $\pi : E \to M$ is a smooth map of laminations. A smooth section of E is a smooth map of laminations $s : M \to E$ such that $\pi \circ s = 1_M$.

A riemannian metric on the lamination M is a smooth and positive definite section of the bundle $S^2 T^* M$ of symmetric bilinear 2-forms on TM. The existence of riemannian metrics is a consequence of the following result proved in [11, p. 44].

PROPOSITION 1.1. − *Every open cover of a lamination has a subordinate smooth partition of unity.*

A smooth map of laminations $f: M \to N$ induces a bundle map $df: TM \to TN$ which, over leaves, is the usual differential: $df(x): T_x L_x \to T_y L_y$ if $f(x) = y$. If g_M, g_N are riemannian metrics on M, N, its norm is:

$$|df(x)| = \sup_{v \in T_x L_x} \frac{g_N(df(x)v, df(x)v)^{1/2}}{g_M(v, v)^{1/2}} .$$

If we have more that one metric on M or N, a subscript will be added to $|df(x)|$.

A tool often used in foliation theory is the normal bundle to a leaf. As a consequence of the following result the same type of structure is available for laminations. The proof, which uses a smooth partition of unity and some elements of Hilbert manifold theory, will not be needed.

PROPOSITION 1.2. − *Let M be a compact lamination. Then there is a smooth embedding of M in a real separable Hilbert space which restricts to a smooth immersion on each leaf.*

From the smooth embedding of a compact lamination in Hilbert space H it follows that a compact submanifold K of a leaf has a tubular neighborhood isomorphic to a neighborhood of the zero section of the normal bundle to K in H. The fibers of the normal bundle define a foliation near K, which is transverse to leaves of the lamination around K. We thus recover the familiar picture we have for foliations.

2. Invariant transverse measures

Invariant transverse measures for foliations were introduced by Plante [13], Ruelle-Sullivan [15] and Sullivan [16]. We refer to [11] for details.

2.1. INVARIANT TRANSVERSE MEASURES. − Let $\mathscr{E}^k(M)$ denote the space k-forms on a lamination M. Exterior differentiation along the leaves $d: \mathscr{E}^k(M) \to \mathscr{E}^{k+1}(M)$ makes it possible to define the de Rham cohomology groups of M as

$$H^k(M, R_l) = \{ \alpha \in \mathscr{E}^k(M); \ d\alpha = 0 \}/d\mathscr{E}^{k-1}(M)$$

These are the cohomology groups of the sheaf R_l on M of germs of continuous real valued functions which are locally constant on the leaves. The weak C_l^∞-topology on $\mathscr{E}^*(M)$ makes these groups topological vector spaces, usually infinite dimensional and non-Hausdorff.

The space of k-currents on M is the continuous dual $\mathscr{D}_k = \text{Hom}(\mathscr{E}^k(M), R)$, doted with the weak star topology. If M is compact, $H_*(M, R_l) \cong \text{Hom}(H^*(M, R_l), R)$.

A transverse measure for M is a measure on the σ-ring of transversals which restricts to a σ-finite measure on each transversal and such that each compact regular transversal has finite mass. It is called invariant if it is invariant by the holonomy transformations acting on transversals. Let $\mathcal{MT}$ (M) denote the vector space of **R**-valued invariant transverse measures for M with the weak topology.

Let M be a compact oriented p-dimensional lamination. To any $\mu \in MT$ (M) we associate a closed current C_μ as follows. Using a smooth partition of unity subordinate to a finite cover $U_i \cong D_i \times T_i$, a p-form ω on M can be decomposed into a finite sum $\omega = \sum_i \omega_i$. Let μ_i be the measure induced by μ on T_i. The formula:

$$C_\mu(\omega) = \sum_i C_\mu(\omega_i) = \sum_i \int_{T_i} \left(\int_{D_i \times \{t\}} \omega_i \right) d\mu_i(t).$$

defines a closed p-current C_μ on M. This correspondence

$$\mu \in \mathcal{MT}(M) \to [C_\mu] \in \mathrm{Hom}(H^p(M, \mathbf{R}_l), \mathbf{R})$$

is a topological isomorphism [11, p. 128] which is called the Ruelle-Sullivan map.

2.2. The Euler class.

The Euler class $e_l(M)$ of a surface lamination M is an element of $H^2(M, \mathbf{R}_l)$. It may be represented by the curvature form of a riemannian metric on M, or, as in [11], it may be obtained by means of the Chern-Weil construction of characteristic classes. In fact, $e_l(M)$ is the image of the Euler class of the 2-plane bundle TM over the space M under the natural homomorphism $H^2(M, \mathbf{R}) \to H^2(M, \mathbf{R}_l)$.

The Euler characteristic of an invariant transverse measure μ for the lamination M is the number

$$\chi(M, \mu) = C_\mu(e_l(M)) \in \mathbf{R},$$

where C_μ is the current corresponding to μ through the Ruelle-Sullivan isomorphism.

2.3. Dirac measures and averaging sequences.

Two geometric constructions of positive invariant transverse measures will be used later.

Dirac measures. The Dirac measure μ_L associated to a compact leaf L is given as follows: If T is a compact transversal, $\mu_{L|T}$ is the sum of the Dirac measures of the points of the intersection $L \cap T$. The corresponding current is given by integration along L.

If L is a leaf of a surface lamination M, Gauss-Bonnet implies $\chi(M, \mu_L) = \chi(L)$, the Euler characteristic of the compact surface L.

Averaging Sequences. This method is due to Plante [13], see also [10]. Fix a riemannian metric on M. A sequence of regions R_i on leaves such that

$$\lim_{i \to \infty} \frac{\mathrm{Length}(\partial R_i)}{\mathrm{Area}(R_i)} = 0$$

is called an averaging sequence. It defines a measure μ_R via the closed current

$$C_{\mu_R}(\omega) = \lim_{i \to \infty} \frac{1}{\text{Area}(R_i)} \int_{R_i} \omega,$$

where, if needed, we pass to a subsequence so that the integrals converge in the weak star topology.

For a surface lamination M the number $\lim_{i \to \infty} \chi(R_i)/\text{Area}(R_i)$ is called the average Euler characteristic of $\{R_i\}$ in [12]; it need not be equal to $\chi(M, \mu_R)$.

3. Riemann surface laminations

We define Riemann surface laminations. A riemannian metric on a surface lamination defines this type of structure; this uses the Riemann mapping theorem for variable metrics [6], [1].

3.1. THE BELTRAMI EQUATION. $-$ Let $D \times T$ be a trivial lamination, D a domain in the plane. A riemannian metric g on $D \times T$ can be written in the form

$$g = \lambda(z, t) \left| dz + \beta(z, t) d\bar{z} \right|^2$$

where $\lambda(z, t)$ is a positive function in $D \times T$, and $\beta(z, t)$ is complex valued with $|\beta(z, t)| < 1$. Let $D' \times T'$ be another trivial lamination and suppose that $f: D \times T \to D' \times T'$ is a diffeomorphism onto its image which is sense preserving on each leaf. If $D' \times T'$ has the euclidean metric $g' = |d\zeta|^2$, then f is a holomorphic map along the leaves with respect to the complex structures determined by the given metrics if and only if it is a solution of the Beltrami equation with parameters

$$\frac{\partial f}{\partial \bar{z}} = \beta(z, t) \frac{\partial f}{\partial z}$$

in $D \times T$. When T is a singleton we have the usual Beltrami equation in D.

A Beltrami coefficient β in a domain $D \subset C$ is an element of $\mathscr{C}^\infty(D, \Delta)$. A β-conformal map is a solution of the Beltrami equation with coefficient β. Every solution is a diffeomorphism of D onto a domain in the plane.

Let β be a Beltrami coefficient in D with $\sup\{|\beta(z)|; z \in D\} \leq k < 1$. If D is the unit disc Δ, the Beltrami equation has a unique solution f_β which is a homeomorphism of the closure of Δ onto itself and leaves -1, 1 and i fixed. If D is the plane C, then there is a unique β-conformal diffeomorphism f_β of the extended complex plane which leaves 0, 1 and ∞ fixed. In both cases f_β is called the normalized solution.

We shall need the following theorem (proved in [6], [1]) about the dependence of f_β on β.

THEOREM 3.1 (Riemann's mapping theorem for variable metrics). − *Let* D *denote either* Δ *or* C. *For each positive number* $k < 1$, *the map* $\beta \mapsto f_\beta$ *is a homeomorphism from the set of Beltrami coefficients* $\mathscr{C}^\infty(D, \Delta)$ *with* $\sup\{|\beta(z)|; z \in D\} \leq k$ *onto its image in* $\mathscr{C}^\infty(D, D)$.

In particular, if $t \in T \mapsto \beta(., t) \in \mathscr{C}^\infty(D, \Delta)$ *is a continuous map from a compact metric space and* f_t *is the normalized solution to the Beltrami equation with coefficient* $\beta(., t)$, *then the map*

$$(z, t) \in D \times T \mapsto (f_t(z), t) \in D \times T$$

is a diffeomorphism of laminations.

3.2. RIEMANN SURFACE LAMINATIONS. − A Riemann surface lamination is a locally compact, separable, metrizable space M with an open cover by flow boxes $\{U_i\}_{i \in I}$ and homeomorphisms $\varphi_i: U_i \to D_i \times T_i$, with D_i an open disc in C, and such that the coordinate changes in $U_i \cap U_j$ are of the form

$$\varphi_j \varphi_i^{-1}(z, t) = (\lambda_{ji}(z, t), \tau_{ji}(t))$$

where the map $z \mapsto \lambda_{ji}(z, t)$ is holomorphic for each t.

Note that the cover U_i gives M an orientation because jacobian determinants of holomorphic maps are positive, and that the coordinate transformations are of class C_l^∞.

A map $f: M \to N$ of Riemann surface laminations is holomorphic if it is continuous and maps each leaf of M holomorphically to a leaf of N. The subset of $\mathscr{C}_l(M, N)$ consisting of holomorphic maps is denoted by $\mathcal{O}(M, N)$. Cauchy's integral formula implies $\mathcal{O}(M, N) \subset \mathscr{C}_l^r(M, N)$ for all $0 \leq r \leq \infty$.

Two riemannian metrics g, g' on a surface lamination M are conformally equivalent if $g' = \eta g$ for some positive function. A diffeomorphism $f: M \to N$ of surface laminations with riemannian metrics g_M and g_N is called conformal if the metrics g_M and $f^* g_N$ on M are conformally equivalent.

Suppose we are given an oriented surface lamination M with a riemannian metric g. A flow box (V, ψ) is isothermal for g if $g = \lambda(z, t)|dz|^2$ in $\psi(V)$. The orientation of M is determined by a (regular) cover $\{U_i, \varphi_i\}$ by flow boxes such that the jacobian determinants of the overlaps are all positive. If the metric in one of the given flow boxes (U, φ), $\varphi(U) = D \times T$, has the form $\lambda(z, t)|dz + \beta(z, t)\, d\bar{z}|^2$, we can always find, by invoking Riemann's mapping theorem for variable metrics, an orientation preserving β-conformal diffeomorphism of laminations $f: D \times T \to D \times T$. Thus $(U, f \circ \varphi)$ is an isothermal flow box, and the cover $\{U_i, f_i \circ \varphi_i\}$ gives M the structure of a Riemann surface lamination because now the coordinate changes are holomorphic. This and other facts are summarized in the following theorem; see also [11, Ap. A].

THEOREM 3.2. − *Let* M *be an oriented surface lamination with a riemannian metric* g. *Then* M *always has a cover (in the given orientation) by isothermal flow boxes for* g. *Any such cover gives* M *the structure of a Riemann surface lamination. Two riemannian metrics define the same Riemann surface structure if and only if they are conformally equivalent.*

A Riemann surface lamination always has metrics inducing its complex structure (conformal metrics).

We say that a Riemann surface lamination is hyperbolic, euclidean or spherical whenever all the leaves are of the respective conformal type. For a compact surface lamination these are topological properties: the conformal type of a leaf is independent of the complex structure on the lamination. Unlike Riemann surfaces, this is not an exhaustive classification of surface laminations.

We conclude this section with some technical results to be used later.

PROPOSITION 3.3. – *Let* M *and* N *be Riemann surface laminations. The space* $\mathcal{O}(M, N)$ *is closed in* $\mathscr{C}_l^r(M, N)$ *for all* $0 \leqq r \leqq \infty$.

Proof. – This is a local question, so we may assume that $M = D \times T$ and $N = D' \times T'$. Let $f_n : M \to N$ be a sequence of holomorphic mappings which converges uniformly to $f : M \to N$. Then f is leafwise continuous. Hence we may further assume that $N = D'$, a lamination with one leaf. Now we can view f_n as a sequence of holomorphic maps from D to D' which converges uniformly to f, so f is also holomorphic. This shows that $\mathcal{O}(M, N)$ is closed in $\mathscr{C}_l^0(M, N)$. The Cauchy integral formula gives the induction step needed to complete the proof.

The next proposition follows from similar considerations.

PROPOSITION 3.4. – *Let* M *and* N *be Riemann surface laminations. Let* g_M *and* g_N *be conformal metrics on* M *and* N. *Let* $\{ f_n \} \subset \mathcal{O}(M, N)$ *be a sequence of holomorphic maps which converges to* $f : M \to N$. *Then* $df_n \to df$ *and* $|df_n(x)| \to |df(x)|$ *for any* $x \in M$.

4. Hyperbolic surface laminations

In this section we prove the following

THEOREM 4.1. – *Let* M *be a compact oriented surface lamination and let* g *be a riemannian metric on* M. *Then* g *is conformal to a metric for which all leaves have curvature* -1 *if and only if* $\chi(M, \mu) < 0$ *for every positive invariant transverse measure* μ.

Remark that the theorem includes the case no invariant transverse measure exists. Thus

COROLLARY 4.2. – *If* M *is a compact Riemann surface lamination with no invariant transverse measure, then* M *has a conformal metric with curvature* -1 *on each leaf.*

The proof has two parts. First we show that under the hypothesis $\chi(M, \mu) < 0$ the universal cover of every leaf is conformal to the unit disc. In this case any given riemannian metric g on M is conformal, on each leaf, to a unique metric of curvature -1. In the second part we show these metrics glue together to define a riemannian metric on M.

Needless to say, the other implication of this theorem is obvious: if g is a metric such that all leaves have constant curvature -1, and if μ is a positive invariant transverse measure, then $\chi(M, \mu) = -\mathrm{mass}(\mu) < 0$.

4.1. HYPERBOLIC SURFACE LAMINATIONS. — From now on M will denote a compact Riemann surface lamination and g a conformal metric on M. First we will show that if M contains a leaf L which is not a hyperbolic Riemann surface, then there is a positive invariant transverse measure μ with $\chi(M, \mu) \geq 0$. This is elementary if L is compact, for then it would be either a sphere or a torus, and the corresponding Dirac measure works. For noncompact L, more general existence theorems like [16, II.8] or [10], even if they apply, are inappropriate as we may not be able to compute $\chi(M, \mu)$ explicitly. However, our euclidean leaf L will always support an averaging sequence, and after some modifications we will be able to compute the Euler characteristic, thus proving the following:

THEOREM 4.3. — *If* $\chi(M, \mu) < 0$ *for every positive invariant transverse measure* μ, *then all leaves are hyperbolic Riemann surfaces. In fact, if* L *is a euclidean leaf, then there exists* μ *with support in* $\bar{L}$ *and* $\chi(M, \mu) = 0$.

There are four possibilities for a non-hyperbolic leaf, two of which, the torus and the sphere, have just been treated. The other two cases are: L is conformal to the euclidean plane or L is conformal to the euclidean cylinder.

Recall that if $f: R \to L$ is a conformal diffeomorphism between riemannian surfaces, then the jacobian $J(f)$ of f is related to the differential df by $J(f) = |df|^2$, where $|df|$ is the norm of the differential of f with respect to the riemannian metrics of R and L. Hence, if D is a regular domain in R with smooth boundary ∂D, the change of variable formula gives

$$\text{Area}(f(D)) = \int_D |df|^2$$

and

$$\text{Length}(\partial f(D)) = \text{Length}(f(\partial D)) = \int_{\partial D} |df|,$$

where the first integral is with respect to the riemannian volume of R and the second with respect to the induced riemannian volume on ∂D.

Suppose our euclidean leaf L is a plane. We have a conformal diffeomorphism $f: C \to L$, where C carries the euclidean metric. Denote by B_r the disc of radius r in C centered at 0 and $L_r = f(B_r)$. Then

$$\text{Area}(L_r) = \int_{B_r} |df|^2 = \int_0^r \left(\int_{\partial B_s} |df|^2 \right) ds.$$

Hence

$$\text{Length}(\partial L_r)^2 = \left(\int_{\partial B_r} |df| \right)^2 \leq 2\pi r \int_{\partial B_r} |df|^2 = 2\pi r \frac{d}{dr} \text{Area}(L_r).$$

If

$$\liminf_{r \to \infty} \frac{\text{Length}(\partial L_r)}{\text{Area}(L_r)} = \alpha > 0,$$

then, for r sufficiently large, say $r \geqq r_0$, the ratio Length $(\partial L_r)/\text{Area}(L_r) \geqq \alpha/2$, and so

$$\infty = \int_{r_0}^{\infty} \frac{dr}{2\pi r} \leqq \int_{r_0}^{\infty} \frac{1}{\text{Length}(\partial L_r)^2} \frac{d}{dr} \text{Area}(L_r)\, dr$$

$$\leqq (4/\alpha^2) \int_{r_0}^{\infty} \frac{1}{\text{Area}(L_r)^2} \frac{d}{dr} \text{Area}(L_r)\, dr = \frac{4}{\alpha^2 \text{Area}(L_{r_0})}.$$

Therefore, to elude a contradiction, there must be a sequence $r_i \to \infty$ such that the corresponding regions $\{L_i,\ i \to \infty\}$ in the leaf L form an averaging sequence as in section 2.3.

When L is a Riemann surface diffeomorphic to a cylinder and holomorphically covered by $\mathbf{C}$, the uniformization theorem and Teichmüller theory imply that L is conformal to the euclidean cylinder $\mathbf{A} = \mathbf{R}/\mathbf{Z} \times (-\infty, \infty)$ with metric $d\theta^2 + dr^2$. We then proceed in exactly the same way, except that now we take the regions $B_r = \mathbf{R}/\mathbf{Z} \times [-r, r]$. If $f: \mathbf{A} \to \mathbf{L}$ is a conformal diffeomorphism and $L_r = f(B_r)$, then we have

$$\text{Length}(\partial L_r)^2 \leqq 2 \frac{d}{dr} \text{Area}(L_r),$$

and the same arguments as above provide the same conclusion.

Thus we see that a noncompact euclidean leaf L supports an averaging sequence $\{L_i\}$. Note that Area $(L_i) \to \infty$, because Area $(L) = \infty$, and, as f is a diffeomorphism, $L = \bigcup_i L_i$. Since all the regions L_i are contained in the same leaf L, the measure μ_L that $\{L_i\}$ defines has support in $\bar{L}$. Moreover, as all the regions L_i are either discs or annuli, μ_L has average Euler characteristic equal to zero. But we cannot conclude that $\chi(M, \mu_L) = 0$ because we have no control on the geodesic curvature of ∂L_i. The next step is to regularize this averaging sequence. For this we need the following result from Phillips-Sullivan [12].

PROPOSITION 4.4. – *There is a finite cover of* M *by closed flow boxes such that plaques of different flow boxes intersect generically, namely*:

(a) *Boundaries of plaques intersect transversely or not at all.*

(b) *There are no triple intersections of boundaries of plaques.*

Remark that their differentiability hypothesis on M is the same as ours, and, although Phillips and Sullivan state and prove this result for foliated manifolds, their proof is also valid for compact laminations.

Let $\{P_i\}$ be the collection of all the plaques of the flow boxes in the proposition above. Then, because each flow box is compact, because there are only finitely many of them, and because the riemannian metric on M is smooth, there exist, as in [12],

(1) A lower bound $\varepsilon_0 > 0$ on the distance, measured along the boundary of a plaque, between intersection points with boundaries of other plaques.

(2) A lower bound $\delta_0 > 0$ on the area of a non-empty sector of a plaque. A sector of a plaque P is a subset of the form

$$P \cap P_{r_1} \cap \ldots \cap P_{r_i} \cap P'_{r_{i+1}} \cap \ldots \cap P'_{r_{i+j}}$$

for $i, j \geq 0$, where P_r are plaques and "prime" denotes complement.

(3) A positive Lebesgue number v for the cover of the lamination by plaques.

(4) Upper bounds κ_0 and K on the absolute value of the geodesic curvature of the boundary of any plaque and of the curvature of the leaves.

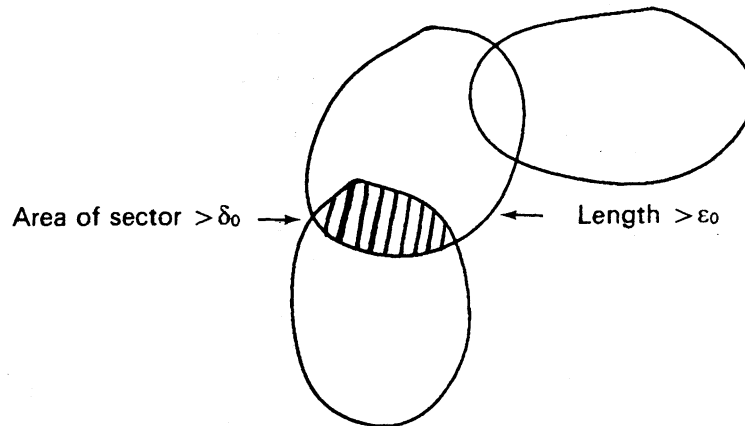(5) Upper bounds A and C for the area and circumference of any plaque.

Fig. 1.

Let R_i be the union of plaques (from the flow boxes in Proposition 4.4) intersecting L_i. To see that $\{R_i\}$ is an averaging sequence and that the invariant transverse measure it defines has the desired property we need some topological estimates in the spirit of [12].

A plaque of R_i contributing to ∂R_i will be called a boundary plaque. Then, by (5):

$$\text{Length}(\partial R_i) \leq C \cdot \#\{\text{Boundary Plaques}\}.$$

Since each boundary plaque of R_i contributes at least one sector to Area$(R_i \setminus L_i)$ not contributed by another plaque, we have

$$\text{Area}(R_i \setminus L_i) \geq \delta_0 \cdot \#\{\text{Boundary Plaques}\}.$$

Therefore:

$$\text{Length}(\partial R_i) \leq \frac{C}{\delta_0} \text{Area}(R_i \setminus L_i).$$

By (2) and (5), any collection of sectors (with disjoint interior) in a plaque contains no more than A/δ_0 elements. By (3), we can cover ∂L_i with $(1/v)$ Length (∂L_i) plaques,

and any other plaque contributing to $R_i \setminus L_i$ intersects one of these. Thus:

$$\frac{1}{A} \text{Area}(R_i \setminus L_i) \leqq \# \{ \text{Plaques meeting } R_i \setminus L_i \} \leqq \frac{A}{v \delta_0} \text{Length}(\partial L_i).$$

All this implies that

$$\text{Length}(\partial R_i) \leqq \frac{A^2 C}{v \delta_0^2} \text{Length}(\partial L_i).$$

Remark that $\text{Area}(R_i) \geqq \text{Area}(L_i) \rightarrow \infty$. Therefore

$$\lim_{i \rightarrow \infty} \frac{\text{Length}(\partial R_i)}{\text{Area}(R_i)} \leqq \lim_{i \rightarrow \infty} \frac{A^2 C}{v \delta_0^2} \frac{\text{Length}(\partial L_i)}{\text{Area}(L_i)} = 0,$$

so $\{ R_i \}$ is an averaging sequence which defines a non-trivial invariant transverse measure μ_R.

To compute $\chi(M, \mu_R)$, note that, by the Gauss-Bonnet Theorem,

$$\int_{R_i} \Omega = 2 \pi \chi(R_i) - \int_{\partial R_i} \kappa_g - \sum_{p \in \partial R_i} \alpha_p$$

where Ω is the curvature form representing the Euler class $e_l(M)$ of TM, κ_g denotes geodesic curvature and α_p is the exterior angle at a corner p of ∂R_i. Then

$$|\chi(M, \mu_R)| = \lim_{i \rightarrow \infty} \frac{1}{\text{Area}(R_i)} \left| \int_{R_i} \Omega \right| \leqq \lim_{i \rightarrow \infty} \frac{1}{\text{Area}(R_i)} \left\{ 2 \pi |\chi(R_i)| + \int_{\partial R_i} |\kappa_g| + | \sum_{p \in \partial R_i} \alpha_p | \right\}.$$

By (1) above, the minimum distance between consecutive corners is ε_0, thus

$$| \sum_{p \in \partial R_i} \alpha_p | \leqq \frac{\pi}{\varepsilon_0} \text{Length}(\partial R_i).$$

By (4),

$$\int_{\partial R_i} |\kappa_g| \leqq \kappa_0 \text{Length}(\partial R_i)$$

Finally, note that all regions R_i are contained in the same leaf L which is either a plane or a cylinder, and that each R_i is connected. Thus, if $\# \{ \pi_0(\partial R_i) \}$ denotes the number of components of ∂R_i, we have

$$\chi(R_i) = 2 - \# \{ \pi_0(\partial R_i) \}$$

By (1), the length of each component of ∂R_i is greater than ε_0 thus

$$|\chi(R_i)| \leqq 2 + \# \{ \pi_0(\partial R_i) \} \leqq 2 + \frac{1}{\varepsilon_0} \text{Length}(\partial R_i).$$

Putting all these estimates together,

$$|\chi(M, \mu_R)| \leq \lim_{i \to \infty} \frac{\text{Length}(\partial R_i)}{\text{Area}(R_i)} \left\{ \frac{2\pi}{\varepsilon_0} + \kappa_0 + \frac{\pi}{\varepsilon_0} \right\} + \lim_{i \to \infty} \frac{4\pi}{\text{Area}(R_i)} = 0,$$

and this concludes the proof of Theorem 4.3.

4.2. THE UNIFORMIZATION MAP. — Let Δ_r be the open disc of radius r in the complex plane. We write Δ for the unit disc. The Poincaré metric on Δ_r is given by:

$$g_P = \frac{r^4}{(r^2 - |z|^2)^2} |dz|^2.$$

If $f : \Delta_r \to L$ is a holomorphic map to a riemannian surface L, then

$$|df(z)|_{\Delta_r} = (1 - |z/r|^2) |df(z)|_e$$

where $|df(z)|_e$ is the norm of $df(z)$ with respect to the euclidean metric $g_e = |dz|^2$ on Δ_r.

The classical Schwarz lemma reads as follows.

LEMMA 4.5 (Schwarz). — *A holomorphic map $f : \Delta \to \Delta$ is distance decreasing for the Poincaré metric, i.e., $|df(z)| \leq 1$ for all $z \in \Delta$. In fact, if $|df(z)| = 1$ for some $z \in \Delta$, then f is an isometry.*

Let L be a hyperbolic Riemann surface with a conformal metric g_L. By the uniformization theorem for Riemann surfaces there exists a holomorphic covering map $u : \Delta \to L$. The lifted metric $u^* g_L$ is conformal to the Poincaré metric g_P, that is, there is a positive function

$$\tilde{\eta} : \Delta \to \mathbf{R}$$

such that

$$u^* g_L = \tilde{\eta}^2 g_P.$$

The function $\tilde{\eta}$ is invariant under the deck transformations of the covering $u : \Delta \to L$ because they are isometries of the Poincaré metric, and so it descends to a positive function $\eta : L \to \mathbf{R}$. Since g_P has constant curvature -4, so does the metric $(1/\eta^2) g_L$ on L.

This function η is called the uniformization map of L. It is independent of the chosen holomorphic covering of L by the unit disc: any two coverings differ by a conformal diffeomorphism of Δ, which is an isometry with respect to the Poincaré metric. Furthermore, if $u_x : \Delta \to L$ is a holomorphic covering map with $u_x(0) = x$, then

$$|du_x(0)| = g_L\left(du_x(0)\frac{\partial}{\partial z}, du_x(0)\frac{\partial}{\partial z} \right)^{1/2} = \eta(x)$$

Note that if $u : \Delta \to L$ is another holomorphic covering map with $u(0) = x$, then Schwarz's lemma implies that $u(z) = u_x(az)$ with $|a| = 1$, and so $|du(0)| = |du_x(0)|$.

If M is a compact oriented surface lamination as in the statement of Theorem 4.1, then all its leaves are hyperbolic Riemann surfaces and we define the uniformization map

$$\eta : M \to (0, \infty)$$

of M by putting together the uniformization maps of the leaves: $\eta(x) = |du_x(0)|$, where u_x is a holomorphic covering of L_x with $u_x(0) = x$.

The proof of Theorem 4.1 would be completed by showing that η is a smooth function on the lamination M. This uniformization map was studied by Verjovsky [18] and by Ghys [9], who also proved the following proposition.

PROPOSITION 4.6. — *The uniformization map is lower semicontinuous.*

Proof. — Let $u : \Delta \to L$ be a holomorphic covering of L with $u(0) = x_0$. For $r < 1$, let $v_r : \bar{\Delta} \to L$ be defined by $v_r(z) = u(rz)$. Since $\bar{\Delta}$ is simply-connected, the local product structure of the lamination and the existence of a normal bundle imply v_r extends to a smooth map

$$\phi : \bar{\Delta} \times T \to M,$$

($\bar{\Delta} \times T$ a trivial lamination, T a compact regular transversal through x_0) which is locally a homeomorphism because u is, and such that:

(1) $\phi|_{\bar{\Delta} \times \{t_0\}} = v_r$,

(2) $\phi(\bar{\Delta} \times \{t\}) \subset L_{\phi(0,\,t)}$,

(3) $\phi(\bar{\Delta} \times \{s\}) \cap \phi(\bar{\Delta} \times \{t\}) = \varnothing$ if $s \neq t$.

Let $\phi^* g$ be the pullback riemannian metric on the trivial lamination $\bar{\Delta} \times T$. Writing

$$\phi^* g = \lambda(z, t) |dz + \beta(z, t) d\bar{z}|^2$$

and applying the Riemann mapping theorem for variable metrics we obtain a diffeomorphism of laminations

$$f : \bar{\Delta} \times T \to \bar{\Delta} \times T,$$

with $f_t = f(., t)$ the normalized $\beta(., t)$-conformal diffeomorphism. Thus, the complex structure determined by $f_t^* |dz|^2$ on Δ and the one determined by $\phi^* g|_{\Delta \times \{t\}}$ coincide. Therefore, there is a smooth function

$$\sigma : \Delta \times T \to \mathbf{R}$$

such that

$$\psi^* g = \sigma^2 g_P$$

where g_P is the Poincaré metric on Δ and $\psi = \phi \circ f^{-1}$. Note that

$$\psi^* g|_{\bar{\Delta} \times \{t\}} = (\phi \circ f_t^{-1})^* g_{L_{\phi(0,\,t)}}.$$

Since ϕ is a locally a homeomorphism and $\phi(0, t_0) = x_0$, there is a neighborhood U of $(0, t_0)$ in $\Delta \times T$ such that $\psi|_U : U \to \psi(U)$ is a homeomorphism. Then the function

$$q = \sigma \circ \psi|_U^{-1}$$

is well defined and continuous on the neighborhood $\psi(U)$ of x_0.

Let $(z, t) \in U$ and $x = \psi(z, t) \in \psi(U)$. Let $\psi_t = \psi|_{\Delta \times \{t\}} : \Delta \to L_x$. Recalling that the uniformization map was determined by

$$u_x^* g_{L_x} = \eta^2 g_P,$$

it easily follows that the lifted map $\tilde\psi_t : \Delta \to \Delta$ satisfies

$$\tilde\psi_t^* g_P = \frac{\sigma^2}{\eta^2 \circ \tilde\psi_t} g_P,$$

so it is holomorphic. By Schwarz's lemma:

$$\left| d\tilde\psi_t(w) \right| = \frac{\sigma(w)}{\eta(\psi_t(w))} \leq 1$$

for all $w \in \Delta$. As every $x \in \psi(U)$ is of the form $x = \psi_t(z)$, we get

$$\eta(x) \geq \sigma(\psi_t^{-1}(x)) = q(x).$$

Note that for $x_0 = \psi(0, t_0)$ we have $\psi_0 = \psi|_{\Delta \times \{0\}} = v_r$ with $v_r^* g_{L_0} = \sigma^2 g_P$. This implies

$$q(x_0) = \sigma(\psi^{-1}(x_0)) = \sigma(0) = |dv_r(0)| = r|du(0)| = r\eta(x_0).$$

Let $\{x_n\} \subset M$ be a sequence converging to x_0. For $r < 1$ construct the corresponding q as above. Since $\psi(U)$ is a neighborhood of x_0, $x_n \in \psi(U)$ for almost all n. Hence

$$\eta(x_n) \geq q(x_n)$$

for large n. Since q is continuous on $\psi(U)$

$$\liminf_{x_n \to x_0} \eta(x_n) \geq \lim_{x_n \to x_0} q(x_n) = q(x_0) = r\eta(x_0).$$

As this holds for arbitrary $r < 1$, it follows that

$$\liminf_{x \to x_0} \eta(x) \geq \eta(x_0),$$

which proves the lower semicontinuity of η at x_0.

To prove the upper semicontinuity of η we need some preliminary facts. First, a lemma from [2].

LEMMA 4.7 (Brody). $-$ *Let $f : \Delta_\rho \to L \subset M$ be a holomorphic map and, for each $r \in [0, 1]$, let $f_r(z) = f(rz)$. Suppose $|df(0)| \geq a > 0$. Then there exists $r \in [0, 1]$ and a conformal*

automorphism T *of* Δ_ρ *such that* $h = f_r \circ T$ *satisfies*:

$$\sup_{|z| < \rho} |dh(z)| = |dh(0)| = a.$$

We need to compare the distance on the leaves with an arbitrary distance on M. A riemannian metric g on M induces a distance d_l on the leaves as follows: if x, y are in the same leaf L then $d_l(x, y)$ is the infimum of the lengths of piecewise smooth curves α in L from x to y, that is

$$d_l(x, y) = \inf_\alpha \int |\alpha'| = \inf_\alpha \int g(\alpha_*(d/dt), \alpha_*(d/dt))^{1/2}.$$

Although we can certainly construct a distance on M for which the inclusions of the leaves are all Lipschitz-1, the referee suggested the following "equicontinuity" should be enough.

LEMMA 4.8. — *Let* d_M *be any distance on* M *compatible with its topology. Given any* $\varepsilon > 0$ *there is* $\delta > 0$ *such that* $B_l(x, \delta) \subset B_M(x, \varepsilon)$ *for any* $x \in M$.

Proof. — The basic fact to note is this: If $D \times T$ is a flow box and α is a path in D, the continuity of the riemannian metric implies that

$$t \in T \mapsto \text{Length of } \alpha \text{ in } D \times \{t\}$$

is a continuous function on T. Let $\{x_n\}$ be a sequence in M, and, by compactness, assume $x_n \to x$. Let $D \times T$ be a flow box around x such that $D \times T \subset B_M(x, \varepsilon/2)$. Here D is a disc in the plane centered at 0 so that $x = (0, t_0)$ and $x_n = (z_n, t_n)$. Then $z_n \to 0$ (both in M and in the plaque through t_0). From the basic fact above, it follows that there is $\delta > 0$ such that, for large n,

$$B_l(x_n, \delta) \subset D \times T \subset B_M(x, \varepsilon/2) \subset B_M(x_n, \varepsilon).$$

From this, arguing by contradiction, the conclusion of the lemma follows.

We now show that there is a uniform bound for the norms of the differentials of holomorphic maps $\Delta \to M$. This is related to [2]. First, let $f_n : \Delta \to M$ be a sequence of holomorphic maps and suppose $|df_n(0)| = r_n$ increases to infinity. By precomposition with $z \in \Delta_{r_n} \mapsto z/r_n \in \Delta$ we may consider a sequence of holomorphic maps $f_n : \Delta_{r_n} \to M$ such that $|df_n(0)|_n = 1$, where $|.|_n$ is the norm with respect to the Poincaré metric on Δ_{r_n}. By Brody's lemma, there exist holomorphic maps $h_n : \Delta_{r_n} \to M$ such that

$$|dh_n(0)|_n = \sup_{|z| < r_n} |dh_n(z)|_n = 1$$

With respect to the euclidean metric we have

$$|dh_n(0)|_e = 1 = \sup_{|z| < r_n} (1 - |z/r_n|^2) |dh_n(z)|_e.$$

Fix m and let $n > m$. Then the restriction of h_n to Δ_{r_m} satisfies

$$\sup_{|z| < r_m} \left| dh_n(z) \right|_m = \sup_{|z| < r_m} \frac{1 - |z/r_m|^2}{1 - |z/r_n|^2} \left| dh_n(z) \right|_n \leq c_m,$$

where c_m is a constant that depends on r_m only. Hence, by the integral expression for the distance on the leaves,

$$d_l(h_n(z), h_n(w)) \leq c_m d_{r_m}(z, w)$$

for all z, w in Δ_{r_m} and $n > m$. Here d_{r_m} is the distance of the Poincaré metric on Δ_{r_m}.

Let $\varepsilon > 0$ be given and let $z \in \Delta_{r_m}$. If δ corresponds to ε as in Lemma 4.8 and if U is the ball in Δ_{r_m} of radius δ/c_m, then

$$d_M(h_n(z), h_n(w)) < \varepsilon$$

for all $w \in U$ and all $n > m$. This means, by definition, that $\{h_n\}_{n>m}$ is equicontinuous on Δ_{r_m}. Since M is compact, Ascoli's theorem implies that $\{h_n\}_{n>m}$ is a relatively compact family in $\mathscr{C}(\Delta_{r_m}, M)$. Define by induction on m a sequence $\{h_{n,m}\}_n$ such that $\{h_{n,m}\}$ is a subsequence of $\{h_{n,m-1}\}$, $h_{n,1} = h_n$ and $\{h_{n,m}\}_n$ converges uniformly in Δ_{r_m}. Then the diagonal sequence $\{h_{m,m}\}$ converges uniformly in every Δ_{r_m} and its limit is a holomorphic map $h: \mathbf{C} \to M$. This map is not constant because, by using Proposition 3.4,

$$\left| dh(0) \right|_e = \lim_{m \to \infty} \left| dh_{m,m}(0) \right|_e = 1.$$

But this, by Liouville's theorem, contradicts the fact that the holomorphic covering of every leaf is the unit disc.

In general, if there are sequences $\{f_n\}$ in $\mathcal{O}(\Delta, M)$ and $\{z_n\}$ in Δ such that $|df_n(z_n)| \to \infty$, let T_n be the conformal automorphism of Δ with $T_n(0) = z_n$. Then the sequence of holomorphic maps $h_n = f_n \circ T_n$ satisfies $|dh_n(0)| = |df_n(z_n)|$, and the previous arguments apply.

This discussion has established that there is a constant C such that $|df(z)| \leq C$ for all $f \in \mathcal{O}(\Delta, M)$ and all $z \in \Delta$. It then follows, by Lemma 4.8, that $\mathcal{O}(\Delta, M)$ is a equicontinuous family of $\mathscr{C}(\Delta, M)$. The Ascoli theorem and Proposition 3.3 imply that:

PROPOSITION 4.9. — *If* M *is compact and all its leaves are hyperbolic, then* $\mathcal{O}(\Delta, M)$ *is compact.*

To conclude the proof of the continuity of the uniformization map, we need the following elementary consequence of Schwarz's lemma and the definitions.

LEMMA 4.10. — *The uniformization map satisfies*

$$\eta(x) = \max\{ |df(0)|; f \in \mathcal{O}(\Delta, M), f(0) = x \}.$$

Finally, if $x_n \to x$ and $u_n: \Delta \to L_{x_n}$, $u_n(0) = x_n$ is a sequence of holomorphic coverings of leaves, then there exists a subsequence of $\{u_n\}$ which converges to a holomorphic

map $u: \Delta \to L_x$. Then $|du_n(0)| \to |du(0)|$, and the previous lemma implies that η is upper semicontinuous. Together with Proposition 4.6 we have

PROPOSITION 4.11. — *The uniformization map η is continuous.*

We shall use the following consequence.

COROLLARY 4.12. — *The family of $\mathcal{O}(\Delta, M)$ consisting of those holomorphic maps which are holomorphic coverings of leaves is compact.*

As remarked at the beginning of this section, η is smooth on each leaf. Next we study how its partial derivatives along the leaves depend on the transverse parameters.

Let $x_0 \in M$ and let (U, φ) be a flow box around x_0, $\varphi: U \to D \times T$ and $\varphi(x_0) = (z_0, t_0)$. We may assume that there is a trivialization Z of TM over U so that $(\varphi^{-1})_*(\partial/\partial z) = Z$. We define a map

$$v: \Delta \times T \to M$$

as follows. For $t \in T$, $v(., t): \Delta \to M$ is the holomorphic covering of the leaf through the point $\varphi^{-1}(z_0, t)$ with $v(0, t) = \varphi^{-1}(z_0, t)$ and such that $dv(., t)(0)(\partial/\partial z)$ and Z differ by a real positive factor. (This factor is $\eta(\varphi^{-1}(z_0, t))/|Z|_{L_{v(0,t)}}$.) This conditions determine v uniquely.

Since each $v(., t)$ is locally a diffeomorphism and their dilatation depends continuously on t, we can find a disc $\Delta_0 \subset \Delta$ centered at 0 such that

$$v: \Delta_0 \times T \to U$$

is a diffeomorphism onto each plaque in U (possibly after replacing U by a smaller flow box). Also, the distance of M in U is approximately the distance on the plaques of U plus the distance on the transversal T. (For the next argument we could assume they are equal.)

PROPOSITION 4.13. — *The map $v: \Delta_0 \times T \to U$ is a holomorphic map of surface laminations.*

Proof. — Only continuity needs to be checked. Let $(z_n, t_n) \in \Delta_0 \times T$ be a sequence converging to (w, s). Then $v(., t_n)$ is a sequence of holomorphic coverings and it has a subsequence $v(., t_{n_k})$ which converges to a holomorphic covering map u_s, and $u_s(0) = v(0, s)$ because

$$v(0, t_n) = \varphi^{-1}(z_0, t_n) \to \varphi^{-1}(z_0, s) = v(0, s).$$

Furthermore, $du_s(0)(\partial/\partial z) = \lim_k dv(0, t_{n_k})(\partial/\partial z)$, so $u_s = v(., s)$. In fact, any subsequence of $v(., t_n)$ has a subsequence which converges uniformly to $v(., s)$. Since the family of holomorphic coverings of leaves is compact, a standard argument shows that $v(., t_n)$ converges uniformly to $v(., s)$. It follows that $v(z_n, t_n) \to v(w, s)$ and this concludes the proof.

For each $x \in U$ there is a unique $(z, t) \in D \times T$ with $x = \varphi^{-1}(z, t)$ and a unique $a \in \Delta_0$ such that $v(a, t) = x$. The original riemannian metric g can be written, when restricted

to U, as $g = \lambda(z, t) |dz|^2$, where λ is a smooth function on $D \times T$. This implies, by the results of section 3.2, that

$$x \in U \mapsto |dv(., t)(v^{-1}(x))|$$

is a smooth map of laminations.

If m_a is the Moebius transformation of Δ taking 0 to a, then $v(., t) \circ m_a$ is a holomorphic covering of L_x with

$$v(., t) \circ m_a(0) = v(a, t) = x.$$

Therefore

$$\eta(x) = |d(v(., t) \circ m_a)(0)|$$
$$= |dv(., t)(a)| \, |dm_a(0)|$$
$$= |dv(., t)(a)|,$$

where the norms are taken with respect to the Poincaré metric on Δ. It follows that the partial derivatives of η with respect to the leaf variables are continuous functions of all variables. That is,

THEOREM 4.14. — *The uniformization map η is a smooth map on the lamination* M.

5. Spherical surface laminations

In this section we prove the following uniformization theorem for spherical surface laminations.

THEOREM 5.1. — *Let* M *be a compact, connected, oriented surface lamination with a nontrivial invariant transverse measure. Then* $\chi(M, \mu) > 0$ *for every positive invariant transverse measure* μ *if and only if all the leaves of* M *are spheres.*

In this situation we also have:

COROLLARY 5.2. — (*a*) M *is the total space of a fiber bundle over a compact space. The fibers, which are the leaves of* M, *are spheres.*

(*b*) *Any riemannian metric g on* M *is conformal to a metric of curvature* 1 *on each leaf. Furthermore, there is a diffeomorphism f of* M *which maps each leaf to itself and such that* $f^* g$ *is the standard metric on the fibers.*

First recall Reeb's stability theorem [14].

THEOREM 5.3 (Reeb). — *The set* M_S *of spherical leaves of a surface lamination* M *is open in* M. *Furthermore, if all leaves are spheres, then* M *is a fiber bundle over the leaf space, the fibers being the leaves.*

Perhaps a word to justify the validity of Reeb's theorem in the laminations context is in order. The first part needs no comment. For the second part one needs the additional fact that a compact leaf has a tubular neighboorhod with a transverse lamination coming from the normal bundle of a smooth embedding of the lamination M

in Hilbert space. This and the local product structure of the lamination permit us to find the locally trivial bundle structure near a spherical leaf.

We now prove Theorem 5.1 and the corollary. If all leaves are spheres, then M is the total space of a fiber bundle over the leaf space. The invariant transverse measures are the Radon measures on the base. Thus, by Gauss-Bonnet for surfaces, $\chi(M, \mu) > 0$ for all positive μ.

Assume now $\chi(M, \mu) > 0$ for all μ. By Theorem 4.3, M has no euclidean leaves. Together with Reeb's theorem, this implies that $M_H = M \setminus M_S$ is a compact hyperbolic surface lamination. Let P denote the union of the supports of all invariant transverse measures. By [16, II.10], P is a compact lamination; by hypothesis, it is non-empty. Furthermore, $M_S \subset P$ because every spherical leaf supports a Dirac measure. If $P \setminus M_S$ is non-empty, then there would be a positive invariant transverse measure μ whose support is contained in M_H. But then Theorem 4.1 would imply $\chi(M, \mu) < 0$, a contradiction. Hence $M_S = P$ and, since M is connected, the theorem follows.

The first part of the corollary is simply Reeb's stability theorem. Hence M is the total space of a fiber bundle over the leaf space B, with fiber S^2 and structure group the orientation preserving diffeomorphisms of the sphere. By Smale's theorem, the rotation group $SO(3)$ is a deformation retract of the group of orientation preserving diffeomorphisms of S^2. As $SO(3)$ is the isometry group of the standard riemannian metric g_0 on the sphere, there is a riemannian metric on M for which every leaf has curvature 1. This is the standard metric on M.

Let $\{T_i\}$ be a cover of the leaf space B so that the fiber bundle M is described by transition functions

$$\phi_{ij} : T_i \cap T_j \to SO(3) \subset \text{Diff}_+(S^2).$$

A riemannian metric on M is given by functions

$$g_i : T_i \to \text{Met}(S^2)$$

such that $\phi_{ij}(t)^* g_j(t) = g_i(t)$ for every $t \in T_i \cap T_j$. As we have seen in section 3.1, there is a continuous map

$$f_i : T_i \to \text{Diff}_+(S^2),$$

where $f_i(t)$ is the normalized diffeomorphism corresponding to Riemann surface structure determined by $g_i(t)$. Thus there are functions

$$\eta_i : T_i \to \mathscr{C}(S^2, (0, \infty))$$

such that $f_i(t)^* g_0 = \eta_i(t) g_i(t)$ for all $t \in T_i$. Then $\eta_i g_i$ is a metric (with curvature 1) for the bundle with coordinate transformations $f_i^{-1} \phi_{ji} f_j$, which is obviously isomorphic to M. This proves the corollary.

There is a theorem of Connes [5, p. 125] related to Theorem 5.1.

THEOREM 5.4 (Connes). — *Let M be a compact Riemann surface lamination. If μ is a positive invariant transverse measure with $\chi(M, \mu) > 0$, then the support of μ intersects the union of spherical leaves in a set of positive measure.*

We construct a foliation by surfaces on $S^2 \times T^2$ having a continuum of spherical leaves and two Reeb components. This will show that the conclusion of Connes' theorem cannot be strengthened to that of Theorem 5.1.

We start by foliating $S^2 \times S^1 \times [0, 1]$. At time $t = 0 \in [0, 1]$, we give $S^2 \times S^1$ the product foliation with leaves $S^2 \times \{\theta\}$. Foliations at time $t \in (0, 1)$ in $S^2 \times S^1$ are constructed by pushing away north and south poles of the spheres, so that as $t \to 1$ the foliations approach the Reeb foliation on $S^2 \times S^1 \times \{1\}$. We picture half of the construction.

t = 0   t = 1/2   t = 1
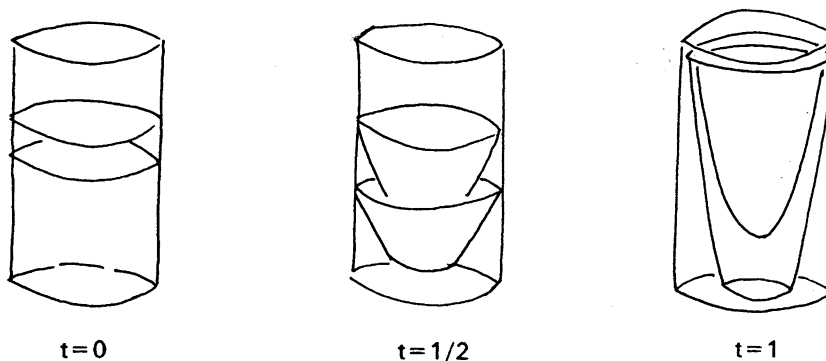
Fig. 2.

In fact, this process describes a cobordism between the product foliation and the Reeb foliation on $S^2 \times S^1$. Gluing two copies of $S^2 \times S^1 \times [0, 1]$ along their boundary produces the desired foliation of $S^2 \times T^2$. This foliation has an invariant measure with positive Euler characteristics, but not all the leaves in its support are spheres.

6. Euclidean surface laminations

It is not difficult to construct a surface lamination M for which all Euler characteristics are zero yet not all its leaves are euclidean. For example, by introducing a Reeb component in a foliated three-manifold with no invariant transverse measure, we obtain a foliation with both euclidean and hyperbolic leaves, and whose only invariant measures are multiples of the Dirac measure on the toroidal leaf.

But even when all leaves are euclidean and all the Euler characteristics are zero there may be no metric for which all leaves are flat. For instance, all the Euler characteristics of the Reeb foliation of the three-sphere are zero. From the usual representation of a planar leaf as a surface of revolution, we see it is conformal to the plane. On the other hand, the volume function of that leaf has linear growth, and this is a fact independent of the riemannian metric on the foliation. Since the volume function of a plane with a flat metric has quadratic growth, we conclude that there is no riemannian metric on the Reeb foliation which is flat along the leaves.

This shows there is no uniformization theorem for euclidean laminations in terms of invariant transverse measures only. In trying to decide when a riemannian metric on a surface lamination is conformal to a flat one, harmonic measures naturally appear.

6.1. LINE BUNDLES AND THE EULER CLASS. — We sketch the sheaf theoretic construction of the Euler class of a Riemann surface lamination M. Let $\mathcal{O}$ be the sheaf of germs of holomorphic functions on M. Let $\mathcal{O}^*$ be the sheaf of nowhere vanishing holomorphic functions. Let $\mathbf{Z}$ be the sheaf of locally constant integer valued functions on M. The exponential sheaf sequence

$$0 \to \mathbf{Z} \stackrel{2\pi i}{\to} \mathcal{O} \stackrel{\exp}{\to} \mathcal{O}^* \to 0$$

is exact and produces the exact cohomology sequence

$$\ldots \to H^1(M, \mathcal{O}) \to H^1(M, \mathcal{O}^*) \stackrel{c}{\to} H^2(M, \mathbf{Z}) \to \ldots$$

Elements of $H^1(M, \mathcal{O}^*)$ are called holomorphic line bundles over M. The homomorphism c associates to a holomorphic line bundle ξ its Chern class $c(\xi)$. If

$$\rho : H^2(M, \mathbf{Z}) \to H^2(M, \mathbf{R}_l)$$

is the map induced in cohomology by the inclusion of sheaves $\mathbf{Z} \to \mathbf{R} \to \mathbf{R}_l$, composition with the Chern homomorphism gives the Euler class of ξ:

$$e_l(\xi) = \rho(c(\xi)) \in H^2(M, \mathbf{R}_l).$$

We obtain the Euler class $e_l(M)$ of M by applying this process to the holomorphic tangent bundle TM. To obtain a representative, let $\{U_i\}$ be a cover of M as in Section 3.2. The cocycle

$$\xi_{ji} = \partial\lambda_{ji}/\partial z_i : U_i \cap U_j \to \mathbf{C}^*$$

represents TM in $H^1(M, \mathcal{O}^*)$. A conformal metric on M is given by positive real valued functions g_i on U_i such that $g_i = g_j |\xi_{ji}|^2$. The curvature 2-form is

$$\Omega = \frac{1}{2\pi i} \partial\bar{\partial} \ln g_i.$$

Here $\partial, \bar{\partial}$ are the usual holomorphic differential operators of complex manifolds extended in the obvious way to Riemann surface laminations. Then Ω is a real differential 2-form on M and $e_l(M) = [\Omega]$ in $H^2(M, \mathbf{R}_l)$.

In the smooth category line bundles are classified by their Chern class, but a topologically trivial line bundle may not be holomorphically trivial because $H^1(M, \mathcal{O})$ need not be trivial. For example, if TM is holomorphically trivial, then M can be parametrized by an action of $\mathbf{R}^2$ and there is a flat conformal metric.

6.2. HARMONIC MEASURES. — Harmonic measures for laminations were introduced and studied by L. Garnett in [8]. Let M be a compact lamination with a riemannian

metric g. Each leaf is a riemannian manifold and it has a Laplace operator. These laplacians glue together to define the leaf laplacian Δ (not to be confused with the unit disc), which acts on smooth functions on the lamination M.

A measure m on M is said to be harmonic, with respect the riemannian metric, if $m(\Delta f) = 0$ for every smooth function f on M. The following result is proved in [8].

THEOREM 6.1 (Garnett). — (a) *A compact riemannian lamination* M *always has a nontrivial harmonic measure.*

(b) *A measure on* M *is harmonic if and only if it locally disintegrates into a transversal sum of leaf measures, where almost every leaf measure is a positive harmonic function times the riemannian leaf measure.*

The second part (which is reminiscent of Weyl's lemma) means that if f is a smooth function with support in a flow box $U = D \times T$, then

$$m(f) = \int_T \left(\int_{D \times \{t\}} u(z, t) f(z, t) v_g \right) d\nu(t),$$

where ν is a measure on the transversal T, $u(., t)$ is a positive harmonic function on the plaque $D \times \{t\}$ for almost all t, and v_g is the volume form of the riemannian metric on the leaves.

From now on M is an oriented surface lamination with riemannian metric. Then M is also a Riemann surface lamination and the laplacian of any conformal metric on M satisfies

$$*\Delta = i \partial \bar{\partial}.$$

Thus harmonic functions on M are intrinsic to the complex structure and not just to a particular conformal metric. This suggests the following interpretation of harmonic measures.

Instead of viewing harmonic measures on a Riemann surface lamination as acting on functions, we consider them as acting on differential 2-forms because the Hodge star operator $* : \mathscr{E}^0(M) \to \mathscr{E}^2(M)$ is an isomorphism. Hence the space $\mathscr{MH}(M)$ of harmonic measures on M is the space of linear functionals on the quotient

$$\mathscr{E}^2(M)/i \partial \bar{\partial} \mathscr{E}^0(M).$$

Here we are considering real differential forms and real harmonic measures because $i \partial \bar{\partial}$ is a real operator.

THEOREM 6.2. — *The space of harmonic measures on the compact Riemann surface lamination* M *is the topological dual of* $\mathscr{E}^2(M)/i \partial \bar{\partial} \mathscr{E}^0(M)$.

Therefore harmonic measures for M are $\partial \bar{\partial}$-closed 2-currents. We previously interpreted invariant transverse measures as d-closed 2-currents. Obviously $\mathscr{MH}(M) \supset \mathscr{MT}(M)$, but in view of Garnett's theorem, these two spaces are generally different.

We need a cohomological interpretation of the space of harmonic measures. Let $\mathscr{H}$ denote the sheaf of germs of harmonic functions on the Riemann surface lamina-

tion M. That is, if $U \cong D \times T$ is a flow box, elements of $\mathscr{H}(U)$ are continuous functions $u: D \times T \to \mathbf{R}$ such that the function $u(., t)$ is harmonic on $D \times \{t\}$ for all $t \in T$.

PROPOSITION 6.3. — *There is an exact sequence of sheaves*

$$0 \to \mathscr{H} \to \mathscr{E}^0 \xrightarrow{\ i\,\partial\bar\partial\ } \mathscr{E}^2 \to 0$$

which is a fine resolution of the sheaf of germs of harmonic functions.

Proof. — A function $u: D \times T \to \mathbf{R}$ is harmonic if and only if $\partial\bar\partial u = 0$. The only point that needs comment is the surjectivity of $\partial\bar\partial$. But that follows from Dolbeault lemma with parameters. All we need is

LEMMA 6.4. — *Let $D \times T$ be a trivial surface lamination and let $f: D \times T \to \mathbf{C}$ be a smooth function with compact support. Then the assignment*

$$(z, t) \in D \times T \mapsto \frac{1}{2\pi i} \int_D \frac{f(z+w, t)}{w} \, dw \, d\bar{w}$$

determines a function $g \in \mathscr{C}_l^\infty(D \times T)$ such that $\partial g / \partial \bar{z} = f$.

Therefore, if $w(z, t) dz \wedge d\bar{z}$ is a differential 2-form on the lamination $U \cong D \times T$, we can find $g \in \mathscr{E}^0(U)$ such that $\partial\bar\partial g = w$ on some open set $V \subset U$.

From this resolution of $\mathscr{H}$ we get

COROLLARY 6.5. — *There is an isomorphism*

$$\mathrm{H}^1(M, \mathscr{H}) \cong \mathscr{E}^2(M) / i\,\partial\bar\partial\mathscr{E}^0(M).$$

We now interpret the Euler class of M as an element in the cohomology group $\mathrm{H}^1(M, \mathscr{H})$. As seen in the previous section, once we choose a cover by flow boxes $\{U_i\}$ compatible with the complex structure, a conformal metric g for M is given by a collection of positive functions g_i on U_i subject to the compatibility condition

$$g_j = g_i |\xi_{ij}|^2,$$

where $\{\xi_{ij}\}$ is the cocycle representing the holomorphic line bundle TM in $\mathrm{H}^1(M, \mathcal{O}^*)$. Then the curvature 2-form of g is

$$\Omega = \frac{1}{2\pi i} \partial\bar\partial \ln g_i.$$

If $h = \{h_i\}$ is another conformal metric on M, then $f = g_i / h_i$ is a smooth positive function on M and

$$\partial\bar\partial \ln g_i = \partial\bar\partial \ln h_i + \partial\bar\partial \ln f.$$

That is, $e_h(M) = [\Omega]$ is a well defined element in $\mathrm{H}^1(M, \mathscr{H})$. Note that there is a surjection $\mathrm{H}^1(M, \mathscr{H}) \to \mathrm{H}^2(M, \mathbf{R}_l)$ taking $e_h(M)$ to $e_l(M)$.

If M is a surface lamination with a riemannian metric g for which $e_h(M)=0$ in $H^1(M, \mathcal{H})$, then the curvature 2-form $\Omega = i\,\partial\bar\partial\sigma$ for some $\sigma \in \mathcal{E}^0(M)$, so that the metric $\exp(2\pi\sigma)g$ is flat along the leaves. This proves the last piece of the uniformization theorem for surface laminations.

THEOREM 6.6. — *Let* M *be a compact oriented surface lamination with riemannian metric* g. *Then* g *is conformal to a flat metric if and only if* $e_h(M)=0$ *in* $H^1(M, \mathcal{H})$.

For a compact Riemann surface S, Hodge theory implies $H^1(S, \mathcal{H}) \cong H^2(S, \mathbf{R})$, so that d-exact 2-forms are also $\partial\bar\partial$-exact. This is far from true for surface laminations. Let Γ be a discrete subgroup of $PSL_2\mathbf{R}$ such that $M=PSL_2\mathbf{R}/\Gamma$ is a rational homology 3-sphere. We can choose Γ so that M has a foliation with all leaves dense and hyperbolic. Then $e_l(M)=0$ in $H^2(M, \mathbf{R}_l)$ because $H^2(M, \mathbf{R})=0$. However, $e_h(M)$ is not zero in $H^1(M, \mathcal{H})$; even more, it is not in the closure of zero. To see why, note that there is a metric giving the leaves curvature -1. If $f_n:M \to \mathbf{R}$ is a sequence of smooth functions on the lamination M such that $2i\Omega = \lim_n \partial\bar\partial f_n$, then $\Delta f_n > 0$ for large n, i.e., the functions f_n are subharmonic on each leaf. Now f_n is continuous, so it reaches a maximum on the compact space M. The function f_n is then constant on the leaf containing the point of maximum value. As every leaf is dense, it is constant on M. Thus, $\Delta f_n = 0$ for all n.

On the other hand, for the Reeb foliation we have $e_l(M)=0$ and $e_h(M)$ belongs to the closure of zero in $H^1(M, \mathcal{H})$, but it is $\neq 0$ there.

7. Questions and comments

To conclude, we would like to call the reader's attention to some of the questions left open. As we have just seen, the vanishing of the Euler characteristics is not enough to characterize euclidean surface laminations. But we could try to do so by imposing some conditions on the surface lamination. A very natural one is to assume M is minimal, i.e., all its leaves are dense, and ask: could M have both euclidean and hyperbolic leaves? If the answer is negative, is there a metric of constant curvature? In relation to this, we may also ask if a minimal lamination with all leaves euclidean could be parametrized by an action of the plane. A modest answer is the following: if M is minimal, has a holomorphic differential and a euclidean leaf, then it can be parametrized by an action of the plane and has a flat metric.

In [9], E. Ghys considered an extended uniformization function. By Reeb's stability theorem we may consider laminations without spherical leaves. Then the extended map is obtained by letting the one we have used to be ∞ on euclidean leaves. He proves it is lower semicontinuous and asks whether it is continuous. A positive answer would imply the set of euclidean leaves is closed, thus giving a nice complement to Reeb's stability theorem.

Classically, open Riemann surfaces were qualified as hyperbolic or parabolic, according to whether the surface has Green's function or not. After this work was completed, we

learned that the argument with the integrals used to construct the averaging sequence within Theorem 4.3 can be generalized to completely characterize simply-connected parabolic Riemann surfaces (*cf.* L. Ahlfors, Collected Papers, vol. I, p. 84, p. 91). Using more machinery from the theory of Riemann surfaces, one can prove that a parabolic leaf always supports an averaging sequence; but, without further geometric information, the Euler characteristic may be impossible to compute. In any case, as the classification parabolic-hyperbolic is more balanced than the euclidean-hyperbolic dichotomy used here, the question: Is there a minimal lamination having parabolic and hyperbolic leaves? may be easier to answer.

We also became aware of the similarities between averaging sequences and certain aspects of Nevanlinna theory. Partly motivated by this we ask: Which compact Riemann surface laminations can be holomorphically embedded in complex projective plane $\mathbf{P}^2$? The meromorphic functions on $M \subset \mathbf{P}^2$ are linear projections onto a hyperplane. Geometrically M is like a foliated bundle over a sphere. For example, if $M \subset \mathbf{P}^2$ has an open neighborhood V carrying a holomorphic foliation which extends M, then it is hyperbolic; in fact, it has no invariant transverse measure. This can be seen directly or by reading the paper of Camacho-Lins-Sad, *Publ. Math. IHES*, **68**.

REFERENCES

[1] L. AHLFORS and L. BERS, *Riemann's Mapping Theorem for Variable Metrics*, Annals of Math., Vol. 72, 1960, p. 385-404.

[2] R. BRODY, *Compact Manifolds and Hyperbolicity*, Trans. Amer. Math. Soc., Vol. 235, 1978, p. 213-219.

[3] G. CAIRNS and E. GHYS, *Totally Geodesic Foliations of Four-Manifolds*, J. Differential Geom., Vol. 23, 1986, p. 241-254.

[4] J. CANTWELL and L. CONLON, *Leafwise Hyperbolicity of Proper Foliations. Comment. Math. Helv.*, Vol. 64, 1989, p. 329-337. A correction. Ibid., Vol. 66, 1991, p. 319-321.

[5] A. CONNES, *Sur la théorie non-commutative de l'intégration, Algèbres d'Opérateurs.* Lecture Notes in Math., Vol. 725, p. 19-143. Springer-Verlag, New York, 1979.

[6] C. EARLE and A. SCHATZ, *Teichmüller Theory for Surfaces with Boundary. J. Differential Geom.*, Vol. 4, 1970, p. 169-185.

[7] D. EPSTEIN, K. MILLETT and D. TISCHLER, *Leaves Without Holonomy. J. London Math. Soc.*, Vol. 16, 1977, p. 548-552.

[8] L. GARNETT, *Foliations, the Ergodic Theorem and Brownian Motion. J. of Funct. Anal.*, Vol. 51, 1983, p. 285-311.

[9] E. GHYS, *Gauss-Bonnet Theorem for 2-Dimensional Foliations. J. of Funct. Anal.*, Vol. 77, 1988, p. 51-59.

[10] S. GOODMAN and J. PLANTE, *Holonomy and Averaging in Foliated Sets. J. Differential Geom.*, Vol. 14, 1979, p. 401-407.

[11] C. MOORE and C. SCHOCHET, *Global Analysis of Foliated Spaces.* Springer-Verlag, New York, 1988.

[12] A. PHILLIPS and D. SULLIVAN, *Geometry of Leaves. Topology*, Vol. 20, 1981, p. 209-218.

[13] J. PLANTE, *Foliations with Measure Preserving Holonomy. Annals of Math.*, Vol. 105, 1975, p. 327-361.

[14] G. REEB, *Sur certaines propriétés topologiques des variétés feuilletées.* Hermann, Paris, 1952.

[15] D. RUELLE and D. SULLIVAN, *Currents, Flows and Diffeomorphisms. Topology*, Vol. 14, 1975, p. 319-327.

[16] D. SULLIVAN, *Cycles for the Dynamical Study of Foliated Manifolds and Complex Manifolds, Invent. Math.*, Vol. 36, 1976, p. 225-255.

[17] D. SULLIVAN, *Bounds, Quadratic Differentials, and Renormalization Conjectures. Mathematics into the 21st Century.* Vol. 2. American Mathematical Society Centennial Publications, Providence, 1991.

[18] A. VERJOVSKY, *A Uniformization Theorem for Holomorphic Foliations. Contemp. Math.*, Vol. 58, III, 1987, p. 233-253.

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