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FIXED POINTS OF POLYNOMIAL MAPS. PART II. FIXED POINT PORTRAITS

BY LISA R. GOLDBERG AND JOHN MILNOR

ABSTRACT. – Douady, Hubbard and Branner have introduced the concept of a "*limb*" in the Mandelbrot set. A quadratic map $f(z)=z^2+c$ belongs to the p/q-*limb* if and only if there exist q external rays of its Julia set which land at a common fixed point of f, and which are permuted by f with combinatorial rotation number $p/q \in \mathbf{Q}/\mathbf{Z}$, $p/q \not\equiv 0$. (Compare Figure 1 and Appendix C, as well as Lemma 2.2.) This note will make a similar analysis of higher degree polynomials by introducing the concept of the "*fixed point portrait*" of a monic polynomial map.

Introduction

The object of this paper is to classify polynomial maps in one complex variable in terms of the *external rays* which land at their fixed points. To each monic polynomial we assign a *fixed point portrait*, which is a list of the angles of the rational external rays which land at the various fixed points. (*See* Section 1 for details). Except in the three appendices, we consider only polynomials with connected Julia set. The paper is organized as follows:

Section 1 contains a more detailed outline of subsequent sections, as well as an overview of the relevant concepts from complex dynamical systems. (A basic reference for this is [M2].)

Section 2 defines the *rational type* T of a fixed point z as the set of all angles of rational external rays which land at z. In the terminology of Part I, such a rational fixed point type $\mathrm{T} \subset \mathbf{Q}/\mathbf{Z}$ is an example of a *rotation set*.

Section 3 shows that the $d-1$ fixed rays cut the plane into some number of *basic regions*, each of which contains exactly one fixed point or fixed parabolic basin (= "virtual fixed point").

In section 4, we introduce the *fixed point portrait* of a polynomial. By definition, this is the collection $\{\mathrm{T}_1, \ldots, \mathrm{T}_k\}$ consisting of all rational types $\mathrm{T}_j \neq \varnothing$ of its fixed points. We outline a set of combinatorial conditions that a fixed point portrait must satisfy, and we formulate our Main Conjecture 4.2: These necessary conditions are also sufficient. In other words, we conjecture that every "candidate" fixed point portrait satisfying certain combinatorial conditions can actually be realized by a polynomial whose filled Julia set is connected. This conjecture has recently been proved by Poirier [Po1].

Sections 5, 6, 7 are devoted to establishing Conjecture 4.2 in the special case of a degree d polynomial which has d distinct repelling fixed points. Our proof relies on the study of the *critical portrait* of a polynomial: This is our name for a basic concept which was introduced and studied in the thesis of Yuval Fisher. (For a more detailed presentation, *see* [BFH].) Fisher gives a set of necessary and sufficient conditions for a collection of sets of angles to be the critical portrait of some critically pre-periodic polynomial. Section 4 summarizes basic facts about critical portraits, and recalls theorems from Fisher's thesis that we use.

Section 6 describes an algorithm that determines the fixed point portrait of a polynomial from its critical portrait.

Section 7 contains our main result. For each fixed point portrait $\{T_1, \ldots, T_d\}$ consisting of d distinct non-vacuous rational types where d is the degree, if the appropriate conditions are satisfied, we construct a compatible critical portrait satisfying Fisher's conditions. It then follows by Fisher's thesis that each such fixed point portrait can be realized by some critically pre-periodic polynomial.

Section 8 discusses further questions and problems.

The paper concludes with three appendices. Appendix A extends the exposition to polynomials whose Julia sets may not be connected. Appendix B considers the transition between different fixed point portraits as we vary the polynomial within parameter space, and Appendix C applies these ideas to prove known results about parameter space in the degree two case.

The authors want to thank A. Douady for suggesting the circle of ideas studied in this paper.

1. Overview

Let $f: \mathrm{C} \to \mathrm{C}$ be a polynomial map of degree $d \geqq 2$, and let $K = K(f)$ be its *filled Julia set*, consisting of all $z \in \mathbf{C}$ for which the orbit of z under f remains bounded. *To simplify the discussion, we will assume that f is monic, and that* $K(f)$ *is connected, or equivalently that the Julia set* $J = \partial K$ is *connected.* (For a discussion of the case where $K(f)$ is not connected, see the three Appendices.) It follows from this assumption that the complement $\mathbf{C} \setminus K(f)$ is isomorphic to the complement of the closed unit disk $\bar{D}$ under a unique conformal isomorphism

$$\psi: \quad \mathbf{C} \setminus \bar{D} \xrightarrow{\cong} \mathbf{C} \setminus K(f)$$

which is asymptotic to the identity map at infinity; and furthermore that

$$(1) \qquad \psi(z^d) = f(\psi(z)) \quad \text{for all } z \in \mathbf{C} \setminus \bar{D}.$$

For each angle $t \in \mathbf{R}/\mathbf{Z}$, the *external ray* $R_t \subset \mathbf{C} \setminus K(f)$ is defined to be the image under ψ of the half-line

$$(1, \infty) e^{2\pi i t} = \{ re^{2\pi i t} : 1 < r < \infty \}$$

which extends from the point $e^{2\pi i t}$ out to infinity in $\mathbf{C} \setminus \bar{D}$. It follows from equation (1) that $f(R_t) = R_{td}$. *In particular, note that* $f(R_t) = R_t$ *if and only if t is a fraction of the form* $j/(d-1)$. In this case, R_t will be called a *fixed ray*. *Similarly, some iterate of f maps* R_t *onto itself if and only if t is rational with denominator prime to d.* In this case, R_t will be called a *periodic ray*. Note that t is rational if and only if some image $f^{\circ n}(R_t)$ is periodic.

We are interested in the limiting values of an external ray R_t as r decreases to 1. By definition, the ray R_t *lands* at a well defined point a_t whenever this limit exists and is equal to a_t. Such a landing point always belongs to the Julia set $J = \partial K$. Putting together results due to Douady, Hubbard, Sullivan, and Yoccoz, we have the following. (Compare [M2]. For definitions, discussion and further references, *see* Section 2.)

THEOREM 1.1. — *If f is a polynomial of degree two or more, with* $K(f)$ *connected, then every periodic external ray* R_t *lands at a well defined periodic point*

$$a_t = \lim_{r \to 1} \psi(re^{2\pi i t}) \in \partial K(f),$$

which is either repelling or parabolic. Conversely, every repelling or parabolic periodic point of f is the landing point of a finite number (not zero) of external rays, all of which are necessarily periodic with the same period.

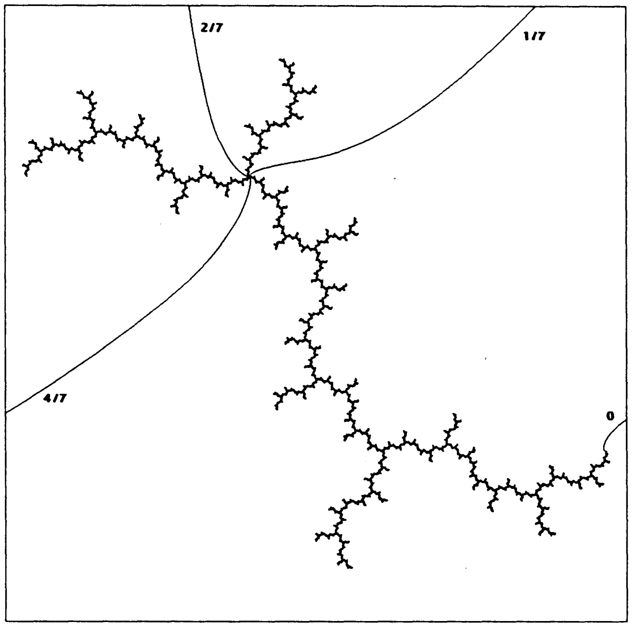
More generally, every rational external ray R_t lands at a well defined point of the Julia set. The landing point a_t is either periodic or pre-periodic according as the angle $t \in \mathbf{Q}/\mathbf{Z}$ is periodic or pre-periodic under multiplication by d. Now consider an arbitrary fixed point $f(z) = z$.

DEFINITION 1.2. — By the rational *type* $T = T(f, z)$ of a fixed point z of a monic polynomial f will be meant the set of angles of the rational external rays of $K(f)$ which land at z. In other words, $T(f, z)$ is the finite subset of $\mathbf{Q}/\mathbf{Z}$ consisting of all rational numbers t modulo 1 for which the landing point a_t of R_t is equal to z.

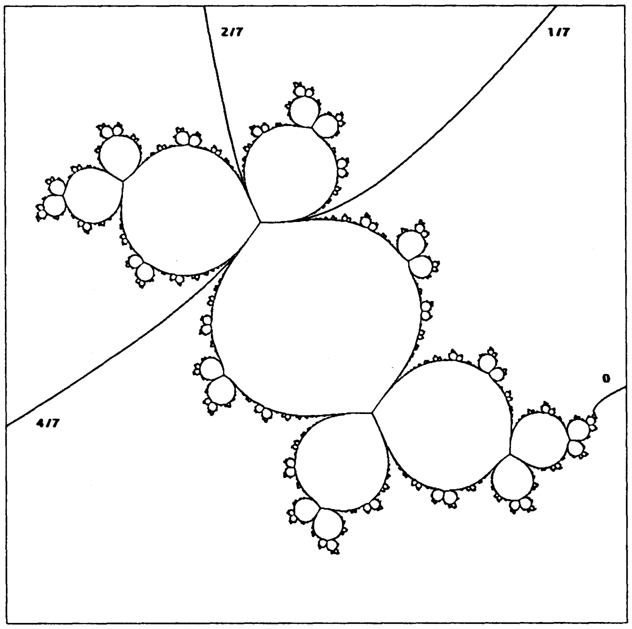
The possible fixed point types fall into three distinct classes, which we briefly describe below. (For further details *see* Part I, as well as Section 2,)

We will say that a fixed point $f(z) = z$ is *rationally invisible* if there are no rational rays at all which land at z, so that the type T is vacuous. Such a point is either attracting, or Cremer, or is surrounded by a Siegel disk. We will largely ignore such points, concentrating rather on the "rationally visible" points.

A fixed point has *rotation number* $\rho = 0$ if it is the landing point of at least one of the fixed rays $R_{j/(d-1)}$. In this case, the type T is some non-vacuous subset of the set of fixed angles $\{0, 1/(d-1), \ldots, (d-2)/(d-1)\}$. It will follow from Theorem 7.1 that all



$$z \mapsto z^2 + i$$



$$z \mapsto z^2 + \omega(2-\omega)/4, \quad \omega = e^{2\pi i/3}$$

Fig. 1. – Julia sets for two quadratic maps in the 1/3-limb. The external rays to the two fixed points have been plotted.

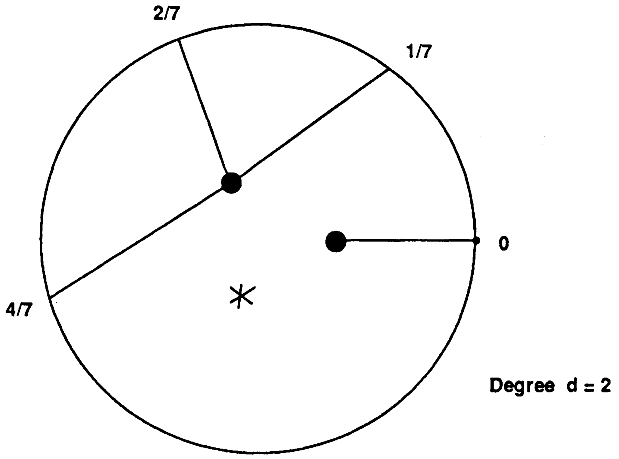


Fig. 2. — Schematic diagram for the fixed point portrait corresponding to Figure 1.
Fixed points are indicated by heavy dots. The location of the critical point is indicated by a star.

$2^{d-1}-1$ such subsets can actually occur. Fixed points of rotation number zero always exist, and will play an organizing role in our discussion.

Finally, if T is non-vacuous and does not consist of fixed angles $j/(d-1)$, then it is uniquely determined by three invariants, namely the *cardinality* $\#T$, the combinatorial *rotation number*

$$0 \neq \rho = p/q \in \mathbf{Q}/\mathbf{Z},$$

and the deployment of the elements of T with respect to the fixed angles $j/(d-1)$. Here we can take $0 < p/q < 1$ to be a fraction in lowest terms. The cardinality $\#T$ can then be expressed as a product of the form kq with $1 \leqq k \leqq d-1$. Thus we can number the elements of T as $0 < t_0 < \ldots < t_{kq-1} < 1$, with $dt_i \equiv t_{i+kp}$ (mod 1). Finally, the deployment of the elements of T with respect to the fixed angles can be described, for example, by specifying the cardinality $s_j = \#(T \cap [0, j/(d-1)))$ of the intersection of T with each half-open interval $[0, j/(d-1))$. When $k > 1$, the resulting sequence $0 \leqq s_1 \leqq \ldots \leqq s_{d-1} = kq$ is subject to certain mild restrictions. (*See* Part I.)

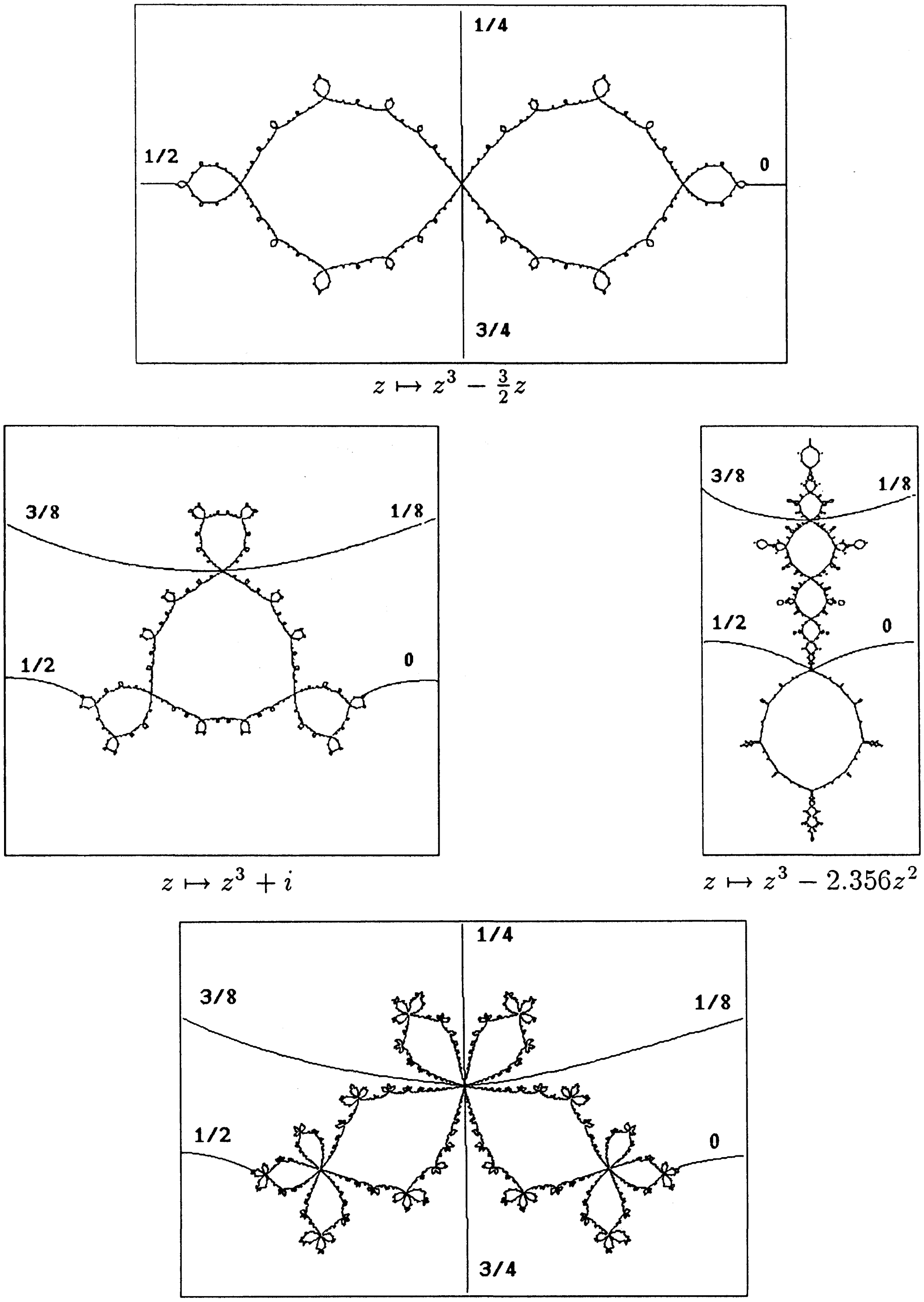
The principal concept which we propose to study is the following.

DEFINITION 1.3. — The *fixed point portrait* of a monic polynomial is the collection of types of its rationally visible fixed points. *Thus two monic polynomials f and g of degree d have the same fixed point portrait if and only if there is a one-to-one correspondence between the rationally visible fixed points of f and the rationally visible fixed points of g which preserves the type.*

As examples, Figure 1 shows the Julia sets for the quadratic polynomials

$$f_1(z) = z^2 + e^{2\pi i/3} z \qquad \text{and} \qquad f_2(z) = z^2 + i.$$

These have the same fixed point portrait, which consists of the type $T_1 = \{0\}$ with rotation number zero and the type $T_2 = \{1/7, 2/7, 4/7\}$ with rotation number 1/3. This



$z \mapsto z^3 - \frac{3}{2}z$

$z \mapsto z^3 + i$

$z \mapsto z^3 - 2.356z^2$

$z \mapsto z^3 - \frac{3}{4}z + \sqrt{-7}/4$

Fig. 3. – Four cubic Julia sets, each with one fixed point of rotation number 1/2.

portrait is indicated schematically in Figure 2. Figure 3 shows the Julia sets for four cubic polynomials. Each of these has one fixed point of rotation number 1/2. The right center Julia set also has one rationally invisible fixed point; while the other three have two distinct fixed points of rotation number zero. Figure 4 shows schematic diagrams for these four fixed point portraits. Note that the last portrait can be described as the union of the first two.

DEFINITION 1.4. — It is often convenient to compactify $\mathbf{C}$ by adding a circle of points at infinity, with one point $\lim_{r\to+\infty} re^{2\pi i t}$ corresponding to each angle $t \in \mathbf{R}/\mathbf{Z}$. We denote this compactified plane by $\copyright$, and denote the circle at infinity by $\partial\copyright \cong \mathbf{R}/\mathbf{Z}$.

In order to understand a general fixed point portrait, first consider the fixed points of rotation number $\rho = 0$. These are precisely the landing points of the $d-1$ fixed rays $R_{j/(d-1)}$. Suppose that there are n such fixed points, and let $T_1, \ldots, T_n$ be their types. Thus the T_n are disjoint non-vacuous sets with union equal to $\{0, 1/(d-1), \ldots, (d-2)/(d-1)\}$. Evidently $1 \leqq n \leqq d-1$. Note that any two of these sets T_h are "unlinked", in the following sense.

DEFINITION 1.5. — We will say that two subsets T and T′ of the circle $\mathbf{R}/\mathbf{Z}$ are *unlinked* if they are contained in disjoint connected subsets of $\mathbf{R}/\mathbf{Z}$, or equivalently if T′ is contained in just one connected component of the complement $\mathbf{R}/\mathbf{Z} \setminus T$. (In particular, T and T′ must be disjoint.) If we identify $\mathbf{R}/\mathbf{Z}$ with the boundary of the unit disk, then an equivalent condition would be that the convex closures of these sets are pairwise disjoint. As an example, if T and T′ are the types for any two distinct fixed points of f, then evidently T and T′ are unlinked.

The $d-1$ fixed rays $R_{j/(d-1)}$ will cut the complex plane into $m = d-n$ connected open subsets, say $U_1, \ldots, U_m$, which we will call *basic regions*. Compare Figure 5, which illustrates the degree six case with $m = n = 3$ and with

$$T_1 = \left\{0, \frac{2}{5}\right\}, \qquad T_2 = \left\{\frac{1}{5}\right\}, \qquad T_3 = \left\{\frac{3}{5}, \frac{4}{5}\right\}.$$

To simplify the discussion, let us assume for now that the d finite fixed points of f are all distinct. The following will be proved in Section 3.

LEMMA 1.6. — *With this hypothesis, each basic region* U_i *contains at least one critical point of* f, *and exactly one fixed point of* f.

Let T'_i be the type of the fixed point in the region U_i. This fixed point may be rationally invisible, so that $T'_i = \varnothing$. However, if $T'_i \neq \varnothing$ then it has a well defined rotation number p_i/q_i, which is an arbitrary non-zero rational modulo 1. In order to describe which fixed point types T'_i are possible for given U_i and given rotation number, we need further definitions.

By the *critical weight* $1 \leqq w(U_i) \leqq d-1$, we mean the number of critical points of f, counted with multiplicity, which are contained in the region U_i. Closely related is the *angular size* $\alpha(U_i)$ of U_i at infinity, which is defined as follows. We think of U_i as a region in the circled plane $\copyright$, and define $\alpha(U_i)$ to be the length of the intersection of ∂U_i

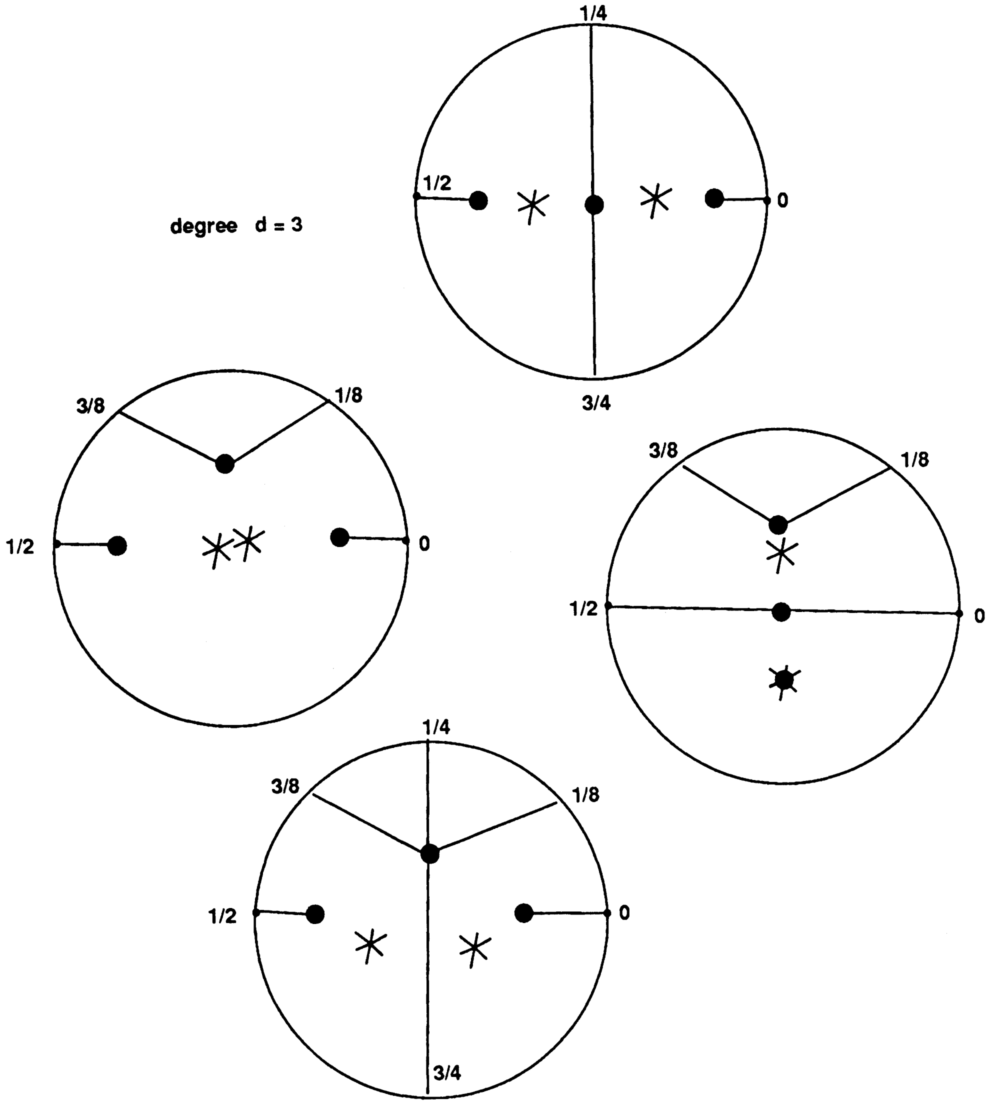


Fig. 4. – Schematic diagrams for the fixed point portraits of Figure 3.

with the circle at infinity, $\partial\mathbb{C}\cong\mathbf{R}/\mathbf{Z}$. By definition, the circle at infinity has total length equal to 1. Thus the sum of the angular sizes of these $m=d-n$ regions is $\sum\alpha(\mathrm{U}_i)=1$, while the sum of the critical weights is $\sum w(\mathrm{U}_i)=d-1$. Note that the intersection $\partial\mathrm{U}_i\cap\partial\mathbb{C}$ corresponds to a union of non-overlapping intervals $\mathrm{I}_j=[j/(d-1),\,(j+1)/(d-1)]$, each of length $1/(d-1)$, in $\mathbf{R}/\mathbf{Z}$.

LEMMA 1.7. – *The number of critical points $w(\mathrm{U}_i)$ is equal to the number of intervals I_j, $0\leqq j<d-1$ which are contained in the boundary of U_i at infinity. Thus the angular size is given by $\alpha(\mathrm{U}_i)=w(\mathrm{U}_i)/(d-1)$. When the critical weight $w=w(\mathrm{U}_i)$ equals one, there is one and only one possible fixed point type T'_i with given rotation number p/q which*

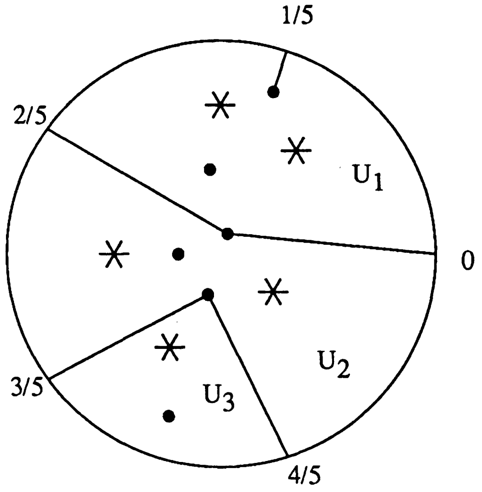


Fig. 5. — Partial schematic diagram for a typical map of degree $d=6$. In this example, the five fixed rays cut the plane into three "basic regions" U_i, each of which contains exactly one interior fixed point (indicated by a solid dot), and as many critical points (stars) as boundary fixed points.

can be placed in the basic region U_i. *However, when* $w=2$ *there are* $q+1$ *possible types of cardinality* q, *and* q *types of cardinality* $2q$.

Compare Part I, as well as Section 2. For each fixed higher value of w, one can show that the number N of distinct types can be expressed analogously as a polynomial $N_w(q)$ of degree $w-1$ in q. Note that the number of possible types is completely independent of the numerator p, the degree d, and the precise shape of the region U_i. It depends only on the denominator q and the weight w. The proof in Part I shows more explicitly that each type T'_i is uniquely determined by its rotation number, together with the cardinalities of the various intersections $T'_i \cap I_j$. Of course only w of these intersections can be non-vacuous.

Example. — If the $d-1$ fixed rays $R_{j/(d-1)}$ all land at distinct points, then there is only one basic region U_1, and its critical weight is $w=d-1$.

The main result of this paper, Theorem 7.1, gives a complete characterization of just which fixed point portraits can occur, providing that we assume that the d fixed points are all distinct and rationally visible. Our proof depends essentially on work by Yuval Fisher and by Bielefeld-Fisher-Hubbard, which is developed in Section 5.

2. Classification of fixed points

We continue to assume that f is a monic polynomial map of degree $d \geqq 2$ with $K(f)$ connected. Recall that the dynamics of f in a neighborhood of a fixed point $f(z)=z$ is controlled by the *eigenvalue* or *multiplier* $f'(z)$. The fixed point is said to be *repelling* if $|f'(z)|>1$, *attracting* if $|f'(z)|<1$, and to be *parabolic* if $f'(z)$ is a root of unity. Combining arguments of Douady, Hubbard, Sullivan, and Yoccoz, we have the following. (Compare 1.1.)

LEMMA 2.1. — *A fixed point is rationally visible* (*that is, admits at least one rational external ray*) *if and only if it is either repelling or parabolic.*

Proof Outline. — In the attracting case the point z cannot be rationally visible since z is in the interior of $K(f)$. Similarly, if there is a Siegel disk around z, then z cannot be rationally visible. If $f'(z)$ is any point on the unit circle, which is not a root of unity (in particular, if z is a Cremer point), then an argument of Douady and Sullivan shows that no *rational* external ray can land at z. Compare [Su], [DH2], p. 70. On the other hand, if z is repelling then an unpublished argument of Douady and Yoccoz shows that at least one rational external ray lands at z (compare [Pe]); and it is not difficult to adapt their methods to prove the corresponding statement in the parabolic case. (*See* [M2].) □

For the rest of this section, we consider only fixed points which are rationally visible.

LEMMA 2.2. — *If at least one rational external ray lands at the fixed point z of f, then there are only finitely many external rays landing at z, and all are rational and are permuted by f. More precisely, if we number these rays as $R_{t(i)}$ where*

$$0 \leqq t(0) < \ldots < t(n-1) < 1,$$

then there is a unique residue class m modulo n so that f maps each ray $R_{t(i)}$ onto $R_{t(i')}$ with $i' \equiv i+m \pmod{n}$.

In practice, we may think of the indices i as integers modulo n, and simply write

$$f(R_{t(i)}) = R_{t(i+m)}.$$

By definition, the ratio m/n in $\mathbf{Q/Z}$ is called the *rotation number* $\rho(f, z)$. Here m and n need not be relatively prime. We will usually write the rotation number as a fraction p/q in lowest terms, where $m=kp$ and $n=kq$, and where $k \geqq 1$ is the greatest common divisor. Note that the collection $T(f, z)$ of external rays landing at z then splits up into k subsets of q rays, where each of these subsets is permuted cyclically by f. The integer $k \geqq 1$ can be described as the *number of cycles of external rays* which land at z. The set $T = T(f, z)$ is called the *type* of the fixed point z.

Remark. — Here $k \leqq d-1$. (Compare [Part I], Cor. 6.) The following more general inequality has been pointed out to us by Jeremy Kahn (U. C. Berkeley): *The number of cycles of external rays landing at any periodic point satisfies $k \leqq hd$, where h is the period.* This inequality is not sharp. His proof is based on Thurston laminations [Th1].

Caution. — By definition, our rotation numbers are always rational. Of course an infinite subset of $\mathbf{R/Z}$ may well have a rotation number which is well defined but irrational. (*Seee* Figure 16, and compare [Ve].) Such rotation numbers are briefly considered in the three Appendices, and are surely worthy of further study. One step in this direction, a study of irrational rotation sets, has been carried out by A. Poirier (unpublished).

Proof of Lemma 2.2. — Clearly the map f carries each ray R_t landing at z to a ray $f(R_t)=R_{td}$ landing at z. Furthermore, since f is a local diffeomorphism near z, this correspondence must preserve the cyclic order of these rays around z, which is the same as the cyclic order of the corresponding angles $t\in\mathbf{R}/\mathbf{Z}$. First suppose that the zero ray R_0 lands at z. Then we claim that any other ray R_t which lands at z must also be mapped into itself by f, and hence must satisfy $td\equiv t$ (mod $\mathbf{Z}$), or in other words have the form $t=j/(d-1)$. For otherwise the successive images $f(R_t)=R_{t'}$, $f(R_{t'})=R_{t''}, \ldots$ would satisfy either $0<t<t'<t''<\ldots<1$ or $0<\ldots<t''<t'<t<1$; since cyclic order is preserved by f. In either case, the angles of these successive images would tend to a limit of the form $j/(d-1)$. But this is impossible, since $j/(d-1)$ is a *repelling* fixed point of the map $t\mapsto td$ (mod 1). Thus the rays which land at z are all rational, and there are at most $d-1$ of them.

Now assume only that some arbitrary rational ray lands at the fixed point z. After applying the map f a sufficient number of times, we may assume that the angle t of this ray has denominator prime to the degree d. In other words, we map assume that this ray R_t is periodic under f, with period say q. Let F be the q-fold iterate $f^{\circ q}$, of degree d^q, so that R_t is fixed by F. Evidently t has the form $j/(d^q-1)$. Now consider the conjugate polynomial map $w\mapsto\lambda^{-1}\mathrm{F}(\lambda w)$, where $\lambda=e^{2\pi it}$. This fixes the zero ray; hence the argument above shows that at most d^q-1 external rays of F, or equivalently of f, land at the point z, and that the corresponding angles are all rational, of the form $j/(d^q-1)$. Further details are straightforward, and will be left to the reader. □

We can restate Lemma 2.2 in the language of Part I of this paper as follows. Recall that a finite subset of $\mathbf{R}/\mathbf{Z}$ with well defined rational rotation number is called a rational *rotation subset*.

COROLLARY 2.3. — *The type* $\mathrm{T}(f, z)$ *of any rationally visible fixed point* z *is a rational rotation subset of the circle.*

A complete combinatorial classification of rotation subsets $\mathrm{T}\subset\mathbf{R}/\mathbf{Z}$ may be found in Theorem 7 of Part I. Such rotation subsets exist for all rotation numbers in all degrees $d\geqq 2$. Furthermore: *The rotation subset* T *is uniquely determined by its rotation number* p/q *and its cardinality* kq, *together with the "deployment" of its elements with respect to the fixed angles* $j/(d-1)$. *When* $k>1$, this deployment is subject to certain restrictions. More explicitly, for small values of the degree d we have the following.

Degree 2. — The rotation number $p/q\in\mathbf{Q}/\mathbf{Z}\setminus\{0\}$ is a complete invariant.

Degree 3. — There are $2q+1$ possible types T for each rotation number $p/q\not\equiv 0$. A convenient complete invariant is the ratio s_1/k, which can be any integer or half-integer between zero and q. Closely related is the ratio s_1/kq, which measures what fraction of the elements of T lie between 0 and 1/2. As examples, for the four maps of Figure 3 with a fixed point of rotation number 1/2, this fraction s_1/kq is respectively 1/2, 1, 1, and 3/4.

Similarly, in higher degrees, the analogous ratios

$$0\leqq s_1/kq\leqq\ldots\leqq s_{d-2}/kq\leqq 1$$

form a complete invariant. Here s_i/kq measures what fraction of the elements of T lie between zero and $i/(d-1)$. *See* Part I for details.

In the parabolic case, there is a very close relationship between multiplier and rotation number, which we explain in the next Lemma.

LEMMA 2.4. — *If z is a parabolic fixed point with multiplier $f'(z)=e^{2\pi ip/q}$, then the rotation number $\rho(f, z)\in\mathbf{Q}/\mathbf{Z}$ is equal to p/q.*

Remark. — In the case of a repelling fixed point, the rotation number p/q is not precisely equal to the argument of the corresponding multiplier $f'(z)$ in most cases. However, the still unpublished *Yoccoz inequality* asserts that $\log f'(z)$ must lie in a certain open disk D_0 in the right half-plane. By definition, D_0 has radius $\log(d)/(kq)$ where kq is the number of rays landing at z, and this disk is tangent to the imaginary axis at the boundary point $2\pi ip/q$. (Compare [Pe].) In particular, suppose that we fix p/q and choose a sequence of maps f_j for which the multiplier $f'_j(z_j)$ at some repelling fixed point of rotation number p/q converges towards the unit circle. Then it follows that these multipliers $f'_j(z_j)$ must converge towards the points $e^{2\pi ip/q}$. Thus Lemma 2.4 can be described as an easy limiting case of the Yoccoz inequality.

Outline Proof of 2.4. — According to the *Leau-Fatou Flower Theorem*, for some integer $r\geqq 1$ there exist rq simply connected regions $U_1, \ldots, U_{rq}$, numbered in counterclockwise order around z, so that $f(V_i)\subset V_j$ with $j\equiv i+rp \pmod{rq}$, and so that an orbit under f converges towards z (without actually hitting z) if and only if it eventually lands in one of the V_i. Compare [M2], [Bl], § 3. Evidently any external ray which lands at z must be disjoint from these V_i. However, since f is an orientation preserving homeomorphism near z, an argument similar to the proof of Lemma 2.2 shows that the combinatorial rotation number for these external rays cannot be different from the combinatorial rotation number p/q for these petals. □

To conclude this section, let us supplement the discussion in Part I, Lemma 3 by describing how rotation sets and their associated external rays are related to critical points and fixed points. Let z be a rationally visible fixed point of type $T=\{t_0, \ldots, t_{n-1}\}$ and rotation number p/q, where $0\leqq t_0<t_1<\ldots<t_{n-1}<1$ and $n=kq$, with $n>1$. Then the external rays R_{t_i} divide the circled plane © into n pie slices $S_1, \ldots, S_n$ which we will call *sectors*. The boundary ∂S_i consists of the two rays $R_{t_{i-1}}$ and R_{t_i} together with an arc A_i on the circle at infinity $\partial©\cong\mathbf{R}/\mathbf{Z}$. It is sometimes convenient to set $t_n=t_0+1$, so that the difference t_i-t_{i-1} measures the length of this arc A_i even when $i=n$. We will call this length the *angular size* $l(S_i)$ of the sector S_i. Thus $0<l(S_i)<1$, with $\sum l(S_i)=1$.

DEFINITION. — By the *critical weight* $w(S_i)$, we mean the numer of critical points of f within S_i counted with multiplicity, so that $\sum w(S_i)=d-1$. This integer $w(S_i)\geqq 0$ can also be described in three other quite different ways:

(1) $w(S_i)$ is equal to the number of fixed rays $R_{j/(d+1)}$ which are contained in the sector S_i (where fixed rays forming part of the boundary of S_i are to be counted with weight 1/2). In other words, it is the number of points t in the interval $[t_{i-1}, t_i)$ satisfying $dt\equiv t \bmod 1$.

(2) $w(S_i)$ is the integer part of the product $l(S_i)d$, where the "angular size" $l(S_i)=t_i-t_{i-1}$ measures the length of the boundary at infinity of S_i.

(3) $w(S_i)$ is equal to the number of fixed points or "virtual fixed points" which are contained inside the sector S_i. (*See* 3.2.)
We will prove (1) and (2) in this section, and (3) in 3.4.

LEMMA 2.5. — *The critical weight $w(S_i)$ is equal to the number of fixed rays $R_{j/(d-1)}$ which are contained in S_i. Here, in the special case of rotation number zero where the two boundary rays of S_i are fixed rays, these two boundary rays are to be counted with weight one half. If the weight $w(S_i)$ is zero, then the polynomial map f carries S_i homeomorphically onto the sector S_j, where $j\equiv i+kp$* mod *kp. On the other hand, if $w(S_i)>0$ then f carries S_i onto the entire plane $\mathbb{C}$.*

Since there are exactly $d-1$ fixed rays, this description gives the correct total count $\sum w(S_i)=d-1$.

REMARK 2.6. — If there is a critical point in the sector S_i, then there must be at least one critical value in the sector S_j, where $j\equiv i+kp$ mod kq. For otherwise, every one of the d branches of f^{-1} would be well defined and smooth throughout S_j, which is clearly impossible.

Proof of 2.5. — Suppose that we traverse the boundary ∂S_i in three steps: first out to infinity along the ray $R_{t_{i-1}}$ then along the arc A_i and then back to the fixed point along R_{t_i}, The image of this loop under f will first follow the boundary of the corresponding S_j out along the ray $R_{t_{j-1}}$ and along A_j. But then it will continue all the way around the circle for some number N of times, where $dl(S_i)=l(S_j)+N$, before returning to the fixed point along R_{t_j}. As noted in Part I Lemma 3, this N is the number of fixed points at infinity in A_i. In fact, as t increases from t_i-a/d to $t_i-(a-1)/d$ the image dt mod 1 increases from t_j through t_{i-1} and t_i to t_j+1 (assuming that $t_j<t_{i-1}$ to fix our ideas). By the intermediate value theorem there must be a fixed point as this image varies between t_{i-1} and t_i.

Let us round off the corners of S_i so that ∂S_i has a smoothly turning tangent vector, which rotates through one full turn as we circumnavigate this boundary. Evidently the tangent vector of the image of ∂S_i under f will rotate through $N+1$ full turns. It follows easily that there are exactly N critical points, counted with multiplicity, in the interior of S_i. □

The proof shows also that

$$dl(S_i)=l(S_j)+w(S_i) \tag{2.7}$$

with $j\equiv i+kp$ mod kp as above. (This is of course just a mild restatement of Lemma 3 of Part I.) In particular, $w(S_i)$ is equal to the integer part of $dl(S_i)$, which proves the statement (2) above. In the special case of rotation number zero, note that equation (2.7) reduces to the formula

$$(d-1)\,l(S_i)=w(S_i). \tag{2.8}$$

3. Counting fixed points

In this section we consider *all* of the fixed points of the polynomial map f of degree d. The first step is to consider the landing points of the $d-1$ external rays $R_{j/(d-1)}$ which are fixed by f. Suppose that n of these landing points, say $z_1, \ldots, z_n$, are distinct. Then the rays $R_{j/(d-1)}$, together with their landing points, cut the plane of complex numbers into $m=d-n$ *basic regions*, which we will denote by $U_1, \ldots, U_m$, in some arbitrary order. Here $1 \leqq m \leqq d-1$. We can roughly locate the critical points of f, and also the m remaining fixed points, as follows. As in Section 1, the *critical weight* $w(U_i)$ will mean the number of critical points in U_i, counted with multiplicity.

Recall that © stands for the compactification of the complex numbers by adding a circle $\partial © \cong \mathbf{R}/\mathbf{Z}$ of points at infinity. This circle at infinity has length $+1$ by definition. The boundary of U_i in this completed plane is made up out of a finite part consisting of rays $R_{j/(d-1)}$, and also a union of one or more arcs on the circle at infinity. (Compare Figure 5.)

LEMMA 3.1. — *The critical weight of each basic region* U_i *is equal to the number of fixed points (necessarily of rotation number zero) on the finite part of* ∂U_i, *or to* $d-1$ *times the length of that part of* ∂U_i *wich which lies on* $\partial ©$.

Proof (*See* the proof of 2.5). — Let N be the number of fixed points on the finite part of ∂U_i. As we traverse the boundary of U_i, starting at one of these finite fixed points, we first travel out along a ray $R_{j/(d-1)}$, then traverse an arc of angle $1/d-1)$ at infinity, and then come in to the next fixed point along $R_{(j+1)/(d-1)}$. This pattern is repeated N times. The image of ∂U_i under f has a similar description. The only change is that each arc of $\partial U_i \cap \partial ©$ of length $1/(d-1)$ is mapped to an arc which wraps all the way around the circle, so as to have total length $d/(d-1)=1+1/(d-1)$. Let us round off the corners of ∂U_i so as to obtain a smooth curve whose tangent vector has winding number $+1$. Evidently the tangent vector of the image of this curve under f will have winding number $N+1$. It then follows easily that the number of critical points $w(U_i)$ enclosed by this curve must be equal to N. □

If the d finite fixed points of f are all distinct, then we will show that each basic region U_i contains exactly one interior fixed point. More generally, we will modify this statement so that it remains correct even when there are multiple fixed points. However, to do this we will need some definitions.

A fixed point $f(z_0)=z_0$ is said to have *multiplicity* μ if the Taylor expansion of $f(z)-z$ about z_0 has the form

$$f(z)-z=a(z-z_0)^\mu+(\text{higher terms}),$$

with $a \neq 0$ and $\mu \geqq 1$. The sum of the multiplicities of the fixed points is always equal to the degree d. By definition, z_0 is a *multiple* fixed point if $\mu \geqq 2$, or equivalently if the multiplier $f'(z_0)$ is equal to 1. Such a multiple fixed point is the center of a *Leau-Fatou flower* with $\mu-1$ attracting petals, each contained in an immediate parabolic basin. (*See* for example [M2].)

DEFINITION 3.2. — Each one of these $\mu-1$ immediate basins, mapped onto itself by f, about a fixed point of multiplicity $\mu \geqq 2$ will be called a *virtual fixed point* of f. (More generally, any immediate parabolic basin which has period p under f may be called a "virtual periodic point" of period p.)

Since the equation $f(z)-z=0$ has exactly d roots, counted with multiplicity, the total number of fixed points and virtual fixed points for f in the finite plane $\mathbf{C}$ *is always equal to the degree d.* For our purposes, virtual fixed points are very much like rationally invisible fixed points: neither one makes any contribution to the fixed point portrait. In fact the following seems very likely:

CONJECTURE. — Any virtual fixed point can be converted to an attracting fixed point by a small perturbation of the polynomial, without affecting the fixed point portrait. Further, we conjecture that it is possible to choose this perturbed polynomial so that, when restricted to its Julia set, it is topologically conjugate to the original polynomial on its Julia set.

The following is an important topological restriction on the distribution of fixed points.

THEOREM 3.3. — *Each one of the basic regions U_i contains exactly one interior fixed point or virtual fixed point.*

Evidently a fixed point has a well defined non-zero rotation number if and only if it is rationally visible and interior to some U_i. As an immediate consequence of 3.3, we see that: *Each basic region U_i contains at most one rationally visible interior fixed point.*

A more or less equivalent fixed point theorem can be stated as follows. As in 2.5, consider a "sector" S_j bounded by two external rays which land at a common fixed point.

COROLLARY 3.4. — *The total number of fixed points and virtual fixed points in S_j is equal to the critical weight $w=w(S_j)$.*

Proof (*assuming* 3.3). — The fixed rays $R_{k/(d-1)}, \ldots, R_{(k+w-1)/(d-1)}$ which lie in S_j land on some number $n \leqq w$ of distinct fixed points, and cut out $w-n$ distinct basic regions which are strictly contained in S_j. Since each of these contains exactly one fixed point or virtual fixed point, the conclusion follows. □

Before proving 3.3, let us state one further consequence, which has been pointed out to us by A. Poirier.

COROLLARY 3.5. — *Let* V *be any bounded invariant Fatou domain for the polynomial f, that is any bounded component of $\mathbf{C} \setminus J(f)$ which is mapped to itself by f. Then any fixed point on the boundary ∂V must be either parabolic or repelling, with rotation number zero. There cannot be any Cremer point on the boundary.*

Proof (*assuming* 3.3). — First recall that the region V must be either a Siegel disk, or the immediate basin of an attractive fixed point, or an immediate basin of a parabolic fixed point. (*See* for example [M2], 13.) In the first two cases, V contains an interior fixed point, while in the parabolic case it contains a virtual fixed point. Evidently V must be contained in some basic region U_i. Hence it follows from 3.3 that any fixed point on the boundary of V must also be in the boundary of U_i. The conclusion follows. □

The proof of 3.3 will depend on the following ideas.

DEFINITION 3.6. – Let $\bar{\Delta} \subset \mathbf{C}$ be a topologically embedded closed disk with interior Δ. A map $f: \bar{\Delta} \to \mathbf{C}$ will be called *weakly polynomial-like of degree* d if $f(\partial\Delta) \cap \Delta = \varnothing$, and if the induced map on integer homology

$$f_*: \quad H_2(\bar{\Delta}, \partial\Delta) \cong \mathbf{Z} \to H_2(\mathbf{C}, \mathbf{C} \setminus \{z_0\}) \cong \mathbf{Z}$$

is multiplication by d. Here z_0 can be any base point in Δ.

Remark. – If f is holomorphic, and satisfies the sharper condition that $f(\partial\Delta) \cap \bar{\Delta} = \varnothing$, then it is called *polynomial-like*. Compare [DH4]

LEMMA 3.7. – *If $f: \bar{\Delta} \to \mathbf{C}$ is weakly polynomial-like of degree d, with isolated fixed points, then each fixed point $f(z_i) = z_i$ can be assigned a* Lefschetz index $\iota(f, z_i) \in \mathbf{Z}$ *which is a local invariant, so that the sum of these Lefschetz indices is equal to the degree d.*

As an example, if f is a polynomial of degree d and if Δ is a large disk centered at the origin, then $f | \bar{\Delta}$ is polynomial-like of degree d, and the Lefschetz index is $+1$ at a simple fixed point and μ at a μ-fold fixed point. Thus the sum of the indices is equal to d, as expected.

Proof of 3.7. – For presentations of the Lefschetz Fixed Point Theorem, *see* for example [Brn], [DGr], [Gr] or [Ji]. In the case of an interior fixed point, the Lefschetz index can be defined as the local degree of the map $z \mapsto f(z) - z$ at the fixed point. That is, if U is a small neighbourhood of z_i, then the induced homomorphism

$$(f - \text{identity})_*: \quad H_2(U, U \setminus \{z_i\}) \to H_2(\mathbf{C}, \mathbf{C} \setminus \{0\})$$

is multiplication by ι. If there are no boundary fixed points, then the sum of these indices is the degree of

$$(f - \text{identity})_*: \quad H_2(\bar{\Delta}, \partial\Delta) \to H_2(\mathbf{C}, \mathbf{C} \setminus \{0\}).$$

But the identity map of Δ is homotopic to the constant map $z \mapsto z_0$, so this sum of indices is equal to d.

If there are boundary fixed points, then we can first modify the map in a neighborhood of each one so as to push all of the fixed points inside, and then apply the construction above. The resulting index does not depend on the local modification, since the global degree cannot change. □

Proof of 3.3. – Let U_i be one of the regions of 3.1, and let Δ be the topological disk which is obtained by intersecting U_i with a large round disk centered at the origin. Then it is easy to check that f restricted to $\bar{\Delta}$ is weakly polynomial-like of degree $w + 1 = w(U_i) + 1$, and that it has exactly w boundary fixed points. If U_i contains no virtual fixed point, then we will show that each of these boundary fixed points has

Lefschetz index $+1$. Therefore, it will follow from 3.7 that there must be an interior fixed point as well.

First consider a boundary fixed point z_j which is repelling, $|f'(z_j)|>1$. Then a small open disk D_ε centered at z_j maps diffeomorphically onto a strictly larger disk. Let Δ_j be that component of $\Delta \cap D_\varepsilon$ which has z_j as boundary point. Then the closure $\bar{\Delta}_j$ is a relative neighborhood of z_j in $\bar{\Delta}$, and the map f restricted to $\bar{\Delta}_j$ is weakly polynomial-like of degree $+1$, with unique fixed point at z_j. Hence by 3.7 the local index $\iota(f|\bar{\Delta}, z_j)=\iota(f|\bar{\Delta}_j, z_j)$ is equal to $+1$, as asserted.

Now suppose that z_j is a parabolic fixed point. Then by 2.4 the multiplier $f'(z_j)$ must be equal to $+1$. The two external rays of ∂U_i which land at z_j must be contained in a common repelling petal as they approach z_j, since otherwise U_i would contain an attracting petal or "virtual fixed point", contrary to our hypothesis. In this case, we let Δ_j be one component of the intersection of Δ with a small repelling petal at z_j. Proceeding just as above, we again see that the Lefschetz index is $+1$.

To complete the proof, we must consider the basic regions which *do* contain virtual fixed points. If there are v such regions, then each of the remaining $m-v$ basic regions contains as interior fixed point by the argument above. Here $m=d-n$ where n is the number of fixed points of rotation number zero. Thus we have accounted for at least $n+(m-v)+v=d$ distinct fixed points or virtual fixed points. Thus all such points have been accounted for, and no one of these basic regions can contain more than one fixed point or virtual fixed point. □

Recapitulating and summarizing the results of Sections 2 and 3, we have the following two statements, which follow easily from the discussion above.

THEOREM 3.8. — *For any "sector", or more generally for any region* S *bounded by two external rays* R_t *and* R_u *landing at a common fixed point, where* $t<u\leqq t+1$, *the following four numbers are equal*:

(1) *the critical weight* $w(S)$, *that is the number of critical points in* S *counted with multiplicity;*

(2) *the number of fixed rays in* S (*where a fixed ray on the boundary of* S *is counted with weight* 1/2);

(3) *the greatest integer less than the product* $d.l(S)=d.(u-t)$;

(4) *the number of fixed points plus virtual fixed points in the interior of* S.

By definition, a "basic region" is bounded by some number of fixed rays, and contains no fixed rays in its interior.

THEOREM 3.9. — *Every basic region* U_i *contains exactly one interior fixed point or virtual fixed point. Furthermore, the following three numbers are equal*:

(1) *the critical weight* $w(U_i)$,

(2) *the number of fixed points on* (*the finite part of*) ∂U_i,

(3) *the product* $(d-1).l(U_i)$.

4. Fixed point portraits

We are now in a position to give a conjectured description of all possible fixed point portraits. Recall from 1.3 that the *fixed point portrait* for a polynomial f which has k rationally visible fixed points is the collection

$$\mathscr{P}=\{\mathrm{T}_1, \ldots, \mathrm{T}_k\}$$

consisting of the *types* of these rationally visible fixed points. Here $1 \leqq k \leqq d$. Assembling previous results, we have the following.

THEOREM 4.1. – *If $\mathscr{P}=\{\mathrm{T}_1, \ldots, \mathrm{T}_k\}$ is the fixed point portrait for some polynomial map of degree d, then the following four conditions must be satisfied.*

P1. *Each T_j is a rational rotation set. In particular, it has a well defined rotation number p_j/q_j.*

P2. *The T_j are disjoint and pairwise unlinked.*

P3. *The union of those T_j which have rotation number zero is precisely equal to the set $\{0, 1/(d-1), \ldots, (d-2)/(d-1)\}$ consisting of all angles which are fixed by the d-tupling map.*

P4. *Each pair $\mathrm{T}_i \neq \mathrm{T}_j$ with non-zero rotation number is separated by at least one T_l with zero rotation number. That is, T_i and T_j must belong to different connected components of the complement $(\mathbf{R}/\mathbf{Z}) \smallsetminus \mathrm{T}_l$.*

Proof. – P1 follows from 2.3, P2 follows from 1.5, P3 is clear from the discussion in Section 1 or above, and P4 is an easy consequence of 3.3. □

MAIN CONJECTURE 4.2. – *These necessary conditions are also sufficient.* In other words, given sets T_i satisfying these four conditions, there exist a polynomial of degree d whose filled Julia set is connected having $\{\mathrm{T}_i\}$ as fixed point portrait.

In the special case where $k=d$ (so that the d fixed points are distinct and rationally visible) a proof will be given in Section 7. A proof in the general case has recently been given by Poirier [Po1]. (*See* also [HJ].)

In order to illustrate 4.2, let us look at the low degree cases.

Degree 2. – In this case, we always have $\mathrm{T}_1=\{0\}$. If there is also a fixed point with rotation number $p/q \not\equiv 0$, then the resulting fixed point portrait might be denoted by the symbol $\mathscr{P}(p/q)$. A corresponding centered polynomial map is said to belong to the *p/q-limb* of the Mandelbrot set. If the other fixed point is invisible (as for $z \mapsto z^2$) or is only a virtual fixed point, then we could use the symbol $\mathscr{P}(\bullet)$. Such maps are said to belong to the *central core* of the Mandelbrot set. For further details, *see* Appendix C.

Degree 3. – Here there are two subcases. If the rays R_0 and $\mathrm{R}_{1/2}$ land at distinct points, then we have $\mathrm{T}_1=\{0\}$, $\mathrm{T}_2=\{1/2\}$, and there is at most one further fixed point. If this further fixed point is distinct and rationally visible, the corresponding portrait might be indicated by the symbol $\mathscr{P}(p/q;\, s_1/k)$. (Compare the discussion following 2.3.) Here $q \geqq 2$, p is relatively prime, and s_1/k is an integer or half-integer between zero and q. With this notation, three of the portraits of Figure 4 would be represented by the symbols $\mathscr{P}(1/2;\, 1)$, $\mathscr{P}(1/2;\, 2)$ and $\mathscr{P}(1/2;\, 3/2)$ respectively. If the third fixed point is rationally invisible (as for $z \mapsto z^3$), or is virtual (as for $z \mapsto z^3 - z^2 + z$), then some notation such as $\mathscr{P}(\bullet;)$ might be used.

On the other hand, if R_0 and $R_{1/2}$ land at a common point of type $T_1 = \{0,\, 1/2\}$, then these two rays divide the plane into an upper half and a lower half. Each of these two halves must contain a fixed point or virtual fixed point. If the upper half contains a fixed point of rotation number p/q and the lower half a fixed point of rotation number p'/q', then the symbol $\mathscr{P}((p/q)/(p'/q'))$ might be used. If either the top or bottom fixed point is rationally invisible or is only a virtual fixed point, then the corresponding rotation number should be replaced by a heavy dot. For example, with this notation, the right hand portrait of Figure 4 would be written as $\mathscr{P}(1/2/\bullet)$; while the portrait for the map $z \mapsto z^3 + z$ with two virtual fixed points, or the map $z \mapsto z^3 + (3/2)z$ with two superattracting fixed points would be written as $\mathscr{P}(\bullet/\bullet)$.

5. Critical portraits: Fisher's thesis

This section will develop a complementary concept of "critical portrait" for certain polynomial maps. The exposition is based on the work of Yuval Fisher and of Bielefeld-Fisher-Hubbard, and will omit most proofs. (*See* [Fi] and [BFH].)

In Section 6 we will show that the fixed point portrait of a polynomial is uniquely determined by its critical portrait, whenever the latter is defined. In fact, we describe an algorithm that effectively computes the fixed point portrait of a polynomial whose critical portrait is given. These results will be used in Section 7 to construct polynomials with specified fixed point portrait.

HYPOTHESIS. — We will assume that f is a monic polynomial of degree $d \geqq 2$ with the property that each critical point is the landing point for at least one external ray R_θ. Choose some fixed numbering for the distinct critical points $\omega_1, \ldots, \omega_k$, and let μ_j be the multiplicity of the critical point ω_j. Thus $\mu_j \geqq 1$ for each j, and the sum $\sum \mu_j$ is equal to $d-1$.

DEFINITION. — By a *critical portrait* for f we will mean a sequence $\boldsymbol{\Theta} = \{\Theta_1, \ldots, \Theta_k\}$ where each $\Theta_j \subset \mathbf{R}/\mathbf{Z}$ is a finite set of angles satisfying three conditions:

(1) *Each ray* R_θ, *with* $\theta \in \Theta_j$, *must land at the critical point* ω_j.

(2) *Any two angles in the same* Θ_j *must be congruent modulo* $1/d$, *so that* Θ_j *maps to a single point under the correspondence* $\theta \mapsto d\theta$ (mod 1).

(3) *Each* Θ_j *must have* (*the largest possible*) *cardinality* $\# \Theta_j = \mu_j + 1$.

Thus for any two angles θ, $\eta \in \Theta_j$, the corresponding rays R_θ and R_η must have the same image $R_{d\theta}$ under the map f. Since the correspondence $z \mapsto f(z)$ preserves external rays, and is exactly $(\mu_j + 1)$-to-one in a neighborhood of ω_j (with the point ω_j itself deleted), if follows that Θ_j is precisely the set of all external rays which land at the critical point ω_j, and which map to one common ray $R_{d\theta}$ landing at the critical value $f(\omega_j)$. *Consequently, the map f possesses a unique critical portrait if and only if each critical value $f(\omega_j)$ is the landing point of one and only one external ray.*

The elements of $\Theta_1 \cup \ldots \cup \Theta_k$ will be called the preferred *critical angles*, and the corresponding R_θ the preferred *critical rays*. Following Fisher, the pair $(f, \boldsymbol{\Theta})$ is called a *marked polynomial*. Evidently any critical portrait $\{\Theta_1, \ldots, \Theta_k\}$ must satisfy the following three conditions:

C1. The Θ_j are pairwise disjoint and unlinked. (*See* 1.5.)

C2. Any two angles in the same Θ_j are congruent modulo $1/d$.

C3. The cardinalities of these sets satisfy $\#\Theta_j \geqq 2$ and

$$\sum (\#\Theta_j - 1) = d - 1.$$

By definition, any collection of sets of angles satisfying these three conditions will be called a *formal critical portrait*. Fisher's Thesis is concerned with formal critical portraits which satisfy the following further condition:

C4. Each critical angle $\theta \in \Theta_1 \cup \ldots \cup \Theta_k$ is *strictly preperiodic* under the map $\theta \mapsto d\theta$ (mod 1). In other words, each such θ is rational, and the denominator of θ is never relatively prime ([1]) to the degree d.

His main theorem asserts that a formal critical portrait satisfying C4 is actually realized by a polynomial map f with $J = J(f)$ connected if and only if it satisfies one further condition C5, which will be described below. Fisher's method is constructive, and has been implemented on a computer by Bielefeld, Fisher and Hubbard as the so-called "spider algorithm". An example of this procedure is illustrated in Figures 6 and 7: If we start with the degree 3 critical portrait which is illustrated schematically in Figure 6, then the spider algorithm yields the cubic polynomial of Figure 7.

Remark. — Fisher's work is based on Thurston's theory of post-critically finite rational maps. Hence condition C4 is essential for his proofs, although his results may actually be true in much greater generality.

Consider a marked polynomial with critical portrait $\{\Theta_1, \ldots, \Theta_k\}$. The critical rays R_θ, $\theta \in \Theta_1 \cup \ldots \cup \Theta_k$, together with their landing points ω_j, cut the plane $\mathbf{C}$ up into d regions Ω_p with boundary. In particular, they cut the Julia set J of f up into d compact connected subsets $J_1, \ldots, J_d$, where $J_p = J \cap \Omega_p$. These d sets are nearly disjoint, in the sense that each intersection $J_p \cap J_q$ consists of at most a single point, which is necessarily one of the critical points ω_j.

LEMMA 5.1. — *The map f carries each of these d subsets J_p homeomorphically onto the entire Julia set* J.

This can be proved by first checking that f is univalent on the interior of each region Ω_p, and maps the closed region onto the entire plane $\mathbb{C}$. In fact the boundary

([1]) For the case of angles θ which are allowed to the periodic, *see* [Po2].

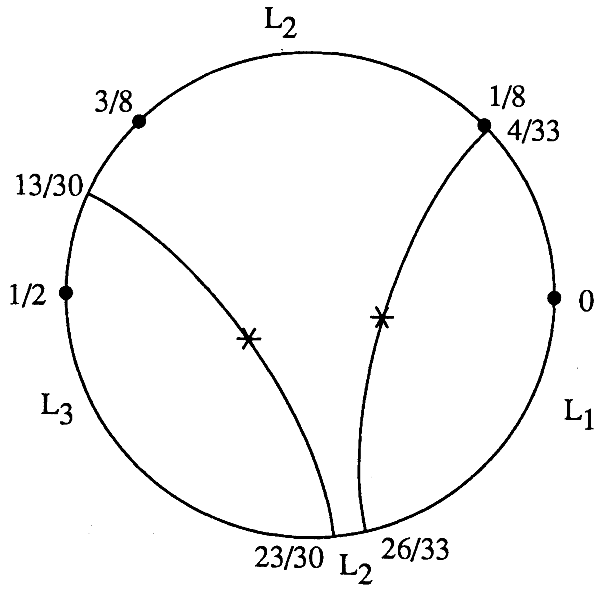


Fig. 6. — Diagram for the cubic critical portrait

$$\Theta=\left\{\left\{\frac{4}{33},\frac{26}{33}\right\},\left\{\frac{13}{30},\frac{23}{30}\right\}\right\}.$$

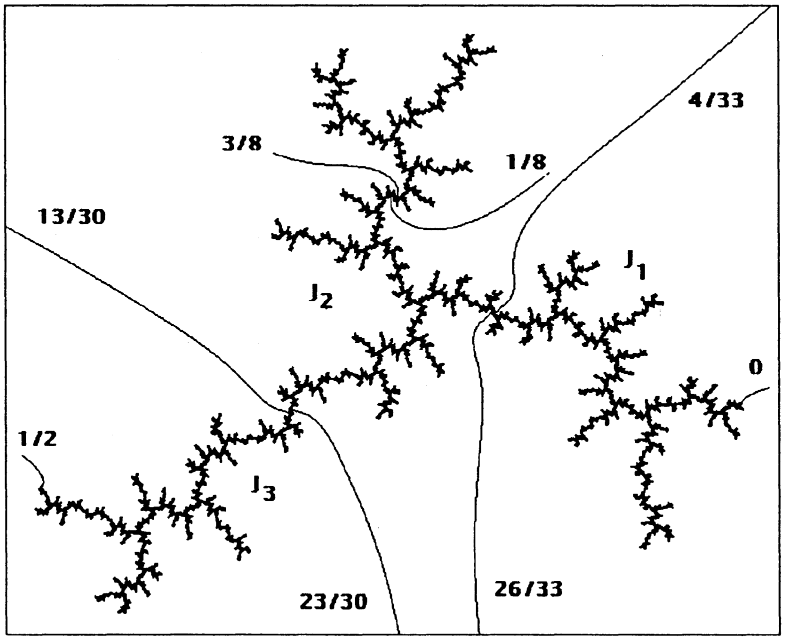


Fig. 7. — Julia set for the associated polynomial
$z \mapsto z^3-(.309+.396\,i)\,z-(.216-.995\,i)$.

$\partial\Omega_p$ maps to a loop which simply traverses the circle at infinity $\partial\mathbb{C}$, with a detour first in and then out along each preferred external ray leading to a critical value $f(\omega_j)$. Here ω_j ranges over those critical points which belong to the boundary of Ω_p: the associated critical values $f(\omega_j)$ are all distinct. Details will be omitted. □

LEMMA 5.2. — *If Condition C4 is satisfied, then this partition of* J *has the following much sharper property: Given an arbitrary sequence* $p_0, p_1, \ldots$ *of indices between* 1 *and* d, *there exists one and only one point* $z = z(p_0, p_1, \ldots)$ *which belongs to the intersection*

$$J_{p_0} \cap f^{-1} J_{p_1} \cap f^{-2} J_{p_2} \cap \ldots,$$

or equivalently satisfies $f^{\circ n}(z) \in J_{p_n}$ *for every* $n \geqq 0$.

Remark. — Here the requirement that the angles $\theta \in \Theta_i$ are rational is surely essential. A well known conjecture asserts the existence of polynomials with locally connected Julia sets having Siegel disks. A corresponding critical portrait would satisfy conditions C1-C3 but not C4. Evidently any two points on the boundary of such a disk must have the same symbol sequence.

Proof of 5.2 (with help from Ben Bielefeld). — We will make use of the Thurston *orbifold metric* associated with f. Since f is post-critically finite without attracting orbit, this is a conformal Riemannian metric which is defined throughout **C**, with singularities exactly at the post-critical points of f. It is *expanding*, in the sense that for any curve segment γ the length $L(f(\gamma))$ with respect to this metric of the image curve is strictly larger than the length $L(\gamma)$. (*See* for example [DH3] or [M2], § 14.5.) If we restrict to some compact subset of **C**, then we can make the sharper statement that there exists a constant $c > 1$ so that

$$L(f(\gamma)) \geqq c\, L(\gamma)$$

for every curve in the subset. In fact, let us work with the compact region M consisting of all $z \in \mathbf{C}$ for which $G(z) \leqq 1$, where G is the canonical potential function (Green's function) which vanishes on the filled Julia set $K(f)$.

Let M′ be the compact surface with boundary which is obtained by cutting open this region M along each of the preferred external rays landing at critical values, and along every forward image of such a ray. Condition C4 guarantees that there are only finitely many such cuts. (Thus each point along such an external ray corresponds to two distinct boundary points of M′.) The landing points of these rays correspond to boundary points at which M′ is usually not smooth. In fact, if more than one such ray lands at one post-critical point, then M′ will consist locally of two or more connected surfaces with boundary which have been pasted together at this common boundary point. However, in spite of the metric singularity and the non-smoothness of the boundary at these landing points, these boundary curves have finite length with respect to the orbifold metric. (More generally, using the expanding condition, we see that the length of any external ray in the compact region M is finite and uniformly bounded.)

Corresponding to the decomposition $\mathbf{C}=\Omega_1 \cup \ldots \cup \Omega_d$ there is a decompositon $M'=M'_1 \cup \ldots \cup M'_d$, where each non-vacuous intersection $M'_p \cap M'_q=\Omega_p \cap \Omega_q \cap M$ contains only a critical point and one or two external rays landing at this critical point. The intersections $J \cap M'_p=J \cap \Omega_p$ are the sets J_p described earlier. These sets M'_p have been constructed in such a way that there is one and only one branch

$$f_p^{-1}: \quad M' \to M'_p$$

of f^{-1} which is defined throughout M' and takes values in M'_p. To see this, recall that f maps the subset Ω_p onto $\mathbf{C}$, and that the restriction $f|\Omega_p$ is one-to-one on the interior of Ω_p and at the critical points, and is two-to-one on the rest of the boundary (which consists of pairs of external rays landing at critical points). If we slit the plane along external rays landing at post-critical points, then the forward map f is no longer well defined. For example rays landing at co-critical points do not have well defined images. However, the inverse branches are well defined, for whenever an image ray $f(R)$ is unslit it follows that R is unslit. The branch of f^{-1} on M' which takes values in M'_p gives the desired map f_p^{-1}.

Define the *distance* $\rho(z, z')$ between two points of M' to be the infimum of the lengths, with respect to the orbifold metric, of rectifiable paths joining z to z' within M'. Note that f_p^{-1} is strictly distance reducing: In fact every curve of length L in M' maps under f_p^{-1} to a curve of length $\leqq L/c$ in $M'_p \subset M'$. Hence the iterated image

$$f_{p0}^{-1} \circ \ldots \circ f_{pn}^{-1}(J)=J_{p0} \cap f^{-1}(J_{p1}) \cap \ldots \cap f^{-n}(J_{pn})$$

is compact and non-vacuous, with diameter less than diam $(J)/c^{n+1}$. Taking the limit as $n \to \infty$, we obtain the required unique point. □

COROLLARY 5.3. — *Still assuming* C4, *each* J_p *contains a unique fixed point of* f.

This follows by taking $p_0=p_1=\ldots=p$. □

The sequence $p_0, p_1, \ldots$ of 5.2 will be called an *itinerary* for the point z with respect to the partition $\{J_1, \ldots, J_d\}$. If there is no critical point in the orbit $\{z, f(z), f^{\circ 2}(z), \ldots\}$, then evidently this itinerary is uniquely determined by z. However, if z is *pre-critical*, that is if there is at least one critical point in its orbit, then z has more than one itinerary.

Corresponding to this partition of the Julia set into nearly disjoint closed subsets $J_1, \ldots, J_d$, there is a partition of the circle $\mathbf{R}/\mathbf{Z}$ into nearly disjoint closed subsets $L_1, \ldots, L_d$, each of total length $1/d$. *By definition*, L_p *consists precisely of those angles* t *such that the ray* R_t *lands on some point of* J_p. In many cases, each L_p will consist of one or more closed intervals. However, if one of the critical points in J_p has higher multiplicity, so that three or more preferred external rays land at this point, then the corresponding L_p will also have isolated points.

More generally, let $\mathbf{\Theta}=\{\Theta_1, \ldots, \Theta_k\}$ be an arbitrary formal critical portrait. First consider two points t and t' in the complement

$$\mathbf{R}/\mathbf{Z} \setminus (\Theta_1 \cup \ldots \cup \Theta_k).$$

By definition, t and t' are "*unlink equivalent*" if they belong to the same connected component of $\mathbf{R}/\mathbf{Z}\setminus\Theta_j$ for each j, so that the $k+1$ sets

$$\Theta_1, \ldots, \Theta_k, \{t, t'\}$$

are pairwise unlinked. (Compare 1.5.) Let $L_1^0, \ldots, L_d^0$ be the resulting "unlink equivalence classes" with union $\mathbf{R}/\mathbf{Z}\setminus(\Theta_1 \cup \ldots \cup \Theta_k)$. It is easy to check that each L_p^0 is a finite union of open intervals with total length $1/d$. Now define L_p to be the union of the closure $\bar{L}_p^0$ together with all of the sets Θ_j which intersect this closure. Thus L_j consists of $\bar{L}_j^0$, which is a finite union of closed intervals, possibly with finitely many isolated points adjoined, as explained above. As an example, consider the critical

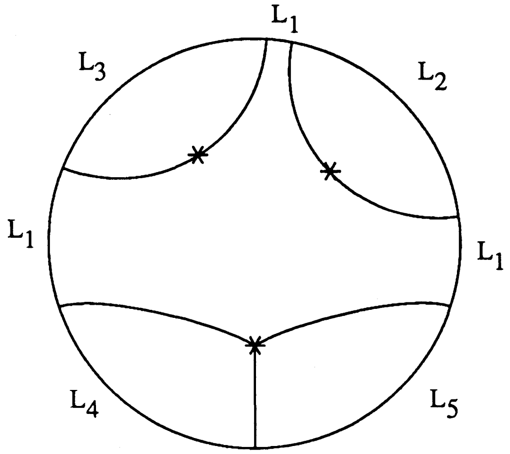


Fig. 8. — A degree 5 critical portrait.

portrait sketched in Figure 8. Here the set L_1 consists of three closed intervals as shown, together with one isolated point at the bottom of the circle.

We will say that the sequence $p_0, p_1, \ldots$ is an *itinerary* for the angle $t \in \mathbf{R}/\mathbf{Z}$ under the map $t \mapsto dt$ (mod 1) if the orbit $t = t_0 \mapsto t_1 \mapsto \ldots$ satisfies the condition that $t_n \in L_{p_n}$ for each $n \geqq 0$. Evidently each angle t has at least one itinerary, and this itinerary is uniquely defined if and only if no t_n belongs to the set $\Theta_1 \cup \ldots \cup \Theta_k$ of critical angles. Using these ideas, Fisher gives a precise criterion in order to decide when two rays land at a common point. We assume that C4 is satisfied, so that all critical orbits are strictly pre-periodic, and we assume that he Julia set is connected. Let s and t be two angles, and let $s = s_0 \mapsto s_1 \mapsto \ldots$ and $t = t_0 \mapsto t_1 \mapsto \ldots$ be their orbits under the map $t \mapsto dt$ (mod 1). Since the itinerary of any angle θ under the d-tupling map must be compatible with the itinerary of the landing point of the corresponding ray R_θ under f, Lemma 5.2 takes the following form.

LEMMA 5.4. — *The two rays* R_s *and* R_t *land at a common point of the Julia set* J *if and only if they have some itinerary in common, or in other words if and only if, for each* $n \geqq 0$, *there exists an index* p_n *for which both* $s_n \in L_{p_n}$ *and* $t_n \in L_{p_n}$.

Fisher's fifth condition is needed in order to guarantee that distinct Θ_j correspond to distinct critical points of f:

C5. If $\theta \in \Theta_h$ and $\theta' \in \Theta_j$ with $h \neq j$, then we require that θ and θ' do not have any itinerary in common.

If all five conditions are satisfied, then he calls $\{\Theta_1, \ldots, \Theta_k\}$ a "polynomial determining family of angles". [Here is an example to show that condition C5 is independent of the other four conditions. In degree $d=4$ let $\Theta_1 = \{1/60, 46/60\}$, $\Theta_2 = \{19/60, 34/60\}$, and $\Theta_3 = \{1/16, 5/16\}$. Then C1 through C4 are satisfied, but C5 is not.]

DEFINITIONS. — Following Branner and Hubbard, we define the degree *d connectedness locus* to be the compact set consisting of all monic centered degree d polynomials with connected Julia set. The polynomial f will be called *critically pre-periodic* if the orbit of every critical point is strictly pre-periodic. That is, each such orbit eventually hits a periodic cycle, but no critical point itself lies on a periodic cycle.

We can now state Fisher's main Theorem.

THEOREM 5.5. — *If a formal critical portrait satisfies all of the conditions* C1 *through* C5, *then there is one and only one polynomial f in the degree d connectedness locus which, when suitably marked, realizes this critical portrait. Furthermore this polynomial f is critically pre-periodic.*

It follows, according to Douady and Hubbard, that f has locally connected Julia set, and has the property that all periodic orbits are strictly repelling. (*See* for example [M2]; 11.6, 14.4 and 17.5.) In particular, f must have d distinct repelling fixed points.

6. From critical portrait to fixed point portrait

Given a critical portrait $\mathbf{\Theta} = \{\Theta_1, \ldots, \Theta_k\}$ satisfying Fisher's five conditions, he constructs a unique associated polynomial $f \in \mathscr{C}_d$ which is critically pre-periodic, and hence has d distinct repelling fixed points (Theorem 5.5). In principle, we can determine the fixed point portrait of f from the given data. In fact it follows easily from 5.4 that:

LEMMA 6.1. — *The fixed point type* T_j *of the unique fixed point which lies in the set* $\mathrm{J}_j \subset \mathrm{J}$ *is just the set of all angles whose orbit under the d-tupling map lies strictly within the corresponding set* $\mathrm{L}_j \subset \mathbf{R}/\mathbf{Z}$.

In this section, we will describe how to effectively compute this fixed point portrait. Our analysis depends on some facts about monotone maps of the circle which we briefly review. (Compare [De], [dM].)

By definition, a continuous self map $\phi : \mathbf{R}/\mathbf{Z} \to \mathbf{R}/\mathbf{Z}$ is *monotone* if some, and hence any lift $\Phi : \mathbf{R} \to \mathbf{R}$ is non-decreasing. Every monotone map ϕ of degree 1 has a well defined *rotation number*

$$\rho(\phi) = \lim_{n \to \infty} \frac{\Phi^{\circ n}(t)}{n} \mod 1$$

which is independent of the choice of $t \in \mathbf{R}$ and of the lift Φ of ϕ.

LEMMA 6.2. — *If* $\phi : \mathbf{R}/\mathbf{Z} \to \mathbf{R}/\mathbf{Z}$ *is monotone of degree* 1, *then:*

(1) *Each* $\phi^{-1}(t)$ *is either a point or a closed interval in* $\mathbf{R}/\mathbf{Z}$.

(2) *The rotation number* $\rho(\phi)$ *is rational if and only if* ϕ *has a periodic point.*

(3) *If* $\rho(\phi) = p/q$, *then every periodic point of* ϕ *has period* q *and rotation number* p/q, *or in other words corresponds to a fixed point of the map* $t \mapsto \Phi^{\circ q}(t) - p$ *for suitable choice of the lift* Φ. *Furthermore, every orbit under* ϕ *is either itself periodic or tends asymptotically to an attracting or one-sided attracting periodic orbit.*

Proof. — If $\phi : \mathbf{R}/\mathbf{Z} \to \mathbf{R}/\mathbf{Z}$ is monotone of degree $n \geqq 1$, then it is easy to check that each pre-image $\phi^{-1}(t) \subset \mathbf{R}/\mathbf{Z}$ has n distinct connected components. Specializing to $n = 1$ we obtain assertion (1). The proofs of assertions (2) and (3) are essentially the same as for circle homeomorphisms. Details may be found in [De] and [dM]. □

Fix $d \geqq 2$ and let $\mathbf{\Theta}$ be a degree d formal critical portrait. In other words, we temporarily assume only conditions C1, C2, C3. As in Section 5, let $L_1^0, \ldots, L_d^0$ be the corresponding unlink equivalence classes, with union equal to $\mathbf{R}/\mathbf{Z} \setminus (\Theta_1 \cup \ldots \cup \Theta_k)$. We associate to each L_j^0 a monotone map ϕ_j from the circle to itself. (Compare Figure 9.)

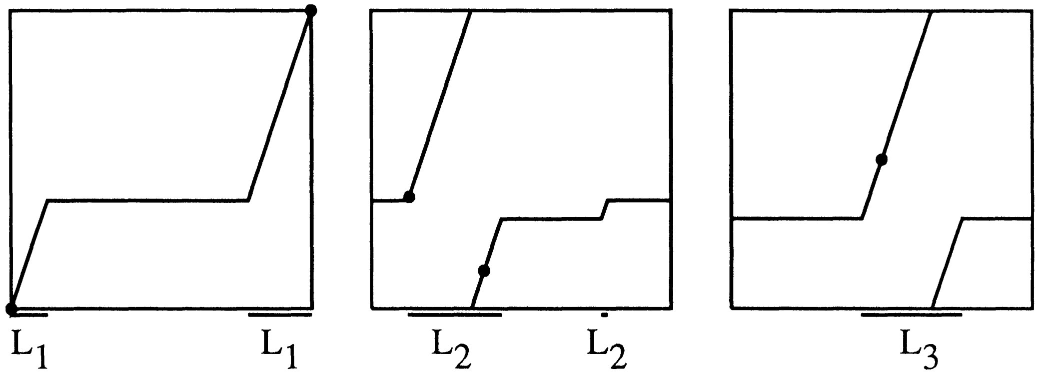


Fig. 9. — Graphs of ϕ_1, ϕ_2 and ϕ_3 for the critical portrait of Figure 6. The repelling periodic points 0, $\{1/8, 3/8\}$ and 1/2 are indicated by heavy dots.

LEMMA 6.3. — *For each* L_j^0 *there is one and only one continuous map* ϕ_j *from* $\mathbf{R}/\mathbf{Z}$ *to itself which is given by the formula*

$$\phi_j(t) \equiv dt \pmod 1 \qquad \text{for} \quad t \text{ in the closure of } L_j^0,$$

and is constant on each component V *of the complement* $\mathbf{R}/\mathbf{Z} \setminus L_j^0$. *Furthermore, this map* ϕ_j *is monotone of degree one. In particular, it has a well defined rotation number.*

The proof is immediate. We need only note that the two endpoints of each such complementary interval V necessarily belong to the same set Θ_i, and hence share a common value of dt (mod 1). The resulting map is piecewise linear, with slope $d > 1$ throughout the open set L_j^0, and with slope zero throughout the complement

$\mathbf{R}/\mathbf{Z} \setminus L_j$. This map has degree 1, since the various components of L_j have total length $1/d$. □

Using these piecewise linear maps ϕ_j, we can compute the fixed point portrait of f as follows. Let z_j be the unique fixed point of f which lies in the subset $J_j \subset J$.

LEMMA 6.4. — *With f as in 5.5, the angles of the external rays which land at the fixed point $z_j \in J_j$ are exactly the repelling periodic points of the associated circle map ϕ_j.*

The proof, based on the following lemma, will give an effective procedure for computing these periodic points.

DEFINITION. — It will be convenient to say that a periodic point $\phi^{\circ q}(t_0) = t_0$ is *ultra-attracting* if the map $\phi^{\circ q}$ is constant throughout some neighborhood of t_0.

Note that ultra-attracting orbits are very easy to find: Every ultra-attracting orbit must intersect some component V of the complement $\mathbf{R}/\mathbf{Z} \setminus L_j^0$. *To locate such an orbit of period q, we can simply start at any point of* V *and iterate the map q times.* We will see that these ultra-attracting orbits can then be used to locate the repelling orbits, which according to 6.4 are those of primary interest.

LEMMA 6.5. — *Suppose that conditions* C1 *through* C4 *are satisfied. Then each monotone map ϕ_j satisfies the following conditions:*

(*a*) *The rotation number of ϕ_j is a rational number p_j/q_j.*

(*b*) *Every periodic point is either repelling or ultra-attracting.*

(*c*) *These two types of periodic points alternate around the circle, and the number k of orbits of each type satisfies $1 \leqq k \leqq d-1$.*

(*d*) *Every point of the circle is either periodic or pre-periodic. In fact, any orbit which is not actually periodic must eventually land on an ultra-attracting periodic orbit.*

Proof. — Recall that there are finitely many disjoint intervals, say $V_1, \ldots, V_r$, on which ϕ_j is constant, and that ϕ_j coincides with the d-tupling map outside of the union $V_1 \cup \ldots \cup V_r$. Let us fix some interval of constancy V_α. Since the endpoints of each V_β are preperiodic under the d-tupling map by C4, it follows that *the forward orbit of* V_α *under ϕ_j is finite.* In fact this orbit either hits some interval of constancy V_β twice and thereafter must repeat periodically, or else hits $V_1 \cup \ldots V_r$ for a last time and thereafter coincides with an eventually periodic orbit under the d-tupling map. [Actually, by assertion (*d*) the latter case cannot occur.]

This proves that ϕ_j has a periodic point. Hence ϕ_j has rational rotation number p_j/q_j and every periodic orbit has period q_j, by Lemma 6.2. Since the slope of the q_j-fold iterate of ϕ_j is alternately zero and $d^{q_j} > 1$, we see that every periodic orbit must be either ultra-attracting, or repelling, or mixed — ultra-attracting on one side and repelling on the other. However, using condition C4 we see easily that such mixed cases cannot occur. This proves assertions (*a*) and (*b*).

Since the graph of $t \mapsto \Phi^{\circ q_j}(t) - p_j$ crosses from above the diagonal to below the diagonal at every ultra-attracting periodic point, and from below to above at every repelling periodic point, these two types of periodic point must alternate around the circle. Evidently the number k of ultra-attracting orbits is dominated by the number r

of intervals of constancy V_α. There are at most $d-1$ such intervals, since they are disjoint and each one has length at least $1/d$. This proves assertion (*c*).

By 6.2(3), every orbit under ϕ_j is either periodic, or tends asymptotically to an attracting periodic orbit. However our attracting periodic orbits are all ultra-attracting, so such a non-periodic orbit must actually land on an ultra-attracting orbit after finitely many iterations. □

Proof of 6.4. – We now suppose that conditions C1 through C5 are satisfied, so that there is an associated map f in the connectedness locus. We can compute the associated fixed point portrait, which we will write simply as $\{T_1, \ldots, T_d\}$, as follows. According to 5.4, the type T_j of the fixed point which belongs to the compact set $J_j \subset J$ consists of all angles t whose orbit under the d-tupling map lies completely within the corresponding closed set $L_j \subset \mathbf{R}/\mathbf{Z}$. Using 6.5, we see that this type consists of the repelling periodic orbits of ϕ_j. There are at most $d-1$ such orbits, and they all have the same rational rotation number, say p_j/q_j. □

Remark 6.6. – Recall that the fixed point type T_j is the set of repelling periodic points of the monotone map ϕ_j. The argument above shows how to compute the rotation number of ϕ_j, and the number of points in T_j, and also shows how to locate these points approximately. To actually compute these repelling periodic points, it is probably easiest to iterate the inverse function ϕ^{-1}, since the points of T_j are strongly attracting fixed points of $\phi_j^{-q_j}$. As a check, one can use the fact that these points are all rational numbers of the form $m/(d^{q_j}-1)$.

If $p_j/q_j \equiv 0$, then the type T_j is precisely the set of all angles $i/(d-1)$ contained in L_j. If p_j/q_j is non-zero, then each of the fixed angles $i/(d-1)$ must be contained in one of the components V of $\mathbf{R}/\mathbf{Z} \setminus L_j$. In this case, the deployment sequence of T_j could be determined, without computing its actual elements, as follows. For each such component V we must check whether the graph of the constant function $\Phi_j^{q_j} - p_j$ on V crosses the diagonal, or lies strictly above or strictly below the diagonal. Further, we must compute all of the ultra-attracting periodic orbits by starting in each component of $\mathbf{R}/\mathbf{Z} \setminus L_j$ and iterating q_j times. Now, proceeding as in 6.5, we can locate each point of T_j with respect to the fixed angles $i/(d-1)$, and hence compute the associated deployment sequence.

Examples. – A formal critical portrait in degree 2 takes an especially simple form; it consists of a single subset $\Theta_1 = \{\theta, \theta+(1/2)\}$ of $\mathbf{R}/\mathbf{Z}$ where $0 \leqq \theta < 1/2$. Condition C4 says that θ must be rational with denominator divisible by 4 (so that $\theta+(1/2)$ also has even denominator), and condition C5 is trivially satisfied. The sets L_1 and L_2 are the closed intervals $[\theta-(1/2), \theta]$ and $[\theta, \theta+(1/2)]$ modulo 1. The map ϕ_j is the doubling map mod 1 on L_j, and takes the constant value 2θ on the complementary interval. The corresponding rotation numbers are $\rho_1 = 0$ and $\rho_2 \neq 0$ respectively. Evidently the corresponding fixed point portrait has the form

$$\{\{0\}, T(p/q)\}$$

where $T(p/q)$ is the unique quadratic rotation cycle with rotation number $p/q \neq 0$.

The possibilities are of course much more diverse in degree 3. (Compare Figures 3, 4, as well as the discussion following 4.2.)

7. Realizing fixed point portraits

Recall that a polynomial map is *critically pre-periodic* if every critical orbit is eventually periodic, but no critical point actually lies in a periodic orbit. In Section 5 we described Fisher's characterization of the *critical portraits* of critically pre-periodic maps, and in Section 6 we showed how to compute the corresponding fixed point portraits. This section will exploit these results to prove the following.

THEOREM 7.1. — *A collection* $\mathscr{P}=\{T_1, \ldots, T_d\}$ *of exactly d non-vacuous subsets of* $\mathbf{Q}/\mathbf{Z}$ *can actually occur as the fixed point portrait of some critically pre-periodic polynomial of degree d if and only if it satisfies the four conditions of Theorem* 4.1, *that is:*

P1. *Each* T_j *is a rational rotation set.*

P2. *The* T_j *are disjoint and pairwise unlinked.*

P3. *The union of those* T_j *which have rotation number zero is precisely equal to* $\{0, 1/(d-1), \ldots, (d-2)/(d-1)\}$.

P4. *Each pair* $T_i \neq T_j$ *with non-zero rotation number is separated by at least one* T_l *with zero rotation number.*

In fact Theorem 4.1 asserts that these four conditions are necessary for any fixed point portrait. In the critically pre-periodic case, there must be d distinct repelling fixed points, so the number of sets T_j in $\mathscr{P}$ must be equal to d.

Conversely, suppose that we start with a collection $\mathscr{P}_0$ of d non-vacuous subsets satisfying all of these conditions. We will call such a $\mathscr{P}_0$ a "*candidate fixed point portrait*". Then we will construct a critical portrait $\boldsymbol{\Theta}$ which satisfies Fisher's five conditions, and hence determines a critically pre-periodic polynomial f. The construction will be carried out in such a way that the associated fixed point portrait $\mathscr{P}(f)$ is equal to the given candidate portrait $\mathscr{P}_0$, thereby completing the proof of 7.1. It should be noted that this construction is not at all unique: there are infinitely many different $\boldsymbol{\Theta}$ which would do the job. Hence, there are infinitely many different critically pre-periodic polynomials with any given fixed point portrait.

The essence of the construction lies in the case where $\mathscr{P}$ has $d-1$ distinct rotation number zero fixed points. A fixed point portrait with this property will be called *elementary*. We first consider the elementary case, and then adapt the argument to the general case.

An elementary fixed point portrait takes the form

$$\mathscr{P}_0=\left\{\left\{\frac{0}{d-1}\right\}, \ldots, \left\{\frac{d-2}{d-1}\right\}, T\right\}$$

where $T=\{t_0, \ldots, t_{kq-1}\}$ is a degree d rotation set with non-zero rotation number p/q. Here $k \leqq d-1$, and T can have any deployment sequence

$$0 \leqq s_1 \leqq s_2 \leqq \ldots \leqq s_{d-1} = kq$$

such that every residue class mod k is realized by at least one of the s_i. (Compare Part I, Lemma 5.)

We recall definitions and notation from Part I. The subset $T \subset \mathbf{R}/\mathbf{Z}$ divides its complement into kq arcs $A_0, A_1, \ldots, A_{kq-1}$ labeled so that A_i is bounded by t_i and $t_{i+1 \bmod kq}$. The *length* of A_i is denoted by $l(A_i)$. Here the whole circle has length 1, so that $\sum l(A_i)=1$. The *weight* $w(A_i)$ is, by definition, equal to the number of points $h/(d-1)$ fixed by the map $t \mapsto dt$ mod 1 which are contained in A_i. Thus $\sum w(A_i)=d-1$.

Let $j(i) \equiv i+kp$ mod kq, so that the d-tupling map carries the end points of the interval A_i onto the end points of the interval $A_{j(i)}$. According to formula (2.7) or Part I, Lemma 3, we have

$$dl(A_i)=l(A_{j(i)})+w(A_i).$$

It follows that the d-tupling map carries A_i homeomorphically onto $A_{j(i)}$ if and only if $w(A_i)=0$. Since $w(A_i)$ is an integer and $0<l(A_j)<1$, the following is an immediate consequence.

LEMMA 7.2. – *The product $dl(A_i)$ necessarily lies strictly between $w(A_i)$ and $w(A_i)+1$. Hence the weight $w(A_i)$ is equal to the integer part of this rational number $dl(A_i)$. If $\theta \in A_i$ is sufficiently close to the left hand endpoint t_i, it follows that A_i contains precisely $w(A_i)+1$ angles of the form $\theta+(h/d)$.*

Choosing $\theta \in A_i$ even closer to t_i if necessary, we can further suppose that the interval $(t_i, \theta]$ is disjoint from any of the p/q rotation cycles in degree d (since there are only finitely many such cycles), and from any points of the form $(p/(d-1))-(h/d)$.

Let $i_1, i_2, \ldots, i_m$ be those indices for which the weight $w(A_{i_j})$ is positive. For each $j=1, \ldots, m$, choose $\theta_j \in A_{i_j}$ subject to the following two conditions:

Θ1. The point θ_j is sufficiently close to the left endpoint t_{i_j} in the senses mentioned above.

Θ2. Under iteration of $t \mapsto dt$ mod 1 the point θ_j eventually maps to a fixed point $p/(d-1)$ which is contained in this same interval A_{i_j}.

Note that these conditions can always be satisfied, since the backward orbit of any point $p/(d-1)$ under the map $t \mapsto dt$ mod 1 is dense in $\mathbf{R}/\mathbf{Z}$. For $j=1, 2, \ldots, m$, let

$$\Theta_j=\left\{\theta_j+\frac{h}{d}: h=0, 1, \ldots, w(A_{i_j})\right\}$$

be the set of all angles of the form $\theta_j+(h/d)$ which are contained in the interval A_{i_j}.

This construction is illustrated in Figure 10. Here the candidate fixed point portrait $\mathscr{P}_0=\{\{0\}, \{1/4\}, \{1/2\}, \{3/4\}, \{69/124, 97/124, 113/124\}\}$ of degree $d=5$ is indicated with solid lines. The set T in this case has rotation number 1/3, and cuts the circle into arcs A_1, A_2, A_3 of weights 3, 1, 0 respectively. Corresponding sets $\Theta_j = \{\theta_j, \theta_j + 1/d, \ldots, \theta_j + w/d\}$, constructed as above, are indicated by dotted lines. (Only the first set Θ_1 has been labelled in the figure.) We will see that this schematic diagram can be realized by an actual polynomial map having a critical point of multiplicity 3 in the upper sector S_1, a simple critical point in the lower left sector S_2, and no critical point in S_3. The main step in the proof is as follows.

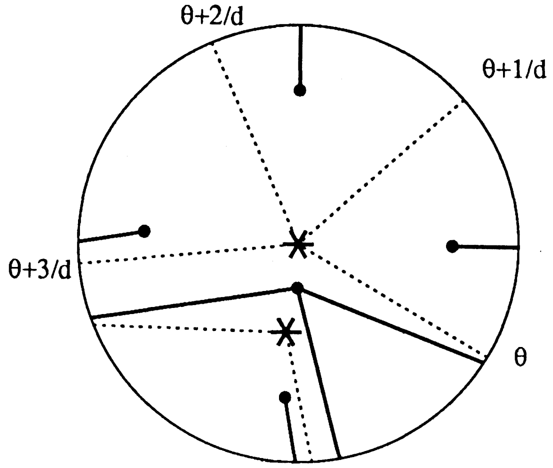


Fig. 10. – Construction of Θ_j in the elementary case.

LEMMA 7.3. – *The collection* $\boldsymbol{\Theta} = \{\Theta_1, \Theta_2, \ldots, \Theta_m\}$ *as constructed above, satisfies all of the conditions* C1 *through* C5 *of Section* 5, *and hence determines a unique critically pre-periodic polynomial* f *of degree* d.

In fact it is straightforward to check that $\boldsymbol{\Theta}$ satisfies the conditions C1 through C4. The proof that it satisfies C5 will depend on a subsidiary lemma. Let $\{L_1^0, \ldots, L_d^0\}$ be the decomposition of

$$\mathbf{R}/\mathbf{Z} \setminus (\Theta_1 \cup \ldots \cup \Theta_m)$$

into d unlink equivalence classes, as discussed in Sections 5 and 6.

LEMMA 7.4. – *Each* L_i^0 *contains precisely one of the fixed point types* T_i *of the given portrait* $\mathscr{P}_0 = \{\{0\}, \{1/(d-1)\}, \ldots, \{(d-2)/(d-1)\}, T\}$.

Proof of 7.4. – Since $\mathscr{P}_0$ is elementary, the unlink equivalence classes determined by $\boldsymbol{\Theta}$ take a special form: Exactly $d-1$ of the L_i^0 are open intervals $(\theta + (h/d), \theta + ((h+1)/d))$, while the d-th is the union of the remaining m disjoint intervals. Each of these d sets has total length $1/d$. Furthermore, the last set L_d^0 contains the specified rotation set T, with rotation number $p/q \not\equiv 0$. Since $\boldsymbol{\Theta}$ satisfies conditions C1 through C4, Lemmas 6.3 and 6.5 imply that there is a well defined rotation number associated with each L_i^0. The last set L_d^0 has rotation number $p/q \not\equiv 0$, and hence cannot contain any point $k/(d-1)$ with rotation number zero. Since none of the other L_i^0 is long enough to contain more than one such point, we conclude that the points $k/(d-1)$ must lie in distinct intervals $L_1^0, \ldots, L_{d-1}^0$. $\square$

Proof of 7.3. — To verify that $\boldsymbol{\Theta}$ satisfies condition C5, we must show that the sets $\Theta_1, \ldots, \Theta_m$ have distinct itineraries. But condition $\Theta 2$ implies that the d-tupling map sends these m sets eventually to distinct fixed points $k/(d-1)$, and these fixed points lie in distinct intervals L_j^0 by 7.4. Thus all of Fisher's conditions C1 through C5 are satisfied. $\square$

Proof of 7.1 *in the "elementary" case.* — In 7.3, we have used Fisher's Theorem to show that $\boldsymbol{\Theta}$ is the critical portrait of a unique critically pre-periodic polynomial f. It remains to show that the corresponding fixed point portrait $\mathscr{P}(f)$ is equal to the required portrait $\mathscr{P}_0$. From Lemma 7.4 we conclude that the fixed rays $R_{j/(d-1)}$ have distinct itineraries with respect to the sets L_i, and so land at distinct fixed points of f by Lemma 6.1. No other rays can land at these points, since we have accounted for all of the rays of rotation number zero. Similarly, the rays R_t with $t \in T$ have a common itinerary and hence land at a common fixed point of f. This proves that the fixed point portrait $\mathscr{P}(f)$ has the form

$$\left\{\{0\}, \left\{\frac{1}{d-1}\right\}, \ldots, \left\{\frac{d-2}{d-1}\right\}, T'\right\},$$

where T' is a rotation set containing T. To complete the proof of Theorem 7.1, we need only show that T' must be precisely equal to T.

Suppose to the contrary that T' were strictly larger than T. Then some of the intervals A_i complementary to T must be split by T' into two or more subintervals. For each such A_i, let A_i' be the rightmost of these subintervals. Thus A_i' is an open interval of the form (t', t_{i+1}) with $t' \in T' \cap A_i$. We claim that the weight $w(A_i')$ of such a subinterval must be zero; or equivalently (by 7.2) that the length $l(A_i')$ must be strictly less than $1/d$. In fact, if A_i itself has weight zero, then this is clear. But if $A_i = A_{i_j}$ has weight $w > 0$, then we have inserted a set $\Theta_j = \{\theta, \theta + (1/d), \ldots, \theta + (w/d)\}$ of $\boldsymbol{\Theta}$ into the arc A_i. By the construction of θ, the point t' cannot lie to the left of θ. (Condition $\Theta 1$.) Furthermore, since T' is unlinked with Θ_j, t' cannot lie between θ and $\theta + (w/d)$. Hence t' must lie in the open interval $(\theta + (w/d), t_{i_j+1})$. This interval has length less than $l(A_i) - w/d$, which is less than $1/d$ by Lemma 7.2. Therefore, the subarc A_i' has length less than $1/d$, and hence has weight zero as asserted. It follows that A_i' maps homeomorphically onto another arc of the same form under the d-tupling map. (Compare 2.5.) Similarly, this image arc must have length less than $1/d$, even though it is strictly longer than A_i'. Continuing this construction q times, we return to our starting point and conclude that A_i' is strictly longer than itself, which is impossible. This completes the proof of 7.1 in the elementary case.

The proof in the general case is essentially the same; however the bookkeeping is a little more complicated. The rotation sets T_j split the circle into equivalence classes $U_1, \ldots, U_m$, where two points of $\mathbf{R}/\mathbf{Z} \setminus (T_1 \cup \ldots \cup T_d)$ belong to the same U_h if and only if they belong to the same component of $\mathbf{R}/\mathbf{Z} \setminus T_j$ for every j. *Note that each such*

U_h *must have exactly one* T_j *with non-zero rotation number intersecting its boundary:* There cannot be more than one by P4, and there must be at least one since otherwise there could not be d distinct sets T_p. (Compare 3.3.)

Evidently this U_h is contained in just one arc A_i of the complement $\mathbf{R}/\mathbf{Z} \setminus T_j$. In fact either $U_h = A_i$, or else U_h can be obtained from this complementary arc A_i by removing one or more (possibly degenerate) intervals of the form $[a/(d-1), b/(d-1)] \subset A_i$, where $a \leqq b$. The weight w of this set U_h can be defined as the number of such missing intervals. If $w > 0$, we can choose a point θ near the left end of U_h exactly as in the argument above. These points $\theta \in U_h$ for different sets U_h must be chosen so that their orbits under the d-tupling map end up on different rotation sets $T_j \in \mathscr{P}_0$. Given such a choice of θ, let Θ_h be the set of all angles of the form $\theta + (p/(d-1))$ which are contained in U_h. Just as in the argument above, this set has cardinality $\# \Theta_h = w(U_h) + 1$. The resulting critical portrait $\mathbf{\Theta} = \{\Theta_1, \ldots, \Theta_d\}$ satisfies Fisher's five conditions, and hence determines a critially pre-periodic polynomial f. Again, it can be shown that the associated portrait $\mathscr{P}(f)$ is equal to the given $\mathscr{P}_0$. Details will be left to the reader. □

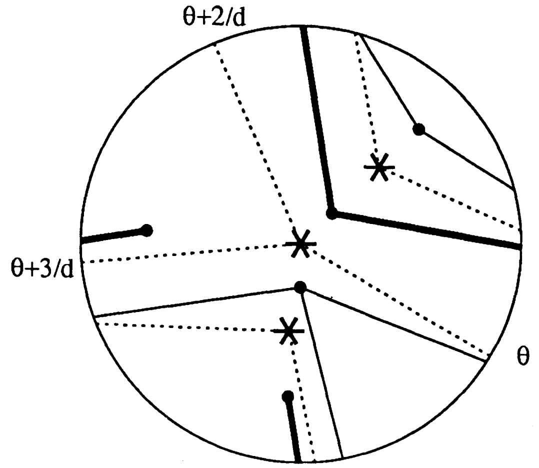


Fig. 11. — Construction of Θ_j, general case.

This construction is illustrated in Figure 11 for the candidate fixed point portrait

$$\mathscr{P}_0 = \left\{ \left\{ 0, \frac{1}{4} \right\}, \left\{ \frac{1}{2} \right\}, \left\{ \frac{3}{4} \right\}, \left\{ \frac{1}{24}, \frac{5}{24} \right\}, \left\{ \frac{69}{124}, \frac{97}{124}, \frac{113}{124} \right\} \right\}.$$

Here one of the fixed points of rotation number zero of Figure 10 has been replaced by a fixed point of rotation number 1/2. The rays to the three rotation number zero fixed points, indicated schematically by heavy lines, now cut the plane into two "basic regions". Each of these contains a unique fixed point, which necessarily has non-zero rotation number. The rays to all five fixed points cut the plane into a number of regions, and correspondingly cut the circle into the same number of equivalence classes. In this example, two of these regions have critical weight zero, two have critical weight one, and the remaining region has critical weight two. The construction of a compatible critical portrait, with just one critical point in each region of positive weight, is illustrated by the dotted lines in the figure.

8. Further discussion

The proof in Section 6 leaves open the problem of establishing Conjecture 4.2 in the case of a portrait $\mathscr{P}$ which contains fewer than d non-empty rotation sets, so that some of the fixed points must be rationally invisible or virtual. However Poirier has carried out a complete proof, based on *Hubbard trees*. (Compare [DH2].) As in the case of Fisher's Theorem, these provide an indirect way of invoking Thurston's theory of post-critically finite rational maps. To each candidate fixed point portrait, Poirier constructs a unique simplest possible Hubbard tree which is *critically periodic*, and whose associated polynomial is shown to have the required fixed point portrait.

An alternative approach, suggested by Hu and Jiang [HJ], would be based on Thurston's method of laminations [Th1].

Still another interesting method would be to build up more complicated polynomials starting with the "elementary" ones by an "intertwining" or "marriage" construction. (Compare [Bi].) Given two monic polynomials of degrees d_1 and d_2, we would like to construct a new polynomial of degree d_1+d_2-1 by cutting each dynamic plane open along its zero ray, and then pasting the two planes together along these rays. It would then be necessary to make further cuts along the iterated pre-images of these zero rays and to put a compatible conformal structure on the resulting topological map. Finally, it would be necessary to prove that the resulting polynomial map has the expected fixed point portrait. This would surely be a useful construction, but we do not know how to carry it out.

There are a number of other loose ends which are left open by this paper. For example, it would be useful to develop the concept of critical portrait for polynomials which are not critically pre-periodic. Also, it would be useful to develop the concept of an irrational rotation set. (Compare [Ve].) This might be helpful in understanding Siegel disks or Cremer points. *Recent work of Yoccoz emphasizes the importance of understanding not only fixed points, but also all of the iterated pre-images of fixed points.* Another natural problem would be to understand how the fixed point portrait for the n-th iterate $f^{\circ n}$ behaves as we increase the integer n.

Here is a final basic problem. (Compare Appendix C.) In the degree d connectedness locus $\mathscr{C}_d$, let $\mathscr{C}_d(\mathscr{P})$ be the subset realizing some given fixed point portrait $\mathscr{P}$. *Is this subset contractible; or even connected? Is its closure a cellular set* (i.e., *is it the intersection of a strictly nested family of cosed topological cells*)?

Appendix A.
Disconnected julia sets

It is frequently useful to consider polynomials which do not belong to the connectedness locus. (*See* for example [At1, At2].) This appendix will briefly describe fixed point theory for such polynomials. For a more complete treatment, *see* [DH2].

Let $f: \mathbf{C} \to \mathbf{C}$ be an arbitrary monic polynomial map of degree $d \geqq 2$. Even if the filled Julia set $K(f)$ is not connected, we can define *external rays*, as the orthogonal trajectories of the level curves for the *Green's function or canonical potential function*

$$G(z) = \lim_{n \to \infty} d^{-n} \log^+ |f^{\circ n}(z)|.$$

This function G is smooth, harmonic, strictly positive outside of $K(f)$, and tends to zero as we approach $K(f)$. If $K(f)$ is not connected, then this potential function will have critical points outside of $K(f)$. In fact G is critical precisely at the *pre-critical* points of f, that is at all points which are critical for some iterate $f^{\circ n} = f \circ \ldots \circ f$. Whenever $K(f)$ [or equivalently $J(f)$] is not connected, there must be at least one critical point of f outside of $K(f)$, and hence infinitely many critical points of G outside of $K(f)$. Evidently critical points of G lead to bad behavior in the external rays. On the other hand, if $K(f)$ is connected, then it contains all of the pre-critical points, and none of this deviant behavior can occur.

Every degree d polynomial g is conjugate to the d-th power map near infinity. That is, there exists a conformal isomorphism $z \mapsto \varphi(z)$, defined throughout a neighborhood of infinity, which satisfies $\varphi(g(z)) = \varphi(z)^d$, with $\log|\varphi(z)| = G(z)$. In general, there are $d-1$ distinct possible choices for φ. However, in the case of a monic polynomial f, there is one preferred $\varphi(z)$ which is asymptotic to z as $|z| \to \infty$. Thus we can label each external ray by an angle $t \in \mathbf{R}/\mathbf{Z}$, just as in Section 1.

As we follow such an external ray R_t, starting out near infinity and working inward by analytic continuation, it may happen that it hits a critical point of G, or equivalently a pre-critical point of f. If this happens, then two or more external rays crash together at this point, and then bounce off in the same number (two or more) of new directions, so that there is no single well defined continuation. However, we can still define the *left hand limit ray* R_{t-} and the *right hand limit ray* R_{t+}. For this purpose, it is convenient to parametrize the subset $R_t \subset \mathbf{C}$. In fact we can use the potential function $G(z) > 0$ as a canonical parameter along each R_t. Hence we can define R_{t+}, for example, as the pointwise limit of the parametrized curve R_s as $s \to t$, $s > t$. These two limit rays R_{t+} and R_{t-} are no longer smooth everywhere, but have abrupt changes in direction at all pre-critical points: one turns always to the left while the other turns always to the right. (Compare Figures 14 and 16 below.) Note again that this behavior occurs whenever the Julia set J of f is not connected.

If the angle t is rational, then just as in [DH2], p. 70 the ray R_t, or the two limit rays R_{t+} and R_{t-} (if R_t bounces off a pre-critical point), tend to well defined limit points in $K(f)$ as the parameter $G(z)$ tends to zero. We will say that the ray or limit ray *lands* at the limit point a_t or $a_{t\pm}$ in $K(f)$. If this landing point is fixed under f, then just as in Lemma 2.2 there is a well defined rotation number in $\mathbf{Q}/\mathbf{Z}$.

In general, as we follow such a ray in from infinity, its set of accumulation points will be a compact and connected subset of J. Here is an important special case: *If the Julia set* J *of* f *is totally disconnected, then every smooth ray, and also every left or right limit ray, must land at a single well defined point of* J. For in this case, any connected set of accumulation points in J must reduce to a single point.

DEFINITION A.1. — Let $\Sigma \subset \mathbf{R}/\mathbf{Z}$ be the set of all of the angles of external rays which crash on critical or pre-critical points of f. Clearly Σ is a countable dense subset of the circle, whenever it is non-vacuous. Let us construct a Cantor set C_Σ out of the circle $\mathbf{R}/\mathbf{Z}$ by cutting the circle open at all points of Σ. In other words, each point $\sigma \in \Sigma$ is to be replaced by two distinct points $\sigma^- < \sigma^+$, and the union $C_\Sigma = (\mathbf{R}/\mathbf{Z} \setminus \Sigma) \cup \{\sigma^-\} \cup \{\sigma^+\}$ is to be topologized as a (locally) ordered set.

LEMMA A.2. — *If the Julia set* J *is totally disconnected, then the correspondence* $t \mapsto a_t$ *which assigns a landing point to each angle in* $\mathbf{R}/\mathbf{Z} \setminus \Sigma$ *extends to a continuous mapping from this Cantor set* C_Σ *onto the Julia set* J. *Hence every point of the Julia set is the landing point of at least one ray or limit ray.*

Proof. — The image of C_Σ in J is a compact fully invariant subset, and hence must coincide with the full Julia set. □

COROLLARY A.3. — *Each fixed point* z_0 *of* f *is the landing point of one or more such rays. These rays are permuted by* f, *preserving their cyclic order; hence they have a well defined rotation number.*

However this rotation number need not be rational: It can be any element of the circle $\mathbf{R}/\mathbf{Z}$. (Compare Appendix C.)

Appendix B.
Transition between fixed point portraits

The concept of fixed point portrait turns out to be a fairly robust one. That is, the fixed point portrait of a polynomial usually does not change as we perturb the polynomial. However, there are exceptions, as detailed in the discussion below.

All of our polynomials are to be monic of some fixed degree d. As in the preceding Appendix, we do not necessarily assume that our Julia sets are connected. Let z_0 be any fixed point of the polynomial f_0. If the multiplier $\lambda_0 = f_0'(z_0)$ satisfies $\lambda_0 \neq 1$, then for all f in some neighborhood of f_0, the implicit function theorem implies that we can solve the equation $f(z) = z$ for the fixed point $z = z(f)$ as a holomorphic function of f, with $z(f_0) = z_0$.

LEMMA B.1. — *Suppose that* $|\lambda_0| > 1$ *so that* z_0 *is a repelling fixed point, and suppose that some rational external ray* $R_t = R_t(f_0)$ *lands at* z_0. *Then for any* f *sufficiently close* *to f_0, the corresponding ray $R_t(f)$ lands at the corresponding fixed point $z(f)$. In particular, it follows that the rotation number $\rho(f, z(f))$ at the fixed point $z(f)$ remains constant as f varies through some neighborhood of f_0.*

Remark B.2. — We cannot weaken the hypotheses of this Lemma. For example, if z_0 is a *parabolic* fixed point, or more generally any fixed point with $|\lambda_0|=1$, then within any neighborhood of f_0 this fixed point can become a parabolic or repelling point with any rotation number ρ' which is sufficiently cose to $\rho(f_0, z_0)$. In particular, there are infinitely many possible choices for ρ'. Similarly, within any neighborhood of f_0, the fixed point can become an attracting or Cremer point or the center of a Siegel disk, and hence rationally invisible.

Outside the connectedness locus, it may well happen that a repelling fixed point admits an external ray R_t with t irrational. [*See* Figure 16. This case cannot occur when $K(f)$ is connected by Theorem 1.1.] Here again, the rotation number $\rho(f, z(f))$ can take on infinitely many distinct values within any neighborhood of f_0. Similarly, whenever a left or right limit ray lands on z_0, the rotation number can change within any neighborhood of f_0 (*Fig.* 14).

Proof of B.1. — By the Koenigs Linearization Theorem (*see* for example [M2]), there exists a local coordinate $\zeta=h(z)$ near z_0 so that $h(f_0(z))=\lambda h(z)$ for all z and so that $h(z_0)=0$. Since the angle t is rational, and since the external ray $R_t(f_0)$ lands at the fixed point z_0, it is easy to check that t is periodic under the d-tupling map, say with period q. (Compare 1.1.) Therefore, we can choose a segment of $R_t(f_0)$ which joins some point z' to $f_0^{\circ q}(z')$, and which lies completely within the domain of h. These conditions will still be satisfied if we perturb f_0 slightly, and it follows that the corresponding ray $R_t(f)$ for the perturbed map f must land at the corresponding fixed point $z(f)$. □

Recall that the *type* $T(f, z)$ is the finite set consisting of all rational angles $t\in\mathbf{Q}/\mathbf{Z}$ for which R_t lands at z. Thus Lemma B.1 asserts that

$$T(f, z(f)) \supset T(f_0, z_0),$$

whenever the appropriate hypotheses are satisfied. In the degree two case it follows that these two sets are equal, since one quadratic rotation set cannot properly contain another. It is natural to ask whether $T(f, z(f))=T(f_0, z_0)$ in all cases. The following shows that this is not true.

Example B.3. — The polynomial $f_0(z)=z+z(z-1)^2$ has connected Julia set, and has a repelling fixed point of rotation number zero and type $T=\{1/2\}$ at the origin. However, polynomials $f(z)=(1+\varepsilon)f_0(z)$ arbitrarily close to f_0 have a fixed point of strictly larger type $T=\{0, 1/2\}$ at the origin. This phenomenon can be explained as follows. The polynomial f_0 has a parabolic fixed point of type $T=\{0\}$ at $z=1$. As we perturb f_0, multiplying it by $1+\varepsilon$, there is a "parabolic implosion" of the filled Julia set. For the perturbed polynomial, the parabolic fixed point splits into two complex fixed points, and the zero ray squeezes between them and continues all the way to the origin.

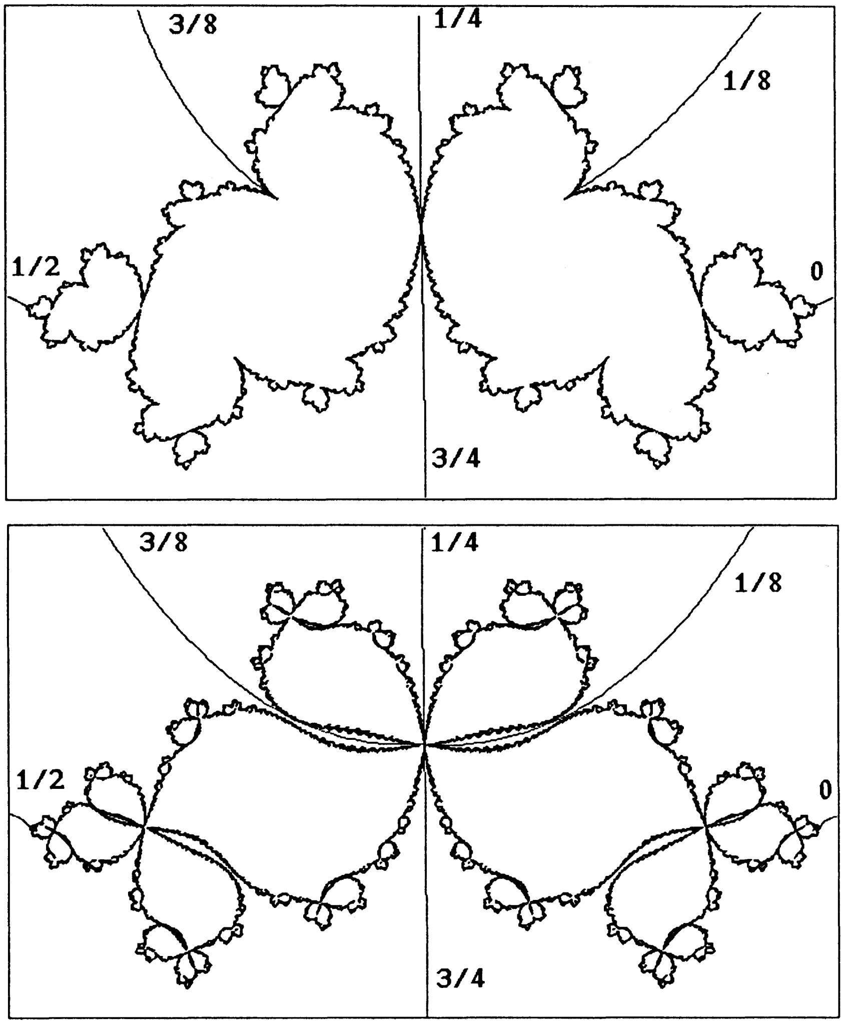


Fig. 12. – Above: Julia set for $f(z)=z^3-z+\sqrt{-4/27}$; the 1/8 and 3/8 rays land on a parabolic period 2 orbit. Below: After an arbitrarily small perturbation of f, these rays land at a repelling fixed point.

Figure 12 shows a similar example for a repelling fixed point of rotation number 1/2. In this case the type jumps from $\{1/4, 3/4\}$ to $\{1/8, 1/4, 3/8, 3/4\}$ under an arbitrarily small perturbation. We show next that such examples are essentially the only possible ones.

Let t be a rational angle, and suppose that for polynomials f arbitrarily close to f_0 the external ray $R_t(f)$ lands at the fixed point $z(f)$. We want to analyze the possible landing points for the external ray $R_t(f_0)$. According to [DH2] this ray must either bounce off a pre-critical point, or land on a parabolic or repelling periodic point. We claim that this last case cannot occur, unless $R_t(f_0)$ lands at the fixed point $z_0=z(f_0)$

itself. For if $R_t(f_0)$ lands on a repelling point $z_1 \neq z_0$, then Lemma B.1 implies that $R_t(f)$ must stay bounded away from $z(f)$ for all f near f_0, contradicting our hypothesis. This proves the following.

LEMMA B.4. — *Fix some rational angle t, and suppose that, for polynomials f arbitrarily close to f_0, the external ray $R_t(f)$ lands at the fixed point $z(f)$. Then either:*

(1) *the ray $R_t(f_0)$ lands at the corresponding fixed point z_0,*

(2) *$R_t(f_0)$ bounces off some pre-critical point, or else*

(3) *$R_t(f_0)$ lands at some parabolic periodic point $z_1 \neq z_0$.*

Remark. — In case (3) above, we conjecture that the period of the point z_1 must be equal to the period q of the angle t under the d-tupling map. The following Lemma implies at least that z_1 must be either a fixed point or a period q periodic point.

LEMMA B.5. — *If a collection of q angles forms a rotation cycle of period q, and if the corresponding rays $R_t(f)$ do not bounce off pre-critical points, then these rays must land either at a single fixed point or at q distinct points.*

Proof. — Let $0 < t(1) < \ldots < t(q) < 1$ be the elements of the rotation cycle, and let $z_1, \ldots, z_q$ be the corresponding landing points. By hypothesis, the d-tupling map permutes these angles $t(i)$ cyclically, while preserving their cyclic order. If $z_1 = z_2$ or $z_1 = z_q$, then it follows easily that $z_1 = z_2 = \ldots = z_q$. On the other hand, if $z_1 = z_h$ with $2 < h < q$, then the rays $R_{t(1)}$ and $R_{t(h)}$ cut the plane into two halves, one containing $R_{t(2)}$ and the other containing $R_{t(2+h)}$. But these last two rays must land at a common point, so it follows that $z_1 = z_2$ and hence $z_1 = z_2 = \ldots = z_q$. □

Appendix C.
The Mandelbrot set

This appendix will describe the "classical" theory of limbs in the Mandelbrot set M. (Compare [Br], [BD], [D2], [At2].)

Let $\mathscr{P}_2 \cong \mathbf{C}$ be the *quadratic parameter space* consisting of all polynomials of the form $f(z) = z^2 + c$, and let $M = \mathscr{C}_2 \subset \mathscr{P}_2$ be the compact subset consisting of those polynomials with connected Julia set (*Fig.* 13). Note that every $f \in M$ is a polynomial map having one and only one fixed point with rotation number zero, namely the landing point of the ray $R_0 = R_0(f)$. If the remaining fixed point is distinct, and is the landing point of at least one rational ray, then it has a well defined rotation number $\rho = p/q \neq 0$ in $\mathbf{Q/Z}$ by Lemma 2.2.

DEFINITION. — Whenever $f \in M$ has a fixed point of rotation number $p/q \neq 0$, we say that f belongs to the *p/q-limb* $M(p/q) \subset M$. Otherwise, if there is no such fixed point, we will say that f belongs to the *central core* $M(\heartsuit) \subset M$.

This last set is quite easy to describe explicitly. It will be convenient to use the notation F_λ for the unique map in $\mathscr{P}_2$ which has a fixed point with multiplier $f'(z)$ equal to λ. A

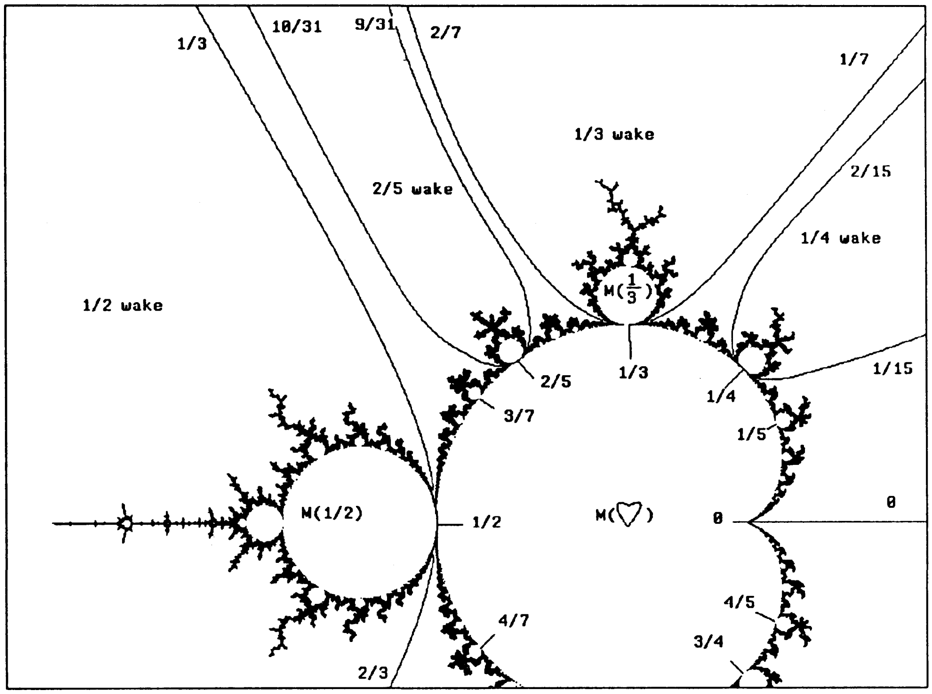


Fig. 13. – Degree 2 parameter space picture, with ∂M emphasized.

brief computation shows that

$$F_\lambda(z) = z^2 + c_\lambda \qquad \text{with} \quad c_\lambda = \frac{1}{4}\lambda(2-\lambda), \tag{2}$$

and that the two fixed points $z=(1/2)\lambda$ and $z=1-(1/2)\lambda$ of F_λ have multipliers equal to λ and $2-\lambda$ respectively. The set

$$M^0(\heartsuit) = \{ F_\lambda : |\lambda| < 1 \}$$

forms an open topological disk consisting exactly of those polynomials $F_\lambda \in M$ which possess an attracting fixed point. Similarly, the F_λ with λ on the unit circle are those which possess an indifferent fixed point. As $\lambda = e^{2\pi it}$ traverses the unit circle, the corresponding values $c_\lambda = e^{2\pi it}(2-e^{2\pi it})/4$ traverse a cardioid, and it follows easily that the closure

$$\bar{M}(\heartsuit) = \{ F_\lambda : |\lambda| \leqq 1 \}$$

is a closed topological disk bounded by this cardioid. The set $M(\heartsuit)$ itself can now be described as the interior $M^0(\heartsuit)$, together with all boundary points $F_{\exp(2\pi it)}$ for which t is either irrational or zero.

Now consider any polynomial $f(z) = z^2 + c$ which does *not* belong to M. Then the Julia set $J(f)$ is totally disconnected. Such an f has two distinct fixed points, each with a well defined rotation number by Corollary A.3. Again, at least one of these two fixed points must have rotation number zero. We let $\rho(f) \in \mathbf{R}/\mathbf{Z}$ be the rotation number of the other fixed point. More generally:

DEFINITION C.1. – For any $f \in \mathcal{P}_2$ which does not belong to the central core $M(\heartsuit)$, let $\rho(f) \in \mathbf{R}/\mathbf{Z}$ be the unique number such that 0 and $\rho(f)$ are the rotation numbers of the two fixed points of f. (Thus we set $\rho(f)=0$ only if *both* fixed points have rotation number zero.)

If $f \in M \setminus M(\heartsuit)$, then $\rho(f)$ must be a rational number $p/q \neq 0$, and, as noted above, we say that f belongs to the p/q-limb. If $f \notin M$, then the number $\rho(f)$ can be *any* element

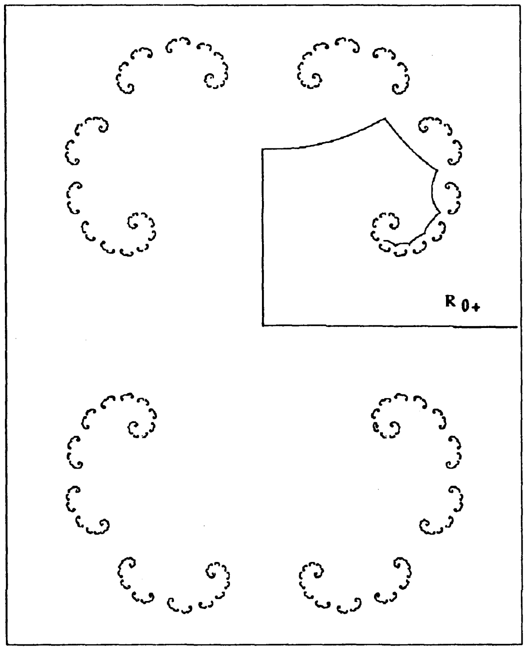


Fig. 14. – Julia set for a polynomial $f(z)=z^2+0.4$ which belongs to the ray $R_0(M)$ in parameter space. The right hand limit ray $R_{0+}(f)$ bounces off infinitely many pre-critical points as it spirals in to the upper fixed point. Both fixed points have rotation number zero.

of $\mathbf{R}/\mathbf{Z}$. The case $\rho(f)=0$ is illustrated in Figure 14. This case occurs whenever the constant $f(0)=c$ is real with $c>1/4$. An example with $\rho(f)$ rational and non-zero is shown in Figure 15, and an example with $\rho(f)$ irrational is shown in Figure 16.

Up to this point, we have considered external rays only in the dynamic plane $\mathbf{C} \setminus K(f)$. Following Douady and Hubbard, we can consider external rays also in the parameter plane $\mathcal{P}_2 \setminus M$. Again these can be described as the orthogonal trajectories of a suitable "canonical potential function", which now vanishes precisely on the Mandelbrot set M. Every polynomial $f \in \mathcal{P}_2 \setminus M$ belongs to some unique external ray $R_t(M)$. Here the angle t is characterized by the fact that the corresponding ray $R_t(f)$ in the dynamic plane passes through the critical value $f(0)=c$. (*See* [DH1] or [DH2].)

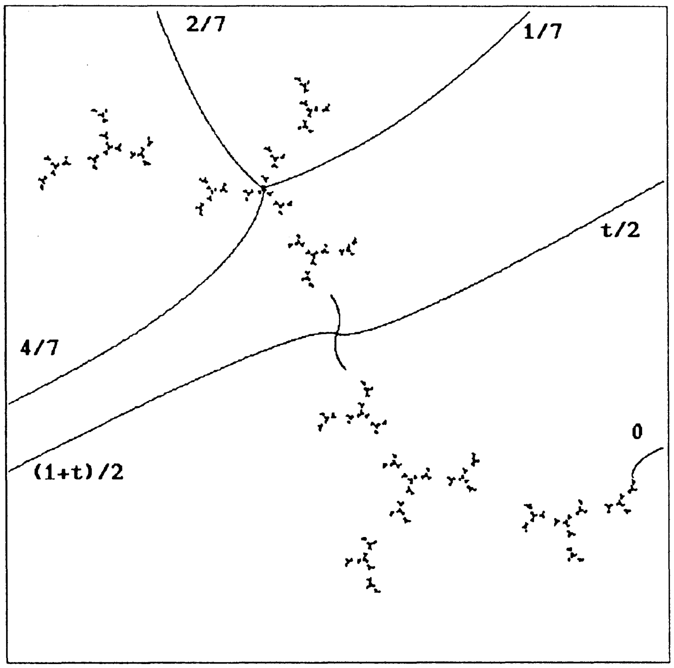


Fig. 15. — Julia set for a polynomial $f(z)=z^2+1.1\,i$ which belongs to the "wake" of the (1/3)-limb in parameter space. (Compare Figure 1.) Here $f\in \mathrm{R}_t(\mathrm{M})$ with $t\approx .1870$.

LEMMA C.2. — *For a polynomial* $f\in\mathscr{P}_2$ *which does not belong to* M, *the rotation number* $\rho(f)$ *depends only on the external ray* $\mathrm{R}_t(\mathrm{M})$ *which contains* f. *Furthermore, the correspondence* $t\mapsto\rho(f)$ *defines a map from the circle* $\mathbf{R}/\mathbf{Z}$ *to itself which is continuous and monotone of degree one.*

Proof. — Since the ray $\mathrm{R}_t(f)$ passes through $f(0)$, it follows that the two pre-images of this ray, namely $\mathrm{R}_{t/2}(f)$ and $\mathrm{R}_{(t+1)/2}(f)$ must crash together at the critical point 0. As in Lemma 5.2, these two rays (truncated at the critical point) cut the plane into two halves, and hence partition the Julia set into two subsets J_0 and J_1. In the present case however, the intersection $\mathrm{J}_0\cap\mathrm{J}_1$ is vacuous, since the critical point is not in the Julia set. It follows that every point of $\mathrm{J}(f)$ has a unique itinerary

$$(i_0, i_1, \ldots)\in \prod_{0\leq n<\infty}\{0, 1\}$$

with respect to this partition, and that f restricted to the Julia set is topologically conjugate to the *one sided 2-shift* $(i_0, i_1, \ldots)\mapsto(i_1, i_2, \ldots)$. As in Lemmas 6.2 through 6.4, we can compute $\rho(f)$ as the rotation number of an associated monotone circle map φ,

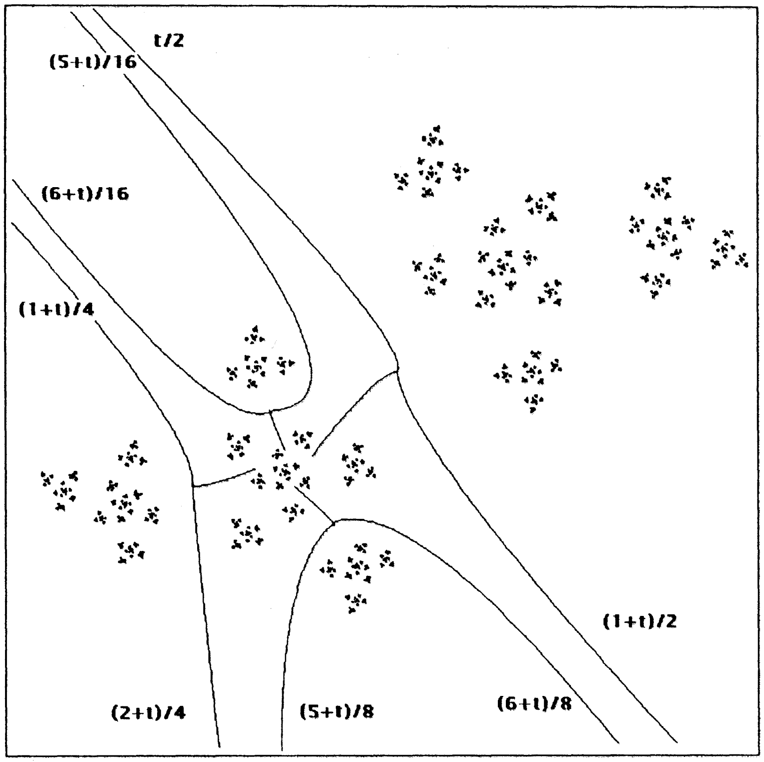


Fig. 16. – Quadratic Julia set with fixed point of rotation number $(\sqrt{5}-1)/2$. Here $t = .70980344$. The corresponding rotation set is a Cantor set obtained from **R/Z** by removing open intervals of lengths 1/2, 1/4, 1/8, ...

which is defined by

$$\varphi(u) \equiv \begin{cases} 2u \pmod 1 & \text{for } t/2 \leqq u \leqq (1+t)/2 \\ t & \text{otherwise.} \end{cases}$$

Further details of the proof are straightforward. □

If a polynomial $f(z) = z^2 + c$ has a fixed point of rotation number p/q, then the q rays landing at this point cut the complex plane into q complementary sectors. According to Corollary 2.3 and Part I, the angles belonging to these rays comprise the unique quadratic rotation set $T(p/q)$ with rotation number p/q. Denote by S_0 the *narrowest* of these complementary sectors, that is the one whose angular width is smallest, and let $S_n = f^{\circ n}(S_0)$ be its n-th forward image for $0 \leqq n \leqq q-1$. It follows from Lemma 2.5 that the sequence of angular widths $l(S_0)$, $l(S_1)$, ..., $l(S_{q-1})$ forms a geometric progression with ratio 2 and sum 1; hence $l(S_n) = 2^n/(2^q - 1)$. Here the widest sector S_{q-1} contains the critical point, and the narrowest sector S_0 contains the critical value. (*See* 2.6.)

Definition. – Let $0 < \theta_-(p/q) < \theta_+(p/q) < 1$ be the angles of the two external rays spanning the sector S_0. Thus each $\theta_\pm(p/q)$ is a rational number of the form $m/(2^q - 1)$, and the difference $\theta_+(p/q) - \theta_-(p/q)$ is equal to $1/(2^q - 1)$.

DEFINITION. — If $\rho(f)$ takes a rational value $p/q \not\equiv 0$ for $f \notin M$, then following Atela, we say that f belongs to the *wake* of the (p/q)-limb.

LEMMA C.3. — *A polynomial f belonging to the external ray $R_t(M)$ belongs to the (p/q)-wake if and only if the angle t lies in the closed interval* $[\theta_-(p/q), \theta_+(p/q)]$.

Proof. — As noted above, for any polynomial having a fixed point of rotation number p/q, the critical value must lie in the narrowest sector S_0. Hence its external angle t must lie in the corresponding interval. Conversely, if t lies in this narrowest interval of $\mathbf{R}/\mathbf{Z} \setminus T(p/q)$, then both of its two pre-images must lie in the corresponding widest interval of $\mathbf{R}/\mathbf{Z} \setminus T(p/q)$. Thus every element of the rotation set $T(p/q)$ lies on just one side of the associated critical portrait $\{t/2, (t+1)/2\}$. Therefore, the corresponding rays land at a single fixed point of f. □

Remark C.4. — The special case of Lemma C.3 in which t is one of the two end points $\theta_\pm(p/q)$ is of particular interest. In this case, the external rays correspond to the angles in $T(p/q)$ all crash into pre-critical points of f. However, the left and right limit rays exist. One of these two sets of limit rays lands on the required fixed point, while the other lands on an orbit of period q.

Remark C.5. — Evidently these intervals $[\theta_-(p/q), \theta_+(p/q)]$ are pairwise disjoint. *Note that their union contains Lebesgue almost every point of the circle.* In other words, the sum

$$\sum_{0<p/q<1} 1/(2^q-1) \tag{3}$$

of their lengths is equal to one. To prove this, we consider the auxiliary sum

$$\sum_{0<m<n} 2^{-n}. \tag{4}$$

If we sum first over n and then over m, we see that this auxiliary sum is equal to $\sum_{m>0} 2^{-m}=1$. On the other hand, if we sum first over all pair $0<m<n$ with some given ratio m/n, expressed as a fraction in lowest terms as p/q, we obtain $2^{-q}+2^{-2q}+2^{-3q}+\ldots=1/(2^q-1)$. Now summing over all such ratios p/q we obtain the required expression (3). It follows that: *For Lebesgue almost every polynomial $f(z)=z^2+c$ in the complement of the Mandelbrot set, the rotation number $\rho(f)$ is rational.* Veerman has proved the sharper assertion that the set of angles t which correspond to irrational rotation numbers under the correspondence $t \mapsto \rho(f)$ of C.2 is a set of Hausdorff dimension zero. Douady and Sentenac (unpublished) have shown that every such angle t is a transcendental number.

Now suppose that we fix some number $\rho \in (0, 1)$ and sum these lengths $1/(2^q-1)$ only over those intervals $[\theta_-(p/q), \theta_+(p/q)]$ for which $p/q \leqq \rho$. Evidently the sum must be equal to $\theta_+(\rho)$ whenever ρ is rational. We take this formula as a definition when ρ is irrational:

$$\theta_+(\rho) = \sum_{0 < p/q \leq \rho} 1/(2^q - 1). \tag{5}$$

The function θ_+ is monotone and continuous from the right, being the inverse of the correspondence $t \mapsto \rho(f)$ of C.2 in the sense that

$$\theta_+(\alpha) = \sup \{ t \in (0, 1) : \rho(\mathrm{R}_t(\mathrm{M})) = \alpha \}.$$

There is an associated function $\theta_-(\rho) = 1 - \theta_+(1 - \rho)$ which is continuous from the left, and coincides with $\theta_+(\rho)$ whenever ρ is irrational.

Proceeding to manipulate this expression (5), just as in the discussion above, we see that $\theta_+(\rho) = \sum_{0 < m \leq \rho n} 2^{-n}$, which yields the following nicely convergent series expansion.

COROLLARY C.6. — *For every* $\rho \in (0, 1)$ *we have*

$$\theta_+(\rho) = \sum_{n=1}^{\infty} [\rho n] 2^{-n},$$

where $[\rho n]$ *stands for the largest integer* $\leq \rho n$.

In the rational case $\rho = p/q$, note that this sum must itself be a rational number of the form $h/(2^q - 1)$.

Let us take a closer look at external rays in parameter space. We next prove an important result of Douady and Hubbard.

THEOREM C.7. — *If* $t \in \mathbf{Q}/\mathbf{Z}$ *is rational with odd denominator, then the external ray* $\mathrm{R}_t(\mathrm{M})$ *for the Mandelbrot set lands at a well defined polynomial* $f \in \mathrm{M}$, *which possesses a parabolic periodic orbit. More precisely: the corresponding ray* $\mathrm{R}_t(f)$ *in the dynamic plane lands at a parabolic periodic point in the Julia set* $\mathrm{J}(f)$.

Remark. — If t is rational with even denominator, then Douady and Hubbard show that $\mathrm{R}_t(\mathrm{M})$ lands at a critically pre-periodic polynomial f, and furthermore that the corresponding ray $\mathrm{R}_t(f)$ lands at the critical value $f(0) \in \mathrm{J}(f)$. We will not try to give a proof of this. For arbitrary values of t there is no known proof that $\mathrm{R}_t(\mathrm{M})$ necessarily lands.

Proof of C.7 (We are indebted to discussions with Hubbard.) — We must compare external rays $\mathrm{R}_t(\mathrm{M})$ in parameter space with external rays $\mathrm{R}_t(f)$ for the Julia set $\mathrm{J}(f)$. Recall from [DH1] or [DH2] that a polynomial $f(z) = z^2 + c$ belongs to the external ray $\mathrm{R}_t(\mathrm{M})$ in parameter space if and only if the corresponding ray $\mathrm{R}_t(f)$ in the dynamic plane passes through the critical value $f(0) = c$. Let $f_0 \in \mathrm{M}$ be any accumulation point for the ray $\mathrm{R}_t(\mathrm{M})$. According to 1.1, the corresponding external ray $\mathrm{R}_t(f_0)$ necessarily lands at a periodic point $z_0 \in \mathrm{J}(f_0)$ which is either parabolic or repelling. Suppose that this point were repelling. Then according to B.1, for any polynomial $f(z) = z^2 + c$ sufficiently close to f_0 the corresponding ray $\mathrm{R}_t(f)$ would land at a periodic point $z(f)$ close to z_0. In particular, this ray $\mathrm{R}_t(f)$ could not bounce off any pre-critical point for f. But if we choose any f belonging to $R_t(M)$, then the ray $R_t(f)$ does bounce off some pre-critical point of f. (In fact it bounces off infinitely many. Compare Figure 14.) For the angle t is periodic under the doubling map, with period say q, and it follows that the forward image $f^{\circ(q-1)}(R_t(f))$ bounces off the critical point zero. Since such an $f \in R_t(M)$ can be chosen arbitrarily close to f_0, this yields a contradiction.

Therefore, z_0 must be a parabolic periodic point for f_0. Since the ray $R_t(f_0)$ is fixed by the q-fold iterate $f_0^{\circ q}$, it follows from 2.4 that its landing point z_0 must be a fixed point of multiplier $+1$ for $f_0^{\circ q}$.

There are only finitely many polynomials $f(z)=z^2+c$ for which $f^{\circ q}$ possesses a fixed point of multiplier one. In fact, the set of all such $c \in \mathbf{C}$ forms an algebraic variety, which is certainly not all of $\mathbf{C}$. Since the set of all limit points of $R_t(M)$ in M is connected, and is contained in this finite set, it follows that $R_t(M)$ must land at a single uniquely defined point $f_0 \in M$. $\square$

Recall that F_λ denotes the unique polynomial in $\mathscr{P}_2$ which has a fixed point of multiplier λ.

THEOREM C.8. — *If either* $t=\theta_-(\rho)$ *or* $t=\theta_+(\rho)$, *then the associated ray* $R_t(M)$ *in parameter space lands at the point* $F_{\exp(2\pi i\rho)}$ *on the cardioid* $\partial M(\heartsuit) \subset M$.

Proof in the rational case. — We first suppose that ρ is a rational angle p/q. Then each $t=\theta_\pm(p/q)$ is a rational number of the form $h/(2^q-1)$, with odd denominator. Hence $R_t(M)$ lands at some point $f_0 \in M$ by Theorem C.7, and furthermore the ray $R_t(f_0)$ lands at a parabolic periodic point of f_0. The orbit of the unique critical point for f_0 must converge to this parabolic orbit; and it follows that f_0 cannot have any Siegel disk or Cremer point, and cannot have a disjoint parabolic orbit. (*See* for example [M2], § 11.) First suppose that f_0 belongs to the cardioid $\partial M(\heartsuit)$. Then f_0 has the form $F_{\exp(2\pi i\eta)}$, where η must be precisely equal to p/q, since otherwise f_0 would have a Siegel disk, Cremer point, or disjoint parabolic fixed point. (Compare Lemma 2.4.)

Now suppose that f_0 lies outside the cardioid, and hence has a repelling fixed point with rotation number $p'/q' \neq 0$. We must have $p'/q' \neq p/q$, since the ray $R_t(f_0)$ of rotation number p/q lands on a parabolic orbit. According to Lemma B.1, it follows that every $f \in \mathscr{P}_2$ which is sufficiently close to f_0 also has a fixed point of rotation number p'/q'. But this is impossible, since by construction there are points $f \in R_t(M)$ arbitrarily close to f_0 with a fixed point of rotation number p/q. This proves C.8 in the rational case.

Before continuing with the proof of C.8, let us prove a closely related result, which is a sharper form of Lemma C.3. Evidently the two rays $R_{\theta_-(p/q)}(M)$ and $R_{\theta_+(p/q)}(M)$, together with their common landing point $F_{\exp(2\pi i p/q)}$, cut the plane $\mathscr{P}_2$ into two halves.

LEMMA C.9. — *One of these two complementary components, together with the common boundary*

$$R_{\theta_-(p/q)}(M) \cup \{F_{\exp(2\pi i p/q)}\} \cup R_{\theta_+(p/q)}(M),$$

consists precisely of all maps $f \in \mathscr{P}_2$ *which possess a fixed point of rotation number* p/q. *The other complementary component consists of all* f *which do not have a fixed point of rotation number* p/q.

Proof of C.9. – For $f \notin \mathrm{M}$, this follows from C.3. For any $f \in \mathrm{M}$ which possesses a repelling fixed point, it follows from Lemma B.1, together with the rational case of C.8. Finally, for f in the closure of the central core $\mathrm{M}(\heartsuit)$, it follows from Theorem 1.1. □

The proof of C.8 continues as follows. We now suppose that ρ is irrational, so that $\theta_+(\rho) = \theta_-(\rho)$. Choose rational numbers $\alpha < \rho < \beta$ which are arbitrarily close to ρ, and let $a = \theta_+(\alpha) < t < b = \theta_-(\beta)$. Then the ray $\mathrm{R}_t(\mathrm{M})$ lies in a region bounded by the rays $\mathrm{R}_a(\mathrm{M})$, $\mathrm{R}_b(\mathrm{M})$ and a short segment of the cardioid. Using Lemma C.9, we see that any limit point must either be on this cardioid segment or on a limb $\mathrm{M}(p/q)$ with $\alpha < p/q < \beta$. Since α and β can be arbitrarily close to ρ, the conclusion follows. □

Corollary C.10. – *The various limbs* $\mathrm{M}(p/q)$ *are disjoint compact connected sets, while the intersection* $\mathrm{M}(p/q) \cap \bar{\mathrm{M}}(\heartsuit)$ *consists of a single point* $\mathrm{F}_{\exp(2\pi i p/q)}$ *on the cardioid.*

Corollary C.11. – *Let* $\mathrm{M}^0(\heartsuit)$ *be the open set consisting of maps in* M *with an attracting fixed point. The correspondence* $f \mapsto \rho(f)$ *of Definition* C.1 *extends to a continuous mapping from the complement* $\mathscr{P}_2 \setminus \mathrm{M}^0(\heartsuit)$ *onto the circle* $\mathbf{R}/\mathbf{Z}$, *taking the values*

$$\rho(\mathrm{F}_{\exp(2\pi i \eta)}) = \eta$$

on the boundary cardioid.

Proofs are easily supplied. □

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