

ANNALES SCIENTIFIQUES DE L'É.N.S.

R. BALASUBRAMANIAN

V. KUMAR MURTY

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Annales scientifiques de l'É.N.S. 4^e série, tome 25, n° 5 (1992), p. 567-615

http://www.numdam.org/item?id=ASENS_1992_4_25_5_567_0

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ZEROS OF DIRICHLET L-FUNCTIONS

BY R. BALASUBRAMANIAN AND V. KUMAR MURTY ⁽¹⁾

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Introduction

Let χ be a Dirichlet character and $L(s, \chi)$ the associated Dirichlet L-function. We are interested in the zeroes of $L(s, \chi)$ in the critical strip $0 < \text{Re}(s) < 1$. In the past, most attention has focussed on this question near $s=1$. We shall be particularly interested in the situation near $s=1/2$.

It follows from classical results of Landau, Page and others (*see* Davenport [D] for example) that the number of real characters χ of conductor $\leq x$ for which $L(s, \chi)$ has a real zero in the region $1 - (1/\log x) \leq \sigma \leq 1$ is $O(\log \log x)$. On the other hand, the situation near $s=1/2$ is more delicate and not as well understood.

⁽¹⁾ Research partially supported by a grant from NSERC.

Several authors have studied the frequency with which $L(1/2, \chi) \neq 0$. In [B] it is shown that there are at least $cq/(\log q)^{1000}$ characters $\chi \pmod{q}$ with $L(1/2, \chi) \neq 0$. (Here, and elsewhere, c is a positive constant, though not necessarily the same constant at different occurrences.) In another direction, we can allow both χ and q to vary while we fix the order of χ . A result of Jutila [J] implies that there are at least $cx/(\log x)$ real characters χ , of conductor at most x , for which $L(1/2, \chi) \neq 0$.

In both of these works, the method is to study the moments

$$\sum |L(1/2, \chi)|^k.$$

For example, by the Cauchy-Schwarz inequality,

$$\#\left\{\chi \pmod{q} : L\left(\frac{1}{2}, \chi\right) \neq 0\right\} \gg \frac{\left(\sum |L(1/2, \chi)|\right)^2}{\sum |L(1/2, \chi)|^2}.$$

From this, we see that it would suffice to have a lower bound for $\sum |L(1/2, \chi)|$ and an upper bound for $\sum |L(1/2, \chi)|^2$.

There are general conjectures which predict, in particular, the asymptotic growth of the above moments. However, even assuming these conjectures, it does not seem possible to use the Cauchy-Schwarz inequality to deduce that $L(1/2, \chi) \neq 0$ for a *positive proportion* of the characters χ to a given modulus, or that $L(1/2, \chi) \neq 0$ for a positive proportion of real characters χ . This result may be viewed as a (partial) q -analogue of theorems of Levinson-Selberg type.

On the other hand, no example is known of a character χ for which $L(1/2, \chi) = 0$. However, Siegel [S] has shown the fundamental result that any point on the line $\sigma = 1/2$ is a limit point of zeroes of the $L(s, \chi)$ as χ ranges over *all* Dirichlet characters.

In this paper, we take a different approach from [B] and [J]. We consider characters to a prime modulus q . Our first main result is the following.

THEOREM. — *Let q be a sufficiently large prime. Then, for a positive proportion of the characters $\chi \pmod{q}$, we have $L(1/2, \chi) \neq 0$.*

Our proof shows that the proportion is $\geq .04$. (Using the explicit formula, Ram Murty [RM] has shown that this proportion can be improved to $\geq .5$ if we assume the Riemann Hypothesis.) Our method actually produces a more general result (Theorem 11.1) which applies to any point $1/2 \leq \sigma < 1$.

Our second main result (Theorem 12.1) gives a non-vanishing theorem which is uniform on a line segment.

THEOREM. — *Let q be a sufficiently large prime. For a positive proportion of the $\chi \pmod{q}$, there are no real zeroes of $L(s, \chi)$ in the region $(1/2) + (c/\log q) \leq \sigma < 1$. Here, $c > 0$ is an absolute constant.*

In proving our results, our new idea is to count the desired characters directly, *without* the intermediary of moments of L-functions. Let χ be a non-trivial character. Using

weights $\{\lambda(n)\}$ first defined by Barban and Vehov [BV], we consider a mollifier polynomial

$$M(s, \chi) = \sum_{n \leq Z} \lambda(n) \chi(n) n^{-s}$$

where $Z = q^{1/2}$. The $\lambda(n)$ (which are closely related to Selberg's sieve) will be chosen with the property that if we set

$$a(n) = \sum_{d|n} \lambda(d),$$

then $a(1) = 1$ and $a(n) = 0$ for $1 < n < Y$ for some $1 \leq Y < Z$. It turns out that to prove our non-vanishing result at a fixed point, the particular choice of Y is not so crucial and we could take $Y = 1$ if we wished. In the proof of the non-vanishing result on an interval, however, we need to take Y to be a power of q . We choose $Y = q^{1/4}$. Then, we consider the integral

$$\frac{1}{2\pi i} \int_{(2)} L(s+w, \chi) M(s+w, \chi) X^w \Gamma(w) dw$$

where we choose $X = q$. On the one hand it is equal to

$$S(s, \chi) = \sum \frac{a(n) \chi(n)}{n^s} e^{-n/X}$$

and on the other, it is

$$L(s, \chi) M(s, \chi) + \frac{1}{2\pi i} \int_{(-\eta)} L(s+w, \chi) M(s+w, \chi) X^w \Gamma(w) dw$$

where $\eta > 0$ is chosen appropriately. Now if χ is a primitive character, we can apply the functional equation to transform the integral into

$$\frac{1}{2\pi i} \int_{(-\eta)} L(1-s-w, \bar{\chi}) M(s+w, \chi) \gamma(s+w, \chi) X^w \Gamma(w) dw$$

where $\gamma(s, \chi)$ is an appropriate quotient of Γ -functions. Now if we have $\eta > \sigma$, we can expand $L(1-s-w, \bar{\chi})$ as a Dirichlet series. Splitting it into a Dirichlet polynomial of length Z and a tail, we get two integrals $I(s, \chi)$ and $J(s, \chi)$. Thus our basic equation is

$$S(s, \chi) = L(s, \chi) M(s, \chi) + I(s, \chi) + J(s, \chi).$$

If $L(s_0, \chi) = 0$ then $S(s_0, \chi)$ is equal to $I(s_0, \chi) + J(s_0, \chi)$. We show that this cannot happen too often by comparing mean-square estimates of $S(s_0, \chi)$, $I(s_0, \chi)$ and $J(s_0, \chi)$. Thus, we obtain a lower bound for the number of $\chi \pmod{q}$ with $L(s_0, \chi) \neq 0$. We then extend this to a lower bound for the number of $\chi \pmod{q}$ for which $L(s, \chi) \neq 0$ in a circle of radius $(\log q)^{-1}$ about s_0 . Equivalently, we obtain an

upper bound for the number of $\chi \pmod{q}$ for which $L(s, \chi)$ does vanish in this circle. This bound decreases exponentially with $(\Re s_0) - (1/2)$. Choosing the point $s_0 = (1/2) + j(\log q)^{-1}$ and summing over j produces our non-vanishing result on an interval.

The estimates for S and J are given in § 3 and § 4. The mean square of I is determined in § 10, after preparations in § 5-§ 9. The main results are proved in § 11 and § 12. For an exposition of some of the results and techniques of this paper, the reader may consult [KM].

It is a pleasure to thank J. Friedlander, M. Jutila, and R. Murty for encouraging and helpful discussions. We would also like to thank the referee for a careful reading of the manuscript.

NOTATION. — $\sum_{\chi \pmod{q}}$ denotes a sum over characters mod q . We denote by $\mathbf{d}(n)$ the number of positive divisors of n and for $r \in \mathbf{R}$, $\sigma_r(n)$ denotes the sum $\sum_{d|n} d^r$.

1. THE BARBAN-VEHOV WEIGHTS. — Let $1 \leq z_1 \leq z_2$. Following Barban and Vehov [BV], we introduce the functions

$$\Lambda_i(n) = \begin{cases} \mu(n) \log(z_i/n) & \text{if } n \leq z_i \\ 0 & \text{if } n > z_i, \end{cases}$$

for $i=1, 2$. We also define

$$(1.1) \quad \begin{aligned} \lambda(n) &= \frac{\Lambda_2(n) - \Lambda_1(n)}{\log(z_2/z_1)} \\ &= \begin{cases} \mu(n) & 1 \leq n \leq z_1 \\ \mu(n) \frac{\log(z_2/n)}{\log(z_2/z_1)} & z_1 \leq n \leq z_2 \\ 0 & n > z_2. \end{cases} \end{aligned}$$

Let us define

$$a(n) = \sum_{d|n} \lambda(d).$$

Graham [Gr] has found asymptotic estimates for the mean square of the $a(n)$. We recall his main result.

PROPOSITION (1.1). — *We have*

$$\sum_{n \leq N} |a(n)|^2 = \begin{cases} \frac{N \log(N/z_1)}{\log^2(z_2/z_1)} + \mathbf{O}\left(\frac{N}{\log^2(z_2/z_1)}\right) & \text{if } z_1 < N < z_2 \\ \frac{N}{\log(z_2/z_1)} + \mathbf{O}\left(\frac{N}{\log^2(z_2/z_1)}\right) & \text{if } z_2 \leq N. \end{cases}$$

Applying the Cauchy-Schwarz inequality and Proposition (1.1), we deduce the following.

PROPOSITION (1.2). — *Let $r \leq N$ and $(b, r) = 1$. We have*

$$\sum_{\substack{n \leq N \\ n \equiv b \pmod{r}}} |a(n)| \ll \frac{N}{\varphi(r)^{1/2} (\log z_2/z_1)^{1/2}}.$$

We next obtain an estimate for a shifted convolution.

PROPOSITION (1.3). — *Let $1 \leq k \in \mathbf{Z}$, $t \in \mathbf{R}$ and $k \leq M < N$. Then we have*

$$\sum_{M < n \leq N} a(n) a(n-k) \left(\frac{n}{n-k}\right)^{it} \ll \frac{k}{\varphi(k)} \left(\frac{N + z_2^2}{(\log z_2/z_1)^2} |P(t)| + \frac{N |t|^4 (\log z_2)^4}{(\log z_2/z_1)^2} \right)$$

where $P(t)$ is a polynomial in t (depending on k) with complex bounded coefficients and of degree ≤ 4 .

The proof will require two preliminary results. We begin by recalling a result from Graham [Gr, Lemma 2].

LEMMA (1.4). — *For any integer r , and any $c > 0$,*

$$\sum_{\substack{n \leq Q \\ (n, r) = 1}} \frac{\mu(n)}{n} \log \left(\frac{Q}{n} \right) = \frac{r}{\varphi(r)} + \mathbf{O}_c(\sigma_{-1/2}(r) \log^{-c}(2Q)).$$

LEMMA (1.5). — *We have for $1 \leq d_1, d_2 \leq z_2$ and $r_1, r_2 \geq 1$ that*

$$\sum_{\substack{1 \leq j_1 \leq z_1/d_1, 1 \leq j_2 \leq z_2/d_2 \\ (j_1, j_2) = (j_1, r_1) = (j_2, r_2) = 1}} \frac{\Lambda_1(d_1 j_1) \Lambda_2(d_2 j_2)}{j_1 j_2} \ll \left(\frac{d_1 r_1}{\varphi(d_1 r_1)} + \sigma_{-1/2}(d_1 r_1) \right) \left(\frac{d_2 r_2}{\varphi(d_2 r_2)} + \sigma_{-1/2}(d_2 r_2) \right).$$

The same estimate holds even if we drop the condition that $(j_1, j_2) = 1$.

Proof. — The sum in question is

$$\sum \frac{\Lambda_1(d_1 j_1) \Lambda_2(d_2 j_2)}{j_1 j_2} \sum_{e | (j_1, j_2)} \mu(e) = \sum_{e \leq z_1/d_1} \mu(e) \sum \frac{\Lambda_1(d_1 j_1) \Lambda_2(d_2 j_2)}{j_1 j_2},$$

the inner sum ranging over j_1, j_2 satisfying

$$1 \leq j_1 \leq z_1/d_1, \quad 1 \leq j_2 \leq z_2/d_2 \\ j_1 j_2 \equiv 0 \pmod{e}, \quad (j_1, r_1) = (j_2, r_2) = 1.$$

Let us set $r = r_1 r_2$ and $d = d_1 d_2$. Then the sum is seen to be

$$\begin{aligned} & \sum_{\substack{e \leq z_1/d_1 \\ (e, r) = 1}} \frac{\mu(e)}{e^2} \sum_{\substack{l_1 \leq z_1/d_1 e, l_2 \leq z_2/d_2 e \\ (l_1, r_1) = (l_2, r_2) = 1}} \frac{\Lambda_1(d_1 e l_1) \Lambda_2(d_2 e l_2)}{l_1 l_2} \\ &= \mu(d_1) \mu(d_2) \sum_{\substack{e \leq z_1/d_1 \\ (e, dr) = 1}} \frac{\mu(e)}{e^2} \left\{ \sum_{\substack{l_1 \leq z_1/d_1 e \\ (l_1, d_1 e r_1) = 1}} \frac{\mu(l_1) \log(z_1/d_1 e l_1)}{l_1} \right\} \\ & \quad \times \left\{ \sum_{\substack{l_2 \leq z_2/d_2 e \\ (l_2, d_2 e r_2) = 1}} \frac{\mu(l_2) \log(z_2/d_2 e l_2)}{l_2} \right\} \\ &= \mu(d_1) \mu(d_2) \sum_{\substack{e \leq z_1/d_1 \\ (e, dr) = 1}} \frac{\mu(e)}{e^2} \prod_{k=1}^2 \left\{ \frac{d_k e r_k}{\varphi(d_k e r_k)} + \mathbf{O}_c \left(\sigma_{-1/2}(d_k e r_k) \log^{-c} \left(\frac{2 z_k}{d_k e} \right) \right) \right\} \end{aligned}$$

using Lemma (1.4).

The main terms contribute an amount

$$\frac{\mu(d_1) \mu(d_2) dr}{\varphi(d_1 r_1) \varphi(d_2 r_2)} \cdot \sum_{\substack{e \leq z_1/d_1 \\ (e, dr) = 1}} \frac{\mu(e)}{\varphi(e)^2} \ll \frac{dr}{\varphi(d_1 r_1) \varphi(d_2 r_2)}.$$

The product of the \mathbf{O} -terms contributes an amount

$$\ll \sum \frac{1}{e^2} \sigma_{-1/2}(d_1 r_1) \sigma_{1/2}(d_2 r_2) \sigma_{-1/2}(e)^2 \ll \sigma_{-1/2}(d_1 r_1) \sigma_{1/2}(d_2 r_2).$$

The cross-terms contribute an amount

$$\begin{aligned} & \ll \sum_{\substack{e \leq z_1/d_1 \\ (e, dr) = 1}} \frac{1}{e^2} \left\{ \frac{d_1 e r_1}{\varphi(d_1 e r_1)} \cdot \sigma_{-1/2}(d_2 e r_2) \log^{-c} \left(\frac{2 z_2}{d_2 e} \right) \right. \\ & \quad \left. + \frac{d_2 e r_2}{\varphi(d_2 e r_2)} \cdot \sigma_{-1/2}(d_1 e r_1) \log^{-c} \left(\frac{2 z_1}{d_1 e} \right) \right\} \\ & \ll \left\{ \frac{d_1 r_1}{\varphi(d_1 r_1)} \sigma_{-1/2}(d_2 r_2) + \frac{d_2 r_2}{\varphi(d_2 r_2)} \sigma_{-1/2}(d_1 r_1) \right\} \end{aligned}$$

since the series $\sum \sigma_{-1/2}(e)/e \varphi(e)$ converges. This proves the first statement. The second statement is easy to verify since there is now no condition relating j_1 and j_2 . We argue as above setting $e = 1$.

Now we are ready to prove the estimate of the shifted convolution.

Proof of Proposition 1.3. – Again, we consider the sum

$$(1.2) \quad \sum_{M < n \leq N} \left(\sum_{d|n} \Lambda_1(d) \right) \left(\sum_{e|n-k} \Lambda_2(e) \right) \left(\frac{n}{n-k} \right)^{it}$$

and we find that it is equal to

$$(1.3) \quad \sum_{d, e} \Lambda_1(d) \Lambda_2(e) \sum_{\substack{M < n \leq N \\ n \equiv 0 \pmod{d} \\ n \equiv k \pmod{e}}} \left(\frac{n}{n-k} \right)^{it}.$$

We see that the inner sum is zero unless $(d, e) | k$. Consider the identity

$$\left(\frac{n}{n-k} \right)^{it} = (1+k)^{it} \left(1 - \frac{k}{1+k} \left(1 - \frac{1}{n-k} \right) \right)^{it}.$$

We have an expansion

$$\left(\frac{n}{n-k} \right)^{it} = (1+k)^{it} \sum_{j=0}^4 P_j(t) (n-k)^{-j} + \mathbf{O}(|t|^4)$$

where $P_j(t)$ is a polynomial in t of degree ≤ 3 with complex coefficients which are absolutely bounded and depend on k . Using this, we see that

$$(1.4) \quad \sum_{\substack{M < n \leq N \\ n \equiv 0 \pmod{d} \\ n \equiv k \pmod{e}}} \left(\frac{n}{n-k} \right)^{it} = (1+k)^{it} \sum_{j=0}^4 P_j(t) \sum_{\substack{M < n \leq N \\ n \equiv 0 \pmod{d} \\ n \equiv k \pmod{e}}} (n-k)^{-j} + \mathbf{O} \left(|t|^4 \sum_{\substack{M < n \leq N \\ n \equiv 0 \pmod{d} \\ n \equiv k \pmod{e}}} 1 \right).$$

Inserting this into (1.3), we get a main term of

$$(1.5) \quad (1+k)^{it} \sum_{j=0}^4 P_j(t) \sum_{d, e} \Lambda_1(d) \Lambda_2(e) \sum_{\substack{M < n \leq N \\ n \equiv 0 \pmod{d} \\ n \equiv k \pmod{e}}} (n-k)^{-j}.$$

If $j=0$, the innermost sum is

$$\frac{N-M}{[d, e]} + \mathbf{O}(1)$$

and if $j=1$, it is

$$\frac{\log((N-k)/(M-k)) + \mathbf{O}(1)}{[d, e]}.$$

For $j \geq 2$ it is $\mathbf{O}(1)$. Thus, (1.5) is

$$(1.6) \quad (1+k)^{it} \left(P_0(t)(N-M) + P_1(t) \log \left(\frac{N-k}{M-k} \right) \right) \sum_{\substack{d,e \\ (d,e)|k}} \frac{\Lambda_1(d)\Lambda_2(e)}{[d,e]} \\ + \mathbf{O}((|t|+1)^3 \sum_{\substack{d,e \\ (d,e)|k}} |\Lambda_1(d)\Lambda_2(e)|)$$

The \mathbf{O} -term is easily seen to be

$$\ll z_1 z_2 (|t|+1)^3.$$

To evaluate the main term, we see that the sum over d, e is

$$(1.7) \quad \sum_{\substack{d,e \\ (d,e)|k}} \frac{\Lambda_1(d)\Lambda_2(e)}{de} \sum_{\substack{m|d \\ m|e}} \varphi(m).$$

This is seen to be equal to

$$\sum_{m|k} \frac{\varphi(m)}{m^2} \sum_{d_0, e_0} \frac{\Lambda_1(m d_0)\Lambda_2(m e_0)}{d_0 e_0}.$$

Here, the inner sum ranges over pairs d_0, e_0 satisfying

$$1 \leq d_0 \leq \frac{z_1}{m}, \quad 1 \leq e_0 \leq \frac{z_2}{m} \\ (d_0, m) = (e_0, m) = 1.$$

Also note that in the outer sum m must be squarefree for otherwise $\Lambda_1(m d_0) = \Lambda_2(m e_0) = 0$. Thus, invoking Lemma (1.5), we find that the main term in (1.7) is

$$\ll \sum_{m|k} \frac{\mu^2(m)\varphi(m)}{m^2} \left(\frac{m}{\varphi(m)} + \sigma_{-1/2}(m) \right)^2 \\ \ll \frac{k}{\varphi(k)}.$$

Hence the main term in (1.6) is

$$\ll \frac{k}{\varphi(k)} (|P_0(t)|N + |P_1(t)| \log N).$$

Summarizing, the main term of (1.4) contributes to (1.3) an amount

$$\ll \frac{k}{\varphi(k)} (|P_0(t)|N + |P_1(t)| \log N) + z_1 z_2 (|t|+1)^3.$$

The error term in (1.4) contributes to (1.3) an amount

$$\ll N |t|^4 \sum_{\substack{d, e \\ (d, e) | k}} \frac{|\Lambda_1(d) \Lambda_2(e)|}{[d, e]} + |t|^4 z_1 z_2.$$

The first term above is estimated by

$$\begin{aligned} \sum_{\substack{d, e \\ (d, e) | k}} \frac{|\Lambda_1(d) \Lambda_2(e)|}{[d, e]} &\ll \sum_{m | k} \frac{\varphi(m) \mu(m)^2}{m^2} \left(\sum_{d_0} \frac{1}{d_0} \log \frac{z_1}{m d_0} \right) \left(\sum_{e_0} \frac{1}{e_0} \log \frac{z_2}{m e_0} \right) \\ &\ll \sum_{m | k} \frac{\varphi(m) \mu(m)^2}{m^2} \left(\log \frac{z_1}{m} \right)^2 \left(\log \frac{z_2}{m} \right)^2 \\ &\ll \frac{k}{\varphi(k)} (\log z_1)^2 (\log z_2)^2. \end{aligned}$$

Summarizing, the error term in (1.4) contributes to (1.3) an amount

$$\ll \frac{k}{\varphi(k)} |t|^4 N (\log z_1)^2 (\log z_2)^2 + z_1 z_2 |t|^4.$$

The Proposition follows.

2. THE MOLLIFIER POLYNOMIAL. — We shall now introduce the following parameters. Let us set

$$\begin{aligned} Y &= (\log q) \\ Z &= q^{1/2} \end{aligned}$$

Corresponding to the choices $z_1 = Y$ and $z_2 = Z$, we have from § 1 the weights

$$\lambda(n) = \frac{\Lambda_2(n) - \Lambda_1(n)}{\log(Z/Y)}.$$

We define the Dirichlet polynomial

$$M(s, \chi) = \sum_{n \leq Z} \frac{\lambda(n) \chi(n)}{n^s}$$

where χ is a Dirichlet character. Then, we have

$$L(s, \chi) M(s, \chi) = \sum_{n=1}^{\infty} \frac{a(n) \chi(n)}{n^s}$$

where

$$a(n) = \sum_{d | n} \lambda(d)$$

satisfies

$$\begin{aligned} a(1) &= 1 \\ a(n) &= 0 \quad \text{for } 1 < n \leq Y. \end{aligned}$$

We record the following estimate.

LEMMA (2.1). — For $|\sigma| < 1/2$, and σ bounded away from $1/2$, we have

$$\sum_{\chi \pmod{q}} |M(s, \chi)|^2 \ll \frac{(q+Z)}{(1-2\sigma)} \cdot \left(q^{1/2-\sigma} \cdot \frac{1}{(\log q)^2} + Y^{1-2\sigma} \right).$$

Proof. — We use the large sieve inequality [D] to get

$$\begin{aligned} \sum_{\chi \pmod{r}} |M(s, \chi)|^2 &\ll (Z+q) \sum_{n \leq Z} \frac{|\lambda(n)|^2}{n^{2\sigma}} \\ &\ll (q+Z) \left\{ \sum_{n \leq Y} \frac{1}{n^{2\sigma}} + \sum_{Y < n \leq Z} \left(\frac{\log Z/n}{\log Z/Y} \right)^2 \cdot \frac{1}{n^{2\sigma}} \right\} \\ &\ll (q+Z) \left\{ \frac{Y^{1-2\sigma}}{1-2\sigma} + \frac{Z^{1-2\sigma}}{1-2\sigma} \cdot \frac{1}{(\log Z/Y)^2} \right\}. \end{aligned}$$

The result follows from our choices of Y and Z .

3. THE BASIC EQUATION. — Let us define

$$S(s, \chi) = S(s, \chi, q) = \sum_{n=1}^{\infty} \frac{a(n) \chi(n)}{n^s} e^{-n/q}.$$

Let $s \in \mathbb{C}$ with $1 > \sigma = \operatorname{Re}(s) \geq 1/2$. Using the well-known identity

$$\frac{1}{2\pi i} \int_{(2)} X^w \Gamma(w) dw = e^{-1/X},$$

we find that for a character χ ,

$$S(s, \chi) = \frac{1}{2\pi i} \int_{(2)} L(s+w, \chi) M(s+w, \chi) q^w \Gamma(w) dw.$$

Moving the line of integration to the left, we find that

$$(3.1) \quad S(s, \chi) = L(s, \chi) M(s, \chi) + \frac{1}{2\pi i} \int_{(-\eta)} L(s+w, \chi) M(s+w, \chi) q^w \Gamma(w) dw$$

where $\sigma < \eta < 1$.

We can decompose the integral along the line $-\eta$ into two parts as follows. Suppose that χ is non-trivial. We apply the functional equation

$$L(s, \chi) = \gamma(s, \chi) L(1-s, \bar{\chi})$$

where

$$\gamma(s, \chi) = \frac{\tau(\chi)}{i^a q^{1/2}} \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{2\pi}{q}\right)^{s-(1/2)} \sin\left(\frac{\pi}{2}(a+s)\right) \Gamma(1-s).$$

[Here $\tau(\chi)$ is the Gauss sum, $a=0, 1$ and $\chi(-1)=(-1)^a$.] Then we truncate the Dirichlet series expansion of $L(1-s-w, \bar{\chi})$ at Z . Let us set

$$I(s, \chi) = \frac{1}{2\pi i} \int_{(-\eta)} \gamma(s+w, \chi) \left\{ \sum_{n < Z} \frac{\bar{\chi}(n)}{n^{1-s-w}} \right\} M(s+w, \chi) q^w \Gamma(w) dw$$

and

$$J(s, \chi) = \frac{1}{2\pi i} \int_{(-\eta)} \gamma(s+w, \chi) \left\{ \sum_{n \geq Z} \frac{\bar{\chi}(n)}{n^{1-s-w}} \right\} M(s+w, \chi) q^w \Gamma(w) dw$$

Thus, we get

$$(3.2) \quad S(s, \chi) = L(s, \chi) M(s, \chi) + I(s, \chi) + J(s, \chi).$$

If $L(s, \chi) = 0$, then $S(s, \chi)$ and $I(s, \chi) + J(s, \chi)$ are equal. We will therefore try to show that, in general, they are *not* equal and for this purpose we study their mean values. We begin with $J(s, \chi)$ which is the easiest of the three to estimate.

PROPOSITION (3.1). — For $|\text{Im } s| < 1$, and $0 \leq \sigma \leq 1$, we have

$$\sum_{1 \neq \chi \pmod{q}} |J(s, \chi)| \ll_{\epsilon} \frac{q^{(3/2)-\sigma}}{\log q}.$$

Proof. — From Stirling's formula, we know that

$$\gamma(s, \chi) \ll (q(|s|+1))^{(1/2)-\sigma}.$$

Using this and the definition, we find that

$$\sum_{1 \neq \chi \pmod{q}} |J(s, \chi)| \ll q^{(1/2)-\sigma+\eta} q^{-\eta} \sum_{\chi \pmod{q}} \int_{(-\eta)} (|w|+1)^{(1/2)-\sigma+\eta} \left| \sum_{n \geq Z} \frac{\bar{\chi}(n)}{n^{1-s-w}} \right| |M(s+w, \chi)| |\Gamma(w)| dw$$

which by a double application of the Cauchy-Schwarz inequality is

$$\begin{aligned} &\ll q^{(1/2)-\sigma} \sum_{\chi \pmod{q}} \left(\int (|w|+1)^{1-2\sigma+2\eta} \left| \sum_{n \geq Z} \frac{\bar{\chi}(n)}{n^{1-s-w}} \right|^2 |\Gamma(w)| |dw| \right)^{1/2} \\ &\qquad \qquad \qquad \times \left(\int |M(s+w, \chi)|^2 |\Gamma(w)| |dw| \right)^{1/2} \\ &\ll q^{(1/2)-\sigma} \left(\sum_{\chi \pmod{q}} \int (|w|+1)^{1-2\sigma+2\eta} \left| \sum_{n \geq Z} \frac{\bar{\chi}(n)}{n^{1-s-w}} \right|^2 |\Gamma(w)| |dw| \right)^{1/2} \\ &\qquad \qquad \qquad \times \left(\sum_{\chi \pmod{q}} \int |M(s+w, \chi)|^2 |\Gamma(w)| |dw| \right)^{1/2}. \end{aligned}$$

Using the large sieve inequality and Lemma (2.1), we find that

$$\begin{aligned} \sum_{1 \neq \chi \pmod{q}} |J(s, \chi)| &\ll q^{(1/2)-\sigma} \left\{ \sum_{n \geq Z} (q+n) n^{2(\sigma-\eta-1)} \right\}^{1/2} \\ &\times \left\{ \frac{(q+Z)}{1-2(\sigma-\eta)} \cdot \left(\frac{q^{(1/2)-\sigma+\eta}}{(\log q)^2} + Y^{1-2(\sigma-\eta)} \right) \right\} \\ &\ll q^{(1/2)-\sigma} Z^{\sigma-\eta} \left\{ \frac{q}{Z} \frac{1}{|2(\sigma-\eta)-1|} + \frac{1}{|\sigma-\eta|} \right\}^{1/2} \\ &\qquad \qquad \qquad \times \left\{ \frac{(q+Z)^{1/2}}{|2(\sigma-\eta)-1|^{1/2}} \frac{q^{1/2((1/2)-\sigma+\eta)}}{\log q} \right\}. \end{aligned}$$

Now, let us choose η so that it satisfies

$$\frac{1}{4} > |\eta - \sigma| > \frac{1}{8} \text{ (say)}$$

if $\sigma < 3/4$.

We would then have

$$(3.3) \quad \sum_{1 \neq \chi \pmod{q}} |J(s, \chi)| \ll \frac{q^{(3/2)-\sigma}}{\log q}$$

which proves the result.

4. THE MEAN AND MEAN SQUARE OF $S(s, \chi)$.

PROPOSITION (4.1). — *For any $\varepsilon > 0$, we have*

$$\sum_{\chi \pmod{q}} S(s, \chi) = \varphi(q) + \mathbf{O}_\varepsilon(q^{1-\sigma+\varepsilon}).$$

Moreover, the same estimate holds if we sum only over non-trivial characters.

Proof. — By definition, we have that

$$\begin{aligned} \sum_{\chi \pmod{q}} S(s, \chi) &= \sum_{n=1}^{\infty} \frac{a(n)}{n^s} e^{-n/q} \sum_{\chi \pmod{q}} \chi(n) \\ &= \varphi(q) \sum_{\substack{n=1 \\ n \equiv 1 \pmod{q}}}^{\infty} \frac{a(n)}{n^s} e^{-n/q}. \end{aligned}$$

Using the bound $|a(n)| \leq d(n) \ll_{\varepsilon} n^{\varepsilon}$, we find that the sum is

$$e^{-1/q} + \mathbf{O}_{\varepsilon} \left(\frac{1}{q^{\sigma-\varepsilon}} \sum_{t=1}^{\infty} t^{\varepsilon-\sigma} \exp(-t) \right).$$

The \mathbf{O} -term is

$$\ll_{\varepsilon} q^{-\sigma+\varepsilon}.$$

It thus follows that

$$\sum_{\chi \pmod{q}} S(s, \chi) = \varphi(q) + \mathbf{O}_{\varepsilon}(q^{1-\sigma+\varepsilon}).$$

Finally,

$$S(s, 1) = \sum_{\substack{(n, q)=1}} \frac{a(n)}{n^s} e^{-n/q} \ll q^{1-\sigma+\varepsilon}$$

as before. This proves the result.

PROPOSITION (4.2). — *We have*

$$\sum_{\chi \pmod{q}} \left| S\left(\frac{1}{2} + it, \chi\right) \right|^2 = \frac{5}{2} \varphi(q) + \mathbf{O}((1+|t|)^4 q (\log q)^{-1/2}) + \mathbf{O}(|t|^4 q (\log q)^{7/2}).$$

For $1/2 < \sigma \leq 1$, we have

$$\begin{aligned} \sum_{\chi \pmod{q}} |S(\sigma + it, \chi)|^2 &= \varphi(q) - \frac{4 \varphi(q) q^{(1/2)-\sigma}}{(1-2\sigma)^2 (\log q)^2} + \frac{4 \varphi(q) Y^{1-2\sigma}}{(\log q)^2 (1-2\sigma)^2} + \frac{2 \varphi(q) q^{1-2\sigma}}{(\log q)(1-2\sigma)} \\ &+ \mathbf{O}\left(\frac{\varphi(q) q^{(1/2)-\sigma}}{(\log q)^2 (1-2\sigma)}\right) + \mathbf{O}\left(\frac{\varphi(q) q^{1-2\sigma}}{\log q}\right) \\ &+ \mathbf{O}\left(\frac{\varphi(q) q^{1-2\sigma} (\log q)^{-\sigma-1}}{1-\sigma} \{(1+|t|)^4 + (|t| \log q)^4\}\right) \end{aligned}$$

where for $\sigma = 1$, we interpret $(1-\sigma)^{-1}$ to be $\log q$.

Proof. — We see that the sum is equal to

$$\sum_{n_1, n_2=1}^{\infty} \frac{a(n_1)a(n_2)}{(n_1 n_2)^{\sigma}} \left(\frac{n_2}{n_1}\right)^{it} \exp(-(n_1+n_2)/q) \sum_{\chi \pmod{q}} \chi(n_1) \bar{\chi}(n_2)$$

which is seen to be

$$(4.1) \quad \varphi(q) \sum_{n_1, n_2=1}^{\infty} \frac{a(n_1)a(n_2)}{(n_1 n_2)^{\sigma}} \left(\frac{n_2}{n_1}\right)^{it} \exp(-(n_1+n_2)/q),$$

where the inner sum ranges over pairs (n_1, n_2) satisfying

$$n_1 \equiv n_2 \pmod{q}, \quad (n_1, q) = (n_2, q) = 1.$$

We split the double sum into three pieces $\Sigma_1 + \Sigma_2 + \Sigma_3$. In Σ_1 we have $n_1 < n_2$, in Σ_2 we have $n_1 > n_2$, and in Σ_3 we have $n_1 = n_2$. The estimation of Σ_1 and Σ_2 is the same, so we only consider Σ_1 . We have

$$(4.2) \quad \Sigma_1 = \sum_{\substack{n_1=1 \\ (n_1, q)=1}}^{\infty} \frac{a(n_1) \exp(-n_1/q)}{n_1^{\sigma}} \sum_{\substack{n_2=1 \\ n_2 \equiv n_1 \pmod{q} \\ n_2 > n_1}}^{\infty} \frac{a(n_2) \exp(-n_2/q)}{n_2^{\sigma}} \left(\frac{n_2}{n_1}\right)^{it}.$$

We begin by considering the sum over n_2 . We must necessarily have $n_2 > q$ for if $n_2 \leq q$, then $n_1 \leq q$ also and so the congruence $n_2 \equiv n_1 \pmod{q}$ would force $n_1 = n_2$. We split Σ_1 into three subsums Σ_{11} , Σ_{12} and Σ_{13} where

in Σ_{11} we have $n_2 \geq q \log q$

in Σ_{12} we have $q \leq n_1 < q \log q$ and $n_1 < n_2 < q \log q$

in Σ_{13} we have $n_1 < q$ and $q < n_2 < q \log q$.

In Σ_{11} , we see, by partial summation, that the sum over n_2 is

$$\ll q^{-1} \int_{q \log q}^{\infty} \left\{ \sum_{\substack{n \leq u \\ n \equiv n_1 \pmod{q}}} |a(n)| \right\} u^{-\sigma} e^{-u/q} du.$$

We have from Proposition (1.2) that

$$\sum_{\substack{n \leq u \\ n \equiv n_1 \pmod{q}}} |a(n)| \ll \frac{u}{\varphi(q)^{1/2} (\log q)^{1/2}}.$$

Thus, we find that the integral is

$$\ll \frac{1}{q^{3/2} (\log q)^{1/2}} \int_{q \log q}^{\infty} u^{1-\sigma} e^{-u/q} du$$

and this is

$$\begin{aligned} &\ll q^{1/2-\sigma} (\log q)^{-1/2} \int_{\log q}^{\infty} v^{1-\sigma} e^{-v} dv \\ &\ll q^{1/2-\sigma} (\log q)^{-1/2-\sigma}. \end{aligned}$$

Inserting this into the n_1 -sum, using Proposition (1.1), the Cauchy-Schwarz inequality and partial summation, we have

$$\begin{aligned} \Sigma_{11} &\ll \frac{q^{1-\sigma}}{(1-\sigma)(\log q)^{1/2}} \frac{(\log q)^{1/2-\sigma}}{q^{1/2+\sigma}} \\ &\ll \frac{q^{1/2-2\sigma} (\log q)^{-\sigma}}{1-\sigma}. \end{aligned}$$

Now we consider the contribution of Σ_{12} . This is

$$\sum_{\substack{q \leq n_1 < q \log q \\ (n_1, q) = 1}} \frac{a(n_1) e^{-n_1/q}}{n_1^\sigma} \sum_{\substack{n_1 < n_2 < q \log q \\ n_2 \equiv n_1 \pmod{q}}} \frac{a(n_2) e^{-n_2/q}}{n_2^\sigma} \left(\frac{n_2}{n_1} \right)^{it}.$$

We split the n_1 sum into $O(\log \log q)$ sums of the form

$$\sum_{\substack{U < n_1 \leq 2U \\ (n_1, q) = 1}} \frac{a(n_1) e^{-n_1/q}}{n_1^\sigma} \sum_{\substack{n_1 < n_2 < q \log q \\ n_2 \equiv n_1 \pmod{q}}} \frac{a(n_2) e^{-n_2/q}}{n_2^\sigma} \left(\frac{n_2}{n_1} \right)^{it}.$$

Let us write $n_2 = n_1 + jq$. The above double sum may therefore be written as

$$(4.3) \quad \sum_{j < \log q} e^{-j} \sum_{\substack{U < n_1 \leq 2U \\ (n_1, q) = 1}} e^{-2n_1/q} \frac{a(n_1) a(n_1 + jq)}{n_1^\sigma (n_1 + jq)^\sigma} \left(\frac{n_1 + jq}{n_1} \right)^{it}$$

If we drop the condition $(n_1, q) = 1$, then we introduce an additional sum

$$(4.4) \quad \sum_{j < \log q} e^{-j} \sum_{U < qk \leq 2U} e^{-2k} \frac{a(kq) a((k+j)q)}{(kq)^\sigma ((k+j)q)^\sigma} \left(\frac{k+j}{k} \right)^{it}.$$

Observe that as q is prime, and $\lambda(n) = 0$ for $n > Z = q^{1/2}$, we have

$$a(kq) = \sum_{d|kq} \lambda(d) = \sum_{d|k} \lambda(d) = a(k).$$

Therefore, we have the estimate

$$|a(kq)| \leq \mathbf{d}(k) \ll_\varepsilon k^\varepsilon.$$

A similar estimate holds for $a((k+j)q)$. Using this in (4.4), we see that it is

$$\ll q^{-2\sigma} \sum_{j < \log q} e^{-j} \sum_{U < qk \leq 2U} \frac{e^{-2k}}{k^{\sigma-\varepsilon} (k+j)^{\sigma-\varepsilon}}$$

and this is

$$\ll q^{-2\sigma}.$$

The sum in (4.3) may thus be replaced by

$$(4.5) \quad \sum_{j < \log q} e^{-j} \sum_{U < n_1 \leq 2U} e^{-2n_1/q} \frac{a(n_1) a(n_1+jq)}{n_1^\sigma (n_1+jq)^\sigma} \left(\frac{n_1+jq}{n_1} \right)^{it}$$

Let us set

$$G(u) = \sum_{U < n_1 \leq u} a(n_1) a(n_1+jq) \left(\frac{n_1+jq}{n_1} \right)^{it}.$$

By Proposition (1.3), we see that for $U < u$,

$$G(u) \ll \frac{j}{\varphi(j)} \left(\frac{(u+(j+1)q) |P(t)|}{(\log q)^2} + (u+jq) |t|^4 (\log q)^2 \right).$$

The sum over n_1 in (4.5) can be estimated using partial summation. We find that it is equal to

$$\frac{G(u) e^{-2u/q}}{u^\sigma (u+jq)^\sigma} \Big|_U^{2U} + \int_U^{2U} G(u) d \left(\frac{e^{-2u/q}}{u^\sigma (u+jq)^\sigma} \right).$$

Using the estimate for $G(u)$ quoted above, we see that for $\sigma \neq 1$, this is

$$e^{-2U/q} \frac{j}{\varphi(j)} \frac{(U+jq)^{1-\sigma}}{U^\sigma} \frac{U}{q(1-\sigma)} (\log q)^{-2} (|P(t)| + (|t| \log q)^4)$$

If $\sigma = 1$, then we can suppress the term $(1-\sigma)^{-1}$. Note that though the coefficients of $P(t)$ depend on j and q , they are absolutely bounded. Thus,

$$|P(t)| \ll (1+|t|)^4.$$

Incorporating these estimates into the sum over j , we find that (4.5) is for $\sigma \neq 1$

$$\ll \sum_{j < \log q} \frac{(U+jq)^{1-\sigma} U^{1-\sigma}}{q(1-\sigma)} e^{-2U/q} (\log q)^{-2} \frac{j}{\varphi(j)} e^{-j} (|P(t)| + (|t| \log q)^4)$$

which is

$$\begin{aligned} &\ll \frac{U^{1-\sigma} e^{-2U/q}}{q(1-\sigma)(\log q)^2} (|P(t)| + (|t| \log q)^4) \sum_{j < \log q} e^{-j} \frac{j}{\varphi(j)} (U+jq)^{1-\sigma} \\ &\ll q^{1-\sigma} (\log q)^{-1-\sigma} \frac{U^{1-\sigma}}{q(1-\sigma)} e^{-2U/q} (|P(t)| + (|t| \log q)^4). \end{aligned}$$

Now summing this over U , we find it is

$$\ll q^{1-2\sigma} (\log q)^{-\sigma-1} (1-\sigma)^{-1} (|P(t)| + (|t| \log q)^4).$$

For $\sigma = 1$, we can suppress the term $(1-\sigma)^{-1}$.

Now we discuss the contribution of Σ_{13} . By the Cauchy-Schwarz inequality, we see that

$$|\Sigma_{13}|^2 \ll \left(\sum_{n_1 < q} \frac{a(n_1)^2}{n_1^{2\sigma}} \exp(-2n_1/q) \right) \left(\sum_{\substack{n_1 \leq q \\ n_2 \equiv n_1 \pmod{q}}} \left| \sum_{q < n_2 < q \log q} \frac{a(n_2) e^{-n_2/q}}{n_2^\sigma} n_2^{it} \right|^2 \right).$$

The first factor above is $O(1)$ as can be seen from our discussion of Σ_3 below. As for the second factor, we see that it is equal to

$$\sum_{\substack{q < n_2, n_2 < q \log q \\ n_2 \equiv n_2' \pmod{q}}} \frac{a(n_2) e^{-n_2/q}}{n_2^\sigma} \frac{a(n_2') e^{-n_2'/q}}{(n_2')^\sigma} \left(\frac{n_2}{n_2'} \right)^{it}.$$

Again, we split this sum into three sums according as $n_2 < n_2'$, $n_2 = n_2'$, and $n_2 > n_2'$.

The third is the same as the first. Also, we note that the first sum is just Σ_{12} which we have estimated above as being (for $\sigma \neq 1$)

$$\ll q^{1-2\sigma} (\log q)^{-\sigma-1} (1-\sigma)^{-1} (|P(t)| + (|t| \log q)^4).$$

If $\sigma = 1$, then as before, we may suppress the $(1-\sigma)^{-1}$ term. As for the second, we see that it is equal to

$$\sum_{q \leq n_2 < q \log q} \frac{a(n_2)^2 e^{-2n_2/q}}{n_2^{2\sigma}}.$$

Using Proposition (1.1) and partial summation, this is

$$\ll \frac{q^{1-2\sigma}}{\log q}.$$

Inserting this into the above, we deduce that

$$\Sigma_{13} \ll q^{(1/2)-\sigma} (\log q)^{-(\sigma+1)/2} (|P(t)| + (|t| \log q)^4)^{1/2}.$$

Finally, we discuss the estimation of Σ_3 , namely the terms with $n_1 = n_2$. Thus,

$$(4.6) \quad \Sigma_3 = \sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{a(n)^2}{n^{2\sigma}} \exp(-2n/q) = \sum_{n \leq Y} + \sum_{Y < n \leq q} + \sum_{n > q}.$$

Since $a(n) = 0$ for $1 < n \leq Y$, we have

$$(4.7) \quad \sum_{n \leq Y} = \begin{cases} 1 & \text{if } r=1 \\ 0 & \text{otherwise.} \end{cases}$$

Also, by partial summation and Proposition (1.1), we find that

$$(4.8) \quad \sum_{n > q} \ll \frac{q^{1-2\sigma}}{\log(z_2/z_1)}.$$

Thus, we see from (4.6)-(4.8) that

$$\Sigma_3 = 1 + \sum_{\substack{Y < n \leq q \\ (n,q)=1}} \frac{a(n)^2}{n^{2\sigma}} \exp(-2n/q).$$

Let us denote the sum on the right by S . We find that

$$S = \sum_{Y < n \leq q} \frac{a(n)^2}{n^{2\sigma}} \left(1 + O\left(\frac{n}{q}\right) \right).$$

Now, the O -term is

$$\begin{aligned} &\ll \frac{1}{q} \sum_{\substack{Y < n \leq q \\ (n,q)=1}} \frac{a(n)^2}{n^{2\sigma-1}} \\ &\ll \frac{1}{q} \frac{1}{\log(z_2/z_1)} \frac{q^{2-2\sigma}}{(1-\sigma)} \\ &\ll \frac{q^{1-2\sigma}}{(1-\sigma) \log q}. \end{aligned}$$

The main term is equal to

$$\sum_{Y < n < q} \frac{a(n)^2}{n^{2\sigma}}.$$

Finally, using Proposition (1.1),

$$\sum_{n < q} \frac{a(n)^2}{n^{2\sigma}} = \sum_{1 \leq n \leq Y} + \sum_{Y < n \leq Z} + \sum_{Z < n < q} \frac{a(n)^2}{n^{2\sigma}}.$$

The first sum is equal to 1 since $a(n)=0$ for $1 < n \leq Y$. Using Proposition (1.1) and partial summation, we see that the second sum is

$$\sum_{Y < n \leq Z} \frac{(\log n/Y)}{(\log Z/Y)^2} \cdot \frac{1}{n^{2\sigma}} + \mathbf{O}\left(\frac{1}{\log Z/Y}\right).$$

If $\sigma = 1/2$ this is

$$\frac{1}{2} + \mathbf{O}\left(\frac{1}{\log q}\right)$$

and if $\sigma > 1/2$, this is

$$\frac{2Z^{1-2\sigma}}{(1-2\sigma)(\log q)} \left(1 - \frac{2}{(1-2\sigma)(\log q)} + \mathbf{O}\left(\frac{1}{\log q}\right)\right) + \frac{4Y^{1-2\sigma}}{(1-2\sigma)^2(\log q)^2}.$$

Similarly, the third sum is

$$\sum_{Z < n < q} \frac{1}{\log Z/Y} \frac{1}{n^{2\sigma}} + \mathbf{O}\left(\frac{1}{\log q}\right)$$

which is

$$= \begin{cases} 1 + \mathbf{O}\left(\frac{1}{\log q}\right) & \text{if } \sigma = \frac{1}{2} \\ \frac{1}{1-2\sigma} \cdot \frac{2}{(\log q)} (q^{1-2\sigma} - Z^{1-2\sigma}) \left(1 + \mathbf{O}\left(\frac{1}{\log q}\right)\right) & \text{if } \sigma > \frac{1}{2}. \end{cases}$$

Putting these together we deduce that

$$\sum_{n < q} \frac{a(n)^2}{n} = \frac{5}{2} \left(1 + \mathbf{O}\left(\frac{1}{\log q}\right)\right)$$

and for $\sigma > 1/2$

$$\begin{aligned} \sum_{n < q} \frac{a(n)^2}{n^{2\sigma}} &= 1 - \frac{4Z^{1-2\sigma}}{(\log q)^2(1-2\sigma)^2} + \frac{4Y^{1-2\sigma}}{(\log q)^2(1-2\sigma)^2} \\ &\quad + \frac{2q^{1-2\sigma}}{(\log q)(1-2\sigma)} + \mathbf{O}\left(\frac{Z^{1-2\sigma}}{(\log q)^2(1-2\sigma)}\right). \end{aligned}$$

This completes the proof of the proposition.

In the next sections, we shall study the mean square of the integral $I(s, \chi)$.

5. THE INTEGRAL $R_a(s, \chi)$. — The purpose of the next few sections is to obtain an asymptotic formula for the mean square of $I(s, \chi)$. Recall that for $\chi \neq 1$, we have

$$I(s, \chi) = \frac{1}{2\pi i} \int_{(-\eta)} \gamma(s+w, \chi) \left\{ \sum_{n < Z} \frac{\overline{\chi(n)}}{n^{1-s-w}} \right\} M(s+w, \chi) q^w \Gamma(w) dw.$$

Thus

$$(5.1) \quad I(s, \chi) = \sum_{m, n < Z} \frac{\overline{\chi(n)} \chi(m) \lambda(m)}{m^s n^{1-s}} \left(\frac{2\pi}{q} \right)^{s-(1/2)} \left(\frac{2}{\pi} \right)^{1/2} \frac{\tau(\chi)}{i^a q^{1/2}} R_a\left(s, \frac{2\pi n}{m}\right)$$

where

$$(5.2) \quad R_a(s, y) = \frac{1}{2\pi i} \int_{(\delta)} y^w \sin\left(\frac{\pi}{2}(s+w+a)\right) \Gamma(1-s-w) \Gamma(w) dw.$$

Here $a=0, 1$, $\chi(-1)=(-1)^a$ and $-1 < \delta < 0$ is arbitrary, $y > 0$ and $0 < \operatorname{Re}(s) < 1$. Notice that

$$(5.3) \quad \overline{R_a(s, y)} = R_a(\bar{s}, y).$$

The integrand has simple poles at $w = -k$ and $w = 1-s+k$ where $0 \leq k \in \mathbf{Z}$. Since $1/2 \leq \operatorname{Re}(s) < 1$, these are distinct points. We have the expansion

$$R_a(s, y) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} y^{-k} \sin\left(\frac{\pi}{2}(s-k+a)\right) \Gamma(1-s+k) \quad \text{for } y \geq 1.$$

Indeed, this is just the sum of the residues at the points $w = -k$, $0 \leq k \in \mathbf{Z}$. The condition $y \geq 1$ ensures that it converges. Indeed, we have the following asymptotic expansion.

LEMMA (5.1). — For $s = \sigma + it$ with $1/2 \leq \sigma < 1$, and $|t| < \sigma/10$, $y \geq 1$, and $0 \leq K \in \mathbf{Z}$, we have

$$R_a(s, y) = \sum_{k=1}^K \frac{(-1)^k}{k!} y^{-k} \sin\left(\frac{\pi}{2}(s-k+a)\right) \Gamma(1-s+k) + \mathbf{O}\left(y^\delta \frac{1}{K^{\sigma/5}} |\Gamma(1-\delta-K-s) \Gamma(1+\delta+K)|\right)$$

for any $\delta \in (-K-1, -K)$.

Proof. — We need only estimate the integral defining R_a along a line $-K-1 < \delta < -K$. We write $w = -K - \eta$, $0 < \operatorname{Re}(\eta) < 1$. Write $s = \sigma + it$ and $\eta = \beta + i\gamma$. Then,

$$|\Gamma(1-s-w) \Gamma(w)| = \prod_{j=1}^K \left| 1 - \frac{s}{j+\eta} \right| \cdot |\Gamma(1+\eta-s) \Gamma(-\eta)|.$$

Now,

$$\left|1 - \frac{s}{j+\eta}\right| \leq \left|1 - \frac{\sigma}{j+\eta}\right| + \frac{|t|}{|j+\eta|}$$

and

$$\left|1 - \frac{\sigma}{j+\eta}\right|^2 = \left(1 - \frac{\sigma(j+\beta)}{(j+\beta)^2 + \gamma^2}\right)^2 + \frac{\sigma^2 \gamma^2}{|j+\eta|^4}.$$

If $j+\beta > 2\gamma$, we see that

$$\left|1 - \frac{\sigma}{j+\eta}\right|^2 \leq \left(1 - \frac{4\sigma}{5(j+\beta)}\right)^2 + \frac{\sigma^2}{4|j+\beta|^2}.$$

Therefore,

$$\left|1 - \frac{s}{j+\eta}\right| \leq 1 - \frac{4\sigma}{5(j+\beta)} + \frac{(1/2)\sigma + |t|}{j+\beta}$$

which simplifies to

$$\left|1 - \frac{s}{j+\eta}\right| \leq 1 - \frac{3\sigma/10 - |t|}{j+\beta}.$$

Let us set

$$u = u(\eta) = \max([2\gamma - \beta], 0) + 1$$

where $[x]$ denotes the greatest integer $\leq x$. We deduce that

$$\prod_{j=u}^K \left|1 - \frac{s}{j+\eta}\right| \leq \prod_{j=u}^K \left(1 - \frac{3\sigma/10 - |t|}{j+\beta}\right) \ll \left(\frac{u}{K}\right)^{(3/10)\sigma - |t|}.$$

Moreover

$$\prod_{j \leq u} \left|1 - \frac{s}{j+\eta}\right| \leq \left(1 + \frac{3\sigma}{5\gamma}\right)^u \ll 1.$$

Note that the sine term in the integrand is bounded as a function of k .

There is a similar expression and estimate when $y \leq 1$.

LEMMA (5.2). — For $s = \sigma + it$ with $1/2 \leq \sigma < 1$, $|t| < \sigma/10$, $0 < y \leq 1$, $0 \leq K \in \mathbf{Z}$ and any $\delta \in (1 - \sigma + K, 2 - \sigma + K)$, we have

$$\begin{aligned} R_a(s, y) = & -\sin\left(\frac{\pi}{2}(s+a)\right) \cdot \Gamma(1-s) \\ & - \sum_{k=1}^K \frac{(-1)^k}{k!} y^{1-s+k} \sin\left(\frac{\pi}{2}(a+k+1)\right) \Gamma(1-s+k) \\ & + \mathbf{O}(y^\delta K^{-\sigma/5} |\Gamma(2-s-\delta+K)\Gamma(\delta-K)|). \end{aligned}$$

In both cases, we see that for s as above (that is, $s = \sigma + it$, and $1/2 \leq \sigma < 1$ and $|t| < \sigma/10$),

$$R_a(s, y) \ll_c 1.$$

Finally, we define

$$(5.4) \quad \omega_k = \omega_k(s) = \frac{(-1)^k}{k!} \sin\left(\frac{\pi}{2}(a+k+1)\right) \Gamma(1-s+k).$$

The argument of Lemma (5.1) shows that for $s = \sigma + it$, with $1/2 \leq \sigma < 1$, we have

$$(5.5) \quad \omega_k(s) \ll k^{-\sigma+|t|}.$$

6. AN EXPRESSION FOR THE MEAN SQUARE OF $I(s, \chi)$. — From (5.1) and (5.3), we see that for a fixed $a=0$ or 1 , and an s , we have

$$\begin{aligned} \sum_{\substack{1 \neq \chi \pmod{q} \\ \chi(-1) = (-1)^a}} |I(s, \chi)|^2 = & \frac{2}{\pi} \left(\frac{2\pi}{q}\right)^{2\sigma-1} \sum_{\substack{m_1, m_2, n_1, n_2 < Z \\ (m_1, q) = \dots = (n_2, q) = 1}} \frac{\lambda(m_1)\lambda(m_2)}{m_1^s m_2^s n_1^{1-s} n_2^{1-s}} \\ & \times R_a\left(s, \frac{2\pi n_1}{m_1}\right) R_a\left(\bar{s}, \frac{2\pi n_2}{m_2}\right) \sum_{\substack{1 \neq \chi \pmod{q} \\ \chi(-1) = (-1)^a}} \chi(m_1 n_2) \overline{\chi(n_1 m_2)}. \end{aligned}$$

Notice that we can drop the condition $(m_1, q) = \dots = (n_2, q) = 1$ since $1 \leq m_1, \dots, n_2 \leq Z < q$. Observe that for $(mn, q) = 1$, and q odd, we have

$$\begin{aligned} \sum_{\substack{1 \neq \chi \pmod{q} \\ \chi(-1) = (-1)^a}} \chi(m) \overline{\chi(n)} &= \frac{1}{2} \sum_{1 \neq \chi} \chi(m) \overline{\chi(n)} + \frac{1}{2} (-1)^a \sum_{1 \neq \chi} \chi(-m) \overline{\chi(n)} \\ &= \frac{1}{2} \sum_{\varepsilon = \pm 1} \varepsilon^a \sum_{1 \neq \chi} \chi(\varepsilon m) \overline{\chi(n)} \\ &= \begin{cases} \frac{1}{2} \varphi(q) - 1 & \text{if } m \equiv \pm n \text{ and } a=0 \\ -1 & \text{if } m \not\equiv \pm n \text{ and } a=0 \\ \frac{1}{2} \varphi(q) & \text{if } m \equiv n \text{ and } a=1 \\ -\frac{1}{2} \varphi(q) & \text{if } m \equiv -n \text{ and } a=1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Applying this to the innermost sum, we see that this is

$$\frac{1}{\pi} (2\pi)^{2\sigma-1} \varphi(q) q^{1-2\sigma} \sum_{\varepsilon} \varepsilon^a \sum_{\substack{m_1, m_2, n_1, n_2 < Z \\ m_1 n_2 \equiv \varepsilon n_1 m_2 \pmod{q}}} \frac{\lambda(m_1) \lambda(m_2)}{m_1^s m_2^s n_1^{1-s} n_2^{1-s}} R_a\left(s, \frac{2\pi n_1}{m_1}\right) R_a\left(\bar{s}, \frac{2\pi n_2}{m_2}\right)$$

minus

$$\delta(a) \frac{2}{\pi} (2\pi)^{2\sigma-1} q^{1-2\sigma} \sum_{\substack{m_1, m_2, n_1, n_2 < Z \\ (m_1 m_2 n_1 n_2, q) = 1}} \frac{\lambda(m_1) \lambda(m_2)}{m_1^s m_2^s n_1^{1-s} n_2^{1-s}} R_0\left(s, \frac{2\pi n_1}{m_1}\right) R_0\left(\bar{s}, \frac{2\pi n_2}{m_2}\right).$$

Here, $\delta(a) = 1 - a$. If we designate the second quantity as $|I(s, 1)|^2$, then setting $a=0$, 1 and adding, we deduce that

$$(6.1) \quad \sum_{\chi \pmod{q}} |I(s, \chi)|^2 = \frac{1}{\pi} (2\pi)^{2\sigma-1} \varphi(q) q^{1-2\sigma} (S^+(s, q) + S^-(s, q))$$

where

$$\begin{aligned} S^{\pm}(s, q) &= \sum_{\substack{m_1, m_2, n_1, n_2 < Z \\ m_1 n_2 \equiv \pm n_1 m_2 \pmod{q}}} \frac{\lambda(m_1) \lambda(m_2)}{m_1^s m_2^s n_1^{1-s} n_2^{1-s}} \\ &\quad \times \left[R_0\left(s, \frac{2\pi n_1}{m_1}\right) R_0\left(\bar{s}, \frac{2\pi n_2}{m_2}\right) \pm R_1\left(s, \frac{2\pi n_1}{m_1}\right) R_1\left(\bar{s}, \frac{2\pi n_2}{m_2}\right) \right]. \end{aligned}$$

Note that if $s=1/2$, this can be rewritten as

$$S^\pm\left(\frac{1}{2}, q\right) = \sum_{\substack{m_1, m_2, n_1, n_2 < Z \\ (m_1 n_2, q) = (m_2 n_1, q) = 1}} \frac{\lambda(m_1)\lambda(m_2)}{(m_1 m_2 n_1 n_2)^{1/2}} \\ \times \left[R_0\left(\frac{1}{2}, \frac{2\pi n_1}{m_1}\right) R_0\left(\frac{1}{2}, \frac{2\pi n_2}{m_2}\right) \pm R_1\left(\frac{1}{2}, \frac{2\pi n_1}{m_1}\right) R_1\left(\frac{1}{2}, \frac{2\pi n_2}{m_2}\right) \right]$$

and

$$(6.2) \quad \sum_{\chi \pmod{q}} \left| I\left(\frac{1}{2}, \chi\right) \right|^2 = \frac{1}{\pi} \phi(q) \left(S^+\left(\frac{1}{2}, q\right) + S^-\left(\frac{1}{2}, q\right) \right).$$

Let us also define, for $a=0, 1$

$$S^\pm(s, q; a) = \sum_{\substack{m_1, m_2, n_1, n_2 < Z \\ m_1 n_2 \equiv \pm n_1 m_2 \pmod{q}}} \frac{\lambda(m_1)\lambda(m_2)}{m_1^s m_2^s n_1^{1-s} n_2^{1-s}} R_a\left(s, \frac{2\pi n_1}{m_1}\right) R_a\left(\bar{s}, \frac{2\pi n_2}{m_2}\right)$$

so that

$$S^\pm(s, q) = S^\pm(s, q, 0) \pm S^\pm(s, q, 1).$$

Our estimations are complicated by the unusual way in which the four indices of summation m_1, m_2, n_1, n_2 are interlaced. Our goal in the next sections will be to show that the main contribution comes from those terms where $m_1 n_2 = n_1 m_2$ and $n_1 \leq (1/2\pi)m_1, n_2 \leq (1/2\pi)m_2$.

7. ESTIMATE OF THE NON-DIAGONAL TERMS. — We wish to show that the terms in $S^-(s, q; a)$ contribute a negligible amount to the right hand side of (6.1). Since $m_1, m_2, n_1, n_2 < Z$ and $m_1 n_2 \equiv -n_1 m_2 \pmod{q}$, this means that $m_1 n_2 = q - m_2 n_1$. (Notice that for the same reason, the indices in $S^+(s, q; a)$ satisfy $m_1 n_2 = m_2 n_1$).

LEMMA (7.1). — For $1/2 \leq \sigma < 1$, we have

$$S^-(s, q; a) \ll \frac{1}{(1-\sigma)^2 (\log q)^2} + \frac{q^{\sigma-1}}{(1-\sigma) \log q}.$$

Proof. — We wish to estimate the sum

$$\sum_{\substack{1 \leq m_1, m_2, n_1, n_2 < Z \\ m_1 n_2 = q - m_2 n_1}} \frac{|\lambda(m_1)| |\lambda(m_2)|}{m_1^\sigma m_2^\sigma n_1^{1-\sigma} n_2^{1-\sigma}} \left| R_a\left(s, \frac{2\pi n_1}{m_1}\right) R_a\left(s, \frac{2\pi n_2}{m_2}\right) \right|.$$

Without loss, we may suppose that $m_2 n_1 < (1/2)q$. A consequence of this is that $m_1 n_2 > (1/2)q$ and so

$$\frac{1}{2}Z < m_1, n_2 < Z.$$

We may also suppose that m_1, m_2, n_1, n_2 are squarefree. Notice that we must have $(m_1, m_2) = 1$. We consider two cases.

Case 1. — $m_2 < n_2$.

In this case, we must have $n_2 \equiv \bar{m}_1 q \pmod{m_2}$ where \bar{m}_1 denotes the inverse of m_1 modulo m_2 . Moreover,

$$\left| R_a \left(s, \frac{2\pi n_2}{m_2} \right) \right| \ll \frac{m_2}{n_2}.$$

Thus, we can rewrite our sum as

$$\begin{aligned} & \sum_{m_2 < Z} \frac{|\lambda(m_2)|}{m_2^\sigma} \sum_{(1/2)Z < m_1 < Z} \frac{|\lambda(m_1)|}{m_1^\sigma} \sum_{\substack{n_2 \equiv q\bar{m}_1 \pmod{m_2} \\ n_2 > m_2 \\ (1/2)Z < n_2 < Z}} \frac{1}{n_2^{1-\sigma}} \left(\frac{q - m_1 n_2}{m_2} \right)^{\sigma-1} \frac{m_2}{n_2} \\ & \ll Z^{\sigma-2} \sum_{m_2 < Z} \frac{|\lambda(m_2)|}{m_2^{2\sigma-2}} \sum_{(1/2)Z < m_1 < Z} \frac{|\lambda(m_1)|}{m_1^\sigma} \sum_{n_2} \left(\frac{1}{q - m_1 n_2} \right)^{1-\sigma} \\ & \ll Z^{\sigma-2} \sum_{m_2 < Z} \frac{|\lambda(m_2)|}{m_2^{2\sigma-1}} \sum_{(1/2)Z < m_1 < Z} \frac{|\lambda(m_1)|}{m_1^{1+\sigma}} \left(q - \frac{m_1 Z}{2} \right)^\sigma \\ & \ll Z^{2\sigma-2} \sum_{m_2 < Z} \frac{|\lambda(m_2)|}{m_2^{2\sigma-1}} \sum_{(1/2)Z < m_1 < Z} \frac{|\lambda(m_1)|}{m_1} \\ & \ll Z^{2\sigma-2} \frac{1}{\log Z/Y} \left(\frac{\log Z}{2-2\sigma} + \frac{Z^{2-2\sigma}-1}{(2-2\sigma)^2} \right) \frac{1}{\log Z/Y} \\ & \ll \left(\frac{q^{\sigma-1}}{\sigma-1} + \frac{1}{(\log q)(1-\sigma)^2} \right) \frac{1}{\log q}. \end{aligned}$$

Case 2. — $m_2 \geq n_2$.

In this case, we write the congruence condition as $m_1 \equiv q\bar{n}_2 \pmod{m_2}$. Since $(1/2)Z < m_1 < Z$, this implies that there are at most two possible values for m_1 . Thus, we see that our sum is

$$\begin{aligned} & \sum_{(1/2)Z < m_2 < Z} \frac{|\lambda(m_2)|}{m_2^\sigma} \sum_{\substack{n_2 \leq m_2 \\ (1/2)Z < n_2 < Z}} \frac{1}{n_2^{1-\sigma}} \sum_{\substack{(1/2)Z < m_1 < Z \\ m_1 \equiv q\bar{n}_2 \pmod{m_2}}} \frac{|\lambda(m_1)|}{m_1^\sigma} \left(\frac{m_2}{q - m_1 n_2} \right)^{1-\sigma} \\ & \ll \sum_{(1/2)Z < m_2 < Z} \frac{|\lambda(m_2)|}{m_2^{2\sigma-1}} \sum_{(1/2)Z < n_2 < Z} \frac{1}{n_2^{1-\sigma}} \frac{1}{\log Z/Y} \frac{1}{Z^\sigma} \frac{1}{(q - Z n_2)^{1-\sigma}} \\ & \ll \frac{1}{Z^{2-\sigma} (\log Z/Y)} \sum_{(1/2)Z < m_2 < Z} \frac{|\lambda(m_2)|}{m_2^{2\sigma-1}} \int_{(1/2)Z}^{m_2} \frac{dt}{(Z-t)^{1-\sigma}} \end{aligned}$$

$$\begin{aligned} &\ll \frac{1}{Z(\log Z/Y)} \sum_{(1/2)Z < m_2 < Z} \frac{\log Z/m_2}{\log Z/Y} \\ &\ll \frac{1}{(\log q)^2}. \end{aligned}$$

This proves the result.

Finally in this section, we shall show that $|I(s, 1)|^2$ is negligible.

LEMMA (7.2). — *We have for $1/2 \leq \sigma < 1$*

$$|I(s, 1)|^2 \ll \frac{q^{1-\sigma}}{(1-\sigma)^2} \left(1 + \frac{q^{1-\sigma}}{(1-\sigma)^2 (\log q)^2} \right).$$

If $\sigma = 1$ we have

$$|I(s, 1)|^2 \ll (\log Z)^2.$$

Proof. — By definition, we have

$$|I(s, 1)|^2 = \frac{2}{\pi} (2\pi)^{2\sigma-1} q^{1-2\sigma} \sum_{\substack{m_1, m_2, n_1, n_2 < Z \\ (m_1 m_2 n_1 n_2, q) = 1}} \frac{\lambda(m_1)\lambda(m_2)}{m_1^s m_2^{\bar{s}} n_1^{1-s} n_2^{1-\bar{s}}} R_0\left(s, \frac{2\pi n_1}{m_1}\right) R_0\left(\bar{s}, \frac{2\pi n_2}{m_2}\right).$$

The n_1 and n_2 sums are estimated as $\ll Z^\sigma/\sigma$. To estimate the sum over m_1 and m_2 we observe that for $\sigma \neq 1$,

$$\sum_m \frac{|\lambda(m)|}{m^\sigma} \leq \sum \frac{\log(Z/m)}{\log(Z/Y)} \cdot \frac{1}{m^\sigma} + O\left(\frac{Y^{1-\sigma}}{(1-\sigma)}\right) \ll \frac{1}{(\log Z/Y)(1-\sigma)} \left(\frac{Z^{1-\sigma}}{1-\sigma} + \log Z \right).$$

Using this estimate, we see that

$$|I(s, 1)|^2 \ll q^{1-2\sigma} \frac{Z^{2\sigma}}{\sigma^2} \left\{ \frac{1}{(\log Z/Y)(1-\sigma)} \left(\frac{Z^{1-\sigma}}{1-\sigma} + \log Z \right) \right\}^2$$

and this simplifies to the stated expression, given our choices of Y and Z .

8. THE DIAGONAL TERMS. — We are now reduced to the study of the sum

$$(8.1) \quad \frac{1}{\pi} (2\pi)^{2\sigma-1} \varphi(q) q^{1-2\sigma} \times \sum_{\substack{m_1, m_2, n_1, n_2 < Z \\ (m_1 n_2, q) = 1 \\ m_1 n_2 = m_2 n_1}} \frac{\lambda(m_1)\lambda(m_2)}{m_1^s m_2^{\bar{s}} n_1^{1-s} n_2^{1-\bar{s}}} \left(\left| R_0\left(s, \frac{2\pi n_1}{m_1}\right) \right|^2 + \left| R_1\left(s, \frac{2\pi n_1}{m_1}\right) \right|^2 \right).$$

Let us define

$$(8.2) \quad D_a(s) = \sum_{\substack{m_1 m_2, n_1, n_2 < Z \\ m_1 n_2 = m_2 n_1}} \frac{\lambda(m_1)\lambda(m_2)}{m_1^s m_2^{\bar{s}} n_1^{1-s} n_2^{1-\bar{s}}} \left| R_a\left(s, \frac{2\pi n_1}{m_1}\right) \right|^2.$$

For future reference let us denote by s the set of quadruples (m_1, m_2, n_1, n_2) included in the above sum. Then, the sum in (8.1) can be written as

$$\frac{1}{\pi} (2\pi)^{2\sigma-1} \frac{\varphi(q)}{q^{2\sigma-1}} (D_0(s) + D_1(s)).$$

We define a splitting

$$D_a(s) = M_a(s) + E_a(s)$$

where in $M_a(s)$, we range over those quadruples (m_1, m_2, n_1, n_2) in (8.2) which in addition satisfy $n_1 \leq (1/2\pi)m_1$. (Note that as $n_1/m_1 = n_2/m_2$, we will then also have $n_2 \leq (1/2\pi)m_2$.)

We shall henceforth assume that $1/2 \leq \sigma \leq 1$. The case $\sigma = 1$ can also be handled, but it will not be necessary for us. Moreover, we suppose that $|t|$ is sufficiently small in the strong sense that

$$|t| \ll \frac{1}{\log q}.$$

We begin our study $M_a(s)$ by replacing $R_a(s, (2\pi n_1/m_1))$ with the Taylor expansion of Lemma (5.2). We find that

$$(8.3) \quad M_a(s) = \sum \frac{\lambda(m_1)\lambda(m_2)}{m_1^s m_2^{\bar{s}} n_1^{1-s} n_2^{1-\bar{s}}} \left| \sin\left(\frac{\pi}{2}(s+a)\right) \Gamma(1-s) + \sum_{k=1}^K \left(\frac{2\pi n_1}{m_1}\right)^{1-s+k} \omega_k + O\left(\left(\frac{2\pi n_1}{m_1}\right)^\delta K^{-\sigma} \Gamma(2-\delta+K-\sigma) \Gamma(\delta-K)\right) \right|^2$$

where we recall (from (5.4)) that

$$\omega_k = \frac{(-1)^k}{k!} \sin\left(\frac{\pi}{2}(a+k+1)\right) \Gamma(1-s+k)$$

and $0 \leq K \in \mathbf{Z}$ and $1-\sigma+K < \delta < 2-\sigma+K$. Expanding, we find that $M_a(s)$ splits into a main term and an O -term. We shall now analyze the O -term with the help of the following lemma.

LEMMA (8.1). — For any $\beta > 0$ we have

$$\sum_{m_1, m_2, n_1, n_2 \in s} \frac{|\lambda(m_1)| |\lambda(m_2)|}{(m_1 m_2)^\sigma (n_1 n_2)^{1-\sigma}} \left(\frac{n_1}{m_1}\right)^\beta \ll \frac{1}{(2\sigma + \beta - 1)} \cdot \frac{1}{(2\pi)^{2\sigma + \beta - 1}} (\log Z)^3.$$

Proof. — We use the fact that

$$m_1 n_2 = m_2 n_1$$

$$n_1 \leq \frac{1}{2\pi} m_1, \quad n_2 \leq \frac{1}{2\pi} m_2$$

and

$$|\lambda(m)| \leq 1 \quad \text{for all } m.$$

Note that $2\sigma + \beta > 1$. Then, denoting $m_1 n_2$ by j , we see that the sum is bounded by

$$(8.4) \quad \sum_{m_1, m_2} \frac{1}{(m_1 m_2)^{2\sigma + \beta - 1}} \sum_j j^{2\sigma + \beta - 2}$$

where the inner sum ranges over integers j satisfying

$$1 \leq j \leq \frac{1}{2\pi} m_1 m_2$$

$$j \equiv 0 \pmod{[m_1, m_2]}.$$

Let us set

$$i = (m_1, m_2).$$

Then $[m_1, m_2] = m_1 m_2 / i$ and (8.4) is

$$\ll \sum \frac{[m_1, m_2]^{2\sigma + \beta - 2}}{(m_1 m_2)^{2\sigma + \beta - 1}} \left(\frac{1}{2\pi} \frac{m_1 m_2}{[m_1, m_2]} \right)^{2\sigma + \beta - 1} \cdot \frac{1}{2\sigma + \beta - 1}$$

$$\ll \frac{1}{2\sigma + \beta - 1} \cdot \frac{1}{(2\pi)^{2\sigma + \beta - 1}} \cdot \sum \frac{i}{m_1 m_2}.$$

Moreover,

$$\sum \frac{i}{m_1 m_2} \ll \sum_{m_1 \leq Z} \frac{1}{m_1} \sum_{i | m_1} \sum_{m \leq (Z/i)} \frac{1}{m}$$

$$\ll (\log Z)^3.$$

This proves the lemma.

Now, the **O**-term in (8.3) is, for any $0 \leq K \in \mathbf{Z}$ and any $1 - \sigma + K < \delta < 2 - \sigma + K$,

$$\begin{aligned} &\ll \sum \frac{|\lambda(m_1)| |\lambda(m_2)|}{(m_1 m_2)^\sigma (n_1 n_2)^{1-\sigma}} \left(\frac{2\pi n_1}{m_1}\right)^\delta K^{-\sigma/5} \Gamma(2-\sigma-\delta+K) \Gamma(\delta-K). \\ &\left(\left| \sin\left(\frac{\pi}{2}(s+a)\right) \Gamma(1-s) \right| + \sum_{k=1}^K \left(\frac{2\pi n_1}{m_1}\right)^{1-\sigma+k} |\omega_k| \right. \\ &\qquad \left. + \left(\frac{2\pi n_1}{m_1}\right)^\delta K^{-\sigma/5} \Gamma(2-\sigma-\delta+K) \Gamma(\delta-K) \right). \end{aligned}$$

Now using (5.5), we find that the above is

$$\begin{aligned} &\ll K^{-\sigma/5} \cdot (\log Z)^3 \cdot |\Gamma(2-\sigma-\delta+K) \Gamma(\delta-K)| (2\pi)^\delta \\ &\quad \times \left\{ \left| \sin\left(\frac{\pi}{2}(s+a)\right) \Gamma(1-s) \right| \frac{1}{2\sigma+\delta-1} \frac{1}{(2\pi)^{2\sigma+\delta-1}} \right. \\ &\quad \left. + \sum_{k=1}^K |\omega_k| (2\pi)^{1-\sigma+k} \cdot \frac{1}{\sigma+\delta+k} \frac{1}{(2\pi)^{\sigma+\delta+k}} \right. \\ &\quad \left. + |\Gamma(2-\sigma-\delta+K) \Gamma(\delta-K)| (2\pi)^\delta K^{-\sigma/5} (2\sigma+2\delta-1)^{-1} (2\pi)^{1-2\sigma-2\delta} \right\}. \end{aligned}$$

Choosing $\delta = (3/2) - \sigma + K$, this is

$$\ll K^{-1-\sigma/5} (\log Z)^3 \left\{ \left| \sin\left(\frac{\pi}{2}(s+a)\right) \Gamma(1-s) \right| + K^{1-\sigma/5} + K^{-\sigma/5} \right\}.$$

Finally, choosing

$$K = (\log q)^{20}$$

shows that the **O**-term in (8.3) is

$$(8.5) \qquad \ll |\Gamma(1-s)| \cdot (\log q)^{-1}.$$

Now we analyze the main term of (8.3), namely

$$(8.6) \qquad \sum \frac{\lambda(m_1)\lambda(m_2)}{m_1^\sigma m_2^\sigma n_1^{1-\sigma} n_2^{1-\sigma}} \left| \sin\left(\frac{\pi}{2}(s+a)\right) \Gamma(1-s) + \sum_{k=1}^K \left(\frac{2\pi n_1}{m_1}\right)^{1-s+k} \omega_k \right|^2.$$

For this purpose we utilise a more refined version of Lemma (8.1).

LEMMA (8.2). — We have for any w with $\beta = \operatorname{Re} w > 0$

$$\sum_{m_1, m_2, n_1, n_2 \in \mathcal{S}} \frac{\lambda(m_1)\lambda(m_2)}{m_1^\sigma m_2^\sigma n_1^{1-\sigma} n_2^{1-\sigma}} \cdot \left(\frac{n_1}{m_1}\right)^w \ll \frac{(2\pi)^{1-2\sigma-\beta}}{1-2\sigma-\beta} \cdot \frac{1}{\log q}.$$

Proof. — We see that the sum is

$$T \stackrel{\text{def}}{=} \sum_{m_1, m_2, n_1, n_2 \in \mathcal{S}} \frac{\lambda(m_1)\lambda(m_2)}{m_1^\sigma m_2^\sigma n_1^{1-\sigma} n_2^{1-\sigma}} \left(\frac{n_1}{m_1}\right)^w = \sum_{1 \leq m_1, m_2 < Z} \frac{\lambda(m_1)\lambda(m_2)}{(m_1 m_2)^{2\sigma+1+w}} \sum_j j^{2\sigma-2+w}$$

where the inner sum ranges over integers j satisfying

$$j \equiv 0 \pmod{[m_1, m_2]}$$

$$1 \leq j \leq \frac{1}{2\pi} m_1 m_2.$$

Setting $j = j_0 [m_1, m_2]$, and $i = (m_1, m_2)$ as before, we see that the sum is

$$T = \sum_{1 \leq m_1, m_2 < Z} \frac{\lambda(m_1)\lambda(m_2)[m_1, m_2]^{w+2\sigma-2}}{(m_1 m_2)^{2\sigma-1+w}} \cdot \sum_{1 \leq j_0 \leq i/2\pi} j_0^{2\sigma-2+w}.$$

Since $[m_1, m_2] = m_1 m_2 / i$, this may be rewritten as

$$(8.7) \quad T = \sum_{1 \leq j_0 \leq (1/2\pi)Z} j_0^{2\sigma-2+w} \sum_{2\pi j_0 \leq i \leq Z} \frac{1}{i^{2\sigma-2+w}} \sum_{\substack{1 \leq m_1, m_2 \leq Z \\ (m_1, m_2) = i}} \frac{\lambda(m_1)\lambda(m_2)}{m_1 m_2}.$$

The innermost sum can be written

$$(8.8) \quad S \stackrel{\text{def}}{=} \sum \frac{\lambda(ij_1)\lambda(ij_2)}{i^2 j_1 j_2}$$

where the summation ranges over pairs (j_1, j_2) satisfying

$$1 \leq j_1 \leq \frac{Z}{i}, \quad 1 \leq j_2 \leq \frac{Z}{i}$$

$$(j_1, j_2) = 1$$

We may suppose that ij_1, ij_2 are squarefree (else $\lambda(ij_1)\lambda(ij_2)$ will be zero). In particular, this implies that $(j_1, i) = 1$ and $(j_2, i) = 1$. Applying Lemma (1.5) to (8.8), we find that

$$(8.9) \quad S \ll \left(\frac{i}{\varphi(i)} + \sigma_{-1/2}(i^2) \right)^2 \cdot \frac{1}{\log^2(Z/Y)} \frac{1}{i^2}.$$

Substituting this estimate into (8.7), we find that

$$\begin{aligned} T &\ll \frac{1}{\log^2(Z/Y)} \cdot \sum_{1 \leq j_0 \leq (1/2\pi)Z} j_0^{2\sigma-2+\beta} \sum_{2\pi j_0 \leq i \leq Z} \frac{1}{i^{2\sigma-2+\beta}} \frac{1}{i^2} \left(\frac{i}{\varphi(i)} + \sigma_{-1/2}(i^2) \right)^2 \\ &\ll \frac{1}{(\log q)^2} \sum_{1 \leq j_0 \leq (1/2\pi)Z} j_0^{2\sigma-2+\beta} \sum_{2\pi j_0 \leq i \leq Z} \frac{1}{i^{2\sigma+\beta}} \\ &\ll \frac{1}{(1-2\sigma-\beta)(\log q)^2} (2\pi)^{1-2\sigma-\beta} \sum_{1 \leq j_0 \leq (1/2\pi)Z} \frac{1}{j_0} \\ &\ll \frac{1}{\log q} \frac{(2\pi)^{1-2\sigma-\beta}}{1-2\sigma-\beta}. \end{aligned}$$

Here, we have used the fact that

$$\left(\frac{i}{\varphi(i)} + \sigma_{-1/2}(i^2) \right)^2$$

is bounded on average. This proves the Lemma.

We now apply Lemma (8.2) to analyze (8.6). We find that it is equal to

$$\begin{aligned} &\sum \frac{\lambda(m_1)\lambda(m_2)}{m_1^s m_2^s n_1^{1-s} n_2^{1-s}} \left\{ \left| \sin\left(\frac{\pi}{2}(s+a)\right) \Gamma(1-s) \right|^2 \right. \\ &\quad + 2 \sum_{k=1}^K \operatorname{Re} \left(\sin\frac{\pi}{2}(s+a) \cdot \Gamma(1-s) \cdot \left(\frac{2\pi n_1}{m_1}\right)^{1-s+k} \bar{\omega}_k \right. \\ &\quad \left. \left. + \sum_{k_1, k_2=1}^K \left(\frac{2\pi n_1}{m_1}\right)^{1-\sigma+k_1} \omega_{k_1} \left(\frac{2\pi n_1}{m_1}\right)^{1-\sigma+k_2} \bar{\omega}_{k_2} \right\} \\ &= \left| \sin\left(\frac{\pi}{2}(s+a)\right) \Gamma(1-s) \right|^2 \sum \frac{\lambda(m_1)\lambda(m_2)}{m_1^s m_2^s n_1^{1-s} n_2^{1-s}} \\ &\quad + \mathbf{O} \left(\sum_{k=1}^K \frac{1}{k^{\sigma-|t|}} (2\pi)^{1-\sigma+k} \left| \sin\left(\frac{\pi}{2}(s+a)\right) \Gamma(1-s) \right| \frac{1}{\log q} \frac{(2\pi)^{1-2\sigma-(1-\sigma+k)}}{1-2\sigma-(1-\sigma+k)} \right) \\ &\quad + \mathbf{O} \left(\sum_{k_1, k_2=1}^K \frac{1}{(k_1 k_2)^{\sigma-|t|}} \cdot (2\pi)^{2-2\sigma+k_1+k_2} \frac{1}{\log q} \frac{(2\pi)^{1-2\sigma-(2-2\sigma+k_1+k_2)}}{1-2\sigma-(2-2\sigma+k_1+k_2)} \right). \end{aligned}$$

By our assumption that $|t| \ll (\log q)^{-1}$ we may ignore $|t|$ in the estimations below. We observe that the sum over k in the first error term is

$$\sum_{k=1}^K \frac{1}{k^\sigma} \frac{(2\pi)^{1-2\sigma}}{(k+\sigma)} \ll 1.$$

Since $\sigma \geq (1/2)$, the double sum over k_1, k_2 is

$$\sum_{k_1, k_2=1}^K \frac{(2\pi)^{1-2\sigma}}{(k_1 k_2)^\sigma (k_1 + k_2 + 1)} \ll \begin{cases} (\log K) & \text{always} \\ (2\sigma - 1)^{-1} & \text{if } \sigma > (1/2) \end{cases}$$

We also note that if σ is close to $1/2$, it is sometimes more convenient to use the first estimate. Recalling that $K = (\log q)^{2\sigma}$, we deduce that

$$(8.10) \quad M_a(s) = \sum_{\substack{(m_1, m_2, n_1, n_2) \in \mathcal{S} \\ n_1 \leq (1/2\pi) m_1}} \frac{\lambda(m_1)\lambda(m_2)}{m_1^s m_2^{\bar{s}} n_1^{1-s} n_2^{1-\bar{s}}} \left| \sin\left(\frac{\pi}{2}(s+a)\right) \Gamma(1-s) \right|^2 \\ + \mathcal{O}\left(\frac{1}{(2\sigma-1)\log q}\right) + \mathcal{O}\left(\frac{1}{\log q} \left| \sin\left(\frac{\pi}{2}(s+a)\right) \Gamma(1-s) \right|\right)$$

where $2\sigma - 1$ is to be interpreted as $(\log \log q)^{-1}$ when $\sigma = 1/2$. By an entirely analogous argument, it can be shown that

$$(8.11) \quad E_a(s) = \mathcal{O}\left(\frac{1}{(2\sigma-1)\log q}\right) + \mathcal{O}\left(\frac{1}{\log q} \left| \sin\left(\frac{\pi}{2}(s+a)\right) \Gamma(1-s) \right|\right)$$

with the same interpretation of $2\sigma - 1$ as above.

To summarize, we deduce from (6.1), Lemma (7.1), Lemma (7.2), (8.1), (8.9) and (8.10), (8.11) that

$$\sum_{1 \neq \chi \pmod{q}} |I(s, \chi)|^2 \\ = \frac{1}{\pi} (2\pi)^{2\sigma-1} \frac{\varphi(q)}{q^{2\sigma-1}} \left(\left| \sin\left(\frac{\pi}{2}s\right) \right|^2 \right. \\ \left. + \left| \sin\left(\frac{\pi}{2}(s+1)\right) \right|^2 \right) |\Gamma(1-s)|^2 \sum_{\substack{(m_1, m_2, n_1, n_2) \in \mathcal{S} \\ n_1 \leq (1/2\pi) m_1}} \frac{\lambda(m_1)\lambda(m_2)}{m_1^s m_2^{\bar{s}} n_1^{1-s} n_2^{1-\bar{s}}} \\ + \mathcal{O}\left(\frac{\varphi(q)}{q^{2\sigma-1} \log q} \min\left(\frac{1}{(2\sigma-1)}, \log \log q\right)\right) + \mathcal{O}\left(\frac{\varphi(q) |\Gamma(1-s)|}{q^{2\sigma-1} (\log q)}\right) + \mathcal{O}\left(\frac{q^{1-\sigma}}{(1-\sigma) \log q}\right) \\ + \mathcal{O}\left(\frac{q^{1-\sigma}}{(1-\sigma)^2}\right) + \mathcal{O}\left(\frac{\varphi(q)}{q^{2\sigma-1}} \frac{1}{(1-\sigma)^4 (\log q)^2}\right).$$

9. ANALYSIS OF THE MAIN TERM. — We shall now analyze the sum in the main term, namely,

$$N(\sigma) \stackrel{\text{def}}{=} \sum \frac{\lambda(m_1)\lambda(m_2)}{m_1^s m_2^{\bar{s}} n_1^{1-s} n_2^{1-\bar{s}}}$$

where the sum ranges over quadruples (m_1, m_2, n_1, n_2) satisfying

$$\begin{aligned} 1 \leq m_1, m_2, n_1, n_2 < Z \\ m_1 n_2 = m_2 n_1 \\ n_1 \leq \frac{1}{2\pi} m_1. \end{aligned}$$

We note that $N(\sigma)$ is well defined since the relation $m_1 n_2 = m_2 n_1$ makes the right hand side independent of the imaginary part of s .

As before, we set $j = m_1 n_2 = m_2 n_1$, $i = (m_1, m_2)$. Note that given m_1, m_2 and j, n_1 and n_2 are uniquely determined. We may thus rewrite $N(\sigma)$ as

$$N(\sigma) = \sum_{1 \leq m_1, m_2 < Z} \frac{\lambda(m_1)\lambda(m_2)}{(m_1 m_2)^{2\sigma-1}} \sum_j j^{2(\sigma-1)}$$

where the inner sum ranges over integers j satisfying

$$\begin{aligned} 1 \leq j \leq \frac{1}{2\pi} m_1 m_2 \\ j \equiv 0 \pmod{[m_1, m_2]}. \end{aligned}$$

We can rewrite $N(\sigma)$ as in (8.7). Thus, setting $j = j_0 [m_1, m_2]$ and $i = (m_1, m_2)$, we get

$$(9.1) \quad N(\sigma) = \sum_{1 \leq j_0 \leq (1/2\pi)Z} j_0^{2\sigma-2} \sum_i i^{-2\sigma} \sum \frac{\lambda(ij_1)\lambda(ij_2)}{j_1 j_2}$$

where the sum over i ranges over

$$2\pi j_0 \leq i \leq Z$$

and the inner sum ranges over pairs (j_1, j_2) satisfying

$$(9.2) \quad \begin{aligned} 1 \leq j_1 \leq \frac{Z}{i}, \quad 1 \leq j_2 \leq \frac{Z}{i} \\ (j_1, j_2) = 1. \end{aligned}$$

Notice that we can also stipulate that

$$(9.3) \quad i \leq \min(m_1, m_2) \leq Z$$

and that

$$(j_1, i) = (j_2, i) = 1.$$

We write the innermost sum of (9.1) as

$$(9.4) \quad \sum \frac{\lambda(ij_1)\lambda(ij_2)}{j_1j_2} \sum_{\substack{e|j_1 \\ e|j_2}} \mu(e) = \sum_e \frac{\mu(e)}{e^2} \sum_{l_1, l_2} \frac{\lambda(iel_1)\lambda(iel_2)}{l_1l_2}$$

where on the right, e ranges over

$$1 \leq e \leq \frac{Z}{i}$$

and l_1, l_2 range over

$$1 \leq l_1 \leq \frac{Z}{ie}, \quad 1 \leq l_2 \leq \frac{Z}{ie}$$

$$(el_1, i) = (el_2, i) = 1.$$

Writing

$$\lambda(m) = \frac{\Lambda_1(m) - \Lambda_2(m)}{(\log Z/Y)},$$

we find that (9.4) breaks up into four subsums of the form

$$(9.5) \quad \frac{1}{(\log Z/Y)^2} \sum \frac{\mu(e)}{e^2} \left\{ \sum_{l_1} \frac{\Lambda_g(iel_1)}{l_1} \right\} \left\{ \sum_{l_2} \frac{\Lambda_h(iel_2)}{l_2} \right\}$$

where $g, h \in \{1, 2\}$, $z_1 = Y$, $z_2 = Z$.

LEMMA (9.1). — *Define*

$$X_{g,h} = \sum \frac{\mu(e)}{e^2} \left\{ \sum_{l_1} \frac{\Lambda_g(iel_1)}{l_1} \right\} \left\{ \sum_{l_2} \frac{\Lambda_h(iel_2)}{l_2} \right\}$$

and let $z = \min(z_g, z_h)$. Then,

$$X = 0 \quad \text{if } i > z.$$

If $i \leq z$, then

$$(9.6) \quad X = \sum \frac{\mu(e)}{e^2} \mu(ie)^2 \cdot \left(\frac{ie}{\varphi(ie)} \right)^2 + O_c \left(\frac{\sigma_{-1/2}(i)i}{\varphi(i)(\log(2Z/i))^c} \right) + O_c \left(\frac{\sigma_{-1/2}(i)^2}{(\log(2Z/i))^{2c}} \right).$$

Proof. — The first assertion is obvious from the definition of Λ_g and Λ_h . Therefore, suppose that

$$1 \leq e \leq \frac{Z}{i}.$$

We have

$$\begin{aligned} X_{g,h} &= \sum \frac{\mu(e)}{e^2} \left\{ \sum_{l_1 \leq z_g/ie} \frac{\mu(iel_1)}{l_1} \log \left(\frac{z_g}{iel_1} \right) \right\} \left\{ \sum_{l_2 \leq z_h/ie} \frac{\mu(iel_2)}{l_2} \log \left(\frac{z_h}{iel_2} \right) \right\} \\ &= \sum \frac{\mu(e)}{e^2} \mu(ie)^2 \left\{ \sum_{\substack{l_1 \leq z_g/ie \\ (l_1, ie)=1}} \frac{\mu(l_1)}{l_1} \log \left(\frac{z_g}{iel_1} \right) \right\} \left\{ \sum_{\substack{l_2 \leq z_h/ie \\ (l_2, ie)=1}} \frac{\mu(l_2)}{l_2} \log \left(\frac{z_h}{iel_2} \right) \right\}. \end{aligned}$$

Using Lemma (1.4), we have for any $c > 0$,

$$X_{g,h} = \sum \frac{\mu(e)}{e^2} \mu(ie)^2 \left\{ \frac{ie}{\varphi(ie)} + O_c \left(\sigma_{-1/2}(ie) \log^{-c} \left(\frac{2z}{ie} \right) \right) \right\}^2.$$

There are two error terms $\mathcal{E}_1, \mathcal{E}_2$ (say). The first is

$$\mathcal{E}_1 \ll_c \sum \frac{1}{e^2} \frac{ie}{\varphi(ie)} \cdot \sigma_{-1/2}(ie) \log^{-c} \left(\frac{2z}{ie} \right) = \Sigma_1 + \Sigma_2$$

where in $\Sigma_1, e < \sqrt{z/i}$ and in $\Sigma_2, \sqrt{z/i} \leq e < z/i$. We have

$$\begin{aligned} \Sigma_1 &\ll_c \sum \frac{i}{\varphi(i)} \sigma_{-1/2}(i) \frac{1}{(\log(2z/i))^c} \frac{\sigma_{-1/2}(e)}{e \varphi(e)} \\ &\ll_c \frac{\sigma_{-1/2}(i) i}{\varphi(i) (\log(2z/i))^c}. \end{aligned}$$

Also,

$$\begin{aligned} \Sigma_2 &\ll_c \sum \frac{i}{\varphi(i)} \sigma_{-1/2}(i) \cdot \frac{\sigma_{-1/2}(e)}{\varphi(e) e} \\ &\ll_c \sigma_{-1/2}(i) \frac{i}{\varphi(i)} \frac{\sqrt{i}}{\sqrt{z}} \\ &\ll_c \frac{\sigma_{-1/2}(i) i}{\varphi(i) (\log(2z/i))^c} \end{aligned}$$

for any $c > 0$. The second error term is

$$\begin{aligned} \mathcal{E}_2 &\ll_c \sum \frac{1}{e^2} \sigma_{-1/2}(ie)^2 \log^{-2c} \left(\frac{2z}{ie} \right) \\ &\ll_c \sigma_{-1/2}(i)^2 \sum \frac{1}{e^2} \sigma_{-1/2}(e)^2 \cdot \log^{-2c} \left(\frac{2z}{ie} \right) \\ &\ll_c \sigma_{-1/2}(i)^2 \left(\log \frac{2z}{i} \right)^{-2c} \end{aligned}$$

for any $c > 0$. The last estimate is obtained by proceeding as with \mathcal{E}_1 . This proves the lemma.

Now, notice that if $i \leq Y$, then

$$X_{1,1} - 2X_{1,2} + X_{2,2} = \sum_{Y/i < e \leq z/i} \frac{\mu(e)}{e^2} \mu(ie)^2 \left(\frac{ie}{\varphi(ie)} \right)^2 + \mathbf{O}_c \left(\frac{\sigma_{-1/2}(i) i}{\varphi(i) (\log(2Y/i))^c} \right) + \mathbf{O}_c \left(\frac{\sigma_{-1/2}(i)^2}{(\log(2Y/i))^c} \right).$$

The contribution of such terms to $N(\sigma)$ is

$$\ll \frac{1}{(\log Z/Y)^2} \sum_{1 \leq j_0 \leq Z/2\pi} j_0^{2\sigma-2} \sum_{2\pi j_0 \leq i \leq Y} i^{-2\sigma} \left(\frac{i^3}{\varphi(i)^2 Y} + \frac{\sigma_{-1/2}(i) i}{\varphi(i) (\log(2Y/i))^c} + \frac{\sigma_{-1/2}(i)^2}{(\log(2Y/i))^c} \right)$$

and this is

$$\begin{aligned} &\ll \frac{\log Z}{(\log Z/Y)^2} \quad \text{if } \sigma = 1/2 \\ &\ll \frac{1}{(\log Z/Y)^2 (2\sigma - 1)} \quad \text{if } \sigma > 1/2. \end{aligned}$$

On the other hand, if $i > Y$, then $X_{1,1} = X_{1,2} = 0$. Since

$$N(\sigma) = \frac{1}{\log^2(Z/Y)} \sum_{1 \leq j_0 \leq (1/2\pi)Z} j_0^{2\sigma-2} \sum_{2\pi j_0 \leq i \leq Z} i^{-2\sigma} (X_{1,1} - 2X_{1,2} + X_{2,2})$$

we deduce that

$$(9.7) \quad N(\sigma) = \frac{1}{\log^2(Z/Y)} \sum_{1 \leq j_0 \leq (1/2\pi)Z} j_0^{2\sigma-2} \sum_{\substack{2\pi j_0 \leq i \leq Z \\ i > Y}} i^{-2\sigma} \sum_{1 \leq e \leq Z/i} \frac{\mu(e)}{e^2} \mu(ie)^2 \left(\frac{ie}{\varphi(ie)} \right)^2 + \mathbf{O} \left(\frac{1}{\log^2(Z/Y)} \left\{ \frac{1}{1-2\sigma} \text{ or } \log Z \right\} \right) + \mathbf{O} \left(\frac{1}{\log^2(Z/Y)} \sum_{1 \leq j_0 \leq (1/2\pi)Z} j_0^{2\sigma-2} \sum i^{-2\sigma} \left\{ \frac{\sigma_{-1/2}(i)}{(\log(2Z/i))^c} + \frac{i}{\varphi(i)} \right\} \frac{\sigma_{-1/2}(i)}{(\log(2Z/i))^c} \right).$$

The first \mathbf{O} -term is

$$\ll \frac{1}{|1-2\sigma| (\log q)^2} \quad \text{if } \sigma > \frac{1}{2}$$

and

$$\ll \frac{1}{(\log q)} \quad \text{if } \sigma = \frac{1}{2}.$$

Let us simplify the second **O**-term. If $\sigma > 1/2$, we interchange the i and the j_0 sums and we find that this **O**-term is

$$\begin{aligned} &\ll \frac{1}{(\log q)^2} \sum i^{-2\sigma} \left(\log \frac{2Z}{i} \right)^{-c} \sum_{1 \leq j_0 \leq i} j_0^{2\sigma-2} \\ &\ll \frac{1}{(\log q)^c} \frac{1}{2\sigma-1}. \end{aligned}$$

If $\sigma = 1/2$, then the **O**-term is

$$\ll \frac{1}{(\log 2Z)^c}.$$

(The value of c is not the same at each occurrence.) Summarizing, we have proved that

$$(9.8) \quad N(\sigma) = \frac{1}{\log^2(Z/Y)} \sum_{1 \leq j_0 \leq (1/2\pi)Z} j_0^{2\sigma-2} \sum_{\substack{2\pi j_0 \leq i \leq z \\ i > Y}} i^{-2\sigma} \sum_{1 \leq e \leq Z/i} \frac{\mu(e)}{e^2} \mu(ie)^2 \left(\frac{ie}{\phi(ie)} \right)^2 + \begin{cases} \mathbf{O}\left(\frac{1}{\log q}\right) & \text{if } \sigma = \frac{1}{2} \\ \mathbf{O}\left(\frac{1}{(\log q)^2} \frac{1}{2\sigma-1}\right) & \text{if } \sigma > \frac{1}{2}. \end{cases}$$

Note that in the above sum, we may suppose that e and i are squarefree.

10. THE MAIN TERM: CONTINUED. — Let us define the constant

$$C_2 = \frac{2}{3\zeta(2)} \prod_{p>2} \left(1 + \frac{2}{(p-2)(p+1)} \right) \prod_{p>2} \left(1 - \frac{1}{(p-1)^2} \right) \approx .45.$$

The main result of this section is the following.

PROPOSITION (10.1). — *We have*

$$\sum_{1 \neq \chi \pmod{q}} |I(s, \chi)|^2 = c(s, q) \phi(q) + \mathcal{E}(\sigma).$$

Here,

$$c\left(\frac{1}{2} + it, q\right) = \frac{C_2}{\pi} \left| \Gamma\left(\frac{1}{2} - it\right) \right|^2 (\cosh \pi t)$$

and for $1 > \sigma > 1/2$,

$$c(\sigma + it, q) = \frac{2C_2}{\pi} \frac{1}{2\sigma-1} \left| \Gamma(1 - \sigma - it) \right|^2 (\cosh \pi t) \frac{q^{1-2\sigma}}{\log q}.$$

Also,

$$\mathcal{E}\left(\frac{1}{2}\right) \ll \frac{q(\log \log q)}{\log q}$$

and

$$\mathcal{E}(\sigma) \ll \frac{q^{2-2\sigma}}{\log q} \min\left(\frac{1}{(2\sigma-1)}, \log \log q\right)$$

if $1/2 < \sigma \leq 3/4$ (say), while

$$\mathcal{E}(\sigma) \ll q^{2-2\sigma} (\log q)^2$$

if $3/4 \leq \sigma < 1 - (1/\log q)$.

Proof. — We saw in (6.1), § 7, and § 8 that

$$\begin{aligned} (10.1) \quad & \sum_{1 \neq \chi \pmod{q}} |I(s, \chi)|^2 \\ &= \frac{1}{\pi} (2\pi)^{2\sigma-1} |\Gamma(1-s)|^2 N(\sigma) \varphi(q) q^{1-2\sigma} \left(\left| \sin\left(\frac{\pi s}{2}\right) \right|^2 + \left| \sin\left(\frac{\pi(s+1)}{2}\right) \right|^2 \right) \\ &+ \mathbf{O}\left(\frac{\varphi(q)}{q^{2\sigma-1} \log q} \min\left(\frac{1}{(2\sigma-1)}, \log \log q\right)\right) + \mathbf{O}\left(\frac{\varphi(q) |\Gamma(1-s)|}{q^{2\sigma-1} (\log q)}\right) \\ &+ \mathbf{O}\left(\frac{q^{1-\sigma}}{(1-\sigma) \log q}\right) + \mathbf{O}\left(\frac{q^{1-\sigma}}{(1-\sigma)^2}\right) + \mathbf{O}\left(\frac{\varphi(q)}{q^{2\sigma-1}} \frac{1}{(1-\sigma)^4 (\log q)^2}\right). \end{aligned}$$

We shall now study the main term of $N(\sigma)$. From (9.8), we see that it is

$$\begin{aligned} (10.2) \quad &= \frac{1}{\log^2(Z/Y)} \sum_{1 \leq j_0 \leq (1/2\pi)Z} j_0^{2\sigma-2} \sum_{\substack{2\pi j_0 \leq i \leq Z \\ i > Y}} \frac{i^{2-2\sigma} \mu(i)^2}{\varphi(i)^2} \sum_{\substack{e \leq Z/i \\ (e, i) = 1}} \frac{\mu(e)}{\varphi(e)^2} \\ &= \frac{4}{(\log q)^2} \sum_{\substack{2\pi \leq i \leq Z \\ i > Y}} \frac{i^{2-2\sigma} \mu(i)^2}{\varphi(i)^2} \left\{ \sum_{1 \leq j_0 \leq i/2\pi} j_0^{2\sigma-2} \right\} \left\{ \sum_{\substack{e \leq Z/i \\ (e, i) = 1}} \frac{\mu(e)}{\varphi(e)^2} \right\}. \end{aligned}$$

We note that

$$\sum_{1 \leq j_0 \leq i/2\pi} j_0^{2\sigma-2} = \begin{cases} \left(\frac{i}{2\pi}\right)^{2\sigma-1} \cdot \frac{1}{2\sigma-1} + \mathbf{O}(i^{2\sigma-2}) & \text{if } \sigma \neq 1/2 \\ \log\left(\frac{i}{2\pi}\right) + \mathbf{O}(1) & \text{if } \sigma = 1/2. \end{cases}$$

We easily check that the contribution of the \mathbf{O} -terms is

$$\ll \frac{1}{(\log q)}$$

which is negligible.

If we replace in (10.2) the sum over e with

$$\sum_{\substack{e=1 \\ (e, i)=1}}^{\infty} \frac{\mu(e)}{\varphi(e)^2},$$

we introduce an error of

$$\ll \frac{1}{(\log q)^2} \frac{1}{2\sigma-1} \quad \text{if } \sigma \neq \frac{1}{2}$$

and of

$$\ll \frac{1}{\log q} \quad \text{if } \sigma = \frac{1}{2}.$$

In any case, it is negligible.

Notice that

$$\sum_{\substack{e=1 \\ (e, i)=1}}^{\infty} \frac{\mu(e)}{\varphi(e)^2} = \begin{cases} \prod_{p \nmid i} \left(1 - \frac{1}{(p-1)^2}\right) & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

We see that for $\sigma > 1/2$,

$$\begin{aligned} N(\sigma) = & \frac{1}{(2\pi)^{2\sigma-1}} \frac{1}{2\sigma-1} \frac{4}{(\log q)^2} \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \sum_{i \text{ even}} \frac{\mu(i)^2 i}{\varphi(i)^2} \prod_{p|(i/2)} \left(1 - \frac{1}{(p-1)^2}\right)^{-1} \\ & + \mathbf{O}\left(\frac{1}{\log q}\right) + \mathbf{O}\left(\frac{1}{(\log q)^2} \frac{1}{2\sigma-1}\right). \end{aligned}$$

On the other hand, for the case $\sigma = 1/2$ we have

$$(10.3) \quad N\left(\frac{1}{2}\right) = \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \frac{4}{(\log q)^2} \sum_{i \text{ even}} \frac{i \mu^2(i)}{\varphi(i)^2} \log\left(\frac{i}{2\pi}\right) \prod_{p|(i/2)} \left(1 - \frac{1}{(p-1)^2}\right)^{-1} + \mathbf{O}\left(\frac{1}{\log q}\right).$$

Here, the sum over i has range

$$\max(Y, 2\pi) \leq i \leq Z, \quad i \text{ even.}$$

Also, note that as i may be assumed squarefree, $i/2$ is odd. We observe that

$$\frac{i\mu^2(i)}{\varphi(i)^2} \cdot \prod_{p| (i/2)} \left(1 - \frac{1}{(p-1)^2}\right)^{-1} = 4 \frac{\mu^2(i)}{i} \cdot \prod_{p| (i/2)} \frac{p^2}{(p-1)^2} \cdot \frac{(p-1)^2}{((p-1)^2-1)} = 2 \frac{\mu^2(i)}{\varphi_1(i/2)},$$

where $\varphi_1(n)$ is defined by

$$\varphi(n) = \sum_{d|n} \varphi_1(n).$$

Thus, the sum over i is

$$2 \sum \frac{\mu^2(i)}{\varphi_1(i/2)} \log\left(\frac{i}{2\pi}\right) \quad \text{if } \sigma = \frac{1}{2}$$

$$2 \sum \frac{\mu^2(i)}{\varphi_1(i/2)} \quad \text{if } \sigma > \frac{1}{2}.$$

This sum can be estimated as follows. We first observe that for $\text{Re}(s) > 1$

$$\sum_{\substack{i=1 \\ i \text{ odd}}}^{\infty} \frac{\mu^2(i)i}{\varphi_1(i)i^s} = \prod_{p>2} \left(1 + \frac{p}{(p-2)p^s}\right).$$

Now,

$$\left(1 + \frac{p}{(p-2)p^s}\right) = \left(1 - \frac{1}{p^{2s}}\right) \left(1 - \frac{1}{p^s}\right)^{-1} \left(1 + \frac{2}{(p-2)(p^s+1)}\right).$$

Thus,

$$\sum_{\substack{i \leq x \\ i \text{ odd}}} \frac{\mu^2(i)i}{\varphi_1(i)} = C_1 x + \mathbf{O}(x^{3/4})$$

where

$$C_1 = \frac{2}{3\zeta(2)} \prod_{p>2} \left(1 + \frac{2}{(p-2)(p+1)}\right).$$

By partial summation, it follows that

$$2 \sum_{\substack{i=1 \\ i \text{ odd}}}^{Z/2} \frac{\mu^2(i) \log i}{\varphi_1(i)} = 2 \int_{1^-}^{Z/2} \left(\frac{\log u}{u}\right) d\left(\sum_{\substack{n \leq u, \\ n \text{ odd}}} \frac{\mu^2(n)n}{\varphi_1(n)}\right)$$

$$= C_1 (\log Z)^2 + \mathbf{O}(\log Z).$$

Similarly,

$$2 \sum_{\substack{i \leq Z/2 \\ i \text{ odd}}} \frac{\mu^2(i)}{\Phi_1(i)} = 2 C_1 \log(Z/2) + \mathbf{O}(1).$$

Substituting this information into (10.3), we find that

$$N\left(\frac{1}{2}\right) = C_2 + \mathbf{O}\left(\frac{1}{\log q}\right)$$

where

$$C_2 = C_1 \cdot \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \approx .45.$$

Similarly, if $\sigma > 1/2$,

$$N(\sigma) = 4 C_2 \cdot \frac{(2\pi)^{1-2\sigma}}{2\sigma-1} \cdot \frac{1}{(\log q)^2} (\log q + \mathbf{O}(1)) + \mathbf{O}\left(\frac{1}{(\log q)^2 (2\sigma-1)}\right).$$

Inserting this into (10.1), and observing that

$$|\sin(x+iy)|^2 + |\cos(x+iy)|^2 = \cosh 2y$$

we deduce that

$$\sum_{1 \neq \chi \pmod{q}} |I(s, \chi)|^2 = c(s, q) \varphi(q) + \mathcal{E}$$

where

$$c(\sigma + it, q) = \begin{cases} \frac{C_2}{\pi} \left| \Gamma\left(\frac{1}{2} - it\right) \right|^2 (\cosh \pi t) & \text{if } \sigma = \frac{1}{2} \\ \frac{4C_2}{\pi} \frac{1}{2\sigma-1} |\Gamma(1-\sigma-it)|^2 (\cosh \pi t) \frac{q^{1-2\sigma}}{\log q} & \text{if } \sigma > \frac{1}{2} \end{cases}$$

and

$$\begin{aligned} \mathcal{E} = \mathbf{O}\left(\frac{q^{2-2\sigma}}{\log q} \min\left(\frac{1}{(2\sigma-1)}, \log \log q\right)\right) + \mathbf{O}\left(\frac{q^{2-2\sigma}}{(1-\sigma)(\log q)}\right) + \mathbf{O}\left(\frac{q^{2-2\sigma}}{(\log q)^2 (1-\sigma)^4}\right) \\ + \mathbf{O}\left(\frac{q^{1-\sigma}}{(1-\sigma)^2}\right) + \mathcal{E}_1(\sigma) \end{aligned}$$

and

$$\mathcal{E}_1\left(\frac{1}{2}\right) \ll \frac{q}{\log q}$$

and for $1/2 < \sigma < 1$,

$$\mathcal{E}_1(\sigma) \ll \frac{q^{2-2\sigma}}{(1-\sigma)^2(2\sigma-1)(\log q)^2}.$$

In particular, we see that if $\sigma = 1/2$

$$\mathcal{E} \ll \frac{q(\log \log q)}{\log q}.$$

If $1/2 < \sigma \leq 3/4$ (say),

$$\mathcal{E} \ll \frac{q^{2-2\sigma}}{\log q} \min\left(\frac{1}{(2\sigma-1)}, \log \log q\right)$$

and if $3/4 \leq \sigma < 1 - (1/\log q)$

$$\mathcal{E} \ll q^{2-2\sigma}(\log q)^2.$$

This proves the result.

11. NON-VANISHING AT A FIXED POINT. — The main result of this section is the following.

THEOREM (11.1). — *Fix a σ in the interval $1/2 \leq \sigma < 1$. Then, for all sufficiently large primes q ,*

$$L(\sigma, \chi) \neq 0$$

for a positive proportion of the characters $\chi \pmod{q}$.

Remark. — The proof will produce a lower bound for this proportion. Notice that how large q must be taken will depend on σ .

Proof. — Let us fix $s_0 \in \mathbb{C}$ with $1/2 \leq \operatorname{Re} s_0 < 1 - (1/\log q)$. We return now to (3.2). For $\chi \neq 1$, we have

$$S(s_0, \chi) = L(s_0, \chi)M(s_0, \chi) + I(s_0, \chi) + J(s_0, \chi).$$

Thus,

$$\sum_{\chi \neq 1} S(s_0, \chi) = \sum' (I(s_0, \chi) + J(s_0, \chi)) + \sum'' S(s_0, \chi)$$

where \sum' ranges over $\chi \neq 1$ such that $L(s_0, \chi) = 0$ and \sum'' over the remaining non-trivial $\chi \pmod{q}$. By Proposition (4.1), we have

$$\sum_{\chi \neq 1} S(s_0, \chi) = \varphi(q) + \mathbf{O}_\varepsilon(q^{1-\sigma+\varepsilon}).$$

Thus, we have

$$\sum'' S(s_0, \chi) = \varphi(q) - \sum' (I(s_0, \chi) + J(s_0, \chi)) + \mathbf{O}_\varepsilon(q^{1-\sigma_0+\varepsilon})$$

and consequently,

$$\sum'' S(s_0, \chi) \geq \varphi(q) - |\sum' (I(s_0, \chi) + J(s_0, \chi))| + \mathbf{O}_\varepsilon(q^{1-\sigma+\varepsilon}).$$

Now, if we assume that $|\operatorname{Im} s_0| < 1$ (say), then

$$\begin{aligned} |\sum' (I(s_0, \chi) + J(s_0, \chi))| &\leq \sum |I(s_0, \chi)| + \sum |J(s_0, \chi)| \\ &\leq \varphi(q)^{1/2} (\sum |I(s_0, \chi)|^2)^{1/2} + \mathbf{O}\left(\frac{q^{(3/2)-\sigma}}{\log q}\right) \end{aligned}$$

by Proposition (3.1). Now using the main result of § 10, namely

$$\sum |I(s_0, \chi)|^2 = c(s_0, q) \cdot \varphi(q) + \mathcal{E}(\sigma)$$

we have

$$|\sum' (I(s_0, \chi) + J(s_0, \chi))| \leq \sqrt{c(s_0, q)} \varphi(q) + \mathbf{O}(\sqrt{\varphi(q) \mathcal{E}(\sigma)}) + \mathbf{O}(q^{3/2-\sigma} (\log q)^{-1}).$$

Thus,

$$|\sum'' S(s, \chi)| \geq (1 - \sqrt{c(s_0, q)}) \varphi(q) + \mathbf{O}(q^{1-\sigma+\varepsilon}) + \mathbf{O}(\sqrt{\varphi(q) \mathcal{E}(\sigma)}) + \mathbf{O}(q^{3/2-\sigma} (\log q)^{-1}).$$

On the other hand, by the Cauchy-Schwarz inequality, setting $\mathcal{N}(s_0, q)$ to be the number of $\chi \pmod{q}$ with $L(s_0, \chi) \neq 0$, we get

$$|\sum'' S(s_0, \chi)|^2 \leq \mathcal{N}(s_0, q) (\sum |S(s_0, \chi)|^2).$$

From now on, we shall assume that $t = \operatorname{Im}(s_0)$ satisfies $|t| \ll (\log q)^{-1}$. Suppose first that $\sigma_0 = 1/2$. We have from Proposition (4.2)

$$\sum_{\chi \pmod{q}} \left| S\left(\frac{1}{2} + it, \chi\right) \right|^2 = \frac{5}{2} \varphi(q) + \mathbf{O}(q (\log q)^{-1/2}).$$

We deduce that

$$\frac{2}{5} \varphi(q) (1 - \sqrt{c(s_0, q)})^2 + \mathbf{O}\left(q \sqrt{\frac{\log \log q}{\log q}}\right) \leq \mathcal{N}(s_0, q) (1 + \mathbf{O}((\log q)^{-1/2})).$$

Thus,

$$\mathcal{N}(s_0, q) \geq \frac{2}{5} \varphi(q) (1 - \sqrt{C_2})^2 + \mathbf{O}\left(q \sqrt{\frac{\log \log q}{\log q}}\right).$$

Now let us set

$$j = \left[\left(\sigma_0 - \frac{1}{2} \right) \log q \right] + 1.$$

Thus,

$$\frac{1}{2} + \frac{j-1}{\log q} < \sigma_0 \leq \frac{1}{2} + \frac{j}{\log q}.$$

We will suppose that q is sufficiently large that $j \geq 2$. Then Proposition (4.2) gives

$$(11.1) \quad \sum_{\chi \pmod{q}} |\mathbf{S}(\sigma_0 + it_0, \chi)|^2 = \varphi(q) \left\{ 1 - \frac{e^{-j+1}}{(j-1)^2} + \frac{Y^{1-2\sigma_0}}{(j-1)^2} - \frac{e^{-2(j-1)}}{j-1} \right. \\ \left. + \mathbf{O}\left(\frac{e^{-2j}}{\log q} \left(1 + \frac{1}{j} + \frac{1}{(\log q)^{\sigma_0}(1-\sigma_0)}\right)\right) \right\} \\ = \varphi(q) \left\{ 1 - \frac{e^{-j+1}}{(j-1)^2} + \frac{Y^{1-2\sigma_0}}{(j-1)^2} - \frac{e^{-2(j-1)}}{j-1} + \mathbf{o}(e^{-j}) \right\}.$$

Also, if σ_0 is bounded away from 1 (say $\sigma_0 \leq 3/4$) then

$$\sum_{\chi \neq 1} |\mathbf{I}(s_0, \chi)|^2 = c(s_0, q) \varphi(q) + \mathbf{O}\left(qe^{-2j} \min\left(\frac{1}{j-1}, \frac{\log \log q}{\log q}\right)\right).$$

If $3/4 \leq \sigma_0 < 1 - (\log q)^{-1}$, Then

$$\sum_{\chi \neq 1} |\mathbf{I}(s_0, \chi)|^2 = c(s_0, q) \varphi(q) + \mathbf{O}(qe^{-2j}(\log q)^2).$$

We see that under our assumption on $|t|$, we have

$$c(s_0, q) = \frac{2C_2}{\pi j} |\Gamma(1-\sigma_0)|^2 e^{-2j} + \mathbf{O}\left(\frac{1}{\log q}\right).$$

Putting all these estimates together, we deduce that

$$\mathcal{N}(s_0, q) \geq (\alpha_j + \mathbf{o}(1)) \varphi(q)$$

Where

$$\alpha_j = \frac{(1 - e^{-j} \sqrt{2C_2/\pi j} |\Gamma(1-\sigma_0)|)^2}{(1 - (j-1)^{-2} (e^{-j+1} - 1) - (j-1)^{-1} e^{-2j+2})}.$$

12. NON-VANISHING AT A VARIABLE POINT. — In the previous section, we showed that a positive proportion of the $L(s, \chi)$ are non-zero at a given real value σ of s in the critical strip. Now we shall refine this to a statement uniform on a line. Up to this point, we have made no significant use of the parameter Y . We shall now choose it to be $Y = q^{1/4}$.

THEOREM (12.1). — *Suppose that q is a sufficiently large prime. For a positive proportion of the $\chi \pmod{q}$, $L(s, \chi)$ does not have a real zero in the region $1/2 + c/\log q \leq \sigma < 1$. Here, $c > 0$ is an absolute constant.*

Proof. By the functional equation, it suffices to concentrate attention on the region $\sigma \geq 1/2$. It is well known that there is at most one χ with a real zero in the range

$$1 - (\log q)^{-1} \leq \sigma \leq 1.$$

Thus we consider

$$\frac{1}{2} + \frac{2}{\log q} \leq \sigma < 1 - (\log q)^{-1}$$

and split it into intervals

$$I_j: \frac{1}{2} + \frac{j}{\log q} < \sigma \leq \frac{1}{2} + \frac{j+1}{\log q},$$

of length $1/\log q$. Here $2 \leq j \leq (1/2)\log q - 2$. We count the number $Z(j, q)$ of $\chi \pmod{q}$ for which $L(s, \chi)$ has a zero in I_j . For each χ , let $\sigma(\chi)$ denote a point in I_j , and let $\sigma = \sigma_j = (1/2) + (j/\log q)$. Let $C = C_j$ denote the circle of radius $r = r_j = 2/\log q$ about σ . We have by Cauchy's theorem

$$\begin{aligned} \sum_{n>Y} \frac{a(n)\chi(n)}{n^{\sigma(\chi)}} e^{-n/q} - \sum_{n>Y} \frac{a(n)\chi(n)}{n^{\sigma}} e^{-n/q} \\ = \frac{1}{2\pi i} \int_C \left\{ \sum_{n>Y} \frac{a(n)\chi(n)}{n^w} e^{-n/q} \right\} \left(\frac{1}{w - \sigma(\chi)} - \frac{1}{w - \sigma} \right) dw. \end{aligned}$$

Let us denote the left hand side by $S_{\text{diff}}(\sigma, \sigma_\chi, \chi)$ and let us write $w = u + iv$. By (a variant of) Proposition (4.2).

$$\begin{aligned} (12.1) \quad \sum_{\chi} \left| \sum_{n>Y} a(n)\chi(n) n^{-w} \exp\left(-\frac{n}{q}\right) \right|^2 \\ = \varphi(q) \sum_{Y \leq n \leq q} \frac{a(n)^2}{n^{2u}} + \mathbf{O}\left(\frac{q^{(3/2)-u}}{(1-2u)(\log q)^2}\right) \\ + \mathbf{O}(q^{1-u}(\log q)^{3/2}) + \mathbf{O}\left(q^{2-2u}(\log q)^{3u-2} \frac{1}{1-u}\right) \end{aligned}$$

for $1 > \sigma > (1/2) - (1/\log q)$. Now

$$\begin{aligned} \sum_{\chi} |S_{\text{diff}}(\sigma, \sigma_\chi, \chi)|^2 \leq \sum_{\chi} \frac{1}{4\pi^2} \left(\int_C \left| \sum_{n>Y} \frac{a(n)\chi(n)}{n^w} e^{-n/q} \right|^2 |dw| \right) \\ \times \left(\int_C \left| \frac{1}{w - \sigma(\chi)} - \frac{1}{w - \sigma} \right|^2 |dw| \right) \end{aligned}$$

and

$$\begin{aligned} \int_C \left| \frac{1}{w - \sigma(\chi)} - \frac{1}{w - \sigma} \right|^2 |dw| &= \int_C \left| \frac{\sigma(\chi) - \sigma}{(w - \sigma(\chi))(w - \sigma)} \right|^2 |dw| \\ &\leq \frac{(r/2)^2}{r^2 (r/2)^2} \cdot 2\pi r \\ &= \frac{2\pi}{r}. \end{aligned}$$

Therefore, for $j \geq 1$, we have by (12.1) (see also (11.1)) that

$$\begin{aligned} \sum_x |S_{\text{diff}}(\sigma, \sigma_x, \chi)|^2 &\leq \frac{1}{2\pi r} \int_C \sum_x \left| \sum_{n>Y} \frac{a(n)\chi(n)}{n^w} e^{-n/q} \right|^2 |dw| \\ &\leq \Phi(q) \left(\sum_{Y \leq n \leq q} \frac{a(n)^2}{n^{2((1/2)+((j-2)/\log q))}} + o(e^{-j}) \right). \end{aligned}$$

We observe that

$$\begin{aligned} \sum_{Y \leq n \leq q} \frac{a(n)^2}{n^{2((1/2)+((j-2)/\log q))}} \\ \sim \sum_{Y \leq n \leq Z} \frac{\log n/Y}{(\log Z/Y)^2} \frac{1}{n^{1+((2j-4)/\log q)}} + \sum_{Z \leq n \leq q} \frac{1}{(\log Z/Y)} \frac{1}{n^{1+((2j-4)/\log q)}} \end{aligned}$$

and this is seen to be

$$\sim \frac{4}{(j-2)^2} \{ Y^{-(2j-4)/\log q} - Z^{-(2j-4)/\log q} \} - \frac{2}{j-2} q^{-(2j-4)/\log q}$$

and this is

$$= \frac{4}{(j-2)^2} (e^{-1/2(j-2)} - e^{-(j-2)}) - \frac{2}{j-2} e^{-(2j-4)}.$$

(Here, we have used the fact that $Y = q^{1/4}$.) Let us denote the above expression by $f(j-2)$. If $j=2$ we have to replace the above by

$$\frac{5}{2} + O((\log q)^{-1}).$$

Then,

$$(12.2) \quad \sum_x \left| \sum_{n>Y} \frac{a(n)\chi(n)}{n^{\sigma(x)}} e^{-n/q} \right|^2 \\ \leq 2 \left\{ \sum_x \left| \sum_{n>Y} \frac{a(n)\chi(n)}{n^{\sigma}} e^{-n/q} \right|^2 + \sum_x |S_{\text{diff}}(\sigma, \sigma_x, \chi)|^2 \right\} \\ \leq 8 \varphi(q) (f(j) + f(j-2) + o(e^{-j})).$$

It is convenient to introduce here the notation

$$S^*(s, \chi) = \sum_{n>Y} \frac{a(n)\chi(n)}{n^s} e^{-n/q}.$$

Clearly, it is equal to $S(x, \chi) - 1$.

Similarly, we have

$$\sum_x |I(\sigma_x, \chi)|^2 \leq 2 \left\{ \sum_x |I(\sigma, \chi)|^2 + \sum_x |I_{\text{diff}}(\sigma, \sigma_x, \chi)|^2 \right\}$$

where now,

$$I_{\text{diff}}(\sigma, \sigma_x, \chi) = \frac{1}{2\pi i} \int_C I(w, \chi) \left(\frac{1}{w - \sigma(x)} - \frac{1}{w - \sigma} \right) dw.$$

As before, if $j \leq \log q / \log \log q$, then by Proposition (10.1), we see that

$$\sum_x |I_{\text{diff}}(\sigma, \sigma_x, \chi)|^2 \leq \frac{1}{2\pi r} \int_C \sum_x |I(w, \chi)|^2 |dw| \\ \leq \frac{1}{2\pi r} \varphi(q) \left(c(s_0, q) + O(qe^{-2j} \min\left(\frac{1}{j}, \frac{\log \log q}{\log q}\right)) \right) 2\pi r \\ = \varphi(q) \left(\frac{C_2}{\pi j} |\Gamma(1 - \sigma_0)|^2 e^{-2j} + O\left(e^{-2j} \frac{\log \log q}{\log q}\right) \right).$$

If $j \geq \log q / \log \log q$, the last estimate above is replaced by

$$\leq \varphi(q) \left(\frac{C_2}{\pi j} |\Gamma(1 - \sigma_0)|^2 e^{-2j} + O(e^{-2j} (\log q)^2) \right).$$

The same estimate holds for

$$\sum_x |I(\sigma, \chi)|^2.$$

Hence we deduce that

$$\frac{1}{\varphi(q)} \sum_x |I(\sigma_x, \chi)|^2 \leq \frac{4C_2}{\pi j} |\Gamma(1 - \sigma_0)|^2 e^{-2j} + \begin{cases} O(e^{-2j} (\log \log q / \log q)) & \text{if } j \leq 3/4 \log q, \\ O(e^{-2j} (\log q)^2) & \text{otherwise.} \end{cases}$$

Finally, a calculation similar to the one above and in Proposition (3.1) shows that

$$\sum_{\chi} |J(\sigma_{\chi}, \chi)|^2 \ll q^{2-2\sigma_0}.$$

With that established, we return to our basic equation

$$S(\sigma_{\chi}, \chi) = L(\sigma_{\chi}, \chi) M(\sigma_{\chi}, \chi) + I(\sigma_{\chi}, \chi) + J(\sigma_{\chi}, \chi)$$

and deduce that

$$(12.3) \quad \sum_{\chi} |L(\sigma_{\chi}, \chi) M(\sigma_{\chi}, \chi) - 1|^2 \leq 3 \sum_{\chi} (|S^*(\sigma_{\chi}, \chi)|^2 + |I(\sigma_{\chi}, \chi)|^2 + |J(\sigma_{\chi}, \chi)|^2).$$

Using the estimates established above, we see that the right hand side is

$$(12.4) \quad \leq 3 \varphi(q) \left(8(f(j) + f(j-2)) + \frac{4C_2}{\pi j} |\Gamma(1 - \sigma_0)|^2 e^{-2j} + \mathbf{o}(e^{-j}) \right).$$

Now let us set $Z(j, q)$ to be the number of characters $\chi \pmod{q}$ such that $L(s, \chi)$ has a real zero in the circle C_j . It follows immediately from (12.3) and (12.4) that

$$Z(j, q) \leq 3 \varphi(q) \left(8(f(j) + f(j-2)) + \frac{4C_2}{\pi j} \left| \Gamma\left(\frac{1}{2} - \frac{j+(1/2)}{\log q}\right) \right|^2 e^{-2j} + \mathbf{o}(e^{-j}) \right).$$

If we sum this over $j \geq j_0$, for some absolute constant j_0 we see that we have

$$\frac{1}{\varphi(q)} \sum_{j \geq j_0} Z(j, q) \leq 3 \sum_{j \geq j_0} \left(\frac{32}{(j-2)^2} (e^{-(1/2)(j-2)} - e^{-(j-2)}) + \frac{32}{j^2} (e^{-(1/2)j} - e^{-j}) + \mathbf{o}(e^{-j}) \right).$$

If we choose j_0 sufficiently large, we see that the right hand side is < 1 . This completes the proof.

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(Manuscript received November 6, 1990,
revised November 19, 1991).

R. BALASUBRAMANIAN,
Institute of Mathematical Sciences,
Madras 600-113, India;

V. KUMAR MURTY,
Department of Mathematics,
University of Toronto,
Toronto, Ontario, M 5S 1A1,
Canada.