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THE INDECOMPOSABLE K_3 OF FIELDS (*)

BY MARC LEVINE

Introduction

In this paper, we extend the theorem of Merkurjev-Suslin (Hilbert's Theorem 90 for K_2) to the relative K_2 of semi-local principal ideal rings (PIR) containing a field. Most of the results Suslin proves for K_2 of fields in [S] then carry over to the relative K_2 of a semi-local PIR, e. g. computation of the torsion subgroup, and the isomorphism $K_2(F)/n \to H^2_{et}(F, \mu_n^{\otimes 2})$. Applying this to the semi-local ring of $\{0, 1\}$ in $\mathbb{A}^1_E$, for a field E, gives a computation of the torsion and co-torsion in $K_3(E)^{ind} := K_3(E)/K_3(E)^{dec}$, where $K_3(E)^{dec}$ is the subgroup of $K_3(E)$ generated by products from $K_1(E)$. Specifically we show

1. The l-primary torsion subgroup of $K_3(E)^{ind}$ is $H^0(E, \mathbb{Q}_l/\mathbb{Z}_l(2))$ for $(l, \mathrm{char}(E)) = 1$; $K_3(E)^{ind}$ has no p-torsion if $\mathrm{char}(E) = p > 0$.

2. $K_3(E; \mathbb{Z}/n)^{ind} \xrightarrow{\sim} H^1(E, \mu_n^{\otimes 2})$ for $(n, \mathrm{char}(E)) = 1$, so $\varprojlim K_3(E)^{ind}/l^n \xrightarrow{\sim} H^1(E, \mathbb{Z}_l(2))$ for $l \neq \mathrm{char}(E)$.

3. $K_3(E)^{ind}$ satisfies Galois descent for extensions of degree prime to $\mathrm{char}(E)$.

4. Bloch's group $B(E)$ is uniquely l-divisible if E contains an algebraically closed field, and $l \neq \mathrm{char}(E)$.

Let F be a number field, l an odd prime number, S the set of places of F lying over l, and $\mathcal{O}_S$ the ring of S-integers in F. Quillen (*see* [Li2]) has conjectured

(Q) There are isomorphisms

$$c_{q,2}: K_{2q-2}(\mathcal{O}_S) \otimes \mathbb{Z}_l \to H^2(\mathrm{Spec}\,\mathcal{O}_S, \mathbb{Z}_l(q))$$
$$c_{q,1}: K_{2q-1}(\mathcal{O}_S) \otimes \mathbb{Z}_l \to H^1(\mathrm{Spec}\,\mathcal{O}_S, \mathbb{Z}_l(q)).$$

(*) Partially supported by the N.S.F.

Borel [Borel] has computed the ranks of the K-groups $K_*(\mathcal{O}_F)$ as

$$K_{2q}(\mathcal{O}_F) \otimes \mathbb{Q} = 0; \qquad \text{for} \quad q \geqq 1$$
$$K_{2q-1}(\mathcal{O}_F) \otimes \mathbb{Q} = \mathbb{Q}^{r_2}; \qquad \text{for} \quad q = 2n, \quad n \geqq 1$$
$$K_{2q-1}(\mathcal{O}_F) \otimes \mathbb{Q} = \mathbb{Q}^{r_1+r_2}; \qquad \text{for} \quad q = 2n+1, \quad n \geqq 1.$$

Quillen [Q3] has shown that the groups $K_*(\mathcal{O}_F)$ are finitely generated. Soulé [So] has constructed Chern classes

$$c_{q,2}: \quad K_{2q-2}(\mathcal{O}_{S'}, \mathbb{Z}/l^\nu) \to H^2(\operatorname{Spec} \mathcal{O}_{S'}, (\mu_{l_\nu})^{\otimes q})$$
$$c_{q,1}: \quad K_{2q-1}(\mathcal{O}_{S'}, \mathbb{Z}/l^\nu) \to H^1(\operatorname{Spec} \mathcal{O}_{S'}, (\mu_{l^\nu})^{\otimes q}),$$

for any set of places S′ containing S, and has verified the surjectivity part of the conjecture (Q), at least for $l > q$, as well as the injectivity modulo torsion.

Soulé has also shown that $K_{2q-1}(F) = K_{2q-1}(\mathcal{O}_F)$ for $q \geqq 2$; it is easily seen that the natural map $H^1(\operatorname{Spec} \mathcal{O}_S, \mathbb{Z}_l(q)) \to H^1(F, \mathbb{Z}_l(q))$ is an isomorphism for $q \geqq 2$. Bass and Tate ([B−T]) have computed the Milnor K-groups of number fields; they show in particular that $K_3^M(E)$ is $(\mathbb{Z}/2)^{r_1}$. This, together with (1) and (2), proves Quillen's conjecture for K_3. In fact, for all prime l, (1) and (2) imply that the Chern class

$$c_{2,1}: \quad K_3(F)^{\text{ind}} \otimes \mathbb{Z}_l \to H^1_{\text{et}}(F, \mathbb{Z}_l(2))$$

is an isomorphism.

For the case $F = \mathbb{Q}$ this gives a new proof of the result of Lee and Szczarba [L-S] that $K_3(\mathbb{Z}) = \mathbb{Z}/48$. Indeed, it follows from our results that $K_3(\mathbb{Q})^{\text{ind}} = \mathbb{Z}/24$; to complete the computation one need only show that the symbol $\{-1, -1, -1\}$ of $K_3(\mathbb{R})$ is non-zero and divisible by 2 in $K_3(\mathbb{Z})$. This is done, for example, in [Igusa]. More generally, this gives the complete determination of $K_3(\mathcal{O}_F)$, F a number field, as

5. $K_3(\mathcal{O}_F) = K_3(\mathcal{O}_F)_{\text{tor}} \oplus \mathbb{Z}^{r_2}$;

$$K_3(\mathcal{O}_F)_{\text{tor}} = \begin{cases} (\mathbb{Z}/2)^{r_1-1} \oplus \mathbb{Z}/2\,w_2(F); & \text{if } r_1 > 0 \\ \mathbb{Z}/2\,w_2(F); & \text{if } r_1 = 0 \end{cases}$$

where $w_q(F)$ denotes the order of the group $H^0_{\text{et}}(F, \mathbb{Q}/\mathbb{Z}(q))$.

Lichtenbaum [Li2] has conjectured that, for F a totally real number field, and q a positive even number,

$$\text{(L1)} \qquad \zeta_F(1-q) = \#(K_{2q-2}(\mathcal{O}_F))/\#(K_{2q-1}(\mathcal{O}_F)),$$

at least up to powers of 2. This follows from the conjecture (Q), and the conjecture of Lichtenbaum [Li2]:

(Li2) Let F be a totally real number field, l an odd prime, q an even positive number. Then

(i) the groups $H^1(\operatorname{Spec} \mathcal{O}_S, j_* \mathbb{Q}_l/\mathbb{Z}_l(q))$ and $H^0(\operatorname{Spec} \mathcal{O}_S, j_* \mathbb{Q}_l/\mathbb{Z}_l(q))$ are finite

(ii) the groups $H^k(\operatorname{Spec}\mathcal{O}_S, j_*\mathbb{Q}_l/\mathbb{Z}_l(q))$ are zero for $k \geqq 2$.

(iii) $|\zeta_F(1-q)|_l = \#(H^1(\operatorname{Spec}\mathcal{O}_S, j_*\mathbb{Q}_l/\mathbb{Z}_l(q))/\#(H^0(\operatorname{Spec}\mathcal{O}_S, j*\mathbb{Q}_l/\mathbb{Z}_l(q))$.

Here $|-|_l$ denotes the l-primary part of a rational number, and $j\colon \operatorname{Spec} F \to \operatorname{Spec}\mathcal{O}_S$ is the inclusion.

Here is a brief history of this conjecture and its proof:

Birch and Tate ([B], [T2]) conjectured that, for all totally real fields F,

$$\text{(BT)} \qquad \#(K_2(\mathcal{O}_F)) = w_2(F)\,\zeta_F(-1).$$

Tate's computation of $K_2(\mathcal{O}_F)$ [T] shows this is equivalent to (Li2) (iii) for $q=2$. Coates and Lichtenbaum ([Li] and [C-L]) then showed conjecture (Li2) follows from the Main Conjecture in Iwasawa theory relating the p-adic interpolation of classical L-functions with Iwasawa's p-adic L-functions constructed from Galois representations arising from the cyclotomic $\mathbb{Z}_p$ extension of F. They also verified the Main Conjecture in some cases. Mazur and Wiles ([M-W]) proved the Main Conjecture (for odd primes) for abelian number fields. Recent work of Wiles has extended this to all totally real fields, completing the proof of (Li2).

Our formula (1) shows that $w_2(F) = \#(K_3(F)^{\text{ind}})_{\text{tor}}$, which proves (Li1) for $q=2$. We can also write this as

$$(6) \qquad \zeta_F(-1) = 2^{?}\#(K_2(\mathcal{O}_F))/\#(K_3(F)^{\text{ind}}).$$

The work of Serre [Se] shows that the exponent ? is non-negative; ? has been shown by Hurrelbrink and Kolster [H-K] to be 0 for the fields

(i) $\mathbb{Q}(\sqrt{d})$, $d=2$, p, or $2p$ with p prime, $p \equiv \pm 3 \bmod 8$

(ii) $\mathbb{Q}(\sqrt{d})$, $d=pq$, with p and q distinct primes p, $q \equiv 3 \bmod 8$, or $d=p$ with p prime, $p=u^2-2w^2$, $u>0$, $u \equiv 3 \bmod 4$, $w \equiv 0 \bmod 4$

(iii) $\mathbb{Q}(\zeta_{2^m})^+$

(iv) $\mathbb{Q}(\zeta_p)^+$, if p and $q=(p-1)/2$ are prime, and 2 is a primitive root mod q.

The conjectures of Lichtenbaum and Quillen were made "up to powers of 2". From (1) we see that the "correct" group for $q=2$ having a good relation with Galois cohomology, including the prime 2, is $K_3(E)^{\text{ind}}$. Let gr_γ^* denote the associated graded with respect to the gamma filtration. As $K_2(\mathcal{O}_E)$ agrees with the $\mathrm{gr}_\gamma^2 K_2(\mathcal{O}_E)$ and $K_3(E)^{\text{ind}}$ agrees with $\mathrm{gr}_\gamma^2 K_3(\mathcal{O}_E)$, at least up to 2-torsion, our results suggest that (Li1) should perhaps be weakened as follows: for F totally real, the value $\zeta_F(1-q)$ is given by the formula

$$(\bigstar\bigstar) \qquad \zeta_F(1-q) = a_q \cdot \prod_{n=0}^{2q-1} \#(\mathrm{gr}_\gamma^q K_n(\mathcal{O}_F))^{(-1)^n}$$

where a_q is a rational number involving only primes less than $2q-1$. More optimistically, Lichtenbaum [Li3] and Beilinson [Be] conjecture the existence of a "bigraded arithmetic cohomology theory over $\mathbb{Z}$", $H^p_{\mathcal{A}}(-, \mathbb{Z}(q))$, which computes $\mathrm{gr}_\gamma^q K_{2q-p}$, up to primes less than $2q-p$, and which has a precise relationship with Galois cohomology. This cohomology theory arises as the hypercohomology of a complex of sheaves $\Gamma(q)$ (for the etale topology in Lichtenbaum's theory, for the Zariski topology in Beilinson's). The value $\zeta_F(1-q)$ should then given as the Euler characteristic

$$\zeta_F(1-q)=\Pi^{\#}(H^p_{\mathscr{A}}(F, \mathbb{Z}(q)))^{(-1)^p}.$$

This is the motivation for the formula (★★).

Lichtenbaum [Li4] has constructed the weight two arithmetic complex $\Gamma(2)$ for fields, which gives

$$H^2_{\mathscr{A}}(F, \mathbb{Z}(2))=K_2(F)$$
$$H^1_{\mathscr{A}}(F, \mathbb{Z}(2))=[K_3(F_s)^{ind}]^{Gal(F_s/F)}.$$

From (3), we have $H^1_{\mathscr{A}}(F, \mathbb{Z}(2))=K_3(F)^{ind}$, at least after inverting char(F). This gives some evidence for the interpretation of $\zeta_E(1-q)$ as an Euler characteristic. One can also unite our results, the Merkurjev-Suslin theorem for K_2, and Suslin's computation of the torsion in K_2 in a way that is suggestive of an arithmetic cohomology theory. In fact, we have the exact sequence

$$0\to H^0_{et}(E, \mu_n^{\otimes 2})\to K_3(E)^{ind}\xrightarrow{\times n}K_3(E)^{ind}\to H^1_{et}(E, \mu_n^{\otimes 2})$$
$$\to K_2(E)\xrightarrow[\times n]{}K_2(E)\to H^2_{et}(E, \mu_n^{\otimes 2})\to 0$$

where E is an arbitrary field, and n is prime to the characteristic of E. This exact sequence arises from the exact triangle

$$\begin{array}{ccc}\Gamma(2) & \xrightarrow{\times n} & \Gamma(2)\\ & \nwarrow \quad \swarrow & \\ & \mu_n^{\otimes 2} & \end{array}$$

together with the computation of $H^*_{\mathscr{A}}(E, \mathbb{Z}(2))$ above. This formulation was pointed out to me by Bruno Kahn.

The proof of Hilbert's Theorem 90 is a modification of the proof used by Suslin in [S]. The analysis of the $H^1(X, \mathscr{K}_2)$ for X a Brauer-Severi scheme over a (equicharacteristic) semi-local PIR R is essentially the same as in the case R a field. Suppose R contains μ_l. Let α be a unit in R, R^α the extension $R[X]/X^l-\alpha$, or the extension $R[X]/X^l-X-\alpha$ if $l=\mathrm{char}(R)$, and J^α the Jacobson radical of R^α. The next step is to show the relation

$$(\bigstar) \qquad \{x, \text{1-Norm}(x)\}\in(1-\sigma)\,\underset{iv}{K_2}(R^\alpha; J^\alpha)$$

for $x\in(1+J^\alpha)^\times$, $\mathrm{Norm}(x)\neq 1$. This is done by the "generic element" method first, where one can assume that R is *local*, in which case the relative K_2 is a subgroup of the usual K_2. One then makes a specialization argument, which is the main technical difficulty. After this point, the proof proceeds essentially as in [S].

The first chapter gives a discussion of the properties of relative K-theory. This is essentially an extension of most of the results of Quillen's *Higher Algebraic* K-*theory I* to the setting of relative K-theory. Most of this chapter is quite straightforward, but as there is no reference for the material in the literature we include it here. The second chapter gives a description of relative K_2 via the symbols of Keune and Loday, and the symbols of Bloch. We also prove some preliminary results required for the construction of the specialization subgroup and homomorphism, which is the main technical construction of the chapter. In chapter three, we apply the generic element method to get the relation described above. We also get simplified generators for $K_2(R^\alpha, J^\alpha)$, assuming as in [S] that R has no prime to l extensions and that the norm map $N: (1+J^\alpha)^* \to (1+J)^*$ is surjective. In chapter four we prove Hilbert's Theorem 90 for relative K_2, and the other results above. In chapter five, we use the continuous cohomology of Jannsen to extend the results of Tate and Merkurjev-Suslin on Galois symbols to the case of relative K_2.

This work was done while I was visiting the M.S.R.I. in Berkeley during the fall of '86; I would like to thank the M.S.R.I. for their hospitality, support and encouragement during my stay. I would especially like to thank Rick Jardine for his help on the topological aspects of K-theory, and Florence Lecomte, Wayne Raskind, Shuji Saito and Christophe Soulé for patiently listening. Altha Blanchet contributed her expertise in central simple algebras. Esther Beneish and David Saltman helped a great deal by finally convincing me that an earlier approach of mine was doomed to failure. Dan Grayson and Dinakar Ramakrishnan went through many of the details of an earlier version of this work; the referee did an exceedingly thorough job of going through the manuscript and gave many valuable suggestions; whatever clarity exists in this version is due to their help.

Hilbert's theorem 90 for relative K_2, and its consequences for K_3 have been proven independently by Merkurjev and Suslin [M-S 2]. Their approach differs from ours in that they derive the behavior of the relative K_2 under extension by a Brauer-Severi scheme by the Galois-theoretic properties of K_2, rather than redoing the argument for fields in the relative case, as we have done. They also acheive some simplification by a judicious use of Karoubi-Villamayor K-theory to define norms. Finally, they prove the relation (★) by a more direct method, avoiding our use of specialization. In addition to the results given here, they show that $K_3(E)^{\text{ind}}$ is uniquely p-divisible in characteristic $p>0$, and they prove a part of a conjecture of Milnor on the relation between Milnor K-theory and the Witt ring of a field.

1. Relative K-theory

1.1. Here we recall the definition and some basic properties of relative K-theory. For a more detailed discussion, *see* [Coombes].

Let $f: \mathscr{A} \to \mathscr{B}$ be an exact functor on exact categories $\mathscr{A}$ and $\mathscr{B}$. Define $(K(f), *)$ to be the homotopy fiber of $BQf: BQ\mathscr{A} \to BQ\mathscr{B}$ with basepoint $*$ coming from the zero objects of $\mathscr{A}$ and $\mathscr{B}$. The K-groups of f are then the homotopy groups of $K(f)$:

$$K_p(f) := \pi_{p+1}(K(f), *).$$

One gets a long exact sequence

$$\to K_p(f) \to K_p(\mathscr{A}) \to K_p(\mathscr{B}) \to K_{p-1}(f) \to$$

from the fibration $K(f) \to BQ\mathscr{A} \to BQ\mathscr{B}$.

Let $f_0 : \mathscr{A}_0 \to \mathscr{B}_0$ be another exact functor on exact categories. Suppose we have a pair of exact functor $G : \mathscr{A}_0 \to \mathscr{A}$, $H : \mathscr{B}_0 \to \mathscr{B}$, and a natural isomorphism $\theta : fG \to Hf_0$. Then θ induces a homotopy $BQ\theta$ between $BQf \circ BQG$ and $BQH \circ BQf_0$, hence the triple (G, H, θ) gives a map $BQ(G, H, \theta) : K(f_0) \to K(f)$. This induces a homomorphism $(G, H, \theta)* : K_p(f_0) \to K_p(f)$, and a commutative ladder

$$\begin{array}{ccccccccc}
\to K_p(f_0) & \to & K_p(\mathscr{B}_0) & \to & K_p(\mathscr{A}_0) & \to & K_{p-1}(f_0) & \to \\
{\scriptstyle (G, H, \theta)^*}\downarrow & & {\scriptstyle G^*}\downarrow & & {\scriptstyle H^*}\downarrow & & {\scriptstyle (G, H, \theta)^*}\downarrow & \\
\to K_p(f) & \to & K_p(\mathscr{B}) & \to & K_p(\mathscr{A}) & \to & K_{p-1}(f) & \to
\end{array}$$

Now let X be a scheme over a ring R, Y a closed subscheme $j_Y : Y \to X$ the inclusion. Let $\mathscr{P}_X$ (resp. $\mathscr{P}_Y$) be the exact category of locally free sheaves on X (resp. Y) of finite rank. Then $j_Y^* : \mathscr{P}_X \to \mathscr{P}_Y$ is exact; let $K(X, Y)$ denote the homotopy fiber $K(j_Y^*)$, and $K_p(X, Y)$ the K-group $K_p(j_Y^*)$. $K_p(X, Y)$ is called the pth K-group of X relative to Y. One defines a relative K′ similarly: let $\mathscr{M}_{(X, Y)}$ be the exact subcategory of coherent sheaves on X, $\mathscr{M}_X$, consisting of sheaves $\mathscr{F}$ with $\mathrm{Tor}_i^{\mathscr{O}} X(\mathscr{F}, \mathscr{O}_Y) = 0$ for $i > 0$. Then $j_Y^* : \mathscr{M}_{(X, Y)} \to \mathscr{M}_Y$ is exact. We let $K'(X, Y)$ denote $K(j_Y^* : \mathscr{M}_{(X, Y)} \to \mathscr{M}_Y)$, and $K'_p(X, Y)$ the pth K-group of $j_Y^* : \mathscr{M}_{(X, Y)} \to \mathscr{M}_Y$. The inclusions $i_X : \mathscr{P}_X \to \mathscr{M}_{(X, Y)}$, $i_Y : \mathscr{P}_Y \to \mathscr{M}_Y$ induce $i : K_p(X, Y) \to K'_p(X, Y)$, and we have the commutative ladder

$$\begin{array}{ccccccc}
\to K_p(X, Y) & \to & K_p(X) & \to & K_p(Y) & \to \\
\downarrow & & \downarrow & & \downarrow & \\
\to K'_p(X, Y) & \to & K'_p(X) & \to & K'_p(Y) & \to
\end{array}$$

Thus, if X and Y are regular, the resolution theorem shows that $K'_p(X, Y) \to K_p(X, Y)$ is an isomorphism. Similarly, if $\mathscr{O}_Y$ has finite Tor dimension over $\mathscr{O}_X$ (e. g. X regular or Y locally principal) the resolution theorem shows that the inclusion $\mathscr{M}_{(X, Y)} \to \mathscr{M}_X$ induces an isomorphism $K_p(\mathscr{M}_{(X, Y)}) \to K_p(\mathscr{M}_X)$.

Let $h : X' \to X$ be a morphism of schemes, and let Y' be a closed subscheme of X' contained in $h^{-1}(Y)$. Then the pair of exact functors $(h^*, h|_{Y'}^*) : (\mathscr{P}_X, \mathscr{P}_Y) \to (\mathscr{P}_{X'}, \mathscr{P}_{Y'})$, together with the natural isomorphism $\theta(h^*) : j_{Y'}^* \circ h^* \to h|_{Y'}^* \circ j_Y^*$, gives a map $(h^*, h|_{Y'}^*, \theta(h^*)) : K_p(X, Y) \to K_p(X', Y')$. We denote this map by h^*. If h is flat, we get a similar functorial pull-back $h^* : K'_p(X, Y) \to K'_p(X', Y')$. The functoriality of h^* is rather difficult to show directly; we briefly describe the method used in [Coombes], as this will also be applicable when we discuss multiply-relative K-theory.

Let $\mathscr{C}$ be a small category. For an object c of $\mathscr{C}$, we let $/c$ denote the category of objects over c, *i.e.* objects are morphisms $f: c' \to c$ in $\mathscr{C}$ and morphisms are commutative triangles and $X: \mathscr{C} \to \mathbf{Schemes}$ a functor, so $\{X(c) \mid c \in \mathscr{C}\}$ is a set of schemes indexes by the category $\mathscr{C}$. Let $\mathscr{P}/X(c)$ be the category in which an object is a set indexed by $/c$:

$$\{P(f, c') \text{ in } \mathrm{Obj}(\mathscr{P}_{X(c')}) \mid f: c' \to c \text{ is a morphism in } \mathscr{C}\}$$

together with a choice of isomorphism $j_h: P(f, c') \to X(h)^*(P(f', c''))$ for each $h: (f, c') \to (f', c'')$. In addition, we require for each $h: (c', f) \to (c'', f')$ and $k: (c'', f') \to (c''', f'')$ the diagram

$$\begin{tikzcd} X(h)^* P(f', c'') \arrow[r, "X(h)^*(j_k)"] & X(h)^* X(k)^* P(f'', c''') \\ P(f, c') \arrow[u, "j_h"] \arrow[r, "j_{kh}"'] & X(kh)^* P(f'', c''') \arrow[u, "\mathrm{Nat}"'] \end{tikzcd}$$

commutes.

Morphisms in $\mathscr{C}$ are maps $g(f, c'): P(f, c') \to Q(f, c')$ so that the obvious diagram commutes. Given a morphism $g: b \to c$ in $\mathscr{C}$, we get a functor

$$g^*: \quad \mathscr{P}/X(c) \to \mathscr{P}/X(b)$$

by restricting to the subcategory $/b$ of $/c$. Coombes then shows that $(gh)^* = h^* g^*$, and that the projection $\mathscr{P}/X(c) \to \mathscr{P}_{X(c)}$ is an equivalence of categories. In addition, enlarging the indexing category is compatible with this equivalence. Thus, replacing the spaces $BQ\mathscr{P}_{X(c)}$ with $BQ\mathscr{P}/X(c)$, we get a functor from $\mathscr{C}$ to **Top**, which makes that functoriality of the homotopy fibers obvious. To avoid overburdening the notation, we will hence forth assume that we have made this construction wherever necessary. A similar construction works for the categories $\mathscr{M}_X$.

If $g: (X', Y') \to (X, Y)$ is finite and $Y' = g^{-1}(Y)$, using the above construction defines a functorial $g_*: K'_p(X', Y') \to K'_p(X, Y)$. Given such a g, and a flat map $h: (Z, W) \to (X, Y)$ with W contained in $h^{-1}(Y)$, let $Z' = Z \times_X X'$, $W' = W \times_Y Y'$, and form the cartesian square

$$\begin{tikzcd} (Z', W') \arrow[r, "h'"] \arrow[d, "g'"'] & (X', Y') \arrow[d, "g"] \\ (Z, W) \arrow[r, "h"'] & (X, Y) \end{tikzcd}$$

Then $h'^{-1}(Y')$ contains W', and $g'^{-1}(W) = W'$, so $h'^*: K'_p(X', Y') \to K'_p(Z', W')$ and $g'^*: K'_p(Z', W') \to K'_p(Z, W)$ are defined and the diagram

$$\begin{tikzcd} K'_p(X', Y') \arrow[r, "h'^*"] \arrow[d, "g^*"'] & K'_p(Z', W') \arrow[d, "g'^*"] \\ K'_p(X, Y) \arrow[r, "h^*"'] & K'_p(Z, W) \end{tikzcd}$$

commutes. If X, Y, Z, and W are smooth, we get a similar commutative diagram for the relative K-theories, for g finite as above, and h an arbitrary morphism. To see this, let $\mathcal{M}^h(X, Y)$ be the subcategory of $\mathcal{M}_{(X, Y)}$ consisting of sheaves $\mathcal{F}$ such that $\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_Z) = \mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_W) = 0$ for $i > 0$, and similarly let $\mathcal{M}_Y^h$ be the subcategory of $\mathcal{M}_Y$ consisting of sheaves $\mathcal{G}$ such that $\mathrm{Tor}_i^{\mathcal{O}_Y}(\mathcal{G}, \mathcal{O}_W) = 0$ for $i > 0$. Then $j_Y^*: \mathcal{M}_{(X, Y)} \to \mathcal{M}_Y$ restricts to $j_{Y*}^h: \mathcal{M}_{(X, Y)}^h \to \mathcal{M}_Y^h$, and $g^*: (\mathcal{P}_{X'}, \mathcal{P}_{Y'}) \to (\mathcal{M}_{X'}, \mathcal{M}_{Y'})$ factors through $(\mathcal{M}_{X'}^h, \mathcal{M}_Y^h)$. Letting $K_p^h(X, Y)$ be the homotopy group $\pi_{p+1}(K(j_{Y*}^h))$, we get as above a commutative diagram

$$\begin{array}{ccc}
K_p(Z, W) & \xrightarrow{h'^*} & K_p(Z', W') \\
\downarrow g^* & & \downarrow g'^* \\
K_p^h(X, Y) & \xrightarrow{h^*} & K_p'(Z, W) \\
\wr\uparrow & & \wr\uparrow \\
K_p(X, Y) & \xrightarrow{h^*} & K_p(Z, W)
\end{array}$$

where the bottom two isomorphisms come from the resolution theorem, and the five lemma.

1.2. Additivity for relative K-theory. — The additivity theorem of Quillen for an exact sequence of exact functors extends to relative K-theory. To see this, it is convenient to use Waldhausen's [W] construction of the homotopy fiber of an exact functor $f: \mathcal{A} \to \mathcal{B}$. This is the simplicial set $F.(f)$, with n-simplices

$$F_n(f) = \{(A_0 \rightarrowtail \ldots \rightarrowtail A_n, B_0 \rightarrowtail \ldots \rightarrowtail B_n, \omega)\}$$

where the A_i's are objects of $\mathcal{A}$, the B_j's are objects of $\mathcal{B}$, and ω is an isomorphism

$$\omega: \ f(A_1/A_0 \rightarrowtail \ldots \rightarrowtail A_n/A_0) \to (B_0 \rightarrowtail \ldots \rightarrowtail B_n).$$

Included in this is the data of compatible choices of the quotients A_i/A_j and B_i/B_j for $i > j$. The boundary maps d_i are "omit the i-th term" for $i \geqq 1$, and d_0 is "mod out by A_0 (resp. B_0)". Given an exact functor $f_0: \mathcal{A}_0 \to \mathcal{B}_0$, a pair of exact functors $G: \mathcal{A}_0 \to \mathcal{A}$, $H: \mathcal{B}_0 \to \mathcal{B}$, and a natural isomorphism $\theta: f \circ G \to H \circ f_0$, we get a map of simplicial sets

$$(G, H, \theta): F.(f_0) \to F.(f)$$

by

$$(G, H, \theta)((A, B, \omega)) = (G(A), H(B), H(\omega) \circ \theta(d_0 A)).$$

In addition, Waldhausen shows that $\Omega\, BQF.(f)$ is a natural model for the homotopy fiber of $BQf: BQ\mathcal{A} \to BQ\mathcal{B}$. We now show

Proposition 1.1. — *Let $f_0: \mathcal{A}_0 \to \mathcal{B}_0$, $f: \mathcal{A} \to \mathcal{B}$ be exact functors, and let*

$$0 \to (G', H', \theta') \to (G' H' \theta) \to (G'', H'', \theta'') \to 0$$

be an exact sequence of functors from $(\mathcal{A}_0, \mathcal{B}_0)$ to $(\mathcal{A}, \mathcal{B})$, with compatible natural isomorphisms. Then

$$(G, H, \theta)^* = (G', H', \theta')^* + (G'', H'', \theta'')^*$$

as maps $K_p(f_0) \to K_p(f)$.

Proof. — Let $E(f)$ be the simplicial set with $E_n(f)$ consisting of short exact sequences E in $F_n(f)$:

$$E = 0 \to s\,E \to t\,E \to q\,E \to 0.$$

The exact functor $(s, q): E_n(f) \to (F_n(f))^2$ induces by [Quillen] a homotopy equivalence $Q(s, q): QE_n(f) \to Q(F_n(f))^2$, hence a homotopy equivalnce $BQE.(f) \to (BQF.(f))^2$. Let $\oplus: (BQF(f))^2 \to BQE(f)$ be the section $(P, Q) \to (0 \to P \to P \oplus Q \to Q \to 0)$ to (s, q). An exact sequence of pairs of exact functors with compatible natural isomorphisms

$$0 \to (G', H', \theta') \to (G' H' \theta) \to (G'', H'', \theta'') \to 0$$

from $(\mathscr{A}_0, \mathscr{B}_0)$ to $(\mathscr{A}, \mathscr{B})$ gives an exact functor $\sigma: F.(f_n) \to E.(f)$. A choice of homotopy from $\mathrm{id}_{BQE.(f)}$ to $\oplus \circ (s, q)$ gives a homotopy from $BQ(G, H, \theta)$ to $BQ(G', H', \theta') \oplus BQ(G'', H'', \theta'')$. This gives the desired additivity. □

1.3. PRODUCTS. — In [Weibel] products in relative K-theory are constructed using the Waldhausen construction above. More specifically, there are functorial products

$$\cup: \quad K_p(X, Y) \otimes K_q(X) \to K_{p+q}(X, Y).$$

Moreover, if $f: X' \to X$ is a finite morphism with $\mathcal{O}_{X'}$ projective as an $\mathcal{O}_X$ module, and $Y' = f^{-1}(Y)$, then we have the projection formulae:

$$(1.4) \qquad \begin{cases} f_*(\alpha \cup f^*(\beta)) = f_*(\alpha) \cup \beta; & \alpha \in K_p(X', Y'),\ \beta \in K_q(X) \\ f_*(f^*(\alpha) \cup \beta) = \alpha \cup f_*(\beta); & \alpha \in K_p(X, Y),\ \beta \in K_q(X'). \end{cases}$$

1.4. The five lemma gives the *homotopy property:* Let $\pi: \mathbb{A}^1_X \to X$ be the projection, Y a closed subscheme of X with $\mathcal{O}_Y$ having finite Tor dimension over $\mathcal{O}_X$. Then $\pi^*: K'_p(X, Y) \to K'_p(\mathbb{A}^1_X, \mathbb{A}^1_Y)$ is an isomorphism. If X and Y are smooth, then $\pi^*: K_p(X, Y) \to K_p(\mathbb{A}^1_X, \mathbb{A}^1_Y)$ is an isomorphism.

1.5. THE LOCALIZATION SEQUENCE. — Let (X, Y) be as above with $\mathcal{O}_Y$ of finite Tor dimension over $\mathcal{O}_X$. Let Z be a closed subscheme of X with $\mathcal{O}_Z$ in $\mathscr{M}_{(X, Y)}$. Let $U = X - Z$, $Y_U = Y \cap U$, $Y_Z = Y \cap Z$. By the resolution theorem, the inclusions $\mathscr{M}_{(X, Y)} \to \mathscr{M}_X$, $\mathscr{M}_{(U, Y_U)} \to \mathscr{M}_U$, and $\mathscr{M}_{(Z, Y_Z)} \to \mathscr{M}_Z$ induce homotopy equivalences on the Q constructions. In addition, the localization theorem of Quillen shows that

$$\begin{array}{ccccc} K'(Z, Y_Z) & \to & K'(X, Y) & \to & K'(U, Y_U) \\ \downarrow & & \downarrow & & \downarrow \\ BQ\mathscr{M}_{(Z, Y_Z)} & \to & BQ\mathscr{M}_{(X, Y)} & \to & BQ\mathscr{M}_{(U, Y_U)} \\ \downarrow & & \downarrow & & \downarrow \\ BQ\mathscr{M}_{Y_Z} & \to & BQ\mathscr{M}_Y & \to & BQ\mathscr{M}_{Y_U} \end{array}$$

is a commutative diagram of homotopy fiber sequences. The Quetzlcoatl lemma then shows that the natural map

$$K'(Z, Y_Z) \to \mathrm{fiber}(K'(X, Y) \to K'(U, Y_U))$$

is a homotopy equivalence. This gives a long exact localization sequence

$$(1.5)\qquad \to K'_p(Z, Y_Z) \to K'_p(X, Y) \to K'_p(U, Y_U) \xrightarrow{\delta} K'_{p-1}(Z, Y_Z) \to.$$

Swan has shown in [Swan] that the localization sequence for K′ theory is natural; the same argument applied to the Waldhausen construction for $K'(X, Y)$, $K'(U, Y_U)$ and $K'(Z, Y_Z)$ shows that (1.5) is natural for pullbacks by flat maps, and pushforward for finite maps.

1.6. QUILLEN SPECTRAL SEQUENCE. — In this section we suppose that Y is a locally principal subscheme of X, defined locally by a non zero-divisor. Then $\mathcal{M}_{(X, Y)}$ is just the category of coherent sheaves having no $\mathcal{I}_Y$-torsion. In particular, if Z is a reduced closed subscheme, then $\mathcal{O}_Z$ is in $\mathcal{M}_{(X, Y)}$ if and only if Z intersects Y properly. Furthermore, if $\mathcal{F}$ is in $\mathcal{M}_{(X, Y)}$, then $\operatorname{supp}(\mathcal{F})$ intersects Y properly.

Let $\mathcal{M}^i_{(X, Y)}$ be the subcategory of $\mathcal{M}_{(X, Y)}$ of sheaves $\mathcal{F}$ with $\operatorname{codim}_X \operatorname{supp}(\mathcal{F}) \geqq i$. Then j^*_Y maps $\mathcal{M}^i_{(X, Y)}$ to $\mathcal{M}^i_Y$; let $K'(X^i, Y^i)$ be the homotopy fiber of $BQj^*_Y\colon \mathcal{M}^i_{(X, Y)} \to \mathcal{M}^i_Y$. By the remarks above, the map

$$(1.6)\qquad \varinjlim_{\substack{Z \subset X\\ \text{Z reduced, closed}\\ \text{subscheme of}\\ \text{codim} \geqq i \text{ with}\\ \text{Z intersecting Y}\\ \text{properly.}}} K'(Z, Z_Y) \to K'(X^i, Y^i)$$

is a homotopy equivalence. Let $\mathcal{M}^{i/k}_{(X, Y)}$ be the direct limit

$$\mathcal{M}^{i/k}_{(X, Y)} = \varinjlim_{\substack{Z \subset X,\\ \text{Z intersecting}\\ \text{Y properly}\\ \operatorname{codim}_X Z \geqq k}} \mathcal{M}^i_{(X-Z, Y-Z)}$$

and let $\mathcal{M}^{i/k}_Y$ be the direct limit

$$\mathcal{M}^{i/k}_Y = \varinjlim_{\substack{Z \subset Y\\ \operatorname{codim}_Y Z \geqq k}} \mathcal{M}^i_{Y-Z}$$

Let $K'(X^{i/k}, Y^{i/k})$ be the homotopy fiber of $BQ\mathcal{M}^{i/k}_{(X, Y)} \to BQ\mathcal{M}^{i/k}_Y$. Then (1.5), (1.6), and a limit argument shows that

$$K'(X^k, Y^k) \to K'(X^i, Y^i) \to K'(X^{i/k}, Y^{i/k})$$

is a homotopy fiber sequence. Let

$$K'_p(X^{i/k}, Y^{i/k}) = \pi_{p+1}(K'(X^{i/k}, Y^{i/k})),$$
$$\bar{K}'_0(X^{i/k}, Y^{i/k}) = \operatorname{Im}(K'_0(X^i, Y^i) \to K'_0(X^{i/k}, Y^{i/k})),$$
$$\bar{K}_0(\mathscr{M}^{i/k}_{(X, Y)}) = \operatorname{Im}(K_0(\mathscr{M}^i_{(X, Y)}) \to K_0(\mathscr{M}^{i/k}_{(X, Y)})).$$

The method of the exact couple gives a spectral sequence

$$E_1^{p,q}(X, Y) \Rightarrow K'_{-p-q}(X, Y),$$

$$(1.7) \qquad E_1^{p,q}(X \mid Y) = \begin{cases} K'_{-p-q}(X^{p/p+1}, Y^{p/p+1}); & -p-q > 0,\ p \leqq \dim Y. \\ \bar{K}'_0(X^{p/p+1}, Y^{p/p+1}); & -p-q = 0. \\ 0; & \text{otherwise} \end{cases}$$

The filtration on $K'_*(X, Y)$ is the "topological" filtration:

$$F^p K'_*(X, Y) = \operatorname{Im}(K'_*(X^p, Y^p) \to K'_*(X, Y)).$$

We denote $F^p K'_*(X, Y)$ by $K'_*(X, Y)^p$ and the E_∞ term $\operatorname{Gr}^p K'_*(X, Y)$ by $K'_*(X, Y)^{p/p+1}$.

We can similarly form an E_1 spectral sequence converging to $K'_*(X)$:

$$E_1^{p,q}(X \mid Y) \Rightarrow K'_{-p-q}(X),$$

$$(1.8) \qquad E_1^{p,q}(X \mid Y) = \begin{cases} K_{-p-q}(\mathscr{M}^{p/p+1}_{(X, Y)}); & -p-q > 0,\ p \leqq \dim Y \\ \bar{K}_0(\mathscr{M}^{p/p+1}_{(X, Y)}); & -p-q = 0, \\ 0; & \text{otherwise,} \end{cases}$$

and an E_1 spectral sequence converging to $K'_*(Y)$:

$$E_1^{p,q}(Y) \Rightarrow K'_{-p-q}(Y),$$
$$E_1^{p,q}(Y) = K_{-p-q}(\mathscr{M}^{p/p+1}_Y).$$

If the maps $K'_0(X^p, Y^p) \to K_0(X^{p/p+1}, Y^{p/p+1})$ and $K_0(\mathscr{M}^p_{(X, Y)}) \to K_0(\mathscr{M}^{p/p+1}_{(X, Y)})$ are surjective for $p = 0, \ldots, \dim Y$, then we get a long exact sequence of E_1 terms:

$$\to E_1^{p, q-1}(Y) \to E_1^{p,q}(X, Y) \to E_1^{p,q}(X \mid Y) \to E_1^{p,q}(Y) \to$$

compatible with the differentials. We also have the usual Quillen spectral sequence on X:

$$E_1^{p,q}(X) \Rightarrow K'_{-p-q}(X),$$
$$E_1^{p,q}(X) = K_{-p-q}(\mathscr{M}^{p/p+1}_X).$$

The inclusion $\mathscr{M}^{p/p+1}_{(X, Y)} \to \mathscr{M}^{p/p+1}_X$ gives a map of E_1 terms compatible with the differentials.

LEMMA 1.2. — *Suppose* X *is quasi-projective over a Noetherian ring, and regular in a neighborhood of* Y. *Then the mapping* $(i_{X-Y})^*: \mathcal{M}^{p/p+1}_{(X,Y)} \to \mathcal{M}^{p/p+1}_{X-Y}$ *induces an isomorphism*

$$K_0(\mathcal{M}^{p/p+1}_{(X,Y)}) \tilde{\to} K_0(\mathcal{M}^{p/p+1}_{X-Y}) \tilde{\to} \bigoplus_{x \in (X-Y)^p} \mathbb{Z}$$

In addition, if X *is a scheme over a field k, then* $(i_{X-Y})^*$ *gives rise to short exact sequences* $(q \geqq 1)$:

$$\begin{array}{ccccccc} O \to K_q(\mathcal{M}^{p/p+1}_{(X,Y)}) \to & K_q(\mathcal{M}^{p/p+1}_{X-Y}) & \xrightarrow{\delta} & K_{q-1}(\mathcal{M}^{p/p+1}_{Y}) & \to 0 \\ & \wr\uparrow & & \wr\uparrow & \\ & \bigoplus_{x \in (X-Y)^p} K_q(k(x)) & & \bigoplus_{y \in Y^p} K_{q-1}(k(y)) & \end{array}$$

Here, δ is the composition

$$\begin{array}{ccccccc} K_q(\mathcal{M}^{p/p+1}_{X-Y}) & \xrightarrow{\text{inc}} & K_q(\mathcal{M}^{p+1/p+2}_{X}) & \xrightarrow{\partial} & K_{q-1}(\mathcal{M}^{p+1/p+2}_{X}) & \xrightarrow{\text{proj}} & K_{q-1}(\mathcal{M}^{p/p+1}_{Y}) \\ \wr\uparrow & & \wr\uparrow & & \wr\uparrow & & \wr\uparrow \\ \bigoplus_{x \in (X-Y)^p} K_q(k(x)) & & \bigoplus_{x \in X^p} K_q(k(x)) & & \bigoplus_{x \in X^{p+1}} K_q(k(x)) & & \bigoplus_{y \in Y^p} K_q(k(y)) \end{array}$$

Proof: For Z a closed subset of X, let $\mathcal{M}_X(Z)$ denote the category of $\mathcal{O}_X$ modules $\mathcal{F}$ with $\text{supp}(\mathcal{F}) \subset Z$. Let $\mathcal{M}^{p/p+1}_{X \mid Y}$ be the direct limit

$$\mathcal{M}^{p/p+1}_{X \mid Y} = \varinjlim_{\substack{W \subset Z \subset X \\ \text{codim}_X Z = p \\ \text{codim}_X W = p+1 \\ W,\ Z \text{ intersect } Y \text{ properly}}} \mathcal{M}_{X-W}(Z-W)$$

The resolution theorem [Quillen] shows that $Q\mathcal{M}^{p/p+1}_{(X,Y)} \to Q\mathcal{M}^{p/p+1}_{X \mid Y}$ is a homotopy equivalence. Indeed, given $\mathcal{F}$ in $\mathcal{M}_{X-W}(Z-W)$, with Z, W as above, we can find a closed subscheme $\mathcal{Z} \subset X-W$, with $\text{codim}_{X-W}\mathcal{Z} = p$, and $\mathcal{O}_{\mathcal{Z}}$ having no $\mathcal{I}_Y$ torsion, such that $\mathcal{F}$ is an $\mathcal{O}_{\mathcal{Z}}$ module. Take a surjection

$$0 \to K \to (\mathcal{O}_{\mathcal{Z}}(N))^n \to \mathcal{F} \to 0,$$

then K is also $\mathcal{I}_Y$ torsion free, hence K and $(\mathcal{O}_{\mathcal{Z}}(N))^n$ determine elements of $\mathcal{M}^{p/p+1}_{(X,Y)}$, and the hypotheses of the resolution theorem are satisfied.

The localization theorem [Quillen] shows that

$$(\bigstar) \qquad Q\mathcal{M}^{p/p+1}_{Y} \to Q\mathcal{M}^{p/p+1}_{X \mid Y} \to Q\mathcal{M}^{p/p+1}_{X-Y}$$

is a homotopy fiber sequence. On the other hand, let x be a generic point of a codimension p irreducible subscheme of Y. Take a closed codimension p reduced irreducible subscheme D of X such that D contains x, D is regular at x, and D intersects Y properly. Ler R be the semi-local ring of $D \cap Y$ in D, R^N the normalization of R.

Then R^N is regular, semi-local, and one dimensional, hence a PIR. In particular, the inclusion map $x \to \operatorname{Spec}(R^N)$ induces the 0 map

$$\mathbb{Z} \cong K_0(k(x)) \to K_0(R^N) \to K_0'(R).$$

Thus $K_0(\mathcal{M}_Y^{p/p+1}) \to K_0(\mathcal{M}_{X|Y}^{p/p+1})$ is the zero map. This, together with the localization sequence derived from ($\bigstar$), shows that the map $K_0(\mathcal{M}_{X|Y}^{p/p+1}) \to K_0(\mathcal{M}_{X-Y}^{p/p+1})$ is an isomorphism, which proves the first assertion.

For the second, as X is a scheme over a field, the ring R^N is regular, semi-local, and contains a field. Thus, by Gersten's conjecture (proved by Quillen [Quillen] in this case), the map

$$K_q(k(x)) \to K_q(R^N) \to K_q'(R)$$

is the zero map. Thus $K_q(\mathcal{M}_Y^{p/p+1}) \to K_q(\mathcal{M}_{X|Y}^{p/p+1})$ is the zero map, and the sequence

$$\begin{array}{l} 0 \to K_q(\mathcal{M}_{X|Y}^{p/p+1}) \to K_q(\mathcal{M}_{X-Y}^{p/p+1}) \to K_{q-1}(\mathcal{M}_Y^{p/p+1}) \to 0 \\ \qquad\qquad \wr\uparrow \\ \qquad K_q(\mathcal{M}_{(X,Y)}^{p/p+1}) \end{array}$$

derived from ($\bigstar$) is exact. This proves the second assertion. $\square$

LEMMA 1.3. — *Suppose* X *is quasi-projective over a Noetherian ring and regular in a neighborhood of* Y. *Then the map*

$$K_0(\mathcal{M}_{(X,Y)}^p) \to K_0(\mathcal{M}_{(X,Y)}^{p/p+1}); \qquad 0 \leqq p \leqq \dim Y$$

is surjective. If Y *is regular, then the map*

$$K_0'(X^p, Y^p) \to K_0(X^{p/p+1}, Y^{p/p+1}); \qquad 0 \leqq p \leqq \dim Y$$

is surjective.

Proof. — By the previous lemma, we have

$$K_0(\mathcal{M}_{(X,Y)}^{p/p+1}) \cong \bigoplus_{x \in (X-Y)^p} \mathbb{Z} \cong \bigoplus_{x \in (X-Y)^p} K_0(k(x)).$$

Similarly,

$$K_0(\mathcal{M}_Y^{p/p+1}) \cong \bigoplus_{y \in Y^p} \mathbb{Z} \cong \bigoplus_{y \in Y^p} K_0(k(y)).$$

In the commutative ladder with exact rows

$$(\bigstar) \qquad \begin{array}{ccc} \to K_0'(X^p, Y^p) & \to K_0(\mathcal{M}_{(X,Y)}^p) & \to K_0(\mathcal{M}_Y^p) \\ \downarrow & \alpha\downarrow & \beta\downarrow \\ \to K_0'(X^{p/p+1}, \underset{Y^{p/p+1}}{Y^p}) & \to K_0(\mathcal{M}_{(X,Y)}^{p/p+1}) & \to K_0(\mathcal{M}_Y^{p/p+1}) \end{array}$$

we have compatible splittings $s_{(X, Y)}$ and s_Y to α and β, where $s_{(X, Y)}$ is defined by sending the vector space of rank n over $k(x)$, for x in $(X-Y)^p$, to the free rank $n\mathcal{O}_{\bar{x}}$ module; s_Y is defined similarly. This proves the first assertion.

We have the exact relativization sequence

$$\begin{array}{c} \to K_1(\mathcal{M}^{p/p+1}_{(X, Y)}) \to K_1(\mathcal{M}^{p/p+1}_{Y}) \to K'_0(X^{p/p+1}, Y^{p/p+1}) \to \\ \uparrow \\ \bigoplus_{y \in Y^p} k(y)^* \end{array}$$

Given y in Y^p, chose a reduced irreducible subscheme D containing y as in lemma 1.2. We retain the notations of that lemma. Given α in $k(y)^*$, we can find a unit u in the semi-local ring R such that u restrict to α at y, and u restrict to 1 at all other closed points of Spec(R). This shows that the map $K_1(\mathcal{M}^{p/p+1}_{(X, Y)}) \to K_1(\mathcal{M}^{p/p+1}_{Y})$ is surjective, and the map γ in (*) is injective. The second assertion follows from this and the existence of the splittings $s_{(X, Y)}$ and s_Y. □

As a consequence of the lemmas above, if X is quasi-projective, and regular in a neighborhood of Y, and Y is regular, then the $E_1^{p,-p}$ terms in the spectral sequences defined above are

$$E_1^{p,-p}(X, Y) = K'_0(X^{p/p+1}, Y^{p/p+1})$$

$$E_1^{p,-p}(X \mid Y) = K_0(\mathcal{M}^{p/p+1}_{(X, Y)}).$$

To end this section, we consider an important case of the above. Let R be a semi-local PIR containing a field k_0. Let $I=(t)R$ be the Jacobson radical of R, $\bar{R}=R/I$. If $g\colon T \to \operatorname{Spec}(R)$ is an R-scheme, we let $\bar{T}$ denote the fiber $g^{-1}(\operatorname{Spec}(\bar{R}))$. R(T) with denote the semi-local ring of $\bar{T}$ in T, I(T) the ideal $(t)R(T)$. We will occasionally abuse standard terminology and refer to the total quotient ring of R as the quotient field of R.

Lemma 1.4. — *Suppose that* X *is a quasi-projective* R*-scheme, smooth over* R, *and* $Y=\bar{X}$. *Then the map* $E_2^{p,q}(X \mid Y) \to E_2^{p,q}(X)$ *is an isomorphism.*

Proof. — We apply lemma 1.2 to describe the E_1 term $E_1^{p,q}(X \mid Y)$:

$$0 \to E_1^{p,q}(X \mid Y) \to \bigoplus_{x \in (X-Y)^p} K_{-p-q}(k(x)) \xrightarrow{\partial} \bigoplus_{y \in Y^p} K_{-p-q-1}(k(y)) \to 0.$$

From this it follows that the E_2 term is given by

$$E_2^{p,q}(X \mid Y) = \frac{\ker\left[\bigoplus_{x \in (X-Y)^p} K_{-p-q}(k(x)) \xrightarrow{\partial} \bigoplus_{x \in X^{p+1}} K_{-p-q-1}(k(x))\right]}{\partial\left[\ker\left(\bigoplus_{x \in (X-Y)^{p-1}} K_{-p-q+1}(k(x)) \xrightarrow{\partial} \bigoplus_{y \in Y^{p-1}} K_{-p-q}(k(y))\right)\right]}.$$

Similarly, the E_2 term $E_2^{p,q}(X)$ is given by

$$E_2^{p,q}(X)=\frac{\ker[\bigoplus_{x\in X^p} K_{-p-q}(k(x))\xrightarrow{\partial}\bigoplus_{x\in X^{p+1}} K_{-p-q-1}(k(x))]}{\partial[\bigoplus_{x\in X^{p-1}} K_{-p-q+1}(k(x))]}.$$

Let ξ be a class in $E_2^{p,q}(X)$. Represent ξ by z,

$$z\in\bigoplus_{x\in X^p} K_{-p-q}(k(x)).$$

We write z as

$$z=\sum_{x\in X^p} z_x;\qquad z_x\in K_{-p-q}(k(x)).$$

If x is a codimension $p-1$ point of Y with $z_x\neq 0$, the take D containing x as in lemma 2.1. By Gersten's conjecture applied to the regular ring $R(D)^N$, we can find η in $K_{-p-q+1}(k_0(D))$ with

$$\partial\eta=z_x+\tau;\qquad \tau=\sum_{x\in X^p\cap D}\tau_x,$$

so that $\tau_y=0$ for all y in Y^{p-1}. Then $z':=z-\partial\eta$ is a new representative for ξ; repeating this for all x in Y^{p-1} with $z_n\neq 0$, we see that ξ is in the image of $E_2^{p,q}(X\mid Y)$. This proves surjectivity; the proof of injectivity is similar and will be left to the reader. $\square$

1.7. Relative K-theory of projective spaces. — Let S be a scheme, $\bar{S}$ a closed subscheme, $\mathscr{V}$ a locally free sheaf of rank n on S, $\bar{\mathscr{V}}$ the restriction to $\bar{S}$, $X=\mathbb{P}(\mathscr{V})$, $\bar{X}=\mathbb{P}(\bar{\mathscr{V}})$. The pair of exact functors

$$(F_i,\bar{F}_i):\ (\mathscr{P}_S,\mathscr{P}_{\bar{S}})\to(\mathscr{P}_X,\mathscr{P}_{\bar{X}})$$
$$(\mathscr{E},\bar{\mathscr{E}})\to(\mathscr{E}\otimes\mathcal{O}_X(-i),\ \bar{\mathscr{E}}\otimes\mathcal{O}_{\bar{X}}(-i)),$$

together with the natural isomorphism

$$\theta_i(\mathscr{E}):\ i_{\bar{X}*}(\mathscr{E}\otimes\mathcal{O}_X(-i))\to i_{\bar{S}*}(\mathscr{E})\otimes\mathcal{O}_{\bar{X}}(-i)$$

gives for each i a homomorphism

$$(F_i,\bar{F}_i,\theta_i)_*:\ K_*(S,\bar{S})\to K_*(X,\bar{X}).$$

Since the maps

$$\sum_{i=0}^{n-1} F_{i*}:\ \bigoplus_{i=0}^{n-1} K_*(S)\to K_*(X),$$

and

$$\sum_{i=0}^{n-1} \bar{F}_{i*}: \bigoplus_{i=0}^{n-1} K_*(\bar{S}) \to K_*(\bar{X})$$

are isomorphisms, we get the following computation of $K_*(\mathbb{P}(\mathscr{V}), \mathbb{P}(\bar{\mathscr{V}}))$:

$$(1.9) \quad \sum_{i=0}^{n-1} (F_i, \bar{F}_i, \theta_i)_*: \bigoplus_{i=0}^{n-1} K_*(S, \bar{S}) \to K_*(\mathbb{P}(\mathscr{V}), \mathbb{P}(\bar{\mathscr{V}})) \qquad \text{is an isomorphism.}$$

Now suppose $S = \mathrm{Spec}(R)$, where R is a semi-local PIR containing an infinite field k_0. We retain the notations of the end of the previous section. C. Sherman [Sherman] has shown that the Quillen spectral sequence converging to $K_*(\mathbb{P}^n_k)$ degenerates at E_2, for k a field or a semi-local ring; we now prove an analogue for the relative situation.

PROPOSITION 1.5. — *Let* $\mathscr{V} = R^{n+1}$. *Let* $(X, \bar{X}) = (\mathbb{P}(\mathscr{V}), \mathbb{P}(\bar{\mathscr{V}}))$. *The spectral sequence*

$$E_1^{p,q}(\mathbb{P}(\mathscr{V}), \mathbb{P}(\bar{\mathscr{V}})) \Rightarrow K_{-p-q}(\mathbb{P}(\mathscr{V}), \mathbb{P}(\bar{\mathscr{V}}))$$

degenerates at E_2. The E_2 term is given by

$$E_2^{p,q}(\mathbb{P}(\mathscr{V}), \mathbb{P}(\bar{\mathscr{V}})) \cong K_{-p-q}(R, \bar{R}).$$

The isomorphism above is given by the composition

$$K_{-p-q}(R, \bar{R}) \underset{\pi^*_p}{\to} K_{-p-q}((\mathbb{P}^{n-p}, \bar{\mathbb{P}}^{n-p}) \to K_{-p-q}(\mathscr{M}^{p/p+1}_{(X, \bar{X})}),$$

where $\mathbb{P}^{n-p}$ is any codimension p linear subspace of $\mathbb{P}(\mathscr{V})$, and

$$\pi_p: \quad \mathbb{P}^{n-p} \to \mathrm{Spec}(R)$$

is the projection. Finally, let γ be the class of $\mathscr{O}_X(-1)$ in $K_0(X)$. Then the topological filtration on $K_*(X, \bar{X})$ is given by

$$K_*(X, \bar{X})^p = \sum_{j \geqq p} (1-\gamma)^p \cup \pi^*(K_*(R, \bar{R})).$$

Proof. — We denote $\mathbb{P}(R^{s+1})$ by $\mathbb{P}^s$; similarly denote the affine space $\mathrm{Spec}(R[X_1, \ldots, X_s])$ by $\mathbb{A}^s$. We first prove the following

CLAIM. — *Let* Z *be a reduced closed codimension p subscheme of* X, *flat over* R. *Then*

$$\mathrm{Im}(K'_*(Z, \bar{Z}) \to K'_*(X^{p-1}, \bar{X}^{p-1})) \subset \mathrm{Im}(K_*(\mathbb{P}^{n-p}, \bar{\mathbb{P}}^{n-p}) \to K'_*(X^{p-1}, \bar{X}^{p-1})),$$

where $\mathbb{P}^{n-p}$ *is any codimension p linear subspace of* $\mathbb{P}^n$.

Proof of claim. — Take a codimension 1 linear subspace $\mathbb{P}^{n-1}$ of $\mathbb{P}^n$ such that no component of $\bar{Z}$ is contained in $\bar{\mathbb{P}}^{n-1}$. Let p be an R-valued point of $\mathbb{P}^{n-1} - Z$. Then projection from p defines a linear map f_p

$$f_p: \underset{\mathbb{A}^n}{\underbrace{\mathbb{P}^n - \mathbb{P}^{n-1}}} \to \mathbb{A}^{n-1}.$$

In addition, letting Z^0 be the intersection $Z \cap \mathbb{A}^n$, the restriction

$$f_{p \mid Z^0}: Z^0 \to \mathbb{A}^n$$

is a finite morphism. Let η be in $K_q(Z, \bar{Z})$, $\eta^0 = \mathrm{res}_{Z^0}(\eta)$ in $K_q(Z^0, \bar{Z}^0)$. We have the diagram

$$\begin{array}{ccc} \mathbb{A}^n & \leftarrow & \mathbb{A}^n \times_{\mathbb{A}^{n-1}} Z^0 \cong \mathbb{A}^1_{Z^0} \\ \downarrow f_p & & g \downarrow \quad \uparrow s \\ \mathbb{A}^{n-1} & \underset{f_{p \mid Z^0}}{\leftarrow} & Z^0 \end{array}$$

where s is the section induced by the inclusion i of Z^0 in $\mathbb{A}^n$. Since $\mathbb{A}^1_{Z^0}$ is the trivial line bundle over Z^0, we can find a regular function f on $\mathbb{A}^1_{Z^0}$ with $s(Z^0)$ defined by the ideal (f). Let W^0 be the image $q(\mathbb{A}^1_{Z^0})$. Then, letting $^-$ denote reduction mod t, we have the exact sequences of functors

$$0 \to q_* g^* \to q_* g^* \xrightarrow{\times f} q_* s_* \to 0; \qquad q_* s_* = i_*,$$

and

$$0 \to \bar{q}_* \bar{g}^* \to \bar{q}_* \bar{g}^* \xrightarrow{\times \bar{f}} \bar{q}_* \bar{s}_* \to 0; \qquad \bar{q}_* \bar{s}_* = \bar{i}_*,$$

from $\mathcal{M}_{(Z^0, \bar{Z}^0)}$ to $\mathcal{M}_{(W^0, \bar{W}^0)}$ and from $\mathcal{M}_{\bar{Z}^0}$ to $\mathcal{M}_{\bar{W}^0}$ respectively. This, together with the natural isomorphisms

$$\theta: \quad j^*_{\bar{W}^0} \circ (q_* g^*) \to (\bar{q}_* \bar{g}^*) \circ j^*_{\bar{Z}^0}$$

and

$$\theta'': \quad j^*_{\bar{W}^0} \circ i_* \to \bar{i}_* \circ j^*_{\bar{Z}^0}$$

gives an exact sequence

$$0 \to (q_* g^*, \bar{q}_* \bar{g}^*, \theta) \xrightarrow{\times (f, \bar{f})} (q_* g^*, \bar{q}_* \bar{g}^*, \theta) \to (i_*, \bar{i}_*, \theta'') \to 0$$

of pairs of functors with compatible natural isomorphisms. Thus

$$i^0_*: \quad K'_*(Z^0, \bar{Z}^0) \to K'_*(W^0, \bar{W}^0);$$

$i^0: Z^0 \to W^0$ the inclusion, is the zero map, by Proposition 1.1.

Let W be the closure of W^0 in $\mathbb{P}^n$. From the localization sequence

$$\to K'_*(W \cap \mathbb{P}^{n-1}, \bar{W} \cap \bar{\mathbb{P}}^{n-1}) \to K'_*(W, \bar{W}) \to K'_*(W^0, \bar{W}^0) \to,$$

we see that $i_*(\eta)$ in $K'_*(W, \bar{W})$ is the image of a class ξ of $K'_*(W \cap \mathbb{P}^{n-1}, \bar{W} \cap \bar{\mathbb{P}}^{n-1})$. By induction on n, there is a $\mathbb{P}^{n-p}$ in $X_1 := \mathbb{P}^{n-1}$, and an element α of $K_*(\mathbb{P}^{n-p}, \bar{\mathbb{P}}^{n-p})$ with the image of ξ in $K'_*(X_1^{p-2}, \bar{X}_1^{p-2})$) equal to the image of α in $K'_*(X_1^{p-2}, \bar{X}_1^{p-2})$). Thus, the image of ξ in $K'_*(X^{p-1}, \bar{X}^{p-1})$) equals the image of α in $K'_*(X^{p-1}, \bar{X}^{p-1})$).

On the other hand, if Y and Y′ are two $\mathbb{P}^{n-p}$'s contained in a $\mathbb{P}^{n-p+1} := Y''$, then the exact sequences

$$0 \to \mathscr{I}_Y \to \mathcal{O}_{Y''} \to \mathcal{O}_Y \to 0$$

and

$$0 \to \mathscr{I}_{Y'} \to \mathcal{O}_{Y''} \to \mathcal{O}_{Y'} \to 0,$$

together with the isomorphisms $\mathscr{I}_Y \cong \mathcal{O}_{Y''}(-1) \cong \mathscr{I}_{Y'}$, and the isomorphism $K_*(\mathbb{P}^q, \bar{\mathbb{P}}^q) \cong [K_*(R, \bar{R})]^{q+1}$, shows that

$$\mathrm{Im}\,[K_*(\mathbb{P}^{n-p}, \bar{\mathbb{P}}^{n-p})) \to K'_*(X^{p-1}, \bar{X}^{p-1})]$$

is independent of the choice of the $\mathbb{P}^{n-p}$ in $\mathbb{P}^n$. This proves the claim. □

Next, we note that $i_*: K_*(\mathbb{P}^{n-p}, \bar{\mathbb{P}}^{n-p}) \to K_*(\mathbb{P}^n, \bar{\mathbb{P}}^n)$ is injective for all linear subspaces $i: \mathbb{P}^{n-p} \to \mathbb{P}^n$. Indeed, we have the localization sequence

$$\begin{tikzcd} \to K_*(\mathbb{P}^{n-1}, \bar{\mathbb{P}}^{n-1}) \arrow[r, "i_*"] & K_*(\mathbb{P}^n, \bar{\mathbb{P}}^n) \arrow[r, "j^*"] & K_*(\mathbb{A}^n, \bar{\mathbb{A}}^n) \to \\ & & K_*(R, \bar{R}) \arrow[ul, "\pi^*"] \arrow[u, "\pi_0^*", "\wr"'] \end{tikzcd}$$

so j^* is split by $\pi_0^*(\pi_0^*)^{-1}$, and i_* is thus injective. The general case follows by induction. As a consequence, the map

$$(\bigstar) \qquad K'_q(X^p, \bar{X}^p) \to \mathrm{Im}\,[K'_q(X^{p/p+2}, \bar{X}^{p/p+2}) \to K'_q(X^{p/p+1}, \bar{X}^{p/p+1})]$$

is surjective. Indeed, let η be in $K'_q(X^{p/p+2}, \bar{X}^{p/p+2})$. Take ξ to be the element $\partial(\eta)$ in $K'_{q-1}(X^{p+2}, \bar{X}^{p+2})$, where ∂ is the boundary in the localization sequence

$$K'_q(X^p, \bar{X}^p) \to K'_q(X^{p/p+2}, \bar{X}^{p/p+2}) \xrightarrow{\partial} K'_{q-1}(X^{p+2}, \bar{X}^{p+2}) \to$$

Then ξ goes to zero in $K'_{q-1}(X^p, \bar{X}^p)$; on the other hand, we can find a τ in $K_{q-1}(\mathbb{P}^{n-p-2}, \bar{\mathbb{P}}^{n-p-2})$ with

$$\mathrm{Im}\,[\xi \to K'_{q-1}(X^{p+1}, \bar{X}^{p+1})] = \mathrm{Im}\,[\tau \to K'_{q-1}(X^{p+1}, \bar{X}^{p+1})].$$

As $K_{q-1}(\mathbb{P}^{n-p-2}, \bar{\mathbb{P}}^{n-p-2}) \to K'_{q-1}(X^p, \bar{X}^p)$ is injective, this forces τ to be zero, hence ξ goes to zero in $K'_{q-1}(X^{p+1}, \bar{X}^{p+1})$. Let δ be the boundary in the localization sequence

$$K'_q(X^p, \bar{X}^p) \to K'_q(X^{p/p+1}, \bar{X}^{p/p+1}) \xrightarrow{\delta} K'_{q-1}(X^{p+1}, \bar{X}^{p+1}) \to,$$

and let η' be the image of η in $K'_q(X^{p/p+1}, \bar{X}^{p/p+1})$. Then the element $\delta(\eta')$ of $K'_{q-1}(X^{p+1}, \bar{X}^{p+1})$ is the image of ξ, hence $\delta(\eta)=0$. Thus there is a σ in $K'_q(X^p, \bar{X}^p)$

with

$$\mathrm{Im}\,[\sigma \to K_q'(X^{p/p+1}, \bar{X}^{p/p+1})] = \eta',$$

hence (★) is surjective, as claimed.

An immediate consequence of the surjectivity of (★) is the degeneration of our spectral sequence at E_2. In addition, the surjectivity of (★), together with the claim proved above, shows that the map

$$s_p:\quad K_{-p-q}(\mathbb{P}^{n-p}, \bar{\mathbb{P}}^{n-p}) \to E_2^{p,q} = E_\infty^{p,q}$$

is surjective. Since the subgroup $i_*(K_{-p-q}(\mathbb{P}^{n-p-1}, \bar{\mathbb{P}}^{n-p-1}))$ of $K_{-p-q}(\mathbb{P}^{n-p}, \bar{\mathbb{P}}^{n-p})$ clearly goes to zero under s_p, for any hyperplane $i: \mathbb{P}^{n-p-1} \to \mathbb{P}^{n-p}$, this proves the statement about the topological filtration on $K_*(\mathbb{P}^n, \bar{\mathbb{P}}^n)$. Finally, let $i_p: \mathbb{P}^{n-p} \to \mathbb{P}^n$ be the inclusion, $\pi_p: \mathbb{P}^{n-p} \to \mathrm{Spec}(R)$ the projection. Since

$$i_{p*}\pi_p^*(\alpha) \equiv (1-\gamma)^p \bmod \sum_{j>p} (1-\gamma)^j \cup K_*(R, \bar{R})$$

for α in $K_*(R, \bar{R})$, we see that $i_{p*}\pi_p^*(\alpha) = 0$ implies $\alpha = 0$, so s_p is an isomorphism, which completes the proof. □

COROLLARY 1.6. — *The sequence*

$$0 \to K_*(R, \bar{R}) \underset{\pi^*}{\to} K_*(R(\mathbb{P}^n), \overline{R(\mathbb{P}^n)}) \underset{d_1^{*,0}}{\to} K_{*-1}((\mathbb{P}^n)^{1/2}, (\bar{\mathbb{P}}^n)^{1/2})$$

is exact.

Proof:

$$\begin{aligned}\ker(d_1^{*,0}) &= E_2^{*,0}\\ &= E_\infty^{*,0}\\ &= K_*(\mathbb{P}^n, \bar{\mathbb{P}}^n)^{0/1}.\end{aligned}$$

On the other hand,

$$\begin{aligned}K_*(\mathbb{P}^n, \bar{\mathbb{P}}^n)^{0/1} &= \bigoplus_{i=0}^{n-1} \gamma^i \cup \pi^* K_*(R, \bar{R})/(1-\gamma) \cup \pi^* K_*(R, \bar{R})\\ &\cong \pi^* K_*(R, \bar{R}). \ \square\end{aligned}$$

1.8. RELATIVE K_1'. — We return briefly to a more general setting. Let X be a smooth scheme over a field k, Y a locally principal closed subscheme of X. We want to compute $K_1'(X^{p/p+1}, Y^{p/p+1})$.

LEMMA 1.8. — *Let Z be a reduced semi-local k-scheme of (Krull) dimension one, $\bar{Z}$ a principal closed subscheme defined by a non zero divisor. If Z_i is an irreducible component of Z, let $\bar{Z}_i$ denote $Z_i \cap \bar{Z}$. Suppose for each closed point z of Z, there is an irreducible* *component* Z_i *of* Z *with localization* $\bar{Z}_{i(z)}$ *of* $\bar{Z}_i$ *at* z *isomorphic to* $\mathrm{Spec}(k(z))$. *Then the two sequences*

$$0 \to K_1'(Z, \bar{Z}) \to K_1'(Z) \to K_1'(\bar{Z}) \to 0$$

and

$$0 \to K_0'(Z, \bar{Z}) \to K_0'(Z) \to K_0'(\bar{Z}) \to 0$$

are exact.

Proof: We need only show that $K_i'(Z) \to K_i'(\bar{Z})$ is surjective for $i=1, 2$. By devissage,

$$K_i'(\bar{Z}) \cong \bigoplus_{z \in \bar{Z}} K_i(k(z)).$$

The assumption $\bar{Z}_{i(z)} \cong \mathrm{Spec}(k(z))$ implies that the composition

$$K_1(Z_i) \to K_1'(Z) \to K_1(k(z))$$

is just the restriction map

$$\Gamma(Z_i, \mathcal{O}_{Z_i}^*) \to k(z)^*.$$

The result for K_1' now follows from the Chinese remainder theorem. Since $K_2(k(z))$ is generated by symbols, a similar argument proves surjectivity for K_2, completing the proof. □

COROLLARY 1.9. — *Let* X *be a smooth scheme over* k, Y *a locally principal closed subscheme. Then the sequence*

$$0 \to K_i'(X^{p/p+1}, Y^{p/p+1}) \to K_i(\mathcal{M}_{(X, Y)}^{p/p+1}) \to K_i(\mathcal{M}_Y^{p/p+1}) \to 0$$

is exact for $i=0, 1$. *In addition, we have the exact sequence*

$$0 \to K_1(\mathcal{M}_{(X, Y)}^{p/p+1}) \to \bigoplus_{x \in (X-Y)^p} k(x)^* \to \bigoplus_{y \in Y^p} \mathbb{Z} \to 0$$

and an isomorphism

$$K_0(\mathcal{M}_{(X, Y)}^{p/p+1}) \xrightarrow{\sim} \bigoplus_{x \in (X-Y)^p} \mathbb{Z}.$$

Proof. — The first statement follows from lemma 1.8 and a limit argument; the second is a special case of lemma 1.2. □

Remark. — One important consequence of corollary 1.9 is encapsulated in the commutative diagram:

$$\begin{array}{ccc} K_2'(X^{p-1/p}, Y^{p-1/p}) & \to & \bigoplus_{x \in X^{p-1}} K_2(k(x)) \\ \downarrow \partial & & \downarrow \partial = T \\ 0 \to K_1'(X^{p/p+1}, Y^{p/p+1}) & \to & \bigoplus_{x \in X^{p}} K_1(k(x)) \end{array}$$

where T is the usual tame symbol map. In words, for η in $K_2'(X^{p-1/p}, Y^{p-1/p})$, we can compute $\partial(\eta)$ as the tame symbol $T(\eta')$, where η' is the image of η in $\bigoplus_{x \in X^{p-1}} K_2(k(x))$.

We also have

COROLLARY 1.10. — *We retain the conventions immediately preceding lemma* 1.4. *Let* $S = \mathrm{Spec}(R)$, *where* R *is a semi-local* PIR *with infinite residue fields,* $\pi: \mathbb{A}_S^1 \to S$ *the affine line over* S. *Let* $s: S \to \mathbb{A}_S^1$ *be a section to* π, B *the semi-local ring of* $s(\bar{S})$ *in* $\mathbb{A}_S^1$, *and let* $L = R(\mathbb{A}_S^1)$ *be the semi-local ring of* $\bar{\mathbb{A}}_S^1$ *in* $\mathbb{A}_S^1$. *Then the map*

$$K_2(B, \bar{B}) \to K_2(L, \bar{L})$$

induced by the inclusion $B \to L$ *is injective.*

Proof. — Let U be an open subset of $\mathbb{A}_S^1$ containing each generic point of $\bar{\mathbb{A}}_S^1$. Then there is a section $\sigma: S \to U$ to $\pi|_U$. Since

$$\pi^*: \quad K_i(R, \bar{R}) \to K_i(\mathbb{A}_S^1, \bar{\mathbb{A}}_S^1)$$

is an isomorphism by the homotopy property (§ 1.4), it follows that the map $K_i(\mathbb{A}_S^1, \bar{\mathbb{A}}_S^1) \to K_i(U, \bar{U})$ is injective. Passing to a suitable limit, we see that the maps

$$K_2(\mathbb{A}_S^1, \bar{\mathbb{A}}_S^1) \to K_2(B, \bar{B}); \qquad K_2(\mathbb{A}_S^1, \bar{\mathbb{A}}_S^1) \to K_2(L, \bar{L})$$

are injective.

Let I_L, I_B be the index sets

$$I_L = \{Z \subset \mathbb{A}_S^1 \mid Z \text{ is reduced, closed, codim } Z = 1, Z \text{ is flat over } S\}$$

$$I_B = \{Z \subset \mathbb{A}_S^1 \mid Z \in I_L \text{ and } Z \cap s(\bar{S}) = \varnothing\}.$$

We have the compatible localization sequences:

$$\begin{array}{ccccccc} 0 \to K_2(\mathbb{A}_S^1, \bar{\mathbb{A}}_S^1) & \to & K_2(L, \bar{L}) & \to & \varinjlim_{Z \in I_L} K_1'(Z, \bar{Z}) & \to & 0 \\ \| & & \uparrow \alpha & & \uparrow \beta & & \\ 0 \to K_2(\mathbb{A}_S^1, \bar{\mathbb{A}}_S^1) & \to & K_2(B, \bar{B}) & \to & \varinjlim_{Z \in I_B} K_1'(Z, \bar{Z}) & \to & 0 \end{array}$$

We can restrict the limits to be over Z's which satisfy the hypotheses of lemma 1.8. If Z is of this type, the argument of lemma 1.8 shows that the map

$$K'_1(Z) \to K_1(k(Z)) = k(Z)^*$$

is injective. Thus the map β is injective, hence α is injective, as desired. □

1.9 Relative K-theory of Brauer-Severi schemes. — Let R be as in § 1.7, with Jacobson radical (t)R, and let $\mathscr{D}$ be an Azumaya algebra over R. Let $S = \mathrm{Spec}(R)$, and let $\pi: X \to S$ be the Brauer-Severi scheme associatied with $\mathscr{D}$. Quillen's computation of the K-theory of X gives pairs of functors

$$(G_i, \bar{G}_i) : (\mathscr{P}_{\mathscr{D}^{\otimes i}}, \mathscr{P}_{\bar{\mathscr{D}}^{\otimes i}}) \to (\mathscr{P}_X, \mathscr{P}_{\bar{X}})$$

by

$$G_i(\mathscr{E}) = \mathscr{I} \otimes_{\mathscr{D}^{\otimes i}} \mathscr{E}; \qquad \bar{G}_i(\bar{\mathscr{E}}) = \bar{\mathscr{I}} \otimes_{\bar{\mathscr{D}}^{\otimes i}} \bar{\mathscr{E}},$$

where $\mathscr{I}$ is a certain vector bundle on X, $\bar{\mathscr{I}}$ the restriction to $\bar{X}$. Letting θ_i be the usual natural isomorphism, we get maps

$$(G_i, \bar{G}_i, \theta_i)_* : K_q(\mathscr{D}^{\otimes i}, \bar{\mathscr{D}}^{\otimes i}) \to K_q(X, \bar{X}).$$

From Quillen's computation of $K_q(X)$ and $K_q(\bar{X})$, together with the five lemma, the map

$$\sum_{i=0}^{\dim X} (G_i, \bar{G}_i, \theta_i)_* : \bigoplus_{i=0}^{\dim_S X} K_q(\mathscr{D}^{\otimes i}, \bar{\mathscr{D}}^{\otimes i}) \to K_q(X, \bar{X})$$

is an isomorphism.

The explicit discription of $K_1(\bar{\mathscr{D}}^{\otimes i})$ as $\bar{\mathscr{D}}^{\otimes i}/[\bar{\mathscr{D}}^{\otimes i}, \bar{\mathscr{D}}^{\otimes i}]$ shows that the restriction map

$$K_1(\mathscr{D}^{\otimes i}) \to K_1(\bar{\mathscr{D}}^{\otimes i})$$

is surjective. This gives the exact sequence

$$\begin{array}{ccccc} 0 \to K_0(X, \bar{X}) & \to & K_0(X) & \to & K_0(\bar{X}) \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ 0 \to \bigoplus_{i=0}^{\dim_S X} K_0(\mathscr{D}^{\otimes i}, \bar{\mathscr{D}}^{\otimes i}) & \to & \bigoplus_{i=0}^{\dim_S X} K_0(\mathscr{D}^{\oplus i}) & \to & \bigoplus_{i=0}^{\dim_S X} K_0(\bar{\mathscr{D}}^{\otimes i}) \end{array}$$

If $\bar{\mathscr{D}}^{\otimes i}$ is split, it is easy to see that the map

$$K_2(\mathscr{D}^{\otimes i}) \to K_2(\bar{\mathscr{D}}^{\otimes i})$$

is also surjective, giving the exact sequence

$$\begin{array}{ccccccccc} 0 \to & K_1(X, \bar{X}) & \to & K_1(X) & \to & K_1(\bar{X}) & \to 0 \\ & \downarrow \wr & & \downarrow \wr & & \downarrow \wr & \\ 0 \to & \bigoplus_{i=0}^{\dim_S X} K_1(\mathcal{D}^{\otimes i}, \bar{\mathcal{D}}^{\otimes i}) & \to & \bigoplus_{i=0}^{\dim_S X} K_1(\mathcal{D}^{\otimes i}) & \to & \bigoplus_{i=0}^{\dim_S X} K_1(\bar{\mathcal{D}}^{\otimes i}) & \to 0 \end{array}$$

As the $\mathcal{D}^{\otimes i}$ are semi-local, the map

$$K_0(\mathcal{D}^{\otimes i}) \to K_0(\bar{\mathcal{D}}^{\otimes i})$$

is injective, hence $K_0(X, \bar{X}) = 0$. We now show a similar vanishing of the E_2 terms in the Quillen spectral sequence for relative $K_*(X, \bar{X})$.

Let $CH^p(X, \bar{X}) := E_2^{p, -p}(X, \bar{X})$. The maps on the E_2 terms give a complex

$$CH^p(X, \bar{X}) \to E_2^{p, -p}(X \mid \bar{X}) \to E_2^{p, -p}(\bar{X}).$$

By lemma 1.4, and a quick look at the E_2 terms, this is

$$CH^p(X, \bar{X}) \to CH^p(X) \to CH^p(\bar{X}).$$

LEMMA 1.11. — *If* $\bar{\mathcal{D}}$ *is split, then* $CH^p(X, \bar{X}) \to CH^p(X)$ *is injective.*

Proof. — This is obvious for $p=0$, so assume that $p \geqq 1$. Let z be in $K_0'(X^{p/p+1}, \bar{X}^{p/p+1})$, *i.e.* z is a codimension p cycle on X, flat over R, with $z.\bar{X}=0$ as a cycle on $\bar{X}$. Suppose the class of z in $E_2^{p, -p}(X \mid \bar{X})$, $[z]$, is zero, *i.e.*

$$z = \sum_i \operatorname{div}(f_i)$$

with the f_i in $k(D_i)^*$ for codimension $p-1$ subvarieties D_i of X, flat over R. Specializing the collection $\sum(f_i, D_i)$ to $\bar{X}$ gives an element $\sum(\bar{f}_i, \bar{D}_i)$

$$\sum(\bar{f}_i, \bar{D}_i) \in \bigoplus_{x \in \bar{X}^p} k(x)^*.$$

Since

$$\sum \operatorname{div}(\bar{f}_i) = (\sum \operatorname{div}(f_i)).\bar{X} = z.\bar{X} = 0,$$

$\sum(\bar{f}_i, \bar{D}_i)$ determines an element ξ of $E_2^{p, -p-1}(\bar{X})$. We note that $E_2^{p, -p-1}(\bar{X})$ is the cohomology group $H^p(\bar{X}, \mathcal{K}_{p+1})$. Since $\bar{\mathcal{D}}$ is split, $\bar{X}$ is just $\mathbb{P}_{\bar{R}}^{n-1}$ [$n = \operatorname{rank}_R(\mathcal{D})$], and $H^p(\bar{X}, \mathcal{K}_{p+1})$ is isomorphic to the group of units $\bar{R}^*$. Let u be the element of $\bar{R}^*$ corresponding to ξ, and lift $\bar{u}$ to u in R^*. We may replace $\sum(f_i, D_i)$ with $\sum(u^{-1} f_i, D_i)$, so we may assume that $\xi = 0$ in $H^p(\bar{X}, \mathcal{K}_{p+1})$; thus if $p \geqq 2$ we can find a reduced closed subscheme $\bar{E}$ of $\bar{X}$ of codimension $p-2$, and $\bar{\eta}$ in $K_2(k(\bar{E}))$ with

$$T(\bar{\eta}) = \sum(\bar{f}_i, \bar{D}_i) \qquad (T = \text{tame symbol}).$$

After adding components to $\bar{\mathrm{E}}$, and extending $\bar{\eta}$ by 1 on these additional components, we may assume there is a reduced closed subscheme E of X, flat over R, with $\bar{\mathrm{E}}=\mathrm{E}\cap\bar{\mathrm{X}}$. Let R(E) denote the semi-local ring of $\bar{\mathrm{E}}$ in E. Since since $\mathrm{K}_2(k(\bar{\mathrm{E}}))$ is generated by symbols, we can lift $(\bar{\eta}, \bar{\mathrm{E}})$ to (η, E), $\eta\in\mathrm{K}_2(\mathrm{R}(\mathrm{E}))$, with E flat over R, and of codimension $p-2$ on X. Modify $\sum(u^{-1}f_i, \mathrm{D}_i)$ by $\mathrm{T}(\eta)$ to get $\sum(f_i', \mathrm{D}_i')$. If $p=1$, then our element $\sum(f_i, \mathrm{D}_i)$ is just an element f of $k(\mathrm{X})^*$, and we take $f'=u^{-1}f$. Then $\sum(f_i', \mathrm{D}_i')$ gives an element of $\mathrm{K}_1(\mathscr{M}_{\mathrm{X}}^{p/p+1})$, *i.e.*, $\sum(f_i', \mathrm{D}_i')$ gives an element τ of $\mathrm{K}_1'(\mathrm{X}^{p/p+1}, \bar{\mathrm{X}}^{p/p+1})$ with $\mathrm{div}(\tau)=z$. Thus

$$[z]=0 \quad \text{in } \mathrm{CH}^p(\mathrm{X}, \bar{\mathrm{X}})$$

as desired. □

COROLLARY 1.12. — *If $\bar{\mathscr{D}}$ is split, and $\mathscr{D}$ has prime rank l over* R, *then* $\mathrm{CH}^p(\mathrm{X}, \bar{\mathrm{X}})=0$ *for all* $p\geqq 0$.

Proof. — Let $h: \mathrm{T}\to\mathrm{S}$ be a finite degree l cover splitting $\mathscr{D}$; we may assume that T is étale over S. Since $\dim(\mathrm{X})=l$, $\mathrm{CH}^p(\mathrm{X})[1/(l-1)!]$ injects into $\mathrm{K}_0(\mathrm{X})[1/(l-1)!]$, which injects into $\mathrm{K}_0(\mathrm{X}_\mathrm{T})[1/(l-1)!]$. Since the kernel of

$$h^*:\quad \mathrm{CH}^p(\mathrm{X})\to\mathrm{CH}^p(\mathrm{X}_\mathrm{T})$$

is l-torsion, h^* is thus injective. Thus

$$h^*:\quad \mathrm{CH}^p(\mathrm{X}, \bar{\mathrm{X}})\to\mathrm{CH}^p(\mathrm{X}_\mathrm{T})$$

is injective, with $h^*(\mathrm{CH}^p(\mathrm{X}, \bar{\mathrm{X}}))$ contained in the kernel of

$$i^*:\quad \mathrm{CH}^p(\mathrm{X}_\mathrm{T})\to\mathrm{CH}^p(\bar{\mathrm{X}}_\mathrm{T}).$$

Since X_T is a projective space over T, i^* is injective, hence $\mathrm{CH}^p(\mathrm{X}, \bar{\mathrm{X}})=0$, as claimed. □

COROLLARY 1.13. — *Assume that $\bar{\mathscr{D}}$ is split, and $\mathscr{D}$ has prime rank l over* R. *Suppose further that* R *contains an infinite field k_0. Let $h: \mathrm{S}'\to\mathrm{S}$ be a finite étale cover. Then*

$$\mathrm{E}_2^{1,-2}(\mathrm{X}, \bar{\mathrm{X}})\to\mathrm{E}_2^{1,-2}(\mathrm{X}_{\mathrm{S}'}, \bar{\mathrm{X}}_{\mathrm{S}'})$$

is injective.

Proof. — Since $\mathrm{CH}^p(\mathrm{X}, \bar{\mathrm{X}})=\mathrm{E}_2^{p,-p}(\mathrm{X}, \bar{\mathrm{X}})=0$, the differentials going out of $\mathrm{E}_r^{1,-2}(\mathrm{X}, \bar{\mathrm{X}})$ are all zero for $r\geqq 2$. There are no differentials going into $\mathrm{E}_2^{1,-2}$ by reasons of dimension, so

$$\mathrm{E}_2^{1,-2}(\mathrm{X}, \bar{\mathrm{X}})=\mathrm{E}_\infty^{1,-2}(\mathrm{X}, \bar{\mathrm{X}})=\mathrm{K}_1(\mathrm{X}, \bar{\mathrm{X}})^{1/2},$$

and similarly for $(\mathrm{X}_{\mathrm{S}'}, \bar{\mathrm{X}}_{\mathrm{S}'})$. We may assume that S' splits $\mathscr{D}$, *i.e.* $\mathrm{X}_{\mathrm{S}'}$ is $\mathbb{P}_{\mathrm{S}'}^{l-1}$. By the computation of $\mathrm{K}_*(\mathrm{X}_{\mathrm{S}'}, \bar{\mathrm{X}}_{\mathrm{S}'})$ in paragraph 1.7, we have

$$\mathrm{K}_1(\mathrm{X}_{\mathrm{S}'}, \bar{\mathrm{X}}_{\mathrm{S}'})=\bigoplus_{i=0}^{l-1}(1+\mathrm{I}_{\mathrm{S}'})^*.\gamma^i;$$

$I_{S'}$ the Jacobson radical of $R_{S'} := \Gamma(S', \mathcal{O}_{S'})$.

By the sequence (1.10)′, the map $K_1(X, \bar{X}) \to K_1(X_{S'}, \bar{X}_{S'})$ is injective. Let N be the subgroup of $R_{S'}^*$ of reduced norms from $\mathscr{D}$. Similarly define $\bar{N}$ as the group of reduced norms from $\bar{\mathscr{D}}$. Then N is isomorphic to $K_1(\mathscr{D})$, $\bar{N}$ is isomorphic to $K_1(\bar{\mathscr{D}})$, the kernel N^0 of $N \to \bar{N}$ is isomorphic to $K_1(\mathscr{D}, \bar{\mathscr{D}})$. Thus N^0 is the subgroup of N of reduced norms $z = \mathrm{Nrd}(x)$ from $\mathscr{D}$ with $z \equiv 1 \mod (t)$. Furthermore, we can identify $K_1(X, \bar{X})$ with the subgroup

$$(1+I_{S'})^* \oplus \bigoplus_{i=1}^{l-1} N^0 . \gamma^i$$

of $\bigoplus_{i=0}^{l-1} [R_{S'}^*] \gamma^i = K_1(X_{S'})$.

The topological filtration on $K_1(X_{S'}, \bar{X}_{S'})$ is given by

$$K_1(X_{S'}, \bar{X}_{S'})^p = \left(\sum_{i=p}^{l-1} (1-\gamma)^i . (1+I_{S'})^* \right)$$

so

$$K_1(X_{S'}, \bar{X}_{S'})^2 \cap K_1(X, \bar{X}) = \sum_{i=2}^{l-1} (1-\gamma)^i . N^0.$$

Take a in N^0, so $a = \mathrm{Nrd}(x)$ for some x in $\mathscr{D}$. Since k_0 is infinite, and $\bar{\mathscr{D}}$ is split, there is an element y of $\bar{\mathscr{D}}$ with $\mathrm{Nrd}(y) = 1$ (*i. e.* $\bar{y} \in SL_l(\bar{R})$), such that the characteristic polynomial of $\overline{xy}$ is separable over $\bar{R}$. Since $\bar{y}$ is in the commutator subgroup of $\bar{\mathscr{D}}^*$, we can lift $\bar{y}$ to an element y of $\mathscr{D}$ with $\mathrm{Nrd}(y) = 1$. Then xy is separable over the quotient field K of R; thus we may assume that x is separable over K. Let E be a maximal separable subfield of $\mathscr{D}_K$ containing x. Let $\mathscr{E}$ be the integral closure of R in E, so we get a finite ring extension $R \to \mathscr{E}$. Let $\hat{\mathscr{E}}$ be the integral closure of R in the Galois closure $\hat{E}$ of E over K. Then $G := \mathrm{Gal}(\hat{\mathscr{E}}/R)$ is a subgroup of Σ_l, and has a subgroup H, corresponding to E, of index l. Thus there is a non-trivial l-Sylow subgroup $G_l \cong \mathbb{Z}/l$ in G. Let $\mathscr{E}_l$ be the subring of $\mathscr{E}$ fixed by G_l, giving a diagram

$$\begin{tikzcd} & \mathscr{E} & \\ \mathscr{E} \arrow[ur] & & \mathscr{E}_l \arrow[ul] \\ & R \arrow[ul] \arrow[ur] & \end{tikzcd}$$

As $[\mathscr{E}_l : R]$ is prime to l, E and $\mathscr{E}_l \otimes_R K$ are disjoint over k, so x has the same conjugates over R and over $\mathscr{E}_l$. Thus

$$a = \mathrm{Nm}_{E/K}(x) = \mathrm{Nm}_{\hat{\mathscr{E}}/\mathscr{E}_l}(x).$$

By applying Hilbert's theorem 90 (*see* lemma 1.14 below) we can modify x by an element y of $\hat{\mathscr{E}}^*$ of norm 1 so that

$$x \in \hat{\mathscr{E}}, \; x \equiv 1 \mod (t).$$

In addition, we have

$$\mathrm{Nm}_{\hat{\mathscr{E}}/\mathrm{R}}(x)=a^d \qquad \text{with} \quad d \mid (l-1)!.$$

Let g: Spec$(\hat{\mathscr{E}}) \to$ Spec(R), g: $X_{\mathscr{E}} \to X$ be the covering maps. Then x lifts to an element $\hat{x}$ of $K_1(\hat{\mathscr{E}}, \hat{\mathscr{E}}/(t))$ with

$$g_*(\hat{x})=a^d,$$

and

$$(1-\gamma)^i a^d = g_*((1-\gamma)^i . \hat{x}).$$

Thus $(1-\gamma)^i . a^d$ is in $g_*(K_1(\hat{X}_{\mathscr{E}}, \bar{X}_{\hat{\mathscr{E}}})^i)$, which is a subgroup of $K_1(X, \bar{X})^i$, hence $(1-\gamma)^1 . a^d$ is in $K_1(X, \bar{X})^2$ for $i \geqq 2$. Thus

$$K_1(X_{S'}, \bar{X}_{S'})^2 [1(l-1)!] \cap K_1(X, \bar{X}) [1/(l-1)!] = K_1(X, \bar{X})^2 [1/(l-1)!],$$

hence the kernel of the map $K_1(X, \bar{X})^{1/2} \to K_1(X_{S'}, \bar{X}_{S'})^{1/2}$ is $(l-1)!$ primary torsion. Since we can split $\mathscr{D}$ by a degree l cover, the above kernel must also be l-primary torsion, hence the map

$$K_1(X, \bar{X})^{1/2} \to K_1(X_{S'}, \bar{X}_{S'})^{1/2}$$

is injective, as desired. □

LEMMA 1.14. — *Let* T *be a semi-local* PIR *containing an infinite field* k_0. *Let* $T \to T'$ *be a cyclic extension of degree* l. *Let* I *be the Jacobson radical of* T, K *the quotient field of* T, K′ *the quotient field of* T′. *Suppose* $a \in (1+I)$ *is a norm:*

$$a = N_{K'/K}(x), \qquad x \in K'.$$

Then

$$a = N_{T'/T}(y)$$

for some y *in* $(1+IT')$.

Proof. — This is an easy consequence of Hilbert's theorem 90 (for K_1); we leave the details to the reader.

1.10. RESTRICTIONS AND NORMS. — We consider the functorial properties of relative K-theory in some greater detail.

Let f: $T \to S$ be a map of schemes $\bar{S} \subset S$ a closed subscheme. Let $\bar{T} = f^{-1}(\bar{S})$, and $\hat{T} \subset \bar{T}$ an closed subscheme. The map f^* is exact on $\mathscr{P}_S$ and $\mathscr{P}_{\bar{S}}$ so the pair of functors

$$(f^*, \bar{f}^*) \colon (\mathscr{P}_S, \mathscr{P}_{\bar{S}}) \to (\mathscr{P}_T, \mathscr{P}_{\bar{T}})$$

together with the usual natural isomorphism θ determines the pull back $f^*: K_q(S, \bar{S}) \to K_q(T, \bar{T})$. Similarly, the diagram

$$\begin{array}{ccc} \mathscr{P}_T & \xrightarrow{\times_T \bar{T}} & \mathscr{P}_{\bar{T}} \\ \| & & \downarrow \times_{\bar{T}} \hat{T} \\ \mathscr{P}_T & \xrightarrow[\times_T \hat{T}]{} & \mathscr{P}_{\hat{T}} \end{array}$$

commutes (up to a natural isomorphism), so we get a homorphism

$$\mathrm{res}_{(\bar{T}, T)}: \quad K_q(T, \bar{T}) \to K_q(T, \hat{T}).$$

The composition gives a pullback

$$f^*: \quad K_q(S, \bar{S}) \to K_q(T, \hat{T}).$$

These constructions are just special cases of the pullback discussed in paragraph 1.1, hence they are functorial.

We now suppose that $f: T \to S$ is finite and flat. Restriction of scalars then gives commutative diagrams (up to natural isomorphisms):

$$\begin{array}{ccc} \mathscr{P}_T & \xrightarrow{\times_T \bar{T}} & \mathscr{P}_{\bar{T}} \\ f_* \downarrow & & f_* \downarrow \\ \mathscr{P}_S & \xrightarrow[\times_S \bar{S}]{} & \mathscr{P}_{\bar{S}} \end{array} \qquad \begin{array}{ccc} \mathscr{M}_{(T, \bar{T})} & \xrightarrow{\times_T \bar{T}} & \mathscr{M}_{\bar{T}} \\ f_* \downarrow & & f_* \downarrow \\ \mathscr{M}_{(S, \bar{S})} & \xrightarrow[\times_S \bar{S}]{} & \mathscr{M}_{\bar{S}} \end{array}$$

inducing $f_*: K_q(T, \bar{T}) \to K_q(S, \bar{S})$ and $f_*: K'_q(T, \bar{T}) \to K'_q(S, \bar{S})$.

We now suppose further that T, S, $\hat{T}$ and $\bar{S}$ are regular. Write $\bar{S}$ as a union of connected components

$$\bar{S} = \bigcup \bar{S}_i,$$

and let $\hat{T}^{i,1}, \ldots, \hat{T}^{i,n_i}$ be the components of $\hat{T}$ lying over $\bar{S}_i$. We also assume that each $\bar{S}_i$ and $\hat{T}_i^j$ are principal:

$$I(\bar{S}_i) = (u_i); \qquad I(\hat{T}^{ij}) = (t_{ij}),$$

and that $\bar{T}$ is a thickening of $\hat{T}$ with components $\bar{T}^{ij}$,

$$I(\bar{T}^{ij}) = (t_{ij})^{e_{ij}}, \qquad e_{ij} > 0.$$

Let $\mathbf{e} = (e_{11}, \ldots, e_{ij}, \ldots)$. We call $\mathbf{e}$ the total ramification index.

Let $\oplus_{\mathbf{e}}: \mathscr{P}_{\hat{T}} \to \mathscr{M}_{\bar{T}}$ be the functor whose restriction to $\mathscr{P}_{\hat{T}^{ij}}$ is $M \to (M)^{e_{ij}}$. Given a module M in $\mathscr{P}_{\bar{T}^{ij}}$, we form the filtration

$$F^*(M): \quad 0 = M_0 \subset (t_{ij}^{e_{ij}-1}) M \subset \ldots \subset (t_{ij}) M \subset M.$$

Since M is a projective $\bar{T}^{ij}$ module, the graded quotients $(t_{ij}^k) M/(t_{ij}^{k+1}) M$ are each isomorphic to $M/(t_{ij}) M$ by the map

$$m \to t_{ij}^k . m \bmod (t_{ij}^{k+1}) M.$$

The "natural filtration" theorem of Quillen then gives a homotopy H between the two maps

$$BQr_{\bar{T}} : BQ\mathcal{P}_{\bar{T}} \to BQ\mathcal{M}_{\bar{T}};$$

$r_{\bar{T}} : \mathcal{P}_{\bar{T}} \to \mathcal{M}_{\bar{T}}$ the natural inclusion and

$$BQ(\oplus_e \circ (- \times_{\bar{T}} \hat{T})) : BQ\mathcal{P}_{\bar{T}} \to BQ\mathcal{M}_{\bar{T}}.$$

We get a similar homotopy H′ between the two maps

$$BQ(r_{\bar{T}} \circ (- \times_T \bar{T})) : BQ\mathcal{P}_T \to BQ\mathcal{M}_{\bar{T}}$$

and

$$BQ(\oplus_e \circ (- \times_T \hat{T})) : BQ\mathcal{P}_T \to BQ\mathcal{M}_{\bar{T}}.$$

In fact, replace $- \times_T \hat{T}$ with the composition $(- \times_{\bar{T}} \hat{T}) \circ (- \times_T \bar{T})$, then we can take H′ to be $H \circ BQ(- \times_T \bar{T})$.

We thus get a diagram

(A)
$$\begin{tikzcd}
K'(T, \bar{T}) \arrow[r] & BQ\mathcal{P}_T \arrow[r, "r_{\bar{T}} \circ (- \times_T \bar{T})"] & BQ\mathcal{M}_{\bar{T}} \\
K(T, \bar{T}) \arrow[u, "(id, r_{\bar{T}})"'] \arrow[r] \arrow[d, "(id, - \times_T \hat{T})"] & BQ\mathcal{P}_T \arrow[u, equal] \arrow[d, equal] \arrow[r, "\times_T \bar{T}"] & BQ\mathcal{P}_{\bar{T}} \arrow[u, "r_{\bar{T}}"'] \arrow[d, "- \times_{\bar{T}} \hat{T}"] \\
K(T, \hat{T}) \arrow[r] \arrow[uu, bend left=60, "(id, \oplus_e)"] & BQ\mathcal{P}_T \arrow[r, "\times_T \hat{T}"] & BQ\mathcal{P}_{\hat{T}} \arrow[uu, bend right=60, "\oplus_e"']
\end{tikzcd}$$

Here we suppress some of the BQ's, and the natural isomorphisms used to get the maps on the homotopy fibers. Since the two homotopies H and H′ are compatible, the triangle

$$\begin{tikzcd} K(T, \bar{T}) \arrow[rr, "(\mathrm{id}, r_{\bar{T}})"] \arrow[dr, "(\mathrm{id}, \times_{\bar{T}} \hat{T})"'] & & K'(T, \bar{T}) \\ & K(T, \tilde{T}) \arrow[ur, "(\mathrm{id}, \otimes_e)"'] & \end{tikzcd}$$

commutes, up to homotopy, inducing a commutative triangle

$$\begin{tikzcd} K_*(T, \bar{T}) \arrow[rr] \arrow[dr, "\mathrm{res}_{(\bar{T}, \hat{T})}"'] & & K'_*(T, \bar{T}) \\ & K_*(T, \tilde{T}) \arrow[ur, "(\oplus_e) H"'] & \end{tikzcd}$$

Since S and $\bar{S}$ are smooth, we have the commutative diagram

$$\begin{tikzcd} K_*(T, \bar{T}) \arrow[r, "r_{\bar{T}}"] \arrow[d, "f_*"'] & K'_*(T, \bar{T}) \arrow[d, "f_*"] \\ K_*(S, \bar{S}) \arrow[r, "\simeq", "r_{\bar{S}}"'] & K'_*(S, \bar{S}) \end{tikzcd}$$

Define $f_*^{\mathrm{H}}: K_*(T, \hat{T}) \to K_*(S, \bar{S})$ to be the composition

$$f_*^{\mathrm{H}} = (r_{\bar{S}})^{-1} \circ f_* \circ (\oplus_e)^{\mathrm{H}}.$$

Let

$$(\times_e): K_*(\hat{T}) \to K_*(\hat{T})$$

be the map restricting to $\times e_{ij}: K_*(\hat{T}^{ij}) \to K_*(\hat{T}^{ij})$ on each factor $K_*(\hat{T}^{ij})$. Let

$$(\bar{f}^{ij})_*: K_*(\hat{T}^{ij}) \to K_*(\bar{S}^i)$$

be the restriction of scalars. Then

$$f_* \circ (\times_e) = \bigoplus_{i, j} (\times e_{ij}) \bar{f}^{ij}.$$

PROPOSITION 1.15. — *The ladder*

$$\begin{tikzcd} \to K_*(T, \hat{T}) \arrow[r] \arrow[d, "f_*^{\mathrm{H}}"] & K_*(T) \arrow[r] \arrow[d, "f_*"] & K_*(\hat{T}) \to \arrow[d, "(f_*) \circ (x_e) = \oplus e_{ij} \bar{f}^{ij}"] \\ \to K_*(S, \bar{S}) \arrow[r] & K_*(S) \arrow[r] & K_*(\bar{S}) \to \end{tikzcd}$$

is commutative. Suppose that $\mathcal{O}_T$ is a free $\mathcal{O}_S$ module (e. g. $\mathcal{O}_S$ semi-local), of rank n. Then the composition

$$K_*(S, \bar{S}) \xrightarrow[f^*]{} K_*(T, \hat{T}) \xrightarrow[f_*^{\mathrm{H}}]{} K_*(S, \bar{S})$$

is multiplication by n.

Proof. — We have the diagram

$$(\bigstar)\qquad \begin{array}{ccccc} \mathrm{K}_*(\mathrm{S}, \bar{\mathrm{S}}) & \xrightarrow{f^*} & \mathrm{K}_*(\mathrm{T}, \bar{\mathrm{T}}) & \xrightarrow{f^*} & \mathrm{K}'_*(\mathrm{T}, \bar{\mathrm{T}}) \\ & {\scriptstyle f^*}\searrow & \downarrow{\scriptstyle \mathrm{res}} & \nearrow{\scriptstyle (\oplus_e)\mathrm{H}} & \\ & & \mathrm{K}_*(\mathrm{T}, \hat{\mathrm{T}}) & & \end{array}$$

with all triangles commuting. Suppose that $\mathcal{O}_\mathrm{T} \cong (\mathcal{O}_\mathrm{S})^n$. Then the composite

$$(\mathscr{P}_\mathrm{S}, \mathscr{P}_{\bar{\mathrm{S}}}) \xrightarrow[(f^*, f^*)]{} (\mathscr{P}_\mathrm{T}, \mathscr{P}_{\bar{\mathrm{T}}}) \xrightarrow[(\mathrm{id}, r_{\bar{\mathrm{T}}})]{} (\mathscr{P}_\mathrm{T}, \mathscr{M}_{\bar{\mathrm{T}}}) \xrightarrow[(f^*, f^*)]{} (\mathscr{P}_\mathrm{S}, \mathscr{M}_{\bar{\mathrm{S}}})$$

sends $(\mathrm{M}, \bar{\mathrm{M}})$ to $(\mathrm{M}^n, \bar{\mathrm{M}}^n)$, hence induces multiplication by n

$$\times n:\quad \mathrm{K}_*(\mathrm{S}, \bar{\mathrm{S}}) \to \mathrm{K}'_*(\mathrm{S}, \bar{\mathrm{S}}) \cong \mathrm{K}_*(\mathrm{S}, \bar{\mathrm{S}}).$$

Thus the composite

$$\mathrm{K}_*(\mathrm{S}, \bar{\mathrm{S}}) \xrightarrow[f^*]{} \mathrm{K}'_*(\mathrm{T}, \bar{\mathrm{T}}) \xrightarrow[f_*]{} \mathrm{K}'_*(\mathrm{S}, \bar{\mathrm{S}}) \xrightarrow{\sim} \mathrm{K}_*(\mathrm{S}, \bar{\mathrm{S}})$$

is multiplication by n. The second assertion follows from this and the commutativity of $(\bigstar)$.

For the first assertion, note that $\oplus_e : \mathscr{P}_{\hat{\mathrm{T}}} \to \mathscr{M}_{\bar{\mathrm{T}}}$ induces $\times\,\mathbf{e}$ under the composition

$$\mathrm{K}_*(\mathscr{P}_{\hat{\mathrm{T}}}) \xrightarrow{(\oplus_e)_*} \mathrm{K}_*(\mathscr{M}_{\bar{\mathrm{T}}}) \overset{\text{devissage}}{\xrightarrow{\sim}} \mathrm{K}_*(\mathscr{P}_{\mathrm{T}}).$$

Since the ladder

$$\begin{array}{ccccccc} \to & \mathrm{K}'_*(\mathrm{T}, \bar{\mathrm{T}}) & \to & \mathrm{K}_*(\mathrm{T}) & \to & \mathrm{K}'_*(\bar{\mathrm{T}}) & \to \\ & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & \\ \to & \mathrm{K}_*(\mathrm{S}, \bar{\mathrm{S}}) & \to & \mathrm{K}_*(\mathrm{S}) & \to & \mathrm{K}_*(\bar{\mathrm{S}}) & \to \end{array}$$

is commutative, the result follows from (A), $(\bigstar)$, and our definitions. □

PROPOSITION 1.16. — *Suppose $f: \mathrm{T} \to \mathrm{S}$ is Galois with group* G. *Then there is a homomorphism*

$$f_*:\quad \mathrm{K}_*(\mathrm{T}, \hat{\mathrm{T}})[1/|\mathrm{G}|] \to \mathrm{K}_*(\mathrm{S}, \bar{\mathrm{S}})[1/|\mathrm{G}|]$$

satisfying

(i) $f_*(\eta^\sigma) = f_*(\eta)$; $\eta \in \mathrm{K}_*(\mathrm{T}, \hat{\mathrm{T}})$

(ii) *the ladder*

$$\begin{array}{ccccccc} \to & \mathrm{K}_*(\mathrm{T}, \hat{\mathrm{T}})[1/|\mathrm{G}|] & \to & \mathrm{K}_*(\mathrm{T})[1/|\mathrm{G}|] & \to & \mathrm{K}_*(\hat{\mathrm{T}})[1/|\mathrm{G}|] & \to \\ & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & \\ \to & \mathrm{K}_*(\mathrm{S}, \bar{\mathrm{S}})[1/|\mathrm{G}|] & \to & \mathrm{K}_*(\mathrm{S})[1/|\mathrm{G}|] & \to & \mathrm{K}_*(\bar{\mathrm{S}})[1/|\mathrm{G}|] & \to \end{array}$$

is commutative

(iii) $f^* \circ f_*(\eta) = \sum_{\sigma \in G} \eta^\sigma$ *in* $K_*(T, \bar{T})$, *for* η *in* $K_*(T, \hat{T})$

(iv) *if* $\mathcal{O}_T$ *is a free* $\mathcal{O}_S$ *module, then*

$$f_* f^* : \quad K_*(S, \bar{S})[1/|G|] \to K_*(S, \bar{S})[1/|G|]$$

is multiplication by $|G|$.

Proof. — The pair of functors (id, $\oplus_e$), together with the homotopy H gives the commutative ladder

$$\begin{array}{ccccccc} \to K'_{q+1}(T, \bar{T}) & \to & K_q(T) & \to & K'_q(\bar{T}) & \to \\ \uparrow \oplus_e H & & \| & & \uparrow \times_e & \\ \to K'_{q+1}(T, \hat{T}) & \to & K_q(T) & \to & K'_q(\hat{T}) & \to \end{array}$$

As $T \to S$ is Galois, the ramification indices e_{ij} all divide $|G|$, so $\oplus_e^H$ is an isomorphism. Symmetrizing $\oplus_e^H$ with respect to G gives the map h:

$$h = 1/|G| . \sum_{\sigma \in G} (\oplus_e^H)^\sigma$$

and a commutative G-equivariant ladder

$$\begin{array}{ccccccc} \to K'_{q+1}(T, \bar{T})[1/|G|] & \to & K_q(T)[1/|G|] & \to & K'_q(\bar{T})[1/|G|] & \to \\ \uparrow h & & \| & & \uparrow \times_e & \\ \to K'_{q+1}(T, \hat{T})[1/|G|] & \to & K_q(T)[1/|G|] & \to & K'_q(\hat{T})[1/|G|] & \to \end{array}$$

Define f_* to be the composition of h with $f_*: K'_q(T, \bar{T})[1/|G|] \to K_q(S, \bar{S})[1/|G|]$. Then (i) and (ii) are clear; (iii) follows from the isomorphisms

$$M \otimes_{\mathcal{O}_S} \mathcal{O}_T \xrightarrow{\sim} \bigoplus_{\sigma \in G} M^\sigma; \qquad \bar{M} \otimes_{\mathcal{O}_S} \mathcal{O}_{\bar{T}} \xrightarrow{\sim} \bigoplus_{\sigma \in G} \bar{M}^\sigma.$$

The statement (iv) follows from Proposition 1.15. $\square$

COROLLARY 1.17. — *Suppose* T, S *and* S′ *are semi-local, one dimensional regular schemes.* $f: T \to S$ *a Galois cover with group* G, *and* $p: S' \to S$ *étale. Let* $f': T' \to S'$ *be the fiber product* $T \times_S S'$, *let*

$$f_*: \quad K_*(T, \hat{T})[1/|G|] \to K_*(S, \bar{S})[1/|G|]$$

and

$$f'_*: \quad K_*(T', \hat{T})[1/|G|] \to K_*(S', \bar{S}')[1/|G|]$$

be given by Proposition 1.16, *where* $\hat{T}$ *and* $\hat{T}'$ *are the respective reduced schems* $\bar{T}_{red}$ *and* $\bar{T}'_{red}$. *Let*

$$p_*: \quad K_*(S', \bar{S}')[1/|G|] \to K_*(S, \bar{S})[1/|G|]$$

and

$$p_*:\quad K_*(T', \bar{T}')[1/|G|] \quad\to\quad K^*(T, \bar{T})[1/|G|]$$

the usual pushforward. Then

$$p_* \circ f'_* = f_* \circ p'_*.$$

Proof. — Since $f^*: K_*(S, \bar{S}) \to K_*(T, \bar{T})$ has $|G|$ torsion kernel, it suffices to show that

$$f^* \circ p_* \circ f'_* = f^* \circ f_* \circ p'_*.$$

We have $f^* \circ p_* = p'_* \circ f'^*$, hence we need only show that

$$p'_* \circ (f'^* \circ f'_*) = (f^* \circ f_*) \circ p'_*.$$

This follows from Proposition 1.16(iii). □

1.11. Iterated relative K-theory. — As if life weren't bad enough already, one can iterate the relativization of K-theory we have considered so far. More precisely, let X be a scheme, Y_1 and Y_2 closed subschemes with inclusions $i_1: Y_1 \to X$, $i_2: Y_2 \to X$. Let Y_{12} be the intersection $Y_1 \cap Y_2$, $i_{12}: Y_{12} \to Y_1$ the inclusion. The restriction map

$$(i_2^*, i_{12}^*):\quad (\mathcal{P}_X, \mathcal{P}_{Y_1}) \to (\mathcal{P}_{Y_2}, \mathcal{P}_{Y_{12}})$$

together with the natural isomorphism θ_2 gives a map of homotopy fibers

$$i_2^*:\quad K(X, Y_1) \to K(Y_2, Y_{12}).$$

We let $K(X, Y_1, Y_2)$ be the homotopy fiber of i_2^*. Similarly, let $K(X, Y_2, Y_1)$ be the homotopy fiber of $i_1^*: K(X, Y_2) \to K(X, Y_{12})$. The Quetzlcoatl lemma shows

(*a*) $K(X, Y_1, Y_2)$ and $K(X, Y_2, Y_1)$ are naturally homeomorphic;

(*b*) if $Y_1 \cap Y_2 = \varnothing$, then $K(X, Y_1, Y_2)$ is naturally homeomorphic to $K(X, Y_1 \cup Y_2)$.

We let $K_p(X, Y_1, Y_2)$ be the homotopy group $\pi_{p+1}(K(X, Y_1, Y_2), *)$. From (*a*), we get a commutative diagram

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
\to & & K_*(X, Y_1, Y_2) & \to & K_*(X, Y_1) & \to & K_*(Y_2, Y_{12}) & \to & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\to & & K_*(X, Y_2) & \to & K_*(X) & \to & K_*(Y_2) & \to & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\to & & K_*(Y_1, Y_2) & \to & K_*(Y_1) & \to & K_*(Y_{12}) & \to & \\
 & & \downarrow & & \downarrow & & \downarrow & & .
\end{array}$$

If one wants to iterate this, one runs into trouble with compatibilities between the homotopies; we therefore use the approach of paragraph 1.1 to replace the categories

$$\mathcal{P}_X,\ \mathcal{P}_{Y_1},\ \mathcal{P}_{Y_2},\ \mathcal{P}_{Y_{12}},\ \text{etc.}$$

with equivalent categories so that the appropriate diagram commutes exactly, not just up to homotopy. In this case the homotopy fibers are again functorial, which enables one to define $K_*(X, Y_1, Y_2, \ldots, Y_n)$ inductively as the homotopy fiber of

$$K(X, Y_1, Y_2, \ldots, Y_{n-1}) \to K(Y_n, Y_{1n}, Y_{2n}, \ldots, Y_{n-1,n})$$
$$Y_{in} = Y_i \cap Y_n.$$

The groups one gets are independent of the order of the Y_i, and there is an isomorphism $K_*(X, Y_1, Y_2, \ldots, Y_n) \xrightarrow{\sim} K_*(X, \bigcup Y_i)$ if the Y_i's are pairwise disjoint. There is an n-dimensional commutative diagram generalizing the two dimensional diagram above.

Returning to the case $n=2$, we have the diagram

$$\begin{tikzcd}
K(X, Y \cup Y') \arrow[r] \arrow[d, "{(id_X, i_Y^*)}"'] & BQ\mathscr{P}_X \arrow[r, "i^*_{Y \cup Y'}"] \arrow[d, equal] & BQ\mathscr{P}_{Y \cup Y'} \arrow[d, "i^*_{Y'}"] \arrow[dd, bend left=60, "j^*_{Y'}"] & \\
K(X, Y) \arrow[r] \arrow[d, "{(i^*_{Y'}, i^*_{Y \cap Y'})}"'] & BQ\mathscr{P}_X \arrow[r, "i^*_Y"] \arrow[dr, "i^*_{Y'}"'] & BQ\mathscr{P}_Y \arrow[dr, "i^*_{Y \cap Y'}"'] & \\
K(Y', Y \cap Y') \arrow[rr] & & BQ\mathscr{P}_{Y'} \arrow[r] & BQ\mathscr{P}_{Y \cap Y'}
\end{tikzcd}$$

($j_{Y'}: Y' \to Y \cup Y'$ the inclusion)

The map $j^*_{Y'}$ gives a contraction of the composition

$$(i^*_{Y'}, i^*_{Y \cup Y'}) \circ (id_X, i^*_Y): K(X, Y \cup Y') \to K(Y', Y \cap Y'),$$

hence a lifting of (id_X, i^*_Y) to a map

$$\Theta_{Y \cup Y'}: \quad K(X, Y \cup Y') \to K(X, Y, Y').$$

Similarly, we get a map

$$\Theta_{1 \ldots n}: \quad K(X, Y_1 \cup \ldots \cup Y_n) \to K(X, Y_1, \ldots, Y_n),$$

inducing

$$\Theta_{1 \ldots n}: \quad K_*(X, Y_1 \cup \ldots \cup Y_n) \to K_*(X, Y_1, \ldots, Y_n)$$

which is an isomorphism if the Y_i's are pairwise disjoint.

1.12. Chern classes. — In this section, we recall Gillet's construction of Chern classes [Gillet] and indicate how one constructs Chern classes for relative K-theory. In fact, Gillet has already given the details for the contruction of "Chern classes with support", which is nothing more than Chern classes for the homotopy fiber of $j^*: BQ\mathscr{P}_X \to BQ\mathscr{P}_U$, where $j: U \to X$ is an open subset of X. As there is no essential difference between the cases of the open immersion and the closed embedding, we will be somewhat sketchy.

We first recall some notions from the theory of sheaves of simplicial sets. We use the notations of [Gillet]; for details we refer the reader to section 1 of that work. For a complex of sheaves of abelian groups F^* on $S_{\mathscr{Z}ar}$, we let $\kappa(F^*, n)$ be the Dold-Puppe construction on F^*. If $\mathscr{S}$ is a sheaf of simplicial sets on $S_{\mathscr{Z}ar}$, there is the notion of the generalized cohomology groups of $\mathscr{S}$ defined by

$$\mathbb{H}^{-p}(X, \mathscr{S}) := \pi_p(R\,\Gamma(X, \mathscr{S}))$$

where $R\,\Gamma$ is the functor described in section 1 of [Gillet]. In particular, we have

$$\mathbb{H}^p(X, \kappa(F^*, n)) = \mathbb{H}^{p+n}_{\mathscr{Z}ar}(X, F^*).$$

Let $\Gamma(*)$ be a twisted duality theory in the sense of [Bloch-Ogus] for schemes over a base scheme S. There is an injective complex of sheaves $\Gamma^*(*)$ on the big Zariski site over S such that for each S-scheme X, we have

$$H^p(X, \Gamma(q)) = \mathbb{H}^p_{\mathscr{Z}ar}(X, \Gamma^*(q)).$$

Let $\mathscr{B}\mathscr{G}l$ be the sheaf of simplicial sets associated the presheaf

$$U \to BGl(\Gamma(U, \mathscr{O}_U)).$$

Gillet constructs a map of sheaves of simplicial sets

$$C_q:\quad \mathbb{Z}_\infty \mathscr{B}\mathscr{G}l \to \mathbb{Z}_\infty \kappa(\Gamma^*(q), dq);$$

$d=1$ or 2, depending on $\Gamma(-)$ which gives rise to the Chern class

$$c_{q,p}:\quad K_{dq-p}(X) \to H^p(X, \Gamma(q))$$

by the composition

$$\begin{array}{ccccc}
K_{2q-p}(X) & \to & \mathbb{H}^{p-dq}(X, \mathbb{Z}_\infty \mathscr{B}\mathscr{G}l) & \to & \mathbb{H}^{p-dq}(X, \mathbb{Z}_\infty \kappa(\Gamma^*(q), dq)) \\
& {\scriptstyle c_{q,p}} \searrow & & & \| \\
& & H^p(X, \Gamma(q)) & = & \mathbb{H}^p_{\mathscr{Z}ar}(X, \Gamma^*(q)).
\end{array}$$

Now suppose we have a closed subscheme $i: Y \to X$ of an S-scheme X. Replacing the appropriate simplicial sheaves with weakly equivalent sheaves, we may assume that the horizontal maps in the commutative square

$$\begin{array}{rccc}
i^*: & \mathbb{Z}_\infty \mathscr{B}\mathscr{G}l_X & \to & i_* \mathbb{Z}_\infty \mathscr{B}\mathscr{G}l_Y. \\
& {\scriptstyle C_q} \downarrow & & \downarrow {\scriptstyle C_q} \\
i^*: & \mathbb{Z}_\infty \kappa(\Gamma^*(q), dq)_X & \to & \mathbb{Z}_\infty i_* \kappa(\Gamma^*(q), dq)_Y
\end{array}$$

are global fibrations, and all the simplicial sheaves are flasque. Let $\mathscr{K}(X, Y)$ be the fiber (hence homotopy fiber) of the first i^*, and let $\mathbb{Z}_\infty \kappa(\Gamma^*(X, Y)(q), dq)$ be the fiber (hence homotopy fiber) of the second i^*. Let,

$$\Gamma^*(X, Y)(q) = \mathrm{Cone}(i^*: \Gamma^*(q)_X \to i_* \Gamma^*(q)_Y)[-1].$$

Then

$$\mathbb{H}^{p-dq}(X, \mathbb{Z}_\infty \kappa(\Gamma^*(X, Y)(q), dq)) = \mathbb{H}^p_{\mathscr{Z}ar}(X, \Gamma^*(X, Y)(q))$$

and

$$H^p(X, Y, \Gamma(q)) = \mathbb{H}^p_{\mathcal{Z}ar}(X, \Gamma^*(X, Y)(q)).$$

C_q lifts to a map

$$C_q\colon\ \mathcal{K}(X, Y) \to \mathbb{Z}_\infty \kappa(\Gamma^*(X, Y)(q), dq).$$

This defines as above the Chern class $c_{q,\,p}$ by the composition

$$\begin{array}{ccc} K_{dq-p}(X, Y) \to \mathbb{H}^{p-dq}(X, \mathcal{K}(X, Y)) & \to & \mathbb{H}^{p-dq}(X, \mathbb{Z}_\infty \kappa(\Gamma^*(X, Y)(q), dq)) \\ {\scriptstyle c_{q,p}} \searrow & & \| \\ H^p(X, Y, \Gamma(q)) & = & \mathbb{H}^p_{\mathcal{Z}ar}(X, \Gamma^*(X, Y)(q)). \end{array}$$

From this construction, we see that the Chern classes are compatible with the long exact relativization sequence and the long exact cohomology sequence for the pair (X, Y):

$$\begin{array}{ccccccccc} \to K_{dp-p}(X, Y) & \to & K_{dq-p}(X) & \to & K_{dq-p}(Y) & \to & K_{qd-p-1}(X, Y) \to \\ {\scriptstyle c_{q,p}} \downarrow & & {\scriptstyle c_{q,p}} \downarrow & & {\scriptstyle c_{q,p}} \downarrow & & {\scriptstyle c_{q,p+1}} \downarrow \\ \to H^p(X, Y, \Gamma(q)) & \to & H^p(X, \Gamma(q)) & \to & H^p(Y, \Gamma(q)) & \to & H^{p+1}(X, Y, \Gamma(q)) \to \end{array}$$

Let x be in $K_a(X, Y)$, y in $K_b(X)$. The formula for $c_{q,\,p}(xy)$ in terms of the Chern classes of x [in $H^*(X, Y, \Gamma(*))$] and the Chern classes of y [in $H^*(X, \Gamma(*))$] is formally the same as given by the product formula for the absolute Chern classes, using the structure of $H^*(X, Y, \Gamma(*))$ as a module over $H^*(X, \Gamma(*))$. The proof is the same as in the case of Chern classes with support, and we refer the reader to [Gillet] for details.

Soulé [So] has defined Chern classes for K-theory with coefficients. Specifically, let n be a positive integer, and X an affine scheme over $\mathbb{Z}[1/n]$. Then there are Chern classes

$$c_{q,\,p}\colon\ K_{2q-p}(X, \mathbb{Z}/n) \to H^p_{et}(X, (\mu_n)^{\otimes q}),$$

compatible with Gillet's Chern classes

$$c_{q,\,p}\colon\ K_{2q-p}(X) \to H^p_{et}(X, (\mu_n)^{\otimes q})$$

via the natural map $K_{2q-p}(X) \to K_{2q-p}(X; \mathbb{Z}/n)$.

2. Specialization in relative K_2

2.1. Symbols for relative K_2. — We first recall the work of Dennis-Stein, Keune and Loday on $K_2(A, I)$. Let A be a ring (with unit), $I \subset A$ an ideal. Keune and Loday define a relative Steinberg group St(A, I), and group of elementary matices E(A, I) with a surjection

$$St(A, I) \underset{\pi}{\to} E(A, I) \to 1.$$

$K_2(A, I)$ is then defined to be the kernel of π. St(A, I) and E(A, I) maps to the usual St(A) and E(A), and the resulting map of $K_2(A, I)$ to $K_2(A)$ fits into a long exact sequence

$$K_3(A)\to K_3(A/I)\to K_2(A, I)\to K_2(A)\to K_2(A/I)\to K_1(A, I)\to K_1(A)\to K_1(A/I)$$

Here $K_1(A, I)=\ker(Gl(A)\to Gl(A/I))/E(A, I)$.

Loday [Loday] has shown that there are natural isomorphisms of the $K_i(A, I)$ defined above with the groups $K_i(A, A/I)$ defined earlier via homotopy theory, so that the maps in the above sequence correspond with the long exact homotopy sequence for relative K-theory.

Keune [Keune] constructs certain explicit elements in $K_2(A, I)$ analogous to the symbols $\{a, b\}$ in K_2 of a field. Let $D(A, I)$ be the group with generators

$$\langle a, b\rangle, \qquad \text{with} \quad (a, b)\in A\times I\cup I\times A, \text{and } 1+ab\in A^*.$$

and relations

(D1) $$\langle a, b\rangle=\langle -b, -a\rangle^{-1}$$

(D2) $$\langle a, b\rangle\langle a, c\rangle=\langle a, b+c+abc\rangle$$

(D3) $$\langle a, bc\rangle=\langle ab, c\rangle\langle ac, b\rangle \quad \text{if} \quad a, b, \text{ or } c \text{ is in I}.$$

There is a functorial group homomorphism

$$\Phi_{A, I}\colon\ D(A, I)\to K_2(A, I);$$

if $A\to B$ is a homomorphism of A to a ring B in which b is a unit, then $\Phi(\langle a, b\rangle)$ goes to the symbol $\{1+ab, b\}$ in $K_2(B)$. We will often denote $\Phi(\langle a, b\rangle)$ by $\langle a, b\rangle$ if there is no cause for confusion. Keune has shown the following result:

THEOREM. — *Let* A *be commutative ring. Suppose* $I\subset A$ *is a radical ideal* ($I\subset \operatorname{Jac}(A)$). *Then*

$$\Phi_{A, I}\colon\ D(A, I)\to K_2(A, I)$$

is an isomorphism.

2.2. BLOCH'S SYMBOLS FOR RELATIVE K_2. — Suppose now that A is a commutative ring without nilpotents, I an ideal of A, and let L be the quotient field of A. Let $\bar{K}_2(A, I)$ be the group

$$\bar{K}_2(A, I)=(1+I)^*\otimes_{\mathbb{Z}} L^*/\{f\otimes(1-f)\mid f\in(1+I)^*, f\neq 1\}.$$

We denote the image of $a\otimes b$ in $\bar{K}_2(A, I)$ by $\{a, b\}$. Bloch has defined a map

$$\psi_{A, I}\colon\ D(A, I)\to \bar{K}_2(A, I)$$

by

$$\psi(\langle a, b\rangle)=\begin{cases}\{1+ab, b\} & \text{for} \quad b\neq 0\\ 0 & \text{if} \quad b=0.\end{cases}$$

Weibel [W2] has constructed an inverse to $\psi_{A, I}$ in certain cases, which include the case

$$\text{(2.1)} \qquad \text{A a semi-local PIR, } I=\operatorname{Jac}(A).$$

Our purpose here is to analyze the case where A is regular and semi-local, and

$$I=\left(\prod_{i=1}^{m} t_i^{e_i}\right) \subset \operatorname{Jac}(A)$$

with the ideals (t_i) relatively prime. We first assume that A contains a field of characteristic zero; the case of positive characteristic is actually easier to handle.

We will *not* show that $\psi_{A, I}$ is an isomorphism, rather we content ourselves with exhibiting a specific element of $D(A, I)$ mapping to certain symbols $\{a, b\}$, with suitable conditions on a and b. This lifting will be compatible with the inverse to $\psi_{A, I}$ given by Weibel if (A, I) satisfies (2.1).

By the Chinese Remainder Theorem, we may choose the t_i's so that

$$t_i \equiv 1 \bmod t_j^{e_j} \qquad \text{for} \quad i \neq j.$$

Let

$$s=\prod t_i^{e_i}; \qquad q_i=\prod_{j \neq i} t_j^{e_j}$$

and let a be in A. Then $1+sa$ is in $(1+I)^*$. Fix an i, and let $q=q_i$, $t=t_i$, etc. Then $1+qas$ is in $(1+I)^*$ and

$$(1+as)(1+e^{-1}qas)^{-e}=1+as(1-q)+s^2 c; \qquad \text{for some } c \text{ in A}.$$

Since $q \equiv 1 \bmod t^e$, we get

$$(1+as)=(1+t^e a' s)(1+e^{-1} qas)^e;$$

$a' \in A$ uniquely determined. We now define a function $\tau=\tau_{A, (t_1, \ldots, t_m)}$ on pairs of the form

$$(1+as, u.\prod t_i^{n_i}); \qquad a \in A, \quad u \in A^*,$$

with values in $D(A, I)$, as follows:

1. Since $\{1+as, u\}=\psi(\langle asu^{-1}, u\rangle)$, let

$$\tau(1+as, u)=\langle asu^{-1}, u\rangle.$$

2. To define $\tau(1+as, t)$, $t=t_i$, write $1+as$ as

$$1+as=(1+t^e a' s)(1+e^{-1} qas)^e,$$

so

$$\begin{aligned}\{1+as,\ t\}&=\{1+t^e a' s,\ t\}\{1+e^{-1}qas,\ t\}^e\\&=\{1+t^e a' s,\ -t^{e-1}a' s\}^{-1}\{1+e^{-1}qas,\ -e^{-1}as^2\}^e\\&=\{1+t^e a' s,\ -t^{e-1}a' s\}^{-1}\{1+e^{-1}qas,\ -e^{-1}as\}^e\{1+e^{-1}qas,\ s\}^e\\&=\psi(\langle -t,\ -t^{e-1}a' s\rangle^{-1}\langle -q,\ -e^{-1}as\rangle^e\langle e^{-1}qa,\ s\rangle^e).\end{aligned}$$

Thus we set

$$\tau(1+as,\ t_i)=\langle -t_i,\ -t_i^{e_i-1}a' s\rangle^{-1}\langle -q_i,\ -e_i^{-1}as\rangle^e\langle e^{-1}q_i a,\ s\rangle^e.$$

3. Define $\tau(1+as,\ u.\prod t_i^{n_i})$ by

$$\tau(1+as,\ u.\prod t_i^{n_i})=(\prod\tau(1+as,\ t_i)^{n_i}).\tau(1+as,\ u).$$

Let G be the subgroup of L^* generated by A^* and the t_i's, and let $\mathbb{Z}[(1+I)^*\times G]$ be the free abelian group on $(1+I)^*\times G$. The above defines τ as a map

$$\tau:\quad \mathbb{Z}[(1+I)^*\times G]\to D(A,\ I)$$

which makes the diagram

$$\begin{tikzcd}\mathbb{Z}[(1+I)^*\times G] \arrow[r, "\tau"] \arrow[dr, "\text{symbol}"'] & D(A,\ I) \arrow[d, "\psi_{A,I}"]\\ & \bar{K}_2(A,\ I)\end{tikzcd}$$

commute. Composing τ with $\Phi_{A,I}$: $D(A,\ I)\to K_2(A,\ I)$, we get a map

$$\eta=\eta_{A,\ (t_1,\ \ldots,\ t_m)}:\quad \mathbb{Z}[(1+I_A)^*\times G_A]\to K_2(A,\ I) \tag{2.2}$$

(note the dependence on the choice of the t's). η_A is functorial for ring homomorphisms π: $A\to A'$ such that A' is semi-local, and where we use the $\pi(t_i)$ for $\eta_{A'}$. In addition, suppose that A' is a semi-local PIR with Jacobson radical I', and π: $A\to A'$ is a ring homomorphism with $\pi(I)\subset I'$. Then using 2.1,

$$\pi(\eta(f,\ g))=\Phi_{A',\ I'}\circ\psi_{A',\ I'}^{-1}(\{\pi(f),\ \pi(g)\})\quad\text{in } K_2(A',\ I'). \tag{2.3}$$

The above construction also works in arbitrary characteristic if we assume that all the e_i's are 1.

If A contains a field of characteristic $p>0$, there are in general obstructions to lifting a symbol $\{a,\ b\}$ in $\bar{K}_2(A,\ I)$ to $D(A,\ I)$. However, for our purposes it suffices to work in $K_2(A,\ I)[1/p]$, where the lifting problem is easy to solve. In fact, suppose that A is semi-local, and $I=(t)A$. Let a be in A, so $1+ta$ is in $(1+I)^*$, and let b be in A with $|\operatorname{div}(b)|\subset\operatorname{supp}(A/I)$. Then for $n\gg 0$, letting $q=p^n$, b divides $(ta)^q$. As

$$\{1+ta,\ b\}^q=\{1+(ta)^q,\ b\}=\psi_{A,I}(\langle(ta)^q/b,\ b\rangle),$$

we define $\eta_{A, I}$ by

$$\eta_{A,I}(1+ta, b)=q^{-1}\Phi_{A,I}(\langle (ta)^q/b, b\rangle) \quad \text{in } K_2(A, I)[1/p]. \tag{2.4}$$

It is easy to check that $q^{-1}\langle (ta)^q/b, b\rangle$ in $D(A, I)[1/p]$ is independent of the choice of n. This defines

$$\eta_{A,I}\colon \quad \mathbb{Z}[(1+I)^*\times G]\to K_2(A, I)[1/p]$$

with functorial properties as above.

2.3. PRODUCTS AND SYMBOLS. — If A is a commutative ring, u and v units in A, then the symbol $\{u, v\}$ in $K_2(A)$ agrees with the cup pruduct $u\cup v$, where we consider u and v as elements of $K_1(A)$ via the canonical inclusion $A^*\to K_1(A)$. We proceed to derive a similar relationship between the symbol $\{f, g\}$ in $K_2(A, I)$, $f\in(1+I)^*$, $g\in A^*$, and the cup product $f\cup g$,

$$\cup\colon \quad K_1(A, I)\otimes K_1(A)\to K_2(A, I).$$

Since $K_1(A, I)=\ker(Gl(A)\to Gl(A/I)/E(A, I))$, the map $Gl_1(A)\to Gl(A)$ induces a homomorphism

$$\iota\colon \quad (1+I)^*\to K_1(A, I).$$

This is split by the determinant map $\det\colon K_1(A, I)\to(1+I)^*$, so ι is injective.

PROPOSITION 2.1. — *Let a be in* I, *let b be a unit in* A, *and suppose that $1+ab$ is a unit. Then*

$$\Phi_{A,I}(\langle a, b\rangle)=(\iota(1+ab))\cup b.$$

Proof. — Let R be the ring $\mathbb{Z}[u, u^{-1}, t, (1+tu)^{-1}]$. Define a ring homorphism $\pi\colon R\to A$ by

$$\pi(u)=b, \qquad \pi(t)=a.$$

Since $R\to R/(t)$ is split by the inclusion $\mathbb{Z}[u, u^{-1}]\to R$, we have the exact sequences

$$0\to K_2(R, (t)R)\to K_2(R)\to K_2(R/(t))\to 0$$

and

$$0\to K_1(R, (t)R)\to K_1(R)\to K_1(R/(t))\to 0.$$

As $\Phi_{R,(t)}(\langle t, u\rangle)$ maps to the symbol $\{1+tu, u\}$ in $K_2(R)$, we get

$$\Phi(\langle t, u\rangle)=\iota(1+tu)\cup u \quad \text{in } K_2(R, (t)R).$$

The result then follows from the functoriality of Φ, $\langle\ ,\ \rangle$ and $\cup$. $\square$

COROLLARY 2.2. — *Let* A *a semi-local* PIR, $I \subset A$ *the Jacobson radical,* $f \in (1+I)^*$, $g \in A^*$. *Then*

$$\{f, g\} = f \cup g \quad \textit{in } K_2(A, I),$$

where we identify $(1+I)^*$ *with* $K_1(A, I)$, A^* *with* $K_1(A)$, *and* $\bar{K}_2(A, I)$ *with* $K_2(A, I)$.

Proof. — Write $f = 1 + a$, a in I. Then

$$\begin{aligned}\{f, g\} &= \psi_{A, I}(\langle ag^{-1}, g \rangle)\\ &= \iota(1+a) \cup g\\ &= f \cup g. \ \square\end{aligned}$$

2.4. MILNOR K_3. — Let F be a field. Bass and Tate have considered the Milnor ring $K_*^M(F)$ of F. This is the tensor algebra on F^*, modulo the ideal generated by tensors $a \otimes (1-a)$, $a \neq 1$. The image of a tensor $a_1 \otimes \ldots \otimes a_n$ in $K_n^M(F)$ is denoted $\{a_1 \ldots a_n\}$. Let R be a Dedekind domain with quotient field F. There is a tame symbol map

$$T_p: \quad K_p(F) \to \bigoplus_{\substack{P \subset R \\ \text{prime}}} K_{p-1}(\mathbf{k}(P))$$

where $\mathbf{k}(P)$ is the residue field of P. One can then define $K_p^M(R)$ to be the kernel of T_p. This is not really the "correct" definition in general; however, if R is semi-local it is reasonable to force a "Gersten's conjecture" for Milnor K-theory by taking this as a definition.

An obvious subgroup of $K_*^M(R)$ is the subgroup generated by the tensor algebra on R^*. Dennis and Stein [D-S] have shown

THEOREM. — $K_2^M(R) \cong K_2(R)$ *if* R *is a semi-local PIR. Furthermore,* $K_2(R)$ *is generated by* $R^* \otimes R^*$.

We now give an extension of the latter statement to K_3^M, with some additional hypotheses on R.

PROPOSITION 2.3. — *Let* R *be a semi-local PIR with infinite residue fields. Then* $K_3^M(R)$ *is generated by* $R^* \otimes R^* \otimes R^*$.

Proof. — Let $(t_1), \ldots, (t_r)$ be the maximal ideals of R. If R is a field there is nothing to prove; we therefore assume the result for $R[t_1^{-1}]$ and proceed by induction. Let $t = t_1$ and let τ be in $K_3^M(R)$. By induction we can write τ as

$$\tau = \prod \{a_i, b_i, c_i\}; \qquad a_i, b_i, c_i \quad \text{in } R[t^{-1}]^*.$$

As $\{t, t\} = \{t, -1\}$, we may also assume that b_i, c_i are in R^* and $a_i = t$ for all i. Let $\eta = \Pi \{b_i, c_i\}$, so $\tau = \{t, \eta\}$. Then

$$T_{(t)}(\tau) = (\bar{\eta})^{\pm 1},$$

where $\bar{\eta}$ is the image of η in $K_2(R/(t))$ and $T_{(t)}$ is the (t) component of T_3. Since τ is in $K_3^M(R)$, $\bar{\eta}=1$.

Suppose that $b_i' b_i''=b_i$ with b_i' and b_i'' in R^*. Then

$$\{b_i, c_i\}=\{b_i', c_i\}\{b_i'', c_i\}$$

so we may assume that

(*) for every t_j, $j=1, \ldots, r$, $b_i \not\equiv 1 \bmod t_j$.

Let u_i, v_i be units in R such that

1. $u_i \equiv v_i \equiv 1 \bmod t$.
2. $u_i \equiv b_i \bmod t_j$; $v_i \equiv c_i \bmod t_j$ for $j=2, \ldots, r$.
3. $u_i = 1-d_i t$ with d_i in R, $d_i \not\equiv 0 \bmod t$.

We may thus assume that $t \equiv 1 \bmod t_j$ for $j=2, \ldots, r$. The (2), (3) and (*) imply

4. d_i is in R^*.

We have

$$\{t, u_i, v_i\}=\{t, 1-d_i t, v_i\}=\{d_i^{-1}, 1-d_i t, v_i\},$$

which is in the image of $(R^*)^{\otimes 3}$, so we may multiply τ by $\Pi\{t, u_i, v_i\}^{-1}$. Thus we may assume that

$$\eta \to 1 \quad \text{in } K_2(R/(t_i)) \qquad \text{for} \quad i=1, \ldots, r.$$

Let $s=\prod t_i$. Then η lifts to an element of $K_2(R, (s))=D(R, (s))$, *i.e.* we can write η as a product

$$\eta=\prod\{1-e_i s, f_i\} \qquad \text{with} \quad e_i, f_i \text{ in } R.$$

This reduces to two types of symbols

(*a*) $\tau=\{t, 1-es, t_i\}$, $\quad i\neq 1$, $\quad e$ in R.

(*b*) $\tau=\{t, 1-es, u\}$, $\quad u$ in R^*, $\quad e$ in R.

For symbols of type (*a*), write $1-es$ as a product

$$1-es=(1-e' tt_i)(1-e'' tt_i)$$

with e', e'', $(1-e' tt_i)$, and $(1-e'' tt_i)$ in R^*, which reduces us to symbols of the form

$$(a') \qquad \tau=\{t, 1-ett_i, t_i\} \qquad i\neq 1,\ e \text{ in } R^*,\ 1-ett_i \text{ in } R^*.$$

But

$$\{t, 1-ett_i, t_i\}=\{t, 1-ett_i, et\}^{-1}=\{t, 1-ett_i, e\}^{-1}\{t, 1-ett_i, -1\}^{-1}$$

reducing us to symbols of the form

$$(a'') \qquad \tau=\{t, 1-ett_i, u\}, \qquad e,\ 1-ett_i,\ u \text{ in } R^*.$$

Writing $1-ett_i$ as

$$1-ett_i=(1-e't)(1-e''t) \qquad \text{with} \quad e',\ e'',\ (1-e't),\ (1-e''t) \text{ in } \mathrm{R}^*$$

reduces to symbols $\{t, 1-et, u\}=\{e, 1-et, u\}$ which is in the image of $(\mathrm{R}^*)^{\otimes 3}$.

For symbols of type (b), write $1-es$ as

$$1-es=(1-e't)(1-e''t) \qquad \text{with} \quad e',\ e'',\ (1-e't),\ (1-e''t) \text{ in } \mathrm{R}^*,$$

reducing us to symbols $\{t, 1-ct, u\}$ as above. This completes the proof. $\square$

2.5. K_2 RELATIVE TO RATIONAL CURVES. — The results in this section are preparation for the specialization homomorphism to be defined in paragraph 2.6.

Let X be a regular scheme over an infinite field k, and let $\mathrm{Y}_1, \ldots, \mathrm{Y}_n$ be smooth irreducible curves on X. Let Y^j be the connected component of $\bigcup \mathrm{Y}_i$ containing Y_j. We say that the Y_i's form a simple rational chain if

(2.4) (1) The Y_i's form a divisor with normal crossing on X.

(2) The dual graph of the Y_i's is a (not necessarily connected) tree.

(3) Each connected component Y^i is a k_i scheme ($k_i \supset k$ a field), and each node p of Y^i is k_i rational. We have $k_i=k(p)$.

(4) Each irreducible component Y_j of Y^i is absolutely irreducible and rational over k_i.

If $\mathrm{X}=\mathrm{Spec}(\mathrm{A})$, and Y_i is defined by the ideal I_i, we say that $\mathrm{I}_1, \ldots, \mathrm{I}_n$ form a simple rational chain of ideals in A. Let s be the number of connected components of $\bigcup \mathrm{Y}_j$. We will always order the Y_i's so that

$$\mathrm{Y}_i \cap \left(\bigcup_{j=1}^{i-1} \mathrm{Y}_j\right)$$

is empty for $i=1, \ldots, s$, and is a single point p_i for $i > s$. We call p_j the j^{th} *node* of the Y_i's.

2.4. LEMMA. — Let A be a semi-local k-algebra, $\mathrm{X}=\mathrm{Spec}(\mathrm{A})$, and $\mathrm{Y}_1, \ldots, \mathrm{Y}_\mathrm{N}$ smooth irreducible subvarieties of X. Let Y be the union of the Y_i's. Let W be a smooth absolutely irreducible curve on X, p a point of W. Let Z be the connected component of Y containing p. Suppose

(i) $\mathrm{W} \cap \mathrm{Y}=p$ (scheme theoretically).

(ii) There is a $k(p)$ map $\mathrm{Z} \cup \mathrm{W} \to p$.

(iii) W is rational over $k(p)$.

Then

$$\delta: \quad \mathrm{K}_3(\mathrm{X}, \mathrm{Y}_1, \ldots, \mathrm{Y}_\mathrm{N}) \to \mathrm{K}_3(\mathrm{W}, p)$$

is surjective.

Proof. — We have the commutative diagram

$$\begin{tikzcd} K_3(X, Y) \arrow[d] \arrow[dr] & \\ K_3(X, Y_1, \ldots, Y_N) \arrow[r] & K_3(W, p) \end{tikzcd}$$

so it suffices to show that

$$K_3(X, Y) \to K_3(W, p),$$

is surjective.

We first note that

$$0 \to K_3(W, p) \to K_3(W) \to K_3(p) \to 0$$

is exact. Indeed, our assumption (ii) implies that W is a $k(p)$ scheme, and the inclusion $p \to W$ is split by a projection $g: W \to p$.

Next, we claim that $K_3(W, p)$ is contained in the image of $K_3^M(W)$ in $K_3(W)$. In fact, let $\bar{k}=k(p)$. Since W is smooth and rational, and X is semi-local, W is the localization of $\mathbb{A}^1_{\bar{k}}$ at a finite set of points S,

$$S=\{p=q_1, \ldots, q_r\}.$$

We may assume that $q_1=0$. We have the localization sequence

$$\begin{tikzcd} 0 \arrow[r] & K_3(\mathbb{A}^1_{\bar{k}}) \arrow[r] & K_3(W) \arrow[r, "\delta"] & \bigoplus_{x \in (\mathbb{A}^1_{\bar{k}})^1 - S} K_2(\bar{k}(x)) \arrow[r] & 0 \\ & K_3(\bar{k}) \arrow[u, "\pi^*", "\wr"'] & & & \end{tikzcd}$$

where $\pi: \mathbb{A}^1_{\bar{k}} \to \operatorname{Spec}(\bar{k})=p$ is the projection. By [Bass-Tate], there is a similar localization sequence for Milnor K-theory:

$$0 \to K_3^M(\bar{k}) \xrightarrow[\pi^*]{} K_3^M(W) \xrightarrow{\delta} \bigoplus_{x \in (\mathbb{A}^1_{\bar{k}})^1 - S} K_2^M(\bar{k}(x)) \to 0$$

compatible with the map of Milnor K-theory to Quillen K-theory. Let $K_3(W)^M$ denote the image of $K_3^M(W)$ in $K_3(W)$.

Now suppose $\eta \in K_3(W)$ restricts to 1 in $K_3(p)$, *i.e.* η is in $K_3(W, p)$. Since $K_2^M(F)=K_2(F)$ for F a field, we can find η^* in $K_3(W)^M$ with

$$\delta(\eta)=\delta(\eta^*).$$

Then η and η^* differ by an element τ of $K_3(\bar{k})$:

$$\pi^*(\tau)\,\eta=\eta^*.$$

But then restricting to p gives

$$\tau=\eta^*|_p,$$

which is in $K_3(\bar{k})^M$. Modifying η^* by $\pi^*(\tau)$ gives $\eta=\eta^*$, hence η is in $K_3(W)^M$, as claimed.

By Proposition 2.3, given η in $K_3(W, p)$, we can then write η as

$$\eta=\prod\{\alpha_i, \beta_i, \gamma_i\}; \qquad \alpha_i, \beta_i, \gamma_i \text{ units on W.}$$

Let $\bar{\alpha}_i$, $\bar{\beta}_i$, and $\bar{\gamma}_i$ denote the restrictions to p. By (ii), the $\bar{k}$-morphism $g: W \to p$ extends to a $\bar{k}$-morphism $f: Z \to p=\operatorname{Spec}(\bar{k})$. Let

$$\bar{a}_i=f^*(\bar{\alpha}_i); \qquad \bar{b}_i=f^*(\bar{\beta}_i); \qquad \bar{c}_i=f^*(\bar{\gamma}_i).$$

By abuse of notation, we let $\{\bar{a}_i, \bar{b}_i, \bar{c}_i\}$ denote the element $\bar{a}_i \cup \bar{b}_i \cup \bar{c}_i$ of $K_3(Z)$. Then

$$\prod\{\bar{a}_i, \bar{b}_i, \bar{c}_i\}=1 \quad \text{in } K_3(Z).$$

Lift $\bar{a}_i$, $\bar{b}_i$, $\bar{c}_i$ to units a_i, b_i, c_i on X with value 1 on $Y-Z$, and value α_i, β_i, γ_i on W. We can do this since $Z \cap W=p$ (scheme theoretically). Let $\Delta=\prod\{a_i, b_i, c_i\} \in K_3(X)$. Then

(*a*) $\Delta|_W=\eta$.

(*b*) $\Delta|_Y=1$.

By (*b*), Δ lifts to an element Δ^* of $K_3(X, Y)$ which by (*a*) restricts to η, completing the proof. □

COROLLARY 2.5. — *Let* X *be a regular semi-local k-scheme,* $Y_1, \ldots, Y_n$ *a simple rational chain on* X. *Let s be the number of connected components of* $\bigcup Y_i$, $p_{s+1}, \ldots, p_n$ *the nodes. Then for* $j > s$ *the sequence*

$$0 \to K_2(X, Y_1, \ldots, Y_j) \to K_2(X, Y_1, \ldots, Y_{j-1}) \to K_2(Y_j, p_j)$$

is exact.

Proof. — Taking $N=j-1$, $W=Y_j$, $p=p_j$, the subvarieties $Y_1, \ldots, Y_N$ and W satisfy the hypotheses of the above lemma, since the Y_i's form a simple rational chain. Since $Y_i \cap Y_j$ is either empty, or is p_j, for $i=1, \ldots, j-1$, we have the long exact relativization sequence

$$\to K_p(X, Y_1, \ldots, Y_j) \to K_p(X, Y_1, \ldots, Y_{j-1}) \to K_p(Y_j, p_j) \to.$$

The corollary follows from this and lemma 2.4. □

COROLLARY 2.6. — *Let* X *be a semi-local k-scheme,* $Y_1, \ldots, Y_N$ *subvarieties of* X. *Let* W *be a smooth curve on* X, *disjoint from the* Y_i's, *and p a closed point of* W *such that* $k(W)$ *contains* $k(p)$ *and* W *is absolutely irreducible and rational over* $k(p)$. *Then the restriction map*

$$K_2(X, Y_1, \ldots, Y_N, W) \to K_2(X, Y_1, \ldots, Y_N, p)$$

is injective.

Proof. — By lemma 2.4, the map

$$K_3(X, Y_1, \ldots, Y_N, p) \to K_3(W, p)$$

is surjective. We have the commutative diagram

$$\begin{array}{ccccccc}
K_3(X, Y_1, \ldots, Y_N) & \to & K_3(p) & \to & K_2(X, Y_1, \ldots, Y_N, p) & \to & K_2(X, Y_1, \ldots, Y_N) \to \\
\| & & {\scriptstyle\alpha}\uparrow & & {\scriptstyle\beta}\uparrow & & \| \\
K_3(X, Y_1, \ldots, Y_N) & \to & K_3(W) & \to & K_2(X, Y_1, \ldots, Y_N, W) & \to & K_2(X, Y_1, \ldots, Y_N) \to \\
\uparrow & & \uparrow{\scriptstyle\delta} & & & & \\
K_3(X, Y_1, \ldots, Y_N, p) & \xrightarrow{\varepsilon} & K_3(W, p) & & & &
\end{array}$$

The map α is surjective since $p \to W$ is split. The surjectivity of ε shows that β is injective. □

LEMMA 2.7. — *Let* X *be a regular, irreducible, semi-local k-scheme, essentially of finite type over k;* $X = \mathrm{Spec}(A)$. *Let* $S = \{p_1, \ldots, p_n\}$ *be a set of closed points of* X, U *an open neighborhood of* S *in* X. *Then the map*

$$K_2(X, S) \to K_2(U, S)$$

is injective.

Proof. — Consider the commutative ladder

$$\begin{array}{ccccccccccc}
\to K_3(U) & \xrightarrow{\text{res}'} & K_3(S) & \to & K_2(U, S) & \to & K_2(U) & \to & K_2(S) \to \\
{\scriptstyle\alpha}\uparrow & & \| & & {\scriptstyle\beta}\uparrow & & {\scriptstyle\gamma}\uparrow & & \| \\
\to K_3(X) & \xrightarrow[\text{res}]{} & K_3(S) & \to & K_2(X, S) & \to & K_2(X) & \to & K_2(S) \to
\end{array}$$

By Gersten's conjecture [Quillen], γ and α are injective, hence we need to show that

$$\mathrm{Im}(\mathrm{res}') \subset \mathrm{Im}(\mathrm{res}).$$

Let $C = X - U$. Let η be in $K_3(U)$, and let $\tau \in K_2'(C)$ be $\partial(\eta)$, where ∂ is the boundary in the localization sequence

$$\to K_3(X) \to K_3(U) \xrightarrow{\partial} K_2'(C) \to.$$

Let Y be an affine regular irreducible k-scheme of finite type, V an open subset of Y such that

1. X is a localization of Y.
2. $U = X \cap V$.
3. There is an element ξ of $K_3(V)$ restricting to η.

Let $W = Y - V$, and let $\nu \in K_2'(W)$ be $\delta(\xi)$, where δ is the boundary in

$$\to K_3(Y) \to K_3(V) \xrightarrow{\delta} K_2'(W) \to.$$

Consider the restriction ν^0 of ν to $K'_2(k(W)) = K_2(k(W))$. Since $K_2(k(W))$ is generated by symbols, there is an open subset W^0 of W such that

$$\nu|_{W^0} = \Pi\{u_i, v_i\}; \qquad u_i, v_i \text{ units on } W^0.$$

Let D be the complement $W - W^0$.

Let $n = \dim_k Y$. Take a morphism $\pi: Y \to \mathbb{A}_k^{n-1}$ such that

(a) the fibers of π are curves

(b) π is smooth in a neighborhood of all the closed points of X

(c) $\pi|_W$ is finite

and

(d) $\pi^{-1}\pi(D) \cap S = \varnothing$.

(As k is infinite, we can take π to be a linear projection.)

Form the fiber square

$$\begin{array}{ccc} Y & \xleftarrow{q} & Y \times_{\mathbb{A}^{n-1}} W = Z \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle p} \uparrow{\scriptstyle s} \\ \mathbb{A}^{n-1} & \leftarrow & W \end{array}$$

where s is the section induced by the inclusion of W in Y. Then q is finite; passing to a suitable open subset Y^0 of Y containing the closed points of X gives a diagram

$$\begin{array}{ccc} Y^0 & \xleftarrow{q} & Y^0 \times_{\mathbb{A}^{n-1}} W = Z^0 \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle p} \uparrow{\scriptstyle s} \\ \mathbb{A}^{n-1} & \leftarrow & W \end{array}$$

with $s(W)$ principal on Z^0, defined by an ideal (t). We may assume that $t=1$ at all points of Z^0 lying over S, shrinking Y^0 if necessary.

Let

$$\sigma = t \cup p^*(\nu) \in K'_3(Z^0).$$

Let

$$\delta: K'_3(Y^0 - W) \to K'_2(W), \qquad \delta': K'_3(Z^0 - s(W)) \to K'_2(s(W))$$

be the boundary maps in the relevant localization sequences. Then $q_*(\sigma)$ in $K'_3(Y^0 - W) = K_3(Y^0 - W)$ has boundary $\delta(q_*(\sigma))$:

$$\delta(q_*(\sigma)) = q_*(\delta'(\sigma)) = q_*(s_*(\nu)) = \nu|_{Y^0 \cap W}.$$

Thus $\eta . q_*(\sigma)^{-1}$ extends to an element $K_3(Y^0)$ which restricts to an element η^* of $K_3(X)$. Since $t=1$ over S, $q_*(\sigma)$ restricts to 0 on S, hence

$$\mathrm{res}'(\eta)=\mathrm{res}(\eta^*)$$

completing the proof. □

COROLLARY 2.8. — *Let* X *be a regular semi-local k-scheme, essentially of finite type over k. Let* W *be a smooth absolutely irreducible curve on* X, *p a point of* W *such that* $k(W)$ *contains* $k(p)$ *and* W *is rational over* $k(p)$. *Let* $p=p_1, \ldots, p_n$ *be closed points of* X, *and w the generic point of* W. *Let* A′ *be the semi-local ring of* $S=\{w, p_2, \ldots, p_n\}$ on X, $X'=\mathrm{Spec}(A')$. *Then the map*

$$\mathrm{res}\colon\ K_2(X, W\cup p_2\cup\ldots\cup p_n)\to K_2(X', w\cup p_2\cup\ldots\cup p_n)$$

is injective.

Proof. — Let η be in the kernel of res. Let $\bar{X}$ be a finite type regular k-scheme such that X is a localization of $\bar{X}$; let $\bar{W}$ be the closure of W in $\bar{X}$, P_i the closure of p_i. We may choose $\bar{X}$, and an open neighborhood U of $w\cup p_2\cup\ldots\cup p_n$ in $\bar{X}$ so that η lifts to an element of the kernel of

$$K_2(\bar{X}, \bar{W}\cup P_2\cup\ldots\cup P_n)\to K_2(U, (\bar{W}\cup P_2\cup\ldots\cup P_n)\cap U).$$

Since W is $k(p)$ rational, the $k(p)$ rational points of $\bar{W}\cap U$ are Zariski dense in $\bar{W}$. Let $V=X\cap U$. Then, replacing X with a larger localization of $\bar{X}$ and changing notation if necessary, we may assume that W contains a closed $k(p)$ point q of V, not among the p_i's, and that η is in the kernel of

$$\mathrm{res}\colon\ K_2(X, W\cup p_2\cup\ldots\cup p_n)\to K_2(V, (W\cup p_2\cup\ldots\cup p_n)\cap V).$$

We have the commutative diagram

$$\begin{array}{ccc} K_2(X, W\cup p_2\cup\ldots\cup p_n) & \to & K_2(V, (W\cup p_2\cup\ldots\cup p_n)\cap V). \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ K_2(X, q\cup p_2\cup\ldots\cup p_n) & \xrightarrow[\gamma]{} & K_2(V, q\cup p_2\cup\ldots\cup p_n) \end{array}$$

By corollary 2.6, α is injective, and γ is injective by the previous lemma. Thus $\eta=0$, as desired. □

PROPOSITION 2.9. — *Let* A *be a semi-local k-algebra,* $x, y, t_1, \ldots, t_r$ *in* Jac(A) *with* $t_i\equiv 1 \bmod t_j$ *for* $i\neq j$, *and* $t_i\equiv 1 \bmod xy$ *for all i. We suppose that* $(t_1), \ldots, (t_r)$, *and* $(x)+(y)$ *are maximal. Let* $s=\prod t_i$. *Take f, g in* A *with* $f\equiv 1 \bmod sxy$, *and*

$$g=ux^n y^m \prod t_i^{n_i};\qquad u \text{ in } A^*.$$

Then there are elements τ_x *in* $K_2(A, xs)$, τ_y *in* $K_2(A, ys)$ *such that*

(*a*) $\tau_x=\{f, g\}$ *in* $K_2(A_{(xs)}, xs)$

(*b*) $\tau_y=\{f, g\}$ in $K_2(A_{(ys)}, ys)$

(c) $\tau_x=\tau_y$ *in* $K_2(A, ((x)+(y))\cap(s))$

Proof. — If $n=m=0$, let τ be the element of $K_2(A, (xy\prod t_i))$

$$\tau=\eta_{A,(xy, t_1, \ldots, t_r)}(f, u\prod t_i^{n_i})$$

and let τ_x, τ_y be the respective images of τ in $K_2(A, xs)$, $K_2(A, ys)$. This reduces us to the two cases $g=x$, $g=y$. We treat the case $g=y$.

Let $f=1+axys$, a in A. Write f as a product

$$1+axys=(1+axys^2)(1+bxy^2 s) \qquad b \text{ in A.}$$

Then in $D(A, ((x)+(y))\cap(s))$, we have

$$\begin{aligned}
\langle axs, y\rangle &=\langle axs^2, y\rangle\langle bxys, y\rangle \quad (D2)\\
\langle axs^2, y\rangle &=\langle -y, -axs^2\rangle^{-1} \quad (D1)\\
&=\langle axys, s\rangle^{-1}\langle -ys, -axs\rangle^{-1} \quad (D3)\\
&=\langle -s, -axys\rangle\langle axs, ys\rangle \quad (D1)\\
\langle -s, -axys\rangle &=\langle axs, ys\rangle\langle -ys^2, -ax\rangle \quad (D3)\\
&=\langle axs, y\rangle\langle ax, ys^2\rangle^{-1} \quad (D1).
\end{aligned}$$

Thus

$$\langle axs, y\rangle=\langle -ys^2, -ax\rangle\langle axs, ys\rangle^2\langle -y, -bxys\rangle^{-1}$$

in $D(A, ((x)+(y))\cap(s))$. The LHS above lifts to $\tilde{\tau}_x$ in $D(A, xs)$, the RHS to $\tilde{\tau}_y$ in $D(A, ys)$. One easily checks that

$$\begin{aligned}
\Phi_{A, xs}(\tilde{\tau}_x)&=\{1+axs, y\} \quad \text{in } K_2(A_{(xs)}, (xs))\\
\Phi_{A, ys}(\tilde{\tau}_y)&=\{1+axs, y\} \quad \text{in } K_2(A_{(ys)}, (ys)).
\end{aligned}$$

Letting $\tau_x=\Phi_{A, xs}(\tilde{\tau}_x)\in K_2(A, xs)$, $\tau_y=\Phi_{A, ys}(\tilde{\tau}_y)\in K_2(A, ys)$ completes the proof. □

2.6. THE SPECIALIZATION SUBGROUP. — Let R be a semi-local PIR containing a field k_0, $I=\mathrm{Jac}(R)$, $S=\mathrm{Spec}(R)$, $\pi: \mathbb{A}^1_S\to S$ the affine line over S, with a section $s: S\to\mathbb{A}^1_S$. Let L be the semi-local PIR $R(\mathbb{A}^1_S)$, *i. e.*, the semi-local ring of $\mathbb{A}^1_{\bar{S}}$ in $\mathbb{A}^1_S$.

Let $\mu=\mu^{\mathbf{p}}: \mathbb{A}^1_{S,\mathbf{p}}\to\mathbb{A}^1_S$ be a sequence of blow-ups of $\mathbb{A}^1_S$ at points $(p_1, \ldots, p_r):=\mathbf{p}$ lying over $s(\bar{S})$. For each partial blow-up $\mu^{(p_1, \ldots, p_i)}$, $i\leq s$, there is a section $s^{(p_1, \ldots, p_i)}: S\to\mathbb{A}^1_{S,\mathbf{p}}$ induced by the section s. We call $\mu^{\mathbf{p}}$ *allowable* if each p_{i+1} is one of the following two types

(a) $p_{i+1}\in s^{(p_1, \ldots, p_i)}(\bar{S})$

(b) p_{i+1} is one of the nodes of the exceptional divisor $E^{(p_1, \ldots, p_i)}$ of $\mu^{(p_1 \ldots, p_i)}$.

Suppose that $\mu^{\mathbf{p}}: \mathbb{A}^1_{S,\mathbf{p}}\to\mathbb{A}^1_S$ is an allowable blow-up. Let F be the proper transform of $\mathbb{A}^1_{\bar{S}}$, and let $E_1, \ldots, E_r$ be the irreducible components of the exceptional divisor of

$\mu^{\mathbf{p}}$. Then the curves

$$F, E_1, \ldots, E_r$$

form a simple rational chain on $\mathbb{A}^1_{\bar{S}, \mathbf{p}}$.

Let F^* be the union of the E_i's which meet $s^{\mathbf{p}}(\bar{S})$. Then F^* is a disjoint union of at most $m\,\mathbb{P}^1$'s, where m is the number of closed points of $\bar{S}$. Let $Q=\{q_1, \ldots, q_m\}$ be the set of closed points of $\bar{S}$, and let $N=\{n_1, \ldots, n_t\}$ be the set of nodes of $F \cup E_1 \cup \ldots \cup E_r$. We let $B^{\mathbf{p}}$ denote the semi-local ring of $N \cup s^{\mathbf{p}}(Q)$ on $\mathbb{A}^1_{\bar{S}, \mathbf{p}}$, and B^* the semi-local ring of $s^{\mathbf{p}}(Q)$ on $\mathbb{A}^1_{\bar{S}, \mathbf{p}}$. We identify L with the semi-local ring of F on $\mathbb{A}^1_{\bar{S}, \mathbf{p}}$; this gives a homomorphism

$$\xi^{\mathbf{p}}: \quad B^{\mathbf{p}} \to L.$$

PROPOSITION 2.10. — *The map*

$$\xi^{\mathbf{p}}: \quad K_2(B^{\mathbf{p}}, F, E_1, \ldots, E_r) \to K_2(L, \bar{L})$$

is injective.

Proof. — By lemma 2.7, the map

$$K_2(B^{\mathbf{p}}, F) \to K_2(L, \bar{L})$$

is injective. The map

$$K_2(B^{\mathbf{p}}, F, E_1, \ldots, E_r) \to K_2(B^{\mathbf{p}}, F)$$

is injective by corollary 2.5. □

For each allowable blow-up $\mu^{\mathbf{p}}: \mathbb{A}^1_{\bar{S}, \mathbf{p}} \to \mathbb{A}^1_{\bar{S}}$, denote the group $K_2(B^{\mathbf{p}}, F, E_1, \ldots, E_r)$ by $K_2(B^{\mathbf{p}}, F, E^{\mathbf{p}})$. If $\mu^{\mathbf{p}'}: \mathbb{A}^1_{\bar{S}, \mathbf{p}'} \to \mathbb{A}^1_{\bar{S}}$ is a blow-up of $\mathbb{A}^1_{\bar{S}}$ factoring through $\mu^{\mathbf{p}}$:

$$\begin{tikzcd} \mathbb{A}^1_{\bar{S}, \mathbf{p}'} \arrow[d, "\eta"'] \arrow[dr, "\mu^{\mathbf{p}'}"] & \\ \mathbb{A}^1_{\bar{S}, \mathbf{p}} \arrow[r, "\mu^{\mathbf{p}}"'] & \mathbb{A}^1_{\bar{S}} \end{tikzcd}$$

then we get a commutative diagram

$$\begin{tikzcd} K_2(B^{\mathbf{p}'}, F, E^{\mathbf{p}'}) \arrow[dr, "\xi^{\mathbf{p}'}"] & \\ K_2(B^{\mathbf{p}}, F, E^{\mathbf{p}}) \arrow[u, "\eta^*"] \arrow[r, "\xi^{\mathbf{p}}"'] & K_2(L, \bar{L}). \end{tikzcd}$$

This enables us to define the *specialization subgroup* $K_2(L, \bar{L})_s$ of $K_2(L, \bar{L})$ by

$$K_2(L, \bar{L})_s = \bigcup_{\substack{\mathbf{p} \\ \mu^{\mathbf{p}} \text{ allowable.}}} \xi^{\mathbf{p}}[K_2(B^{\mathbf{p}}, F, E^{\mathbf{p}})].$$

We define a homomorphism $\Psi_s\colon K_2(L, \bar{L})_s \to K_2(S, \bar{S})$ by

$$\Psi_{s\,|K_2(B^p,\,F,\,E^p)} = s^{p*}\colon K_2(B^p, F, E^p) \to K_2(S, \bar{S}). \tag{2.5}$$

Since the ξ^p are all injective, and each two allowable blow-ups can be dominated by a third, Ψ_s is well-defined. We now give a simple sufficient criterion for an element of $K_2(L, \bar{L})$ to be in $K_2(L, \bar{L})_s$.

PROPOSITION 2.11. — *Let $s\colon S \to \mathbb{A}^1_S$ be a section, and let η be in $K_2(L, \bar{L})$. Let B be the semi-local ring of $s(\bar{S})$ in $\mathbb{A}^1_S$, $U = \mathrm{Spec}(B)$. Suppose there is a reduced closed curve $Z \subset U$, and an element z of B, with $z \equiv 1 \bmod IB$, such that the tame symbol $T(\eta)|_U$ is given by*

$$T(\eta)|_U = z|_Z.$$

Then η is in $K_2(L, \bar{L})_s$.

Proof. — Take an allowable blow-up $\mu^p\colon \mathbb{A}^1_{S,\,p} \to \mathbb{A}^1_S$ so that the proper transform $Z^p := \mu^{p\,-1}[Z]$ is disjoint from the nodes of the exceptional divisor E^p of μ^p, and disjoint from $s^p(\bar{S})$. We may assume that each component of $\mu^{p\,-1}[\mathbb{A}^1_{\bar{S}}]$ intersects E^p. Blowing-up points away from the nodes of E^p and away from $s^p(\bar{S})$, by

$$\mu^q\colon\quad \mathbb{A}^1_{S,\,p,\,q} \to \mathbb{A}^1_{S,\,p}$$

we can separate Z^p from E^p, *i.e.*

$$\mu^{q\,-1}[Z^p] \cap \mu^{q\,-1}[E^p] = \varnothing.$$

Let $\mu\colon \mathbb{A}^1_{S,\,p,\,q} \to \mathbb{A}^1_S$ be the composition $\mu^p \circ \mu^q$. Let F_1 be the proper transform $\mu^{-1}[\mathbb{A}^1_{\bar{S}}]$, Z_1 the proper transform $\mu^{q\,-1}[Z^p]$. By the Remark following Corollary 1.9, we can compute the tame symbol of

$$\mu^*(\eta) \in K_2(\mathbb{A}^1_{S,\,p,\,q}, F_1)$$

as the tame symbol of the image $\bar{\mu}^*(\eta)$ of $\mu^*(\eta)$ in $K_2(\mathbb{A}^1_{S,\,p,\,q})$. Let $\bar{\eta}$ be the image of η in $K_2(\mathbb{A}^1_S)$. Then

$$\begin{aligned} T(\bar{\mu}^*(\eta)) &= \mu^*(T(\bar{\eta})) \\ &= \mu^*(z) \text{ on the cycle } \mu^*(Z). \end{aligned}$$

Now, $\mu^*(Z) = Z_1 + Z_{exc}$, where Z_{exc} is a cycle supported on the exceptional divisor of μ. Since

$$\mu^*(z)|_{Z_{exc}} \equiv 1,$$

it follows that $T(\mu^*(\eta)) = 0$ in a neighborhood of $E_1 := \mu^{q\,-1}[E^p]$.

First of all, this implies by a localization sequence that $\mu^*(\eta)$ extends to an element η_1 of $K_2(U_1, F_1 \cap U_1)$, for some neighborhood U_1 of E_1. Next, E_1 is a union of $\mathbb{P}^1$'s, and each connected component of E_1 intersects F_1. Since η_1 restricted to the generic points of F_1 is zero, the restriction of η_1 to each irreducible component E^i of E_1 goes to zero in $H^0(E^i, \mathscr{K}_2) \cong K_2(k_i)$, where k_i is the field of constants of E^i.

Let $U_2 \subset U_1$ be a neighborhood of $E_1 \cap F_1$ such that

(i) U_2 contains $s^{\mathbf{p}, \mathbf{q}}(\bar{S})$

(ii) U_2 contains all the nodes of E_1

(iii) $\mu^{\mathbf{q}}$ maps U_2 isomorphically onto $\mu^{\mathbf{q}}(U_2)$

(iv) $U_2 \cap E^i$ is *not* complete, for each irreducible component E^i of E_1.

By (iv), we see that the restriction of η_1 to $U_2 \cap E^i$ is zero. Thus, by lemma 2.6, η_1 determines a unique element η_2 of $K_2(B^{\mathbf{p}}, F, E^{\mathbf{p}})$. Clearly $\xi^{\mathbf{p}}(\eta_2) = \eta$, which completes the proof. □

PROPOSITION 2.12. — *Let* s, S, L, B *and* U *be as in Proposition* 2.11. *Let* f *be in* B *with* $f \equiv 1 \bmod I$. *Let* M *denote the quotient field of* L, *and let* g *be in* M^* *such that* $s(S)$ *is not a component of* $\operatorname{div}(g)$. *Then* $\{f, g\} \in K_2(L, \bar{L})$ *is in* $K_2(L, \bar{L})_s$, *and*

$$\Psi_s(\{f, g\}) = \{s^*(f), s^*(g)\}.$$

Proof. — Let $\mu^{\mathbf{p}}: \mathbb{A}^1_{S, \mathbf{p}} \to \mathbb{A}^1_S$ be an allowable blow-up such that $\mu^{-1}[|\operatorname{div}(g)|]$ is disjoint from the nodes of the exceptional divisor E of $\mu^{\mathbf{p}}$, disjoint from $s^{\mathbf{p}}(\bar{S})$, and disjoint from $E \cap F$, where F is the proper transform of $\mathbb{A}^1_{\bar{S}}$. We may assume that each connected component of $F \cup E$ has dual graph a straight line, *i.e.* a tree with exactly two end vertices, and that one end lies in F, and the other end is the unique irreducible component of E passing through $s^{\mathbf{p}}(\bar{S})$.

As the tame symbol of $\mu^{\mathbf{p}*}(\{f, g\})$ in a neighborhood of E is

$$T(\mu^{\mathbf{p}*}(\eta)) = f\,|_{\mu^{\mathbf{p}*}(\operatorname{div}(g))}$$

we see as in the proof of Proposition 2.11 that $\{f, g\}$ extends to an element η of $K_2(B^{\mathbf{p}}, F, E^{\mathbf{p}})$.

Write F as a union of components:

$$F = F_1 \amalg \ldots \amalg F_m$$

with F_i lying over $q_i \in \bar{S}$. We can write E as a disjoint union:

$$E = E_1 \amalg \ldots \amalg E_m$$

where E_i is the component of E intersecting F_i. Then $F_i \cup E_i$ is a connected simple rational chain on $\mathbb{A}^1_{S, \mathbf{p}}$. Let $E^1_i, \ldots, E^{r_i}_i$ be the irreducible components of E_1. Since there is the section $s^{\mathbf{p}}$ to $\pi \circ \mu^{\mathbf{p}}$ the unique component, say $E^{r_i}_i$, passing through $s^{\mathbf{p}}(q_i)$ appears with multiplicity 1 in $\mu^{\mathbf{p}-1}(\mathbb{A}^1_{\bar{S}})$.

Let $\nu^{\mathbf{p}}: \mathbb{P}^1_{S, \mathbf{p}} \to \mathbb{P}^1_S$ be the extension of $\mu^{\mathbf{p}}$ to a blow-up of $\mathbb{P}^1_S$. Let $\bar{F}_i$ denote the closure of F_i in $\mathbb{P}^1_{S, \mathbf{p}}$. Let $q = q_i$. Then the total transform $\nu^{\mathbf{p}-1}(\mathbb{P}^1_q)$ is connected, both $\bar{F}_i$ and $E^{r_i}_i$ appear with multiplicity 1 in this divisor, and are at the "ends", *i.e.*

$$E^{r_i}_i \cdot (\nu^{\mathbf{p}-1}(\mathbb{P}^1_q) - E^{r_i}_i) = \bar{F}_i \cdot (\nu^{\mathbf{p}-1}(\mathbb{P}^1_q) - \bar{F}_i) = 1.$$

In particular, $(\nu^{\mathbf{p}-1}(\mathbb{P}^1_q)-E_i^{r_i})$ is an exceptional curve of the first kind, and can be blown down to form a regular surface

$$\mu^+:\quad Y\to \operatorname{Spec}(S)$$

flat over S and having a smooth $\mathbb{P}^1$ as fiber over q. In particular, Y is a $\mathbb{P}^1$ bundle over a neighborhood of q in Spec(S). Thus, if we let

$$\mu_i:\quad \mathbb{A}^1_{S,\,i}\to\mathbb{A}^1_S$$

be the blow-up of all the p_j's in $\mathbf{p}$ lying over q, we see that $\mathbb{A}^1_{S,\,i}-(F_i\cup E_i^1\cup\ldots\cup E_i^{r_i-1})$ is isomorphic to $\mathbb{A}^1_S$, with fiber $E_i^{r_i}-E_i^{r_i-1}$ over q. Let L_1 be the semi-local ring of $E_1^{r_1}\amalg F_2\amalg\ldots\amalg F_m$, in $\mathbb{A}^1_{S,\,\mathbf{p}}$. We claim it suffices to show that

($\bigstar_1$) the image of η in $K_2(L_1,\bar{L}_1)$ is given by the symbol $\{\mu^{\mathbf{p}*}(f),\,\mu^{\mathbf{p}*}(g)\}$.

Indeed, it then follows by induction that, letting L_m be the semi-local ring of $E_1^{r_1}\amalg\ldots\amalg E_m^{r_m}$ in $\mathbb{A}^1_{S,\,\mathbf{p}}$, we have

($\bigstar_m$) the image of η in $K_2(L_m,\bar{L}_m)$ is given by the symbol $\{\mu^{\mathbf{p}*}(f),\,\mu^{\mathbf{p}*}(g)\}$.

Let B_m be the semi-local ring of $s^{\mathbf{p}}(\bar{S})$ in $\mathbb{A}^1_{S,\,\mathbf{p}}$. Arguing as above, B_m is isomorphic to the semi-local ring of the zero section in $\mathbb{A}^1_S$, hence the map

$$K_2(B_m,\bar{B}_m)\to K_2(L_m,\bar{L}_m)$$

is injective, by corollary 2.8. In addition, letting t_i be a generator for the ideal of $E_i^{r_i}$ in B_m with $t_i\equiv 1 \bmod t_j$ for $i\neq j$, we can write $\mu^{\mathbf{p}*}(f)$ and $\mu^{\mathbf{p}*}(g)$ as

$$\mu^{\mathbf{p}*}(f)=1+a\,\Pi\,t_i;\qquad \mu^{\mathbf{p}*}(g)=u\,.\,\Pi\,t_i^{n_i},\qquad \text{with}\quad a \text{ in } B_m,\quad u \text{ in } B_m^*,$$

since $\mu^{\mathbf{p}-1}[\operatorname{div}(g)]$ is disjoint from $\operatorname{Spec}(B_m)$. Let η_m be the image of η in $K_2(B_m,\bar{B}_m)$, and let η' be the element of $K_2(B_m,\bar{B}_m)$:

$$\eta'=\eta_{B_m,\,(t_1,\,\ldots,\,t_m)}(\mu^{\mathbf{p}*}(f),\,\mu^{\mathbf{p}*}(g)).$$

Then $\eta'=\eta_m$, since both have the same image in $K_2(L_m,\bar{L}_m)$. Finally,

$$\begin{aligned} s^{\mathbf{p}*}(\eta')&=\{s^{\mathbf{p}*}(\mu^{\mathbf{p}*}(f)),\,s^{\mathbf{p}*}(\mu^{\mathbf{p}*}(g))\}\\ &=\{s^*(f),\,s^*(g)\}\end{aligned}$$

the first equality following from the functorial properties of the maps $\eta_{*,\,*}$ defined in paragraph 2.2. Thus $\Psi_s(\eta)=\{s^*(f),\,s^*(g)\}$, as desired. We now prove ($\bigstar_1$).

We order the E_1^i's so that F_1 intersects E_1^1, at say n^1, and E_1^i intersects E_1^{i-1} at n^i, for $i=2,\ldots,r_1$. Let N be the set of nodes $n^1,\ldots,n^{r_1}$, and let A be the semi-local ring of $N\amalg\{s^{\mathbf{p}}(q_1)\}\amalg F_2\amalg\ldots\amalg F_m$ in $\mathbb{A}^1_{S,\,\mathbf{p}}$. Let A_i be the semi-local ring of $E_1^i\amalg F_2\amalg\ldots\amalg F_m$ in $\mathbb{A}^1_{S,\,\mathbf{p}}$, and let $A_0=L$, $E_1^0=F_1$. We will show by induction on i that

($\bigstar\bigstar$) the image η_i of η in $K_2(A, E_1^i \amalg F_2 \amalg \dots \amalg F_m)$ has image $\{\mu^{p^*}(f), \mu^{p^*}(g)\}$ in $K_2(A_i, \bar{A}_i)$.

This is true for $i=0$ by construction of η. Assume ($\bigstar\bigstar$) for $i < r_1$; then by proposition 2.9 there exist elements τ_i, τ_{i+1}

$$\tau_i \in K_2(A, E_1^i \amalg F_2 \amalg \dots \amalg F_m); \qquad \tau_{i+1} \in K_2(A, E_1^{i+1} \amalg F_2 \amalg \dots \amalg F_m)$$

such that

(*a*) τ_i has image $\{\mu^{p^*}(f), \mu^{p^*}(g)\}$ in $K_2(A_i, \bar{A}_i)$

(*b*) τ_{i+1} has image $\{\mu^{p^*}(f), \mu^{p^*}(g)\}$ in $K_2(A_{i+1}, \bar{A}_{i+1})$

and

(*c*) $\operatorname{Im}(\tau_i) = \operatorname{Im}(\tau_{i+1})$ in $K_2(A, n \amalg F_2 \amalg \dots \amalg F_m)$; $n = n^{i+1}$.

By corollary 2.8, the map

$$K_2(A, E_1^i \amalg F_2 \amalg \dots \amalg F_m) \to K_2(A_i, \bar{A}_i)$$

is injective, so (*a*) and our inductive assumption implies

(*d*) $\tau_i = \eta_i$ in $K_2(A, E_1^i \amalg F_2 \amalg \dots \amalg F_m)$.

On the other hand, we have the commutative diagram

$$\begin{tikzcd}
& K_2(A, E_1^i, E_1^{i+1}, F_2 \amalg \dots \amalg F_m) \arrow[dd, "\mathrm{res}_n"] \arrow[dr, "\mathrm{res}_{i+1}"] & \\
K_2(A, E_1^i \amalg F_2 \amalg \dots \amalg F_m) \arrow[ur, "\mathrm{res}_i"] \arrow[dr, "\mathrm{res}_n^i"'] & & K_2(A, E_1^{i+1} \amalg F_2 \amalg \dots \amalg F_m) \arrow[dl, "\mathrm{res}_n^{i+1}"] \\
& K_2(A, n \amalg F_2 \amalg \dots \amalg F_m) &
\end{tikzcd}$$

Thus, letting $\eta_{i,\,i+1}$ be the image of η in $K_2(A, E_1^i, E_1^{i+1} \amalg F_2 \amalg \dots \amalg F_m)$, we have

$$\mathrm{res}_n(\eta_{i,\,i+1}) = \mathrm{res}_n^i(\eta_i) = \mathrm{res}_n^{i+1}(\eta_{i+1})$$

so

$$\mathrm{res}_n^i(\tau_i) = \mathrm{res}_n^{i+1}(\eta_{i+1}); \qquad \text{since } \tau_i = \eta_i,$$

and

$$\mathrm{res}_n^i(\tau_i) = \mathrm{res}_n^{i+1}(\tau_{i+1}); \qquad \text{by construction of } \tau_i \text{ and } \tau_{i+1}.$$

Thus

$$\mathrm{res}_n^{i+1}(\eta_{i+1}) = \mathrm{res}_n^{i+1}(\tau_{i+1}).$$

Since res_n^{i+1} is injective by corollary 2.6, this gives

$$\eta_{i+1} = \tau_{i+1},$$

hence

$$\operatorname{Im}(\eta_{i+1}) = \operatorname{Im}(\tau_{i+1}) = \{\mu^{p^*}(f), \mu^{p^*}(g)\} \text{ in } K_2(A_{i+1}, \bar{A}_{i+1}\},$$

and the induction goes through. This completes the proof. $\square$

Let $S \to S'$ be a finite étale extension of semi-local PIR's; J the Jacobson radical of S. Given a section $s: S \to \mathbb{A}^1_S$ let s' be the induced section $s': S \to \mathbb{A}^1_{S'}$. Each allowable blow-up of $\mathbb{A}^1_S$ gives by pull-back an allowable blow-up of $\mathbb{A}^1_{S'}$; conversely, each allowable blow-up of $\mathbb{A}^1_{S'}$ is dominated by an allowable blow-up which is pulled back from $\mathbb{A}^1_S$. This shows that the norm map

$$N: \quad K_2(R(\mathbb{A}^1_{S'}), \bar{R}(\mathbb{A}^1_{S'})) \to K_2(R(\mathbb{A}^1_S), \bar{R}(\mathbb{A}^1_S))$$

restricts to

$$N: \quad K_2(R(\mathbb{A}^1_{S'}), \bar{R}(\mathbb{A}^1_{S'}))_{s'} \to K_2(R(\mathbb{A}^1_S), \bar{R}(\mathbb{A}^1_S))_s$$

satisfying

$$\Psi_s(N(\eta)) = N(\Psi_{s'}(\eta))$$

for η in $K_2(R(\mathbb{A}^1_{S'}), \bar{R}(\mathbb{A}^1_{S'}))_{s'}$.

Let $\{x, b\}$ be in $K_2(S', \bar{S}')$ with x in $(1+J')^*$, b in L^* where L is the quotient field of S. We claim that

$$(2.6) \qquad N(\{x, b\}) = \{N(x), b\} \quad \text{in } K_2(S, \bar{S}).$$

Indeed, we may assume that b is in S. Write $\mathbb{A}^1_S = \mathrm{Spec}(S[u])$ and consider the symbol $\{x, u\}$ in $K_2(R(\mathbb{A}^1_{S'}), \bar{R}(\mathbb{A}^1_{S'}))$. Since u is a unit in $R(\mathbb{A}^1_{S'})$, we have

$$N(\{x, u\}) = \{N(x), u\} \quad \text{in } K_2(R(\mathbb{A}^1_S), \bar{R}(\mathbb{A}^1_S)),$$

by Corollary 2.2, and the projection formula. Specializing via the section s with $s^*(u) = b$, and applying Proposition 2.12 proves (2.6).

3. Some relations in relative K_2

Let S be a semi-local PIR with Jacobson radical I. We suppose that S contains a field k_0 containing $\boldsymbol{\mu}_l$. Let α be in S^*, let $S_\alpha = S[X]/X^l - \alpha$, if $\mathrm{char}(k_0) \neq l$; if $\mathrm{char}(k_0) = l$, let $S_\alpha = S[X]/X^l - X - \alpha$. Let

$$N: \quad K_*(S_\alpha, \bar{S}_\alpha) \to K_*(S, \bar{S})$$

be the norm map, and let σ be a generator of $\mathrm{Gal}(S_\alpha/S)$. Our first object is to show

$$(A) \qquad \{x, 1-N(x)\} \text{ is in } (1-\sigma)K_2(S_\alpha, \bar{S}_\alpha) \quad \text{for all } x \in (1+IS_\alpha)^*.$$

In [M] and [S], this is done by an easy direct computation. We proceed here by a "generic element" method, coupled with the specialization techniques developed in chapter 2.

3.1. THE GENERIC ELEMENT. — Fix a prime l, and let F_0 be the prime field. If $F_0=\mathbb{Q}$, let $R=\mathbb{Q}(\zeta_l)[t]_{(t)}$; if $F_0=\mathbf{F}_p$, let $R=\mathbf{F}_p(\zeta_l, t_0)[t]_{(t)}$, with t_0 and t independent variables. Let $I=(t)$, and let k be the quotient field of R. If E is an extension ring of $\mathbb{Q}(\zeta_l)$, or of $\mathbf{F}_p$, let $R_E=E[t]_{(t)}$, $I_E=(t)R_E$ and k_E the quotient field of R_E. We let k_0 be the ground field $\mathbb{Q}(\zeta_l)$ or $\mathbf{F}_p(\zeta_l, t_0)$, and let p be the characteristic of k_0.

LEMMA 3.1. — *Let* E *be an extension field of* k_0. *If* $l\neq p$, *then* ${}_lK_2(R_E, I_E)$ *is generated by the symbols* $\{f, \zeta_l\}$ *with* f *in* $(1+I_E)^*$. ${}_pK_2(R_E, I_E)=0$ *if* $p>0$.

Proof. — Since the surjection $R_E\to R_E/I_E=E$ is split, we have the short exact sequence

$$0\to K_2(R_E, I_E)\to K_2(R_E)\to K_2(E)\to 0.$$

In addition, the map

$$K_2(R_E)\to K_2(k_E)$$

is injective. Suppose $l\neq p$. Let η be an l-torsion element of $K_2(R_E, I_E)$, so

$$\mathrm{Im}(\eta)=\{g, \zeta_l\}\quad\text{in}\ K_2(k_E),$$

for some g in k_E^*, by Suslin [S]. Since η maps to $K_2(k_E)$ via $K_2(R_E)$, the tame symbol $T_{(t)}(\mathrm{Im}(\eta))$ vanishes, *i.e.*

$$\zeta^{\mathrm{ord}_{(t)}(g)}=1.$$

Thus $g=t^{al}.u$, for some integer a, and some unit u in R_E. Then $\{g, \zeta\}=\{u, \zeta\}$. In addition, since η is in $K_2(R_E, I_E)$, $\{u, \zeta\}$ restricts to 1 in $K_2(R_E/I_E)=K_2(E)$. Let $\mathrm{res}: R_E\to E$ be the canonical surjection; then $\{\mathrm{res}(u), \zeta\}=1$ in $K_2(E)$. Thus

$$\mathrm{Im}(\eta)=\{u/\mathrm{res}(u), \zeta\}\quad\text{in}\ K_2(R_E).$$

Letting $f=u/\mathrm{res}(u)$ completes the proof in this case. If $l=p$, we use the same proof, together with the result of Suslin that ${}_pK_2(k_E)=0$. □

Let $x_0, \ldots, x_{l-1}$, v be independent variables over k, let $u=v^l$ if $l\neq p$; if $l=p$, let $u=v^p-v$. Let A and B be the rings

$$A=k_0[x_0, x_1, \ldots, x_{l-1}, x_{l-1}^{-1}, u],$$
$$B=k_0[x_0, x_1, \ldots, x_{l-1}, x_{l-1}^{-1}, v],$$

so $B=A[v]$. Let x be the element

$$x=1+t\Sigma x_i v^i\in R_B,$$

so x is the "generic element" of the universal Kummer extension (or Artin-Schreier extension if $l=p$) R_B/R_A having x_{l-1} invertible, and with $x\equiv 1 \bmod t$. Let L be the quotient field of B, E the quotient field of A.

Let $N: R_B\to R_A$ be the norm, σ the generator of $\mathrm{Gal}(R_B/R_A)$ with $\sigma(v)=\zeta v$ for $l\neq p$, $\sigma(v)=v+1$ for $l=p$. Let $X^{1/l}=\mathrm{Spec}(R_B)$, $X=\mathrm{Spec}(R_A)$. Let W be the closed subscheme of $X^{1/l}$ defined by the ideal $((1-N(x))/t)$, W' the subscheme defined by (x). We note that W and W' are reduced and irreducible. Write x as $x=1+ty$.

On $X^{1/l}-(W\cup W')$, both $(1-N(x))/t$ and x are units, so the symbols $\langle ty[(1-N(x))/t]^{-1}, (1-N(x))/t\rangle$ and $\langle y, t\rangle$ define elements

$$\eta_1=\Phi(\langle ty[(1-N(x))/t]^{-1}, (1-N(x))/t\rangle)$$
$$\eta_2=\Phi(\langle y, t\rangle)$$

of $K_2(X^{1/l}-(W\cup W'), \bar{X}^{1/l}-\bar{W})$, satisfying

$$\{x, 1-N(x)\}=\mathrm{Im}(\eta_1+\eta_2) \quad \text{in } K_2(R_L, I_L). \tag{3.1}$$

Abusing notation, we will denote the element $\eta_1+\eta_2$ of $K_2(X^{1/l}-(W\cup W'), \bar{X}^{1/l}-\bar{W})$ by $\{x, 1-N(x)\}$.

Let z be the regular function on W defined by

$$z=(1/l)\left(1+\sum_{i=1}^{l-1}(x_W)^{\sigma^{-1}+\dots+\sigma^{-i}}\right), \qquad \text{for } l\neq p \tag{3.2}$$

$$z=-v^{p-1}+\sum_{i=1}^{p-1}(-v^{p-1})^{\sigma^{-i}}.(x_W)^{\sigma^{-1}+\dots+\sigma^{-i}}), \qquad \text{for } l=p \tag{3.2$'$}$$

where x_W is the restriction of x to W. Then $z\equiv 1 \bmod t$; in particular z is not identically zero. Let $Z\subset W$ be the locus $\{z=0\}$. Then

$$x_W=z^\sigma/z \quad \text{on } W-Z. \tag{3.3}$$

Let $A^0=k_0[x_0, \dots, x_{l-1}, x_{l-1}^{-1}]$, L^0 the quotient field of A_0, and let $X^0=\mathrm{Spec}(R_{A^0})$. Then

$$X^{1/l}=\mathrm{Spec}(R_{A^0}[v]), \qquad X=\mathrm{Spec}(R_{A^0}[u]).$$

We form relative compactifications of $X^{1/l}$ and X over X^0 by introducing new variables v_0 and $u_0:=(v_0)^l$, and defining

$$X^*=\mathrm{Proj}_{R_{A^0}}R_{A^0}[uu_0, u_0];$$
$$X^{1/l*}=\mathrm{Proj}_{R_{A^0}}R_{A^0}[vv_0, v_0].$$

Let W^1 be the closure of W in $X^{1/l*}$, and W^* the normalization of W^1.

LEMMA 3.2. — *The element* x_W *of* $k_0[W]$ *extends to a regular function* x_{W^*} *on* W^*, *with* $x_{W^*}\equiv 1 \bmod t$.

Proof. — Let $X^{1/l*}(\infty)$ be the locus $\{v_0=0\}$. $W\subset X^{1/l}$ is defined by the equation

$$0=(N(x)-1)/t=\alpha_0+\sum_{i=1}^{l-1}\alpha_i u^i; \qquad \alpha_i \text{ in } A^0, \tag{3.4}$$

with

$$\alpha_0 \equiv l x_0 \bmod t\,A^0; \qquad \alpha_{l-1} = (-1)^{l-1} . t^{l-1} (x_{l-1})^l, \qquad \text{for } l \neq p$$
$$\alpha_0 \equiv x_{p-1} \bmod t\,A^0; \qquad \alpha_{p-1} = t^{p-1} . (x_{p-1})^p, \qquad \text{for } l = p.$$

Since $t x_{l-1}$ is a unit on $X^0 - \bar{X}^0$, $W - \bar{W}$ is finite over $X^0 - \bar{X}^0$, so $W^1 - W$ is contained in $\bar{X}^{1/l}(\infty)$. Extend x_W to a rational function x_{W^*} on W^*. Then each component C of $(x_{W^*})_\infty$ dominates $\bar{X}^0$. Pass to the semi-local ring $R^0 := R_{L^0}$ of $\bar{X}^0$ in X^0. Then $W^*_{R^0} := W^* \times_{X^0} \operatorname{Spec}(R_{L^0})$ is proper over R_{L^0}, and is irreducible; since R_{L^0} is a DVR, this implies that $W^*_{R^0}$ is finite over R^0.

Let ν be an extension of the valuation of R^0 defined by (t) to a valuation on $k_0(W)$. One easily checks that

$$\nu(\alpha_i) \geqq i, \qquad \text{for } i = 0, \ldots, l-1, \; \nu(\alpha_{l-1}) = l-1.$$

Let $a = -\nu(v)$. Since $x = 1 + t \Sigma x_i v^i$, we need only show that $a \leqq 1/l$. Assume that $a > 1/l$. As

$$\nu(u^i \alpha_i) \geqq -ila + i \qquad \text{and} \qquad \nu(u^{l-1} \alpha^{l-1}) = -(l-1)\, la + l - 1 < 0,$$

we see that

$$\nu(u^{l-1} \alpha_{l-1}) < \nu(u^i \alpha_i) \qquad \text{for } i = 0, \ldots, l-2.$$

But then (3.4) shows that

$$0 \leqq \nu(u^{l-1} \alpha_{l-1}) = -a(l-1)\, l + l - 1,$$

hence $a \leqq 1/l$, contrary to our assumption. □

COROLLARY 3.3. — *The function z defined in* (3.2) *extends to a regular function* z_{W^*} *on* W^*, *with* $z_{W^*} \equiv 1 \bmod t$. *The divisor* (z_{W^*}) *is disjoint from* $\bar{W}^*$, *so the divisor* $Z = (z)$ *on* W *is proper over* X^0.

Proof. — Immediate from the lemma and (3.2). □

We now proceed to explicitly solve the equation

$$\{x, 1 - N(x)\} = a^\sigma / a \quad \text{in } K_2(U, \bar{U})$$

for a particular open subset U of $X^{1/l}$. The final result (3.7) includes an additonal term of the form $\{g, \zeta_l\}$, but we will absorb this factor later on.

Let $W^* \to W^0 \to X^0$ be the Stein factorization of $W^* \to X^0$. Then, as W is finite over $X^0 - \bar{X}^0$, we have

$$W^0 - \bar{W}^0 \xrightarrow{\sim} W - \bar{W}.$$

Since z_{W^*} is identically 1 on $\bar{W}^*$, z_{W^*} defines a regular function z_{W^0} on W^0, with $z_{W^0} \equiv 1 \bmod t$. Let $Z^0 \subset W^0$ be the locus $\{z_{W^0} = 0\}$; then $Z^0 \cap \bar{W}^0 = \varnothing$.

Form the pullbacks

$$X^{1/l}_{W^0}:=X^{1/l}\times_{X^0}W^0;\qquad X^{1/l^*}_{W^0}:=X^{1/l^*}\times_{X^0}W^0.$$

Then

$$p_1:\quad X^{1/l}_{W^0}\to X^{1/l}\qquad\text{and}\qquad p_1:\quad X^{1/l^*}_{W^0}\to X^{1/l^*}$$

are finite, $X^{1/l}_{W^0}$ is isomorphic over W^0 to $\mathbb{A}^1_{W^0}$, and $X^{1/l^*}_{W^0}$ is isomorphic over W^0 to $\mathbb{P}^1_{W^0}$.

Homogenizing the equation (3.4) gives the equation for W^1 in $X^{1/l*}$. Thus $\bar{W}^1\to\bar{X}^0$ is finite over the locus $x_0\neq 0$ (if $l=p$, $\bar{W}^1\to\bar{X}^0$ is finite). In particular, there is a codimension two subset T^0 of W^0, $T^0\subset\bar{W}^0$, and a closed subset T of W^1, $T\subset\bar{W}^1$, such that the birational map $W^0\to W^1$ defines a birational finite morphism

$$p:\quad W^0-T^0\to W^1-T.$$

The map p composed with the inclusion $W^1-T\to X^{1/l^*}$ gives a section

$$s:\quad W^0-T^0\to X^{1/l^*}_{W^0}\cong\mathbb{P}^1_{W^0}.$$

Let F′ be a section of $\mathcal{O}(1)$ on $\mathbb{P}^1_{W^0-T^0}$ with divisor

$$\operatorname{div}(F')=s(W^0-T^0).$$

Since T^0 has codimension at least two, and W^0 is normal, F′ extends to a section F of $\mathcal{O}(1)$ on $\mathbb{P}^1_{W^0}$ with

$$\operatorname{div}(F)=\text{closure of } s(W^0-T^0).$$

Let f be the restriction of F to $X^{1/l}_{W^0}\cong\mathbb{A}^1_{W^0}$, considered as a regular function on $\mathbb{A}^1_{W^0}$. Let Y^0 be the closure in $\mathbb{A}^1_{W^0}$ of $s(W^0-T^0)\cap\mathbb{A}^1_{W^0}$. Then

$$\operatorname{div}(f)=Y^0.$$

In addition, the restriction of p_1

$$p_{1\,|\,Y^0}:\quad Y^0\to X^{1/l}$$

gives a finite birational morphism from Y^0 onto W. Let $p_2: X^{1/l}_{W^0}\to W^0$ be the second projection and let $D^0=p_2^{-1}(Z^0)$; we note that $D^0\cap\mathbb{A}^1_{\bar{W}^0}=\varnothing$.

Consider the symbol

$$\{p_2^*(z_{W^0}),\ f\}\in K_2(R(\mathbb{A}^1_{W^0}),\ I(\mathbb{A}^1_{W^0})).$$

On $\mathbb{A}^1_{W^0}-Y^0-D^0$, $p_2^*(z_{W^0})$ is a unit, $p_2^*(z_{W^0})\equiv 1 \bmod t$, and f is a unit. Writing $p_2^*(z_{W^0})$ as $1+ta$, for some regular function a on $\mathbb{A}^1_{W^0}$, the symbol

$$\mu^0:=\Phi(\langle taf^{-1},\ f\rangle)\in K_2(\mathbb{A}^1_{W^0}-Y^0-D^0,\ \mathbb{A}^1_{\bar{W}^0}-\bar{Y}^0)$$

has image $\{p_2^*(z_{W^0}), f\}$ in $K_2(R(\mathbb{A}^1_{W^0}), I(\mathbb{A}^1_{W^0}))$. The tame symbol of μ^0 is

$$T(\mu^0)=z_{W^0} \quad \text{on } Y^0.$$

Let $D=p_1(D^0)$, and let μ be the element $p_{1*}(\mu^0)$ of $K_2(X^{1/l}-W-D, \bar{X}^{1/l}-\bar{W})$. Then μ has tame symbol

$$T(\mu)=p_{1*}(T(\mu^0))=(z \text{ on } W) \tag{3.5}$$

in $K_1'(X^{1/l}-D, \bar{X}^{1/l})^{1/2}$. In addition, $D=p_{X^0}{}^{-1}(p_{X^0}(D))$, *i.e.*, D consists of fibers of $p_{X^0}: X^{1/l}\to X^0$.

We now pass to $R^0:=R_{L^0}$. $X^{1/l}_{R^0}$ is isomorphic to $\mathbb{A}^1_{R^0}$. The element

$$\{x, 1-N(x)\}.\mu^\sigma/\mu\in K_2(\mathbb{A}^1_{R^0}-(W\cup W'), \bar{\mathbb{A}}^1_{R^0}-\bar{W})$$

has trivial tame symbol by construction, hence determines a unique element α of $K_2(\mathbb{A}^1_{R^0}, \bar{\mathbb{A}}^1_{R^0})\cong K_2(R^0, I^0)$. On the other hand, under the norm map

$$N: \quad K_2(R_L, I_L)\to K_2(R_E, I_E)$$

(recall that E is the quotient field of $A=k_0[x_0, \ldots, u]$ and L is the quotient field of $A[v]$) we have

$$\begin{aligned}\alpha^l&=N(\alpha)\\&=N(\{x, 1-N(x)\}\mu^\sigma/\mu)\\&=\{N(x), 1-N(x)\}\\&=1 \quad \text{in } K_2(R_E, I_E).\end{aligned}$$

Since the map

$$K_2(R_{L^0}, I_{L^0})\to K_2(R_E, I_E)$$

is injective, we have

$$\alpha^l=0 \quad \text{in } K_2(R_{L^0}, I_{L^0}).$$

By lemma 3.1, we have

$$\begin{cases}\{x, 1-N(x)\}\mu^\sigma/\mu=\{g, \zeta_l\} \quad \text{in } K_2(\mathbb{A}^1_{R^0}, \bar{\mathbb{A}}^1_{R^0}), & \text{if } l\neq p,\\ \{x, 1-N(x)\}\mu^\sigma/\mu=1 \quad \text{in } K_2(\mathbb{A}^1_{R^0}, \bar{\mathbb{A}}^1_{R^0}), & \text{if } l=p.\end{cases} \tag{3.6}$$

Here g is an element of $(1+I_{L^0})^*$, $g=1+tb$.

Since K-theory commutes with direct limits, we have proved the

PROPOSITION 3.4. — *There is an affine open subset* U *of* $X^{1/l}-W-W'$, *containing the generic point of* $\bar{X}^{1/l}$, *such that* b *is regular on* U, μ *extends to an element of* $K_2(U, \bar{U})$

and

$$(3.7)\qquad \begin{cases} \{x,\ 1-\mathrm{N}(x)\}\,\mu^{\sigma}/\mu=\{g,\ \zeta_l\} & \textit{in } \mathrm{K}_2(\mathrm{U},\ \bar{\mathrm{U}}), \qquad \textit{if } l\neq p,\\ \{x,\ 1-\mathrm{N}(x)\}\,\mu^{\sigma}/\mu=1 & \textit{in } \mathrm{K}_2(\mathrm{U},\ \bar{\mathrm{U}}), \qquad \text{if } l=p.\end{cases}$$

Here $\{g,\ \zeta_l\}$ *stands for the element* $\Phi(\langle tb\zeta_l^{-1},\ \zeta_l\rangle)$ *of* $\mathrm{K}_2(\mathrm{U},\ \bar{\mathrm{U}})$, *and as explained above,* $\{x,\ 1-\mathrm{N}(x)\}$ *stands for the image of the element* $\eta_1+\eta_2$ *of* $\mathrm{K}_2(\mathrm{X}^{1/l}-\mathrm{W}-\mathrm{W}',\ \bar{\mathrm{X}}^{1/l}-\mathrm{W})$ *defined in* (3.1).

3.2. THE ELEMENT $\{x,\ 1-\mathrm{N}(x)\}$. — We retain the notations X^0, X, $\mathrm{X}^{1/l}$, U, W, W′, x, μ and g of the previous section. Let S be a semi-local PIR containing k_0, J the Jacobson radical of S, $\mathrm{J}=(\mathrm{t})\,\mathrm{S}$. Extend the inclusion $k_0\to\mathrm{S}$ to a ring homomorphism $g^*:\mathrm{R}\to\mathrm{S}$ by $g^*(t)=\mathrm{t}$. Letting $\mathrm{T}=\mathrm{Spec}(\mathrm{S})$, we get a smooth R scheme $g:\mathrm{T}\to\mathrm{Spec}(\mathrm{R})$ with $\bar{\mathrm{T}}=\mathrm{Spec}(\mathrm{S}/\mathrm{J})$. Let

$$\mathrm{t}=\prod_{i=1}^{r}\mathrm{t}_i$$

be a prime factorization of t in S.

If $\mathrm{A}\to\mathrm{Spec}(\mathrm{R})$, $\mathrm{B}\to\mathrm{Spec}(\mathrm{R})$ are R-schemes, we let $\mathrm{A}_{\mathrm{B}}\to\mathrm{B}$ be the B-scheme $p_2:\mathrm{A}\times_{\mathrm{Spec}\,(\mathrm{R})}\mathrm{B}\to\mathrm{B}$.

Take α in S^*, fix a prime l, and let S^{α} be the étale cyclic extension of S:

$$\begin{aligned}\mathrm{S}^{\alpha}&=\mathrm{S}[\mathrm{X}]/(\mathrm{X}^{l}-\alpha)=S[\beta], \qquad \textit{if } l\neq p\\ &=\mathrm{S}[\mathrm{X}]/(\mathrm{X}^{p}-\mathrm{X}-\alpha)=\mathrm{S}[\beta], \qquad \text{if } l=p.\end{aligned}$$

Let $\mathrm{J}^{\alpha}=\mathrm{JS}^{\alpha}$, $\mathrm{T}^{\alpha}=\mathrm{Spec}(\mathrm{S}^{\alpha})$. Let $\mathrm{X}_{\mathrm{T}}(\alpha)$ be the subscheme of X_{T} defined by the ideal $(u-\alpha)$, and let $\mathrm{X}_{\mathrm{T}}^{1/l}(\alpha)$ be the subscheme of $\mathrm{X}_{T}^{1/l}\alpha$ defined by the ideal $(v-\beta)$. We get the commutative diagram

$$\begin{tikzcd} \mathrm{X}_{\mathrm{T}}^{1/l}(\alpha) \arrow[rr] \arrow[dr] \arrow[dd] & & \mathrm{X}_{\mathrm{T}}(\alpha) \arrow[dd] \\ & \mathrm{X}_{\mathrm{T}}^{0} \arrow[ur] \arrow[dr] & \\ \mathrm{T}^{\alpha} \arrow[rr] & & \mathrm{T} \end{tikzcd}$$

We note that $\mathrm{X}_{\mathrm{T}}^{1/l}(\alpha)$ is isomorphic to the fiber product $(\mathrm{X}_{\mathrm{T}}^{1/l})\times_{\mathrm{T}}(\mathrm{X}_{\mathrm{T}}(\alpha))$. If we modify α by the l-th power of a unit in S, $\alpha'=\mathrm{v}^{l}\alpha$ (or $\alpha'=\alpha+a^{p}+a$ if $l=p$) then $\mathrm{T}^{\alpha'}$ and T^{α} are isomorphic as T-schemes, thus we may assume that

$(3.10)_{\mathrm{T}}$ each component of $\bar{\mathrm{X}}_{\mathrm{T}}^{1/l}(\alpha)$ has non-empty intersection with $\mathrm{U}_{\mathrm{T}^{\alpha}}$.

If $\mathrm{Z}\to\mathrm{T}$ is a finite étale T-scheme, let $\mathrm{Z}^{\alpha}=\mathrm{T}^{\alpha}\times_{\mathrm{T}}\mathrm{Z}$. For all such Z, the condition $(3.10)_{\mathrm{Z}}$ is satisfied.

If $y=1+t\Sigma y_i\beta^i$ is in $(1+J^\alpha)^*$, y_i in S^α and y_{l-1} in $S^{\alpha *}$, then y determines a pair of compatible sections $\mathcal{Y}$ and $\mathcal{Y}^\alpha$:

$$\begin{array}{ccc} X_T^{1/l}(\alpha) & \to & X_T^0 \\ \downarrow \uparrow \mathcal{Y}^\alpha & & \downarrow \uparrow \mathcal{Y} \\ T^\alpha & \to & T \end{array}$$

by $\mathcal{Y}=(y_0, \ldots, y_{l-1})$, $\mathcal{Y}^\alpha=(y_0, \ldots, y_{l-1}, \beta)$.

Let $p\colon \mathbb{A}^1_{T^\alpha}\to\mathbb{A}^1_T$ be the map induced by $T^\alpha\to T$. For Z a closed subset of $\mathbb{A}^1_T$, we let Z^α denote $p^{-1}(Z)$.

LEMMA 3.5. — *Let $g=g(z)$ be in $S^\alpha[z]_{(z)}$, z an indeterminant, with $g\equiv 1 \bmod t$, such that*

$$\mathrm{ord}_{t_i}(1-N(g))=\mathrm{ord}_{t_i}(1-N(g(0)));\qquad \text{for}\quad i=1, \ldots, r$$

where we consider g as an element of $S^\alpha[z]_{(J^\alpha S^\alpha[z])}$ in the LHS.

Let C be a closed subset of $\mathbb{A}^1_T$. Then there is a closed subset D of $\mathbb{A}^1_T$, with $D\cap\{z=0\}=\varnothing$, and a proper map

$$F\colon\ \mathbb{A}^1_T-D\to\mathbb{A}^1_T$$

such that

(*a*) $\bar{F}\colon \mathbb{A}^1_{\bar{T}}-\bar{D}\to\mathbb{A}^1_{\bar{T}}$ *is étale at each generic point*

(*b*) $F^{-1}(0)=\{z=0\}\cup Z$, with $Z\cap(C\cup\{z=0\})=\varnothing$

(*c*) $\pi_Z\colon Z\to T$ *is finite and étale*

(*d*) *g is regular on* $\mathbb{A}^1_{T^\alpha}-D^\alpha$.

Let $i\colon Z\to\mathbb{A}^1_T$, $j\colon Z^\alpha\to\mathbb{A}^1_{T^\alpha}$ be the inclusions. For each section $s\colon T\to\mathbb{A}^1_T$, let $Z(s)=F^{-1}(s(T))$, and let $i_s\colon Z(s)\to\mathbb{A}^1_T-D$ be the inclusion. Let $j_s\colon Z(s)^\alpha\to\mathbb{A}^1_{T^\alpha}-D^\alpha$ denote the inclusion of $Z(s)^\alpha$, and let

$$\pi_s\colon\ Z(s)\to T;\qquad \pi^\alpha_s\colon\ Z(s)^\alpha\to T^\alpha;\qquad \pi\colon\ Z\to T;\qquad \pi^\alpha\colon\ Z^\alpha\to T^\alpha$$

denote the projections. Then

(*e*) *If F is étale over a neighborhood of $s(T)$, then*

$$\{g(0),\ 1-N(g(0))\}=(\pi^\alpha_s)_*(\{j^*_s(g),\ 1-N(j^*_s(g))\}).\,\pi^\alpha_*(\{j^*(g),\ 1-N(j^*(g))\}^{-1}).$$

Proof. — Our assumption on $N(g)-1$ can be rephased as $1-N(g)=\nu.\Pi\, t_i n_i$; ν a unit on an affine neighborhood V of $\{z=0\}$ in $\mathbb{A}^1_{T^\alpha}$.
Then

$$\Xi:=\eta_{A,\, t_1\ldots t_r}((g,\ 1-N(g))\in K_2(V, \bar{V});\qquad V=\mathrm{Spec}(A),$$

is defined, and for each map $q\colon R\to V$, with R a regular, semi-local curve with $\bar{R}$ reduced,

$$q^*(\Xi)=\{q^*(g),\ 1-q^*(N(g))\}\quad\text{in}\ K_2(R, \bar{R}).$$

Extend $p\colon \mathbb{A}^1_{T^\alpha} \to \mathbb{A}^1_T$ to $p\colon \mathbb{P}^1_{T^\alpha} \to \mathbb{P}^1_T$. Let $D^{\alpha *} = \mathbb{P}^1_{T^\alpha} - V$, $D^* = p(D^{\alpha *})$. Shrinking V if necessary, we may assume that $D^{\alpha *} = p^{-1}(D^*)$. Let m be the degree of D^* over T. We note that D^* contains no component of $\mathbb{P}^1_T$. Let s_∞ be a section of $\mathcal{O}(m)$ on $\mathbb{P}^1_T$ with $(s_\infty) = D^*$. Let s_0 be a section of $\mathcal{O}(m)$ such that

$$(s_0) = 1.\{z=0\} + Z,$$

with

$$(\bigstar) \qquad Z \subset \mathbb{A}^1_T - (D^* \cup C \cup \{z=0\}), \quad Z \to T \text{ étale, and } Z \text{ reduced.}$$

Let $D = D^* \cap \mathbb{A}^1_T$.

Let $F\colon \mathbb{A}^1_T - D \to \mathbb{A}^1_T$ be the restriction to $\mathbb{A}^1_T - D$ of the map

$$(id_T, (s_0 : s_\infty))\colon \mathbb{P}^1_T \to \mathbb{P}^1_T.$$

Then F is a finite degree m map with $F^{-1}(0) = (s_0)$. From $(\bigstar)$ it follows that F satisfies (*a*)-(*d*). Let $F^\alpha\colon \mathbb{A}^1_{T^\alpha} - D^\alpha \to \mathbb{A}^1_{T^\alpha}$ be the map induced by F.

Let $\tau = F^{\alpha *}(\Xi) \in K_2(\mathbb{A}^1_{T^\alpha}, \bar{\mathbb{A}}^1_{T^\alpha})$. Let

$$q\colon \ \mathbb{A}^1_T \to T; \qquad q^\alpha\colon \ \mathbb{A}^1_{T^\alpha} \to T^\alpha$$

be the projections. By the homotopy property, $\tau = q^{\alpha *}(\xi)$ for some ξ in $K_2(T^\alpha, \bar{T}^\alpha)$. Then for sections $s, s'\colon T \to \mathbb{A}^1_T$, we get induced sections $s^\alpha, s'^\alpha\colon T^\alpha \to \mathbb{A}^1_{T^\alpha}$ and

$$\begin{aligned}\pi^{\alpha *}_s(j^{\alpha *}_s(\Xi)) &= s^{\alpha *} \circ (F|_{Z(s)}\alpha)_*(j^{\alpha *}_s(\mu))\\ &= s^{\alpha *}(F_*(\mu))\\ &= s^{\alpha *} \circ q^{\alpha *}(\xi)\\ &= \xi,\end{aligned}$$

and similarly for s'. Taking s' to be the zero section, we find

$$\begin{aligned}\pi^{\alpha *}_{s'}(j^{\alpha *}_{s'}(\Xi)) &= \pi^{\alpha *}(j^*(\Xi)) + s'^*(\Xi)\\ &= \pi^{\alpha *}(\{j^*(g), 1 - N(j^*(g)\}).\{g(0), 1 - N(g(0)\}\end{aligned}$$

which completes the proof. $\square$

LEMMA 3.6. — *The subgroup of* $K_2(T^\alpha, \bar{T}^\alpha)/(1-\sigma) K_2(T^\alpha, \bar{T}^\alpha)$ *generated by the symbols* $\{y, 1-N(y)\}$ *with* $y = 1 + t\Sigma y_i \beta^i$ *in* $(1+J^\alpha)^*$ *is the same as the subgroup generated by elements of the form*

$$N_{Q^\alpha/T^\alpha}(\{w, 1-N(w)\}),$$

where $Q \to T$ *range over finite étale* T-*schemes, and* $w \in (1+J^{Q^\alpha})^*$, $w = 1 + t\Sigma w_i \beta^i$, $w_i \in \Gamma(Q, \mathcal{O}_Q)$ *satisfies*

(*a*) w_{l-1} *and* w_0 *are units in* S^α

(*b*) $\mathcal{W}(x)$ *is in* U_Q *for each generic point* x *of* Q.

Here $\mathscr{W}: Q \to X_Q^{1/l}$ *is the section determined by* w.

Proof. — Given an open subset V of $\mathbb{A}^1_T$, we can find elements $y' = 1 + t \Sigma y'_i \beta^i$ and z of $(1+J^\alpha)^*$ with

$$y'_i(T) \subseteqq V \quad \text{for } i = 1, \ldots, l-1; \qquad y' = z^\sigma / z,$$

so $N(y') = 1$. Take V so that $a \in V$ implies that $a + y_0$ and $a + y_{l-1}$ are units in S^α. Then $y'' = yy'$ satisfies (a), and

$$\begin{aligned}\{y, 1-N(y)\} &= \{y, 1-N(y'')\} \\ &= \{y'', 1-N(y'')\}\{y', 1-N(y)\}^{-1} \\ &= \{y'', 1-N(y'')\}\{z, 1-N(y)\}^{(1-\sigma)}\end{aligned}$$

which proves (a), with $Q = T$ and $w = y''$.

For (b), let N be the maximum of $\mathrm{ord}_b(1-N(y))$, as b ranges over the closed points of T. Let $P_0(z), \ldots, P_{l-1}(z)$ be in $S[z]$, z an indeterminant, such that

$$(\bigstar) \qquad \begin{cases} P_i(z) \equiv 0 \bmod t^N & \text{for } i = 0, \ldots, l-1, \\ & P_i(0) = 0. \end{cases}$$

Let $g(z) = y + \Sigma P_i(z) \beta^i$. Let M denote the total quotient field of S. Choosing the P_i sufficiently general, we may assume that the $S[z]$-valued point $\mathscr{G}$ of $X^{1/l}$ determined by g satisfies

$$\mathscr{G}(x) \in U_{M(z)} \qquad \text{for each generic point } x \text{ of } \mathrm{Spec}(M(z)).$$

By ($\bigstar$), g satisfies the hypotheses of Lemma 3.2; applying that lemma, with C being the closure of the points of $\mathscr{G}^{-1}(X^{1/l} - U)$ lying over Spec(M), and s being any sufficiently general section, proves (b). □

LEMMA 3.7. — *Let* $\mathscr{S}: T \to \mathbb{A}^1_T$ *be a section,* $\mathscr{S}^\alpha: T^\alpha \to \mathbb{A}^1_{T^\alpha}$ *the induced section. Let* Q *be a neighborhood of* $\mathscr{S}(T)$ *in* $\mathbb{A}^1_T$, Q^α *the neighborhood of* $\mathscr{S}^\alpha(T^\alpha)$ *lying over* Q. *Let* $R(Q)$ (*resp.* $R(Q^\alpha)$), *be the semi-local ring of* $\bar{Q}$ *in* Q (*resp.* $\bar{Q}^\alpha$ *in* Q^α). *Suppose there are elements* $\beta \in K_2(Q^\alpha, \bar{Q}^\alpha)$ *and* $f \in (1 + JR(Q^\alpha))^*$ *with*

$$\mathrm{Im}(\beta) = \{f, \zeta_l\} \quad \text{in } K_2(R(Q^\alpha), \bar{R}(Q^\alpha)).$$

Then $\mathscr{S}^{\alpha *}(\beta) = \{g, \zeta_l\}$ *in* $K_2(T^\alpha, \bar{T}^\alpha)$, *for some* $g \in (1 + J^\alpha)^*$.

Proof. — Let $p: \mathbb{A}^1_{T^\alpha} \to \mathbb{A}^1_T$ be the obvious map. As in lemma 3.5, we construct a proper map

$$F: \quad Q \to \mathbb{A}^1_T$$

with $F^{-1}(0) = s(T) + Z$, $Z \to T$ étale, and $Z \subset Q - p(\mathrm{div}(f)) - s(T)$. Since $f = 1 + at$ on Q^α, with a regular on Q^α, the symbol $\{f, \zeta_l\}$ is the image in $K_2(R(Q^\alpha), R(Q^\alpha))$ of the element $\Phi(\langle at\zeta^{-1}, \zeta \rangle)$ of $K_2(Q^\alpha, \bar{Q}^\alpha)$. Then, retaining the notations of lemma 3.5, we have

$$\begin{aligned}\pi^{\alpha *}(j^{\alpha *}(\beta)) &= \pi^{\alpha *}(\{j^{\alpha *}(f), \zeta_l\}), \quad \text{by functoriality of } \Phi\\ &= \{\pi^{\alpha *} j^{\alpha *}(f), \zeta_l\} \qquad \text{(projection formula)}\\ &= \{g', \zeta_l\}, \qquad g' \in (1+\mathrm{J}^{\alpha})^*.\end{aligned}$$

Similarly, for a sufficiently general section $s: \mathrm{T} \to \mathbb{A}^1_{\mathrm{T}}$, we have

$$\begin{aligned}\pi^{\alpha *}_s(j^*_s(\beta)) &= \pi^{\alpha *}_s(\{j^*_s(f), \zeta_l\}), \quad \text{by functoriality of } \Phi\\ &= \{\pi^{\alpha *}_s \circ j^*_s(f), \zeta_l\} \qquad \text{(projection formula)}\\ &= \{g'', \zeta_l\}, \qquad g'' \in (1+\mathrm{J}^{\alpha})^*.\end{aligned}$$

Using the homotopy property applied to $\tau = \mathrm{F}^{\alpha *}(\beta)$ as in the proof of lemma 3.5, we have

$$\begin{aligned}\mathscr{S}^{\alpha *}(\beta) &= \{g'', \zeta_l\}\{g', \zeta_l\}^{-1}\\ &= \{g''/g', \zeta_l\},\end{aligned}$$

as desired. $\square$

PROPOSITION 3.8. — *Let y be in $(1+\mathrm{J}^{\alpha})^*$. Then $\{y, 1-\mathrm{N}(y)\}$ is in $(1-\sigma)\mathrm{K}_2(\mathrm{T}^{\alpha}, \bar{\mathrm{T}}^{\alpha})$.*

Proof. — By lemma 3.6, replacing T with a finite étale cover and changing notation, we may assume that $y = 1 + t\Sigma y_i \beta^i$ with the y_i in S, and (*a*) and (*b*) of that lemma satisfied. We have the diagram

$$\begin{array}{ccc} \mathrm{X}^{1/l}_{\mathrm{T}}(\alpha) & \to & \mathrm{X}^0_{\mathrm{T}} \subset \mathbb{A}^{l-1}_{\mathrm{T}}\\ \downarrow\uparrow \mathscr{Y}^{\alpha} & & \downarrow\uparrow \mathscr{Y}\\ \mathrm{T}^{\alpha} & \to & \mathrm{T}\end{array}$$

Take a linear projection $p: \mathbb{A}^{l-1}_{\mathrm{T}} \to \mathbb{A}^{l-2}_{\mathrm{T}}$ which then induces projections $p_0: \mathrm{X}^0_{\mathrm{T}} \to \mathbb{A}^{l-2}_{\mathrm{T}}$, and $p^{\alpha}: \mathrm{X}^{1/l}_{\mathrm{T}}(\alpha) \to \mathbb{A}^{l-2}_{\mathrm{T}}$. Choose p so that the fiber $\mathrm{Q}^{\alpha} := p^{\alpha-1} p^{\alpha}(\mathscr{Y}^{\alpha}(\mathrm{T}))$ satisfies

(3.12) each component of $\bar{\mathrm{Q}}^{\alpha}$ intersects $\mathrm{U}_{\mathrm{T}^{\alpha}}$.

Let Q be the fiber $p_0^{-1} p_0(\mathscr{Y}(\mathrm{T}))$. Let $q: \mathrm{Q} \to \mathrm{X}$, $q^{\alpha}: \mathrm{Q}^{\alpha} \to \mathrm{X}^{1/l}$ be the respective compositions

$$\mathrm{Q} \to \mathrm{X}^0_{\mathrm{T}} \to \mathrm{X}^0; \qquad \mathrm{Q}^{\alpha} \to \mathrm{X}^{1/l}_{\mathrm{T}^{\alpha}} \to \mathrm{X}^{1/l}.$$

Let x, μ and g be as given in (3.7), and let x^q, μ_q and g^q be the pullbacks of x, μ and g via q to $(1+\mathrm{JR}(\mathrm{Q}^{\alpha}))^*$, $\mathrm{K}_2(\mathrm{R}(\mathrm{Q}^{\alpha}), \bar{\mathrm{R}}(\mathrm{Q}^{\alpha}))$ and $(1+\mathrm{J}^{\alpha})^*$, respectively. From (3.7) we get the equation

(3.13) $$\{x^q, 1-\mathrm{N}(q^*(x))\}\,\mu^{\sigma}_q/\mu_q = \{g^q, \zeta_l\} \quad \text{in } \mathrm{K}_2(\mathrm{R}(\mathrm{Q}^{\alpha}), \bar{\mathrm{R}}(\mathrm{Q}^{\alpha}))$$

where we take $g=1$ if $l=p$. Here we have used the functoriality of the map $\Phi_{*,*}$ and the functoriality of the symbols $\langle\ ,\ \rangle$. In addition, letting W^q be the divisor $q^*(\mathrm{W})$, $\mathrm{W} = \{(1-\mathrm{N}(x))/t = 0\}$, and z^q the pullback $q^*(z)$, where z is the function constructed in paragraph 3.1, we have from (3.5) the computation of the tame symbol of μ_q:

$$T(\mu_q)=z^q \quad \text{on } W^q.$$

In particular, z^q the restriction to W^q of the regular function

$$Z^q=b+\Sigma b^{\sigma^{-i}}(x^q)^{\sigma^{-1}+\ldots+\sigma^{-i}}$$

where $b=1/l$ if $l\neq p$, and $b=(-\beta^{p-1})$ if $l=p$. Z^q is thus defined in a neighborhood of $\mathscr{Y}^\alpha(T^\alpha)$, with $Z^q\equiv 1 \bmod t$. By Proposition 2.11, μ_q is in the specialization subgroup $K_2(R(Q^\alpha), \bar{R}(Q^\alpha))_{\mathscr{Y}^\alpha}$. By Proposition 2.12, $\{x^q, 1-N(x^q)\}$ is also in $K_2(R(Q^\alpha), \bar{R}(Q^\alpha))_{\mathscr{Y}^\alpha}$ and

$$\Psi_{\mathscr{Y}^\alpha}(\{x^q, 1-N(x^q)\})=\{y, 1-N(y)\}.$$

Thus $\{g^q, \zeta_l\}$ is also in $K_2(R(Q^\alpha), \bar{R}(Q^\alpha))_{\mathscr{Y}^\alpha}$, and we have

$$\Psi_{\mathscr{Y}^\alpha}(\{g^q, \zeta_l\})=\{y, 1-N(y)\}\, v^\sigma/v \quad \text{in } K_2(T^\alpha, \bar{T}^\alpha),$$

where $v=\Psi(\mu_q)$. This completes the proof in case $l=p$.

For $l\neq p$, the tame symbol of $\{x^q, 1-N(q^*(x))\}\, \mu_q^\sigma/\mu_q$ vanishes in a neighborhood of $\mathscr{Y}^\alpha(T^\alpha)$, so $\{g^q, \zeta_l\}$ extends to an element γ of $K_2(V, \bar{V})$, for some neighborhood V of $\mathscr{Y}^\alpha(T^\alpha)$ in Q^α. By lemma 3.7, we have

$$\Psi_{\mathscr{Y}^\alpha}(\{g^q, \zeta_l\})=\{h, \zeta_l\}, \qquad \text{for some } h \text{ in } (1+J^\alpha)^*.$$

But

$$\begin{aligned}\{h, \beta\}^\sigma/\{h, \beta\}&=\{h, \beta^\sigma/\beta\}\\ &=\{h, \zeta_l\},\end{aligned}$$

which completes the proof. □

3.3. GENERATORS FOR RELATIVE K_2. — We now consider a filtering direct system of semi-local PIR's: $\{S_i \mid i\in I\}$, where each S_i contains k_0. We assume there is an initial element 0 of I. Let J_i be the Jacobson radical of S_i. Let S_∞, J_∞ be the direct limits. Since K-theory commutes with direct limits, we have

$$K_2(S_\infty, J_\infty)=\varinjlim K_2(S_i, J_i).$$

Similarly, for α in S_0^*, let S_i^α be $S_i[X]/X^l-\alpha$ (or $S_i[X]/X^p-X-\alpha$ if $l=p$), so S_i^α is étale over S_i, and has Jacobson radical $J_i^\alpha=J_i S_i^\alpha$. Also, letting S_∞^α, J_∞^α be the direct limits, we have

$$K_2(S_\infty^\alpha, J_\infty^\alpha)=\varinjlim K_2(S_i^\alpha, J_i^\alpha).$$

Let L_i denote the quotient field of S_i, L_∞ the direct limit of the L_i, and similarly define L_i^α and L_∞^α. Let σ be the generator of $\mathrm{Gal}(S_i^\alpha/S_i)$, $\sigma(\beta)=\zeta_l\,\beta$ (or $\beta+1$ if $l=p$), where β is the image of X in S_i^α.

We suppose that $\{S_i \mid i \in I\}$ satisfies

(I) Every x in $1+J_\infty$ is a norm from S_∞^α.

(II) If $P(u)$ is a separable polynomial with coefficients in S_∞ and has degree $d < l$, then $P(u)$ factors completely in $S_\infty[u]$.

Using Hilbert's theorem 90, we can replace (I) with

(I)′ Every x in $1+J_\infty$ is a norm from $1+J_\infty^\alpha$.
Our object here is to show

PROPOSITION 3.9. — *Assuming* (I) *and* (II), *the quotient group*

$$K_2(S_\infty^\alpha, J_\infty^\alpha)/(1-\sigma)K_2(S_\infty^\alpha, J_\infty^\alpha)$$

is generated via symbols by $(1+J_\infty^\alpha)^* \otimes L_\infty^*$.

The proof proceeds in a series of steps:

Let $G_1 \subset K_2(S_\infty^\alpha, J_\infty^\alpha) := G$ be the subgroup generated by $(1+J_\infty^\alpha)^* \otimes L_\infty^*$, G_2 the subgroup $(1-\sigma)G$.

STEP 1. — G/G_1 is generated by symbols of the form

$$\{1+a\beta, b+c\beta\} \qquad \text{and} \qquad \{1+a, b+c\beta\}$$

with a in J_∞ and b, c in S_∞^*.

Proof. — G is generated by symbols $\{f, g\}$ with f in $(1+J_\infty^\alpha)^*$, g in $L_\infty^{\alpha *}$. Write g as

$$g = \sum g_i \beta^i \qquad \text{with} \quad g_i \text{ in } L_j \text{ for some } j.$$

Let $p_1, \ldots, p_r$ be the closed points of S_j, $\nu_1, \ldots, \nu_r$ the associated valuations. Take h in L_j so that

$$\nu_s(h) = \min_i \{\nu_s(g_i)\} \qquad \text{for} \quad s = 1, \ldots, r.$$

Then g_i/h is in S_j for each i, and at each p_s, at least one of the g_i/h is a unit. Since $\{f, g\} \equiv \{f, g/h\} \bmod G_1$, we may replace g with g/h, and changing notation, assume that each p_s, at least one of the g_i is a unit.

Assume that $l \neq p$. Take units u_1, u_2 in S_j. Note that $g(u_1 + u_2\beta^j) = \sum g_i' \beta^i$ with

$$g_i' = \begin{cases} u_1 g_i + u_2 g_{i-j} & \text{for} \quad i \geqq j \\ u_1 g_i + u_2 \alpha g_{j-i} & \text{for} \quad i < j. \end{cases}$$

Since α is a unit, it is easy to see from this that we can find units $u_{1\,k}$, $u_{2\,k}$ and integers j_k, $0 < j_k < l$ so that

$$g' = g \cdot \prod_k (u_{1\,k} + u_{2\,k}\beta^{j_k})$$

is of the form $g'=\sum g'_i \beta^i$ with g'_i in S^*_j. Replacing $u_{1k}+u_{2k}\beta^{j_k}$ with $u_{1k}+u_{2k}\beta^{j_k}+v_k\beta^{l-1}+w_k\beta$, it follows that

$$g''=g.\prod(u_{1k}+u_{2k}\beta^{j_k}+v_k\beta^{l-1}+w_k\beta)$$

is also of the form

$$g''=\sum g''_i\beta^i \qquad \text{with} \quad g''_i \text{ in } S^*_j,$$

if we choose the v_k and w_k to be sufficiently general units in S_j. Changing notation, we may therefore assume that g_0 and g_{l-1} are units in S_j. The proof of this fact in case $l=p$ is similar and will be left to the reader.

Write f as

$$f=1+\sum f_i\beta^i \quad \text{with the } f_i \text{ in } J^\alpha_j,$$

increasing j if necessary. Arguing as above, we may assume the polynomials

$$P(u)=\sum g_i u^i; \qquad Q(u)=1+\sum f_i u^i$$

are separable, hence by (II) factor in S_n for some $n>j$; changing notation we may assume that $n=j$, and

$$P(u)=\prod(a_i+b_i u); \qquad Q(u)=\prod(c_i+d_i u)$$

with a_i, b_i, c_i, and d_i in S_j. Since $g_0=\prod a_i$, $g_{l-1}=\prod b_i$, and $1+f_0=\prod c_i$, the elements a_i, b_i, and c_i are all units in S_j. Let $c=\prod c_i$, $d'_i=d_i/c_i$, so c is in $(1+J_j)$ and

$$Q(u)/c=\prod(1+d'_i u).$$

As the coefficient f_i/c of u^i in $Q(u)/c$ is the i-th symmetric function of $d'_1, \ldots, d'_{l-1}$, and $f_i(p)$ vanishes for all closed points p of $\mathrm{Spec}(S_j)$, it follows that $d'_i(p)$ also vanishes for all i, and all closed points p. Thus d'_i is in J_j, and the symbol $\{f, g\}$ can be written as

$$\{f, g\}=\prod_{i,j}\{1+d'_i\beta, c_j+d_j\beta\}\{c, c_j+d_j\beta\}$$

completing the proof of step 1.

STEP 2. — G/G_1 is generated by symbols of the form $\{1+a, b+c\beta\}$, with a in J_∞, and b, c in S^*_∞.

Proof. — By step 1, we need only consider symbols of the form $\{1+a\beta, b+c\beta\}$; a in J_n; b and c in S^*_n, some n.

Let u be an indeterminant, let $S=S_n$, $J=J_n$, let $\bar{S}=S/J$ and let v be the indeterminant with

$$v^l=u \quad \text{if } l\neq p; \qquad v^l+v=u \quad \text{if } l=p.$$

Let $L(v)$ denote the semi-local ring of $\mathbb{A}^1_{\bar{S}}$ in $\mathbb{A}^1_S := \mathrm{Spec}(S[v])$, $\bar{L}(v) = L(v)/JL(v)$. Consider the symbol

$$\eta = \{1+av,\ b+cv\} \quad \text{in } K_2(L(v), \bar{L}(v)).$$

Let $Z_1, Z_2 \subset \mathbb{A}^1_S$ be the curves defined by the ideals $(1+av)$, $(b+cv)$ respectively. Then the projection

$$\pi:\ \mathbb{A}^1_S \to T := \mathrm{Spec}(S)$$

restricts to an isomorphism $Z_2 \to T$ and a generically $1-1$ map $Z_1 \to T$. In addition $Z_1 \cap \mathbb{A}^1_{\bar{S}} = \varnothing$, so $\pi: Z_1 \to T$ defines an isomorphism

$$Z_1 \xrightarrow{\sim} \mathrm{Spec}(L),$$

where L is the quotient field of S. Furthermore $Z_2 \cap \{v=0\}$ is empty, since b and c are units.

The tame symbol $T(\eta)$ is given by

$$T(\eta) = (1+av) \text{ on } Z_2 - (b+cv) \text{ on } Z_1.$$

Then we can find f in $1+J$, g in L^* such that

$$\pi^*(f)|_{Z_2} = (1+av)|_{Z_2}; \qquad \pi^*(g)|_{Z_1} = (b+cv)|_{Z_1}.$$

Thus the product $\{\pi^*(f), b+cv\}\{1+av, \pi^*(g)\}$ has the same tame symbol as η, so there is a τ in $K_2(S, J)$ with

$$(\bigstar) \qquad \eta = \{\pi^*(f), b+cv\}\{1+av, \pi^*(g)\}\pi^*(\tau) \quad \text{in } K_2(L(v), \bar{L}(v)).$$

Let $s: \mathrm{Spec}(S^\alpha) \to \mathbb{A}^1_{S^\alpha}$ be the section (over S) with $s^*(v) = \beta$, $p: \mathbb{A}^1_{S^\alpha} \to \mathbb{A}^1_S$ the obvious map. Since $\pi^*(f)$ and $1+av$ are units in a neighborhood of $p(s(\mathrm{Spec}(S^\alpha))$, Proposition 2.11 implies that the terms in $(\bigstar)$ pulled black to $\mathbb{A}^1_{S^\alpha}$ are in the specialization subgroup $K_2(L^\alpha(v), \bar{L}^\alpha(v))_s$. Thus, using Proposition 2.12,

$$\{1+a\beta,\ b+c\beta\} = \{f,\ b+c\beta\}\{1+a\beta,\ g\}.\tau \quad \text{in } K_2(S^\alpha, J^\alpha).$$

This completes the proof of Step 2. □

STEP 3. — $\{1+a,\ b+c\beta\}$ is in $G_1 G_2$ for a in J_∞, b, c in S^*_∞.

Proof. — By (I)′, we can find x in $(1+J^\alpha_\infty)^*$ with $1+a = N(x)$.

We claim that

$$(\bigstar) \qquad \{1+a,\ b+c\beta\} \equiv \{x,\ N(b+c\beta)\} \bmod G_1 G_2.$$

Since $\{x, N(b+c\beta)\}$ is in G_1, this would complete the proof.

Write x as

$$x = 1 + \sum f_i \beta^i.$$

As in step 1, we can factor this as

$$x=(1+d).\prod(1+d_i\beta^i) \qquad \text{with} \quad d,\ d_i \text{ in } J_\infty.$$

Since $1+a=N(x)=(1+d)\prod N(1+d_i\beta^i)$, we need only show that

$$(\bigstar\bigstar) \qquad \{N(y),\ b+c\beta\}\equiv\{y,\ N(b+c\beta)\} \bmod G_1G_2,$$

for $y=1+a_0+a_1\beta$, a_0 and a_1 in J_∞.

We proceed as in step 2, retaining the notations from that step. We assume all the elements defined above lie in $S_n^\alpha:=S^\alpha$.

Let W^1, W^2 be the curves on $\mathbb{A}_S^1$ defined by ideals $(1+a_0+a_1v)$, $(b+cv)$, respectively (note that $W^1=\varnothing$ if $a_1=0$). As above, $\pi\colon W^2\to \mathrm{Spec}(S)$ is an isomorphism, and $\pi\colon W^1\to\mathrm{Spec}(L)$ is an isomorphism if $a_1\neq 0$. Let V^1 and V^2 be the subschemes of $\mathbb{A}_S^1$ defined by ideals $(N(1+a_0+a_1v))$ and $(N(b+cv))$, respectively. Then V^1 and V^2 are the unions

$$V^1=\bigcup\sigma^i(W^1); \qquad V_2=\bigcup\sigma^i(W^2)$$

Since b and c are units,

$$\sigma^i(W^2)\cap\sigma^j(W^2)=\varnothing \qquad \text{if} \quad i\neq j.$$

Since W^1 is disjoint from $\mathbb{A}_{\bar{S}}^1$,

$$\sigma^i(W^1)\cap\sigma^j(W^2)=\varnothing \qquad \text{for all } i \text{ and } j.$$

Thus V^1 and V^2 are regular, disjoint, and étale over $\mathrm{Spec}(S)$.

Let η be the element of $K_2(L(v),\ \bar{L}(v))$:

$$\eta=\{N(1+a_0+a_1v),\ b+cv\}\{1+a_0+a_1v,\ N(b+cv)\}^{-1}.$$

Then η has tame symbol

$$T(\eta)=(h_2 \text{ on } V^2)-(h_1 \text{ on } V^1),$$

with

$$h_1\in k(V^1)^*; \qquad h_2\in\Gamma(V^2,\ \mathcal{O}_{V^2}), \qquad h_2\equiv 1 \bmod J.$$

Clearly

$$N(h_1)=N(h_2)=1,$$

so we can write

$$h_1=z_1^\sigma/z_1; \qquad h_2=z_2^\sigma/z_2 \quad \text{with } z_1\in k(V^1)^*;\ z_2\in\Gamma(V^2,\ \mathcal{O}_{V^2}).$$

Taking

$$z_2 = b + \sum b^{\sigma^{-i}} (h_2)^{\sigma^{-1} + \ldots + \sigma^{-i}}$$

as in the proof of Proposition 3.8, we may assume that $z_2 \equiv 1 \bmod \mathrm{J}$. Let z_1^i be the restriction of z_1 to $\sigma^i(\mathrm{W}^1)$, and similarly define z_2^i. Then there are elements $h_1^i \in \mathrm{L}^*$, $h_2^i \in (1+\mathrm{J})^*$ with

$$\pi^*(h_j^i)\,|_{\mathrm{W}^j} = z_j^i.$$

Let ω_i and δ_i be the symbols

$$\omega_i = \{1 + a_0 + a_1\,\sigma^i(v),\ h_1^i\}; \qquad \delta_i = \{h_2^i,\ b + c\,\sigma^i(v)\}.$$

Then $\lambda := \prod \omega_i\,\delta_i$ has tame symbol

$$\mathrm{T}(\lambda) = (z_1 \text{ on } \mathrm{V}_1) - (z_2 \text{ on } \mathrm{V}_2),$$

so

$$\eta\,.\,(\lambda^\sigma/\lambda)^{-1} = \pi^*(\tau) \quad \text{in } \mathrm{K}_2(\mathrm{L}(v),\ \bar{\mathrm{L}}(v)),$$

for some τ in $\mathrm{K}_2(\mathrm{S}, \mathrm{J})$. Specializing as in step 2 gives (**), completing the proof of step 3, and the proposition. □

4. Main Theorems

4.1. HILBERT'S THEOREM 90 FOR RELATIVE K_2. — We now follow the proof of Suslin in [S] to prove Hilbert's Theorem 90 for relative K_2. Let S be a semi-local PIR containing the field k_0. Let α be a unit in S; we retain the notations J, S^α, J^α, T, T^α, etc. from part 3. In particular, S^α is an étale cyclic Galois extension of S, of prime degree l, with Galois group generated by σ. For a flat S-algebra W, with W a semi-local PIR, let $\mathrm{W}^\alpha = \mathrm{W} \otimes_{\mathrm{S}} \mathrm{S}^\alpha$, $\mathrm{J}(\mathrm{W}) \subset \mathrm{W}$ the Jacobson radical, $\mathrm{J}(\mathrm{W})^\alpha = \mathrm{J}(\mathrm{W})\,\mathrm{W}^\alpha$. We have the complex

$$\mathrm{M}(\mathrm{W})_*:\quad \mathrm{K}_2(\mathrm{W}^\alpha, \mathrm{J}(\mathrm{W})^\alpha) \xrightarrow[(1-\sigma)]{} \mathrm{K}_2(\mathrm{W}^\alpha, \mathrm{J}(\mathrm{W})^\alpha) \xrightarrow[\mathrm{N}]{} \mathrm{K}_2(\mathrm{W}, \mathrm{J}(\mathrm{W})).$$

Let $\mathrm{V}(\mathrm{W})$ be the homology $\mathrm{H}_1(\mathrm{M}(\mathrm{W})_*)$. If $g: \mathrm{W} \to \mathrm{W}'$ is an inclusion of semi-local PIR's, then g induces $g^*: \mathrm{V}(\mathrm{W}) \to \mathrm{V}(\mathrm{W}')$; if W′ is finite and étale over W we have $g_*: \mathrm{V}(\mathrm{W}') \to \mathrm{V}(\mathrm{W})$. Since $\mathrm{V}(\mathrm{W}^\alpha) = 0$, and $g_* \circ g^* = \deg(g).\,id$, we get

1. $\mathrm{V}(\mathrm{W})$ is an l-torsion group for every W.

If $g: \mathrm{W} \to \mathrm{W}'$ is finite and Galois, and of degree d prime to l, then using the maps

$$g_*:\quad \mathrm{K}_2(\mathrm{W}'^\alpha, \mathrm{J}(\mathrm{W}'^\alpha))\,[1/d] \to \mathrm{K}_2(\mathrm{W}^\alpha, \mathrm{J}(\mathrm{W}^\alpha))\,[1/d]$$

and

$$g_*:\quad \mathrm{K}_2(\mathrm{W}^\alpha, \mathrm{J}(\mathrm{W}^\alpha))\,[1/d] \to \mathrm{K}_2(\mathrm{W}, \mathrm{J}(\mathrm{W}))\,[1/d]$$

defined is paragraph 1.10, we see that

2. $g^*: V(W) \to V(W')$ is injective.

Let x be in $1+J(W)$, let $\mathscr{D}$ be the Azumaya algebra constructed as a crossed product algebra from the Hilbert symbol $(\alpha, x)_l$ (or the symbol $[\alpha, x)_p$ if $l=p$) as in Serre [Se], let $g: X \to \operatorname{Spec}(W)$ be the Brauer-Severi variety associated to $\mathscr{D}$. We let $\bar{W}=W/J(W)$, $\bar{X}=g^{-1}(\operatorname{Spec}(\bar{W}))$, and let $R(X)$ denote the semi-local ring of $\bar{X}$ in X, with radical $J(X)$. We note that $\bar{X}$ is a projective space over $\bar{W}$ ($\bar{\mathscr{D}}$ is split) as $x \equiv 1 \bmod J(W)$. Let $X^\alpha = X \times_W W^\alpha$, and let $f: X^\alpha \to X$, $f: \operatorname{Spec}(W^\alpha) \to \operatorname{Spec}(W)$ be the covering maps. X^α is also a projective space over W^α.

PROPOSITION 4.1. — *The map* $g^*: V(W) \to V(R(X))$ *is injective.*

Proof. – Let η be in $K_2(W^\alpha, \bar{W}^\alpha)$ with $N(\eta)=1$, and suppose that $g^*(\eta)=\lambda^\sigma/\lambda$ for some λ in $K_2(R(X^\alpha), J(X^\alpha))$. Let $z=\partial(\lambda)$, where ∂ is the boundary in the localization sequence

$$\to K_2(R(X^\alpha), J(X^\alpha)) \underset{\partial}{\to} K'_1((X^\alpha)^{1/2}, (\overline{X^\alpha})^{1/2}) \to K_1((X^\alpha)^{0/2}, (\overline{X^\alpha})^{0/2}) \to$$

Then $z^\sigma/z=\partial(g^*(\eta))=0$. By our computation of $K'_1(X^{1/2}, \bar{X}^{1/2})$ and $K'_1((X^\alpha)^{1/2}, (\overline{X^\alpha})^{1/2})$ in paragraph 1.8, the map

$$f^*: \quad K'_1(X^{1/2}, \bar{X}^{1/2}) \to K'_1((X^\alpha)^{1/2}, (\overline{X^\alpha})^{1/2})$$

is injective, so z can be considered as an element of

$$K'_1(X^{1/2}, \bar{X}^{1/2}) \subset K'_1((X^\alpha)^{1/2}, (\overline{X^\alpha})^{1/2}).$$

As $\partial(z)=0$ in $K'_0((X^\alpha)^{2/3}, (\overline{X^\alpha})^{2/3})$, and since

$$f^*: \quad K'_0(X^{2/3}, \bar{X}^{2/3}) \to K'_0((X^\alpha)^{2/3}, (\overline{X^\alpha})^{2/3})$$

is injective (Corollary 1.12), z defines a class $[z]$ in $E_2^{1,-2}(X, \bar{X})$. Since $g^*([z])$ clearly dies in $E_2^{1,-2}(X^\alpha, \bar{X}^\alpha)$, and $\bar{\mathscr{D}}$ is split, Corollary 1.13 implies that $[z]=0$ in $E_2^{1,-2}(X, \bar{X})$. Thus $z=\partial(\tau)$ for some τ in $K_2(R(X), J(X))$. Modifying λ by $f^*(\tau)$, we may assume that $\partial(\lambda)=0$. By Corollary 1.6, we have $\lambda=g^*(\xi)$ for some ξ in $K_2(W^\alpha, \bar{W}^\alpha)$, and thus $g^*(\eta)=g^*(\xi^\sigma/\xi)$. As X^α is a projective space over W^α, g^* is injective, and we get $\eta=\xi^\sigma/\xi$, completing the proof. □

We now define a direct system $\{S_i \mid i \in \mathscr{S}\}$ of S-algebras with $S_0=S$. For each x in $1+J(S)$, we let $S_x=R(X)$, where X is the Brauer-Severi variety over with symbol $(\alpha, x)_l$ (or $[\alpha, x)_p$ if $l=p$), and for each irreducible separable polynomial P of degree $< l$, let S_P be the normalization of S in the splitting field of P. For each finite set of P's and x's, we form the tensor product of the S_P's and S_x's over S and normalize, giving an S-algebra T. We then localize T with respect to JT, forming T′. Let $\mathscr{S}_1$ be the set of such T′. We note that each element of $\mathscr{S}_1$ is a PIR, and is flat as an S-algebra. Repeating this for each T in $\mathscr{S}_1$, and taking localizations of normalizations of all finite tensor products gives $\mathscr{S}_2$, etc. We let $\mathscr{S}$ be the union of all the $\mathscr{S}_i$. Then $\{S_i \mid i \in \mathscr{S}\}$ is a direct, filtering system of PIR's which are flat S-algebras. Let S_∞ be direct limit of the S_i, and J_∞ the direct limit of the $J(S_i)$, S^α_∞, J^α_∞ defined as in paragraph 3.3. Then S_∞ satisfies (I) and (II) of paragraph 3.3. Let L_∞ be the direct limit of the quotient fields L_i of S_i. Let $G=K_2(S^\alpha_\infty, J^\alpha_\infty)$.

Let a be in $1+J_\infty$, b in L^*_∞. Then $a=N(x)$ for some x in $1+J^\alpha_\infty$. Suppose $a\neq 1$. Then

$$\{x, 1-a\}=\{x, 1-N(x)\}=0 \quad \text{in } G/(1-\sigma)G$$

by Proposition 3.8. If $a=1$, then $x=z^\sigma/z$, so $\{x, b\}\equiv 0 \bmod (1-\sigma)G$. Thus the map

$$\begin{gathered}(1+J_\infty)^*\otimes L^*_\infty \to G/(1-\sigma)G\\ a\otimes b\to\{x, b\} \bmod (1-\sigma)G\end{gathered}$$

defines a homomorphism $\Theta: K_2(S_\infty, J_\infty)\to G/(1-\sigma)G$. By Proposition 3.9, Θ is surjective; clearly $N\circ\Theta=id$. Thus $V(S_\infty)=0$. Since $V(S)\to V(S_i)$ is injective for all i in $\mathscr{S}$ by (1) and (2), this implies that $V(S)=0$. Thus we have shown

THEOREM 4.2. — *Let* S *be a semi-local PIR containing* k_0. *Let* α *be a unit in* S, *and* S^α *the extension ring* $S[X]/X^l-\alpha$ *if* $l\neq p=\mathrm{char}(k_0)$, $S[X]/X^p-X-\alpha$ *if* $l=p$. *Let* J *be the Jacobson radical of* S, $J^\alpha=JS\alpha$. *Let* σ *be a generator of* $\mathrm{Gal}(S^\alpha/S)$. *Then*

$$K_2(S^\alpha, J^\alpha)\xrightarrow[(1-\sigma)]{} K_2(S^\alpha, J^\alpha)\xrightarrow[N]{} K_2(S, J)$$

is exact.

4.2. TORSION IN RELATIVE K_2 AND K_3^{ind}. — Using Hilbert's theorem 90 we compute the torsion in $K_2(S, J)$ and in $K_3(E)^{\mathrm{ind}}$, where E is a field.

THEOREM 4.3. — *Let* S *be a semi-local PIR containing a field* k *which contains* μ_n *for* n *prime to the characteristics* p *of* k. *Let* J *denote the radical of* S. *Then* ${}_nK_2(S, J)$ *is generated by symbols* $\{f, \zeta_n\}$, *with* f *in* $1+J$. $K_2(S, J)$ *has no* p-*torsion if* $p>0$.

Proof. — Suppose l is a prime dividing n. Suppose the theorem is true for $n=l$. Let η be an n-torsion element in $K_2(S, J)$. Then η^l is n/l torsion, so by induction we may assume that $\eta^l=\{g, \zeta_n^l\}$ for some g in $1+J$, so $\eta\{g, \zeta_n\}^{-1}$ is l-torsion, thus is of the form $\{h, \zeta_l\}$. Then $\eta=\{g(h)^{n/l}, \zeta_n\}$, as desired.

Consider the generic Kummer extension $S(v)/S(u)$ with $v^l=u$; here $S(v)$ is the semi-local ring of $J[v]$ in $S[v]$, and similarly define $S(u)$. Let $f^*: S\to S(v)$, $h^*: S\to S(u)$ be the inclusions. Let η be in ${}_lK_2(S, J)$. Then

$$N_{S(v)/S(u)}(f^*(\eta))=h^*(\eta^l)=1$$

so we can write $f^*(\eta)$ as

$$f^*(\eta)=\tau^\sigma/\tau \quad \text{for some } \tau \text{ in } K_2(S(v), JS(v)).$$

Let

$$g:\ \mathbb{A}^1_S=\mathrm{Spec}(S[v])\to \mathbb{A}^1_S=\mathrm{Spec}(S[u])$$

be the l-fold cover. Then $\partial(\tau^\sigma/\tau)=\partial(f^*(\eta))=0$ in $K'_1(S[v], JS[v])^1$, so there is an element z of $K'_1(S[u])$ with $g^*(z)=\partial(\tau)$. The image of z in $K'_1((S/J)[u])$ is 1 at all points of $\mathbb{A}^1_{\bar{S}}-\{0\}$, since g is étale away from 0. Thus we can add an element z_0 of the form

$$z_0=(a \text{ on } \{u=0\}), \qquad a\in S^*$$

to z so that $z+z_0$ lands in the subgroup $K'_1(S[u], JS[u])$ of $K'_1(S[u])$.

Take ξ in $K_2(S(u), JS(u))$ with

$$\partial(\xi)=z+z_0.$$

Then $\tau.g^*(\xi)^{-1}:=\tau'$ has $\partial(\tau')$ supported on $\{v=0\}$, and

$$\eta=\tau'^\sigma/\tau'.$$

Since $Z_0:=\{v=0\}$ is smooth, $\partial(\tau')=(f^*(x) \text{ on } Z_0)$, for some x in $1+J$. But then $\tau'.\{f^*(x), v\}^{-1}$ has $\partial(\tau'.\{f^*(x), v\}^{-1})=0$, so

$$\tau'=\{f^*(x), v\}.f^*(\beta) \text{ for some } \beta \text{ in } K_2(S, J).$$

Thus

$$\begin{aligned} f^*(\eta)&=\tau'^\sigma/\tau' \\ &=(\{f^*(x), v\}.\beta)^\sigma/\{f^*(x), v\}.\beta \\ &=\{f^*(x), \zeta_l\} \\ &=f^*(\{x, \zeta_l\}), \end{aligned}$$

hence $\eta=\{x, \zeta_l\}$, as desired.

If $l=p$, we use the generic Artin-Schreier extension $S(v)/S(u)$ where $v^p-v=u$. Since $S[v]/S[u]$ is étale, the above argument shows that there is no p-torsion in $K_2(S, J)$. This completes the proof. □

For a ring R, we let $K_3(R)^{dec}$ be the subgroup of $K_3(R)$ generated by products from $K_1(R)$; $K_3(R)^{ind}$ will denote the quotient group

$$K_3(R)^{ind}=K_3(R)/K_3(R)^{dec}.$$

Let E be a field, and let $E(t)$ be a purely transcendental extension of E; then the map

$$K_3(E)^{ind}\to K_3(E(t))^{ind}$$

is an isomorphism. Indeed, the map is clearly injective. We have the exact localization sequence

$$\begin{array}{c} 0\to K_3(E[t])\to K_3(E(t)) \xrightarrow{\delta} \bigoplus_{g \text{ prime}} K_2(E[t]/(g))\to 0 \\ \| \\ K_3(E) \end{array}$$

and the exact sequence of Milnor K-theory [Bass-Tate]

$$0\to K_3^M(E)\to K_3^M(E(t)) \xrightarrow{\delta} \bigoplus_{g \text{ prime}} K_2^M(E[t]/(g))\to 0$$

compatible with the Quillen localization sequence. As $K_2(F)=K_2^M(F)$, and $K_3(F)^{dec}$ is the image of $K_3^M(F)$ for fields F, we see the map $K_3(E)^{ind}\to K_3(E(t))^{ind}$ is surjective. Similarly, if R is a semi-local ring containing E, with quotient field $E(t)$, then $K_3(E)^{ind}\to K_3(R)^{ind}$ is an isomorphism.

COROLLARY 4.4. — *Let* E *be a field containing* $\boldsymbol{\mu}_n$, $(n, \mathrm{char}(E))=1$. *Then the n-torsion subgroup of* $K_3(E)^{ind}$ *is a quotient of* $\mathbb{Z}/n$. *If* E *has characteristic* $p>0$, *then* $K_3(E)^{ind}$ *has no p-torsion.*

Proof. — Let R be the semi-local ring of $\{0, 1\}$ on $\mathbb{A}^1_E$, J the Jacobson radical of R. We have the exact sequence

$$\to K_3(R)\to K_3(\bar{R})\to K_2(R, J)\to K_2(R)\to$$

which gives the exact sequence

$$\to K_3(R)^{ind}\to K_3(\bar{R})^{ind}\to K_2(R, J)\to K_2(R)\to.$$

Since

$$K_3(R)^{ind}=K_3(E)^{ind};\ K_3(\bar{R})^{ind}=K_3(E(0))^{ind}\oplus K_3(E(1))^{ind}$$

we get the exact sequence

$$0\to K_3(E)^{ind}\to K_2(R, J)\to K_2(R)\to.$$

From this and Corollary 4.3, it follows that ${}_n(K_3(E)^{ind})$ is generated by symbols of the form $\{f, \zeta_n\}$, $f\in(1+J)^\times/((1+J)^\times)^n$, such that the symbol $\{f, \zeta_n\}=0$ as an element of $K_2(R)$. In particular, the tame symbol $T(\{f, \zeta_n\})$ is zero, hence the divisor of f on $\mathbb{A}^1_E$ is divisible by n.

Thus we can write f as an n-th power:

$$f=g^n \quad \text{some } g \text{ in } (\bar{E}\otimes_E R)^*.$$

We normalize g so that $g(0)=1$. Let σ be an element of $\mathrm{Gal}(\bar{E}/E)$. Then

$$g^\sigma=\lambda g$$

for some λ in $\boldsymbol{\mu}_n$; evaluating at 0 shows that $\lambda=1$. Thus g is in R^*. The class of f mod $((1+J)^\times)^n$ is then determined by the value $g(1)\in\boldsymbol{\mu}_n$, proving the corollary. □

Now we can show

THEOREM 4.5. — *Let* E *be a number field. The Chern class*

$$c_{2,1}:\ K_3(E)^{ind}\otimes\mathbb{Z}_l\to H^1_{\text{ét}}(E, \mathbb{Z}_l(2))$$

is an isomorphism, so the l-primary torsion in $K_3(E)^{ind}$ *is isomorphic to* $H^0(E, \mathbb{Q}_l/\mathbb{Z}_l(2))$.

Proof. — We may assume that E contains $\boldsymbol{\mu}_l$. From [Q] and the vanishing of K_2 for finite fields, $K_3(E)$ is finitely generated. From the above, the l-torsion in $K_3(E)^{ind}$ is cyclic, hence the l-primary torsion is also cyclic. By [B-T], $K_3^M(E)$ is a torsion group; by [Borel] the rank of $K_3(E)$ is r_2. Thus $K_3(E)^{ind}/l$ is a $\mathbb{Z}/l$ vector space of dimension between r_2 and $1+r_2$. In addition, the Chern class vanishes on the Milnor K_3 (this follows from the integral product formula for Chern classes).

Let symb: $H^1(E, \mu_l^{\otimes 2}) \to {}_lK_2(E)$ be the map

$$H^1(E, \mu_l^{\otimes 2}) \xrightarrow{\sim} (E^\times/(E^\times)^l) \otimes \mu_l \to {}_lK_2(E),$$

and let H be the kernel of symb. Tate [T] has shown that H is $(\mathbb{Z}/l)^{1+r_2}$. Soulé [So] has shown that $c_{2,1}$ is surjective for $l > 2$; we give here a proof of surjectivity for all prime l: Let R be the semi-local ring of $\{0, 1\}$ on $\mathbb{A}_E^1$. By ([So], Prop. 2) we have the commutative ladder

$$\begin{array}{ccc} K_3(R; \mathbb{Z}/l) & \xrightarrow{\delta} \bigoplus_x^1 K_2(E(x); \mathbb{Z}/l) & \to 0 \\ c_{2,1}\downarrow & \downarrow -c_{1,0} & \\ H^1(R, \mu_l^{\otimes 2}) & \xrightarrow[\delta]{} \bigoplus_x^1 H^0(E(x), \mu_l) & \to 0; \end{array}$$

where $\bigoplus^1$ means the sum over codimension one points of $\mathbb{A}_E^1 - \{0, 1\}$, and the rows are respectively the localization sequence and Bloch-Ogus sequence for the open subset $\mathrm{Spec}(R)$ of $\mathbb{A}_E^1$. As $c_{1,0}$ induces the isomorphism ${}_l(E(x))^* \to H^0(E(x), \mu_l)$, the map

$$K_3(R; \mathbb{Z}/l)/\pi^* K_3(E, \mathbb{Z}/l) \to H^1(R, \mu_l^{\otimes 2})/\pi^* H^1(E, \mu_l^{\otimes 2})$$

is surjective. We have the commutative square

$$(\bigstar\bigstar) \qquad \begin{array}{ccc} K_3(R; \mathbb{Z}/l) & \xrightarrow{\delta_K} & K_3(E; \mathbb{Z}/l) \\ c_{2,1}\downarrow & & \downarrow c_{2,1} \\ H^1(R, \mu_l^{\otimes 2}) & \xrightarrow[\delta_H]{} & H^1(E, \mu_l^{\otimes 2}) \end{array}$$

where δ_K is the composition

$$K_3(R; \mathbb{Z}/l) \xrightarrow[\text{reduce mod } J]{} K_3(R/J; \mathbb{Z}/l) = K_3(E(0); \mathbb{Z}/l) \oplus K_3(E(1); \mathbb{Z}/l) \xrightarrow[(x,\, y) \to y-x]{} K_3(E; \mathbb{Z}/l)$$

and similarly for δ_H. Since $H^1(R, \mu_l^{\otimes 2}) = R^\times \otimes \mu_l$ and $H^1(E, \mu_l^{\otimes 2}) = E^\times \otimes \mu_l$, δ_H is surjective.

Since δ_K kills $\pi^* K_3(E, \mathbb{Z}/l)$ and δ_H kills $\pi^* H^1(E, \mu_l^{\otimes 2})$, it follows that $c_{2,1}: K_3(E; \mathbb{Z}/l) \to H^1(E, \mu_l^{\otimes 2})$ is surjective. This incidently shows that $c_{2,1}: K_3(R; \mathbb{Z}/l) \to H^1(R, \mu_l^{\otimes 2})$ is also surjective. The surjectivity of $c_{2,1}: K_3(E, \mathbb{Z}/l) \to H^1(E, \mu_l^{\otimes 2})$, together with the computation of $K_3(E)^{ind}/l$ and H implies that the Chern class map

$$c_{2,1}: \quad K_3(E; \mathbb{Z}/l)^{ind} \to H^1(E, \mu_l^{\otimes 2})$$

is an isomorphism. The commutative square ($\bigstar\bigstar$), together with the surjectivity of $c_{2,1}: K_3(R; \mathbb{Z}/l) \to H^1(R, \mu_l^{\otimes 2})$ and δ_H then implies that δ_K is surjective (δ_K is obviously surjective on the Milnor K_3), and hence $K_2(R, J; \mathbb{Z}/l) \to K_2(R, \mathbb{Z}/l)$ is injective, hence $K_2(R, J)/l \to K_2(R)/l$ is injective.

Let L be the quotient field of R, i: Spec(L) $\to$ Spec(R) the inclusion. Let $i_!$ be the functor "extension by zero", from sheaves on L to sheaves of R (for the étale topology). The construction of Chern classes for relative K-theory in paragraph 1.12 gives the Chern classes

$$c_{p,q}:\quad K_{2p-q}(R, J) \to H^q(\mathrm{Spec}(R), i_!(\mu_{l^n})^{\otimes p}),$$

together with the commutative ladder

$$(\bigstar)\qquad \begin{array}{ccccccc} K_3(E)/l^n & \to & K_2(R, J)/l^n & \to & K_2(R)/l^n & \to & (K_2(E)/l^n)^2 \to 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^1(E, \mu_{l^n}{}^{\otimes 2}) & \to & H^2(R, i_!(\mu_{l^n}{}^{\otimes 2})) & \to & H^2(R, \mu_{l^n}{}^{\otimes 2}) & \to & (H^2(E, \mu_{l^n}{}^{\otimes 2}))^2 \to 0, \end{array}$$

the horizontal line coming from the relativization sequence, and the vertical arrows Chern classes. For all n, the Chern classes for $K_2(R)/l^n$ and $K_2(E)/l^n$ are isomorphisms. The surjectivity of δ_H shows that $H^2(R, i_!(\mu_l^{\otimes 2})) \to H^1(R, \mu_l^{\otimes 2})$ is injective, hence the second vertical arrow is an isomorphism for $n=1$.

We define the map symb: $H^1(R, \mu_l^{\otimes 2}) \to {}_lK_2(R, J)$ by

$$f \otimes \zeta_l \to \{f, \zeta_l\}; \qquad f \in R^\times,$$

where we identify $H^1(R, \mu_l^{\otimes 2})$ with $R^\times \otimes \mu_l$ via the Chern class $c_{1,1}$. From the product formula for Chern classes, we have

$$c_{2,2}(a.b) = -c_{1,1}(a) \cup c_{1,1}(b)$$

for a in $K_1(R, J) = (1+J)^\times$, b in $K_1(R) = R^\times$. This gives the commutative ladder

$$\begin{array}{ccccccc} {}_lK_2(R, J) & \to & K_2(R, J)/l^n & \to & K_2(R, J)/l^{n+1} & \to & K_2(R, J)/l \to 0 \\ {\scriptstyle -\mathrm{symb}} \uparrow & & \downarrow & & \downarrow & & \downarrow \\ H^1(R, i_!(\mu_l^{\otimes 2})) & \to & H^2(R, i_!(\mu_{l^n}{}^{\otimes 2})) & \to & H^2(R, i_!(\mu_{l^{n+1}}{}^{\otimes 2})) & \to & H^2(R, i_!(\mu_l^{\otimes 2})) \end{array}$$

with the second row exact, and the first row exact, except possibly at $K_2(R; J)/l^n$. This and induction shows that the Chern class for $K_2(R; J)/l^n$ is an isomorphism for all n.

From the localization sequence on $\mathbb{A}^1_E$, together with a knowledge of $K_2(E)$, and K_1 of number fields, it follows that $K_2(R)$ has no divisible subgroups. As $K_3(E)$ is finitely generated, $K_2(R, J)$ has no divisible subgroups as well. Thus for n suficiently large, the l-primary torsion in $K_3(E)^{\mathrm{ind}}$ injects into $K_2(R, J)/l^n$. From the ladder ($\bigstar$), it follows that the Chern class $c_{2,1}$: $K_3(E)^{\mathrm{ind}} \to H^1(E, \mu_{l^n}{}^{\otimes 2})$ is injective on the l-primary torsion for large n. From this, the surjectivity of $c_{2,1}$, and the computation of the ranks of $K_3(E)^{\mathrm{ind}}$ and $H^1(E, \mathbb{Z}_l(2))$ (the latter due to Tate [T]) it follows that the Chern class gives an isomorphism on the limits

$$c_{2,1}:\quad K_3(E)^{\mathrm{ind}} \otimes \mathbb{Z}_l \to H^1(E, \mathbb{Z}_l(2))$$

proving the theorem. $\square$

Using this result, we can refine the statement of Corollary 4.4.

COROLLARY 4.6. — *Let* E *be a field, l a prime with* $(l, \mathrm{char}(E))=1$. *Then the l-primary torsion in* $K_3(E)^{\mathrm{ind}}$ *is isomorphic to* $H^0(E, \mathbb{Q}_l/\mathbb{Z}_l(2))$. *If* F *is an extension field of* E, *then the map*

$$K_3(E)^{\mathrm{ind}} \to K_3(F)^{\mathrm{ind}}$$

is injective.

Proof. – The second statement follows from the first. If E is a finite field, the computation of the torsion is due to Quillen [Q2]; for E a number field this is part of Theorem 4.5. In particular, if $E \to F$ is a map of fields which are finite over the prime field, the induced map

$$K_3(E)^{\mathrm{ind}} \to K_3(F)^{\mathrm{ind}}$$

is injective.

In the general case, since K-theory commutes with direct limits, we may assume that E is finitely generated over the prime field F_0.

Let k be the field of constants in E. Let η be an l-primary torsion element of $K_3(k)^{\mathrm{ind}}$. Let $g^*: k \to E$ denote the inclusion, and suppose that $g^*(\eta)=0$ in $K_3(E)^{\mathrm{ind}}$. Then there is a regular k-algebra A of finite type, A a domain with quotient field $E_0 \subset E$, such that $h^*(\eta)=0$ in $K_3(A)^{\mathrm{ind}}$. Here $h^*: k \to A$ is the inclusion. Taking an F-valued point $j^*: A \to F$ of A, with F a finite extension of k, we see that $j^* h^*(\eta)=0$, contradicting the injectivity of $K_3(k)^{\mathrm{ind}} \to K_3(F)^{\mathrm{ind}}$. Thus there is a natural inclusion

$$\Phi: \quad H^0(E, \mathbb{Q}_l/\mathbb{Z}_l(2)) \to K_3(E)^{\mathrm{ind}}\{l\}.$$

To show that Φ is surjective, we may assume that E contains $\boldsymbol{\mu}_l$. Then ${}_lK_3(E)^{\mathrm{ind}}$ is cyclic by Corollary 4.4, hence Φ is surjective. □

4.3. CO-TORSION IN K_3^{ind}. – We now compute K_3^{ind}/n for fields. Let E be a field, R the semi-local ring of $\{0, 1\}$ in $\mathbb{A}^1_E$, J the Jacobson radical of R, $\bar{R}$ the quotient R/J. For an R-scheme T let $\bar{T}=T \times_{\mathrm{Spec}(R)} \mathrm{Spec}(\bar{R})$. Let $R(T)$ denote the semi-local ring of $\bar{T}$ in T. We consider a chain of R-schemes

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = \mathrm{Spec}(R)$$

such that X_{i+1} is the Brauer-Severi scheme over $\mathrm{Spec}(R(X_i))$ associated to a central simple algebra $\mathscr{D}_{i+1}$ over $R(X_i)$, with $\bar{\mathscr{D}}_{i+1}$ split.

LEMMA 4.7. — *For each* $i=1, \ldots, n$ *there is field* $E_i \supset E$, *a smooth* E_i-*scheme* Y_i, *with*

$$Y_i \cong \mathbb{P}^{n_i}_{E_i} \times_{E_i} \mathrm{Spec}(E_i[t]_{\{0, 1\}}),$$

and finite maps

$$X_i \leftarrow Y_i;\ R(X_{i-1}) \to E_i[t]_{\{0,1\}} \quad \textit{such that the diagram}$$

$$(\bigstar)\qquad \begin{tikzcd} X_i \arrow[d] & Y_i \arrow[l] \arrow[d, "p_2"'] \arrow[dr, "p_1"] & \\ X_{i-1} & \mathrm{Spec}(E_i[t]_{\{0,1\}}) \arrow[l] \arrow[dr] & \mathbb{P}^{n_i} \arrow[d] \\ & & \mathrm{Spec}(E_i) \end{tikzcd}$$

commutes.

Proof. — By Tsen's theorem, $\mathscr{D}_1 \otimes_E E_0$ is split for some finite extension E_0 of E. Let $R \to E_0[t]_{\{0,1\}}$ the natural inclusion and Y_1 the fiber product $X_1 \times_E E_0$. In general, suppose we have the diagram ($\bigstar$). Let F_i be the function field $E_i(\mathbb{P}^{n_i})$. Then the semi-local ring $R(Y_i)$ of $\bar{Y}_i$ in Y_i is $F_i[t]_{\{0,1\}}$ and $R(Y_i)$ is finite over $R(X_i)$. Take the fiber product X'_{i+1}

$$\begin{array}{c} X'_{i+1} = X_{i+1} \times_{R(X_i)} R(Y_i) \to \mathrm{Spec}(R(Y_i)) \\ \qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\ \qquad\qquad\qquad\qquad\qquad\qquad \mathrm{Spec}(F_i) \end{array}$$

Then X'_{i+1} is split by a finite extension E_{i+1} of F_i. Letting Y_{i+1} be the fiber product

$$Y_{i+1} = X'_{i+1} \times_{F_i} E_{i+1} \to E_{i+1}[t]_{\{0,1\}}$$

continues the induction. □

LEMMA 4.8. — *The map*

$$K_2(R(X_i), J(X_i)) \to K_2(R(X_{i+1}), J(X_{i+1}))$$

is injective.

Proof. — Let $X = X_i$, $X' = X_{i+1}$, $Y = Y_i$, $F = E_i$. We have the commutative ladder with exact rows

$$\begin{tikzcd} \arrow[r] & K_3(R(Y)) \arrow[r] & K_3(\bar{R}(Y)) \arrow[r] & K_2(R(Y), J(Y)) \arrow[r] & K_2(R(Y)) \arrow[r] & {} \\ \arrow[r] & K_3(R(X)) \arrow[r] \arrow[u] & K_3(\bar{R}(X)) \arrow[r] \arrow[u] & K_2(R(X), J(X)) \arrow[r] \arrow[u] & K_2(R(X)) \arrow[r] \arrow[u] & {} \end{tikzcd}$$

Since each $\bar{\mathscr{D}}_j$ is split, $\bar{R}(X)$ is a pure transcendental extension of $\bar{R}$, hence the map

$$K_3(E(0))^{\mathrm{ind}} \oplus K_3(E(1))^{\mathrm{ind}} \cong K_3(\bar{R})^{\mathrm{ind}} \to K_3(\bar{R}(X))^{\mathrm{ind}}$$

is an isomorphism. Similarly, the map

$$K_3(F(0))^{\mathrm{ind}} \oplus K_3(F(1))^{\mathrm{ind}} \to K_3(\bar{R}(Y))^{\mathrm{ind}}$$

is an isomorphism. Let X_0 and X_1 denote the two irreducible components of $\bar{X}$, and similarly define Y_0 and Y_1. Y_0 and Y_1 are both projective spaces over F; and X_0 and X_1 are projective spaces over a subfield k of F, so we can identify X_0 and X_1, Y_0 and

Y_1. Then the image of

$$K_3(R(X)) \to K_3(\bar{R}(X)) = K_3(k(X_0)) \oplus K_3(k(X_1))$$

and

$$K_3(R(Y)) \to K_3(\bar{R}(Y)) = K_3(F(Y_0)) \oplus K_3(F(Y_1))$$

contain the respective diagonals. By taking the difference maps

$$K_3(k(X_0)) \oplus K_3(k(X_1)) \to K_3(k(X_0))$$

and

$$K_3(F(Y_0)) \oplus K_3(F(Y_1)) \to K_3(F(Y_0)),$$

and noting that the maps

$$K_3^M(R(X')) \to K_3^M(\bar{R}(X')); \; K_3^M(R(X)) \to K_3^M(\bar{R}(X))$$

are surjective, we can rewrite the ladder above as

$$(\bigstar) \qquad \begin{array}{ccccccccc} \to K_3(R(Y))^{ind} & \to & K_3(F(Y_0))^{ind} & \to & K_2(R(Y), J(Y)) & \to & K_2(R(Y)) & \to \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & \\ \to K_3(R(X)^{ind} & \to & K_3(k(X_0))^{ind} & \to & K_2(R(X), J(X)) & \to & K_2(R(X)) & \to \end{array}$$

Let $L(X)$, $L(Y)$ be the quotient fields of $R(X)$, $R(Y)$. Then by Proposition 2.3, the maps

$$K_3(R(X))^{ind} \to K_3(L(X))^{ind}; \qquad K_3(R(Y))^{ind} \to K_3(L(Y))^{ind}$$

are injective. Thus from Corollary 4.6, the vertical arrows in left-hand the commutative square of ($\bigstar$) are injective. On the other hand, since $R(Y) = F[t]_{\{0, 1\}}$, the image of $K_3(R(Y))$ in $K_3(F(Y_0))$ is exactly $K_3(F(Y_0))^{dec}$, *i. e.* the map $K_3(R(Y))^{ind} \to K_3(F(Y_0))^{ind}$ is the zero map. Thus $K_3(R(X))^{ind} \to K_3(k(X_0))^{ind}$ is the zero map as well, and we have the exact sequence

$$0 \to K_3(k(X_0))^{ind} \to K_2(R(X), J(X)) \to K_2(R(X)).$$

By a similar argument, we have the exact sequence

$$0 \to K_3(k(X'_0))^{ind} \to K_2(R(X'), J(X')) \to K_2(R(X')).$$

By Suslin (Theorem 3.6 [S]) the map $K_2(L(X)) \to K_2(L(X'))$ is injective. This implies that $K_2(R(X)) \to K_2(R(X'))$ is injective; the map $K_3(k(X_0))^{ind} \to K_3(k(X'_0))^{ind}$ is also injective by Corollary 4.6, hence

$$K_2(R(X), J(X)) \to K_2(R(X'), J(X'))$$

is injective, completing the proof. □

THEOREM 4.9. — *Let* $X = X_i$, $X' = X_{i+1}$, $\pi : X' \to \operatorname{Spec}(R(X))$ *the projection. The map*

$$\pi^* : \quad K_2(R(X), J(X)) \to E_2^{0,-2}(X', \bar{X}') \subset K_2(R(X'), J(X'))$$

is an isomorphism.

Proof. — We recall from paragraph 1.6 that $E_2^{0,-2}$ is the kernel of

$$K_2(R(X'), J(X')) \to K_1'(X^{1/2}, \bar{X}^{1/2}) \subset K_1'(X)^{1/2}.$$

The injectivity of π^* follows from the previous lemma. We have the commutative diagram with exact rows:

$$\begin{array}{cccccccccc}
 & K_2(\mathscr{M}^{1/2}_{(X', \bar{X}')}) & \to & K_2(\mathscr{M}^{1/2}_{\bar{X}'}) & \to & K_1'(X'^{1/2}, \bar{X}'^{1/2}) & \to & K_1(\mathscr{M}^{1/2}_{(X, \bar{X})}) & & \\
 & \uparrow & & \uparrow{\scriptstyle\alpha} & & \uparrow & & \uparrow & & \\
\to & K_3(R(X')) & \to & K_3(\bar{R}(X')) & \to & K_2(R(X'), J(X')) & \to & K_2(R(X')) & \to & K_2(\bar{R}(X')) \\
 & {\scriptstyle\pi^*}\uparrow & & {\scriptstyle\pi^*}\uparrow & & {\scriptstyle\pi^*}\uparrow & & {\scriptstyle\pi^*}\uparrow & & {\scriptstyle\pi^*}\uparrow \\
\to & K_3(R(X)) & \to & K_3(\bar{R}(X)) & \to & K_2(R(X), J(X)) & \to & K_2(R(X)) & \to & K_2(\bar{R}(X)) \\
 & & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & & & 0 & & 0 & & 0 & & 0
\end{array}$$

The columns are all complexes. The second and fifth columns are exact since $\bar{X}'$ is a projective space over $\bar{R}(X)$; the fourth column is exact since Suslin has shown that

$$K_2(k(X)) \to H^0(X'_{k(X)}, \mathscr{K}_2)$$

is an isomorphism, and $R(X)$ is semi-local. In addition, the image of the tame symbol α is the same as the image of α restricted to $K_3^M(\bar{R}(X'))$. Since we can lift $K_3^M(\bar{R}(X'))$ to $K_3(R(X'))$, the surjectivity of $\pi^* : K_2(R(X), J(X)) \to E_2^{0,-2}(X', \bar{X}')$ follow from a diagram chase. $\square$

Let $k_2(S, \bar{S})$ denote $K_2(S, \bar{S})/l$, where l is a prime different from $\operatorname{char}(E)$, S an E-scheme with closed subscheme $\bar{S}$.

COROLLARY 4.10. — *Suppose that* E *contains* μ_l, *and that the division algebra* $\mathscr{D} = \mathscr{D}_{i+1}$ *is the crossed product algebra coming from the symbol* $(a, b)_l$, $a \in (1 + J(X))^*$, $b \in L(X)^*$. *The kernel of*

$$\pi^* : \quad k_2(R(X), J(X)) \to k_2(R(X'), J(X'))$$

is the subgroup generated by $\{a, b\}$. *If* $\mathscr{D}$ *is split, then* π^* *is injective.*

Proof. — This follows from Theorem 5.9 and a diagram chase as in [M-S]. $\square$

THEOREM 4.11. — *The Chern class map*

$$c_{2,2} : \quad K_2(R, J)/l^n \to H^2(R(X), i_!(\mu_{l^n})^{\otimes 2})$$

is an isomorphism.

Proof. — We prove the stronger result that

$$c_{2,2} : \quad K_2(R(X), J(X))/l^n \to H^2(R(X), i_!(\mu_{l^n})^{\otimes 2})$$

is an isomorphism. An argument as in the case of K_2 of fields reduces to the case $n=1$; we may also assume that E contains $\boldsymbol{\mu}_l$. In this case, for $X=X_i$, $H^2(R(X), i_!(\boldsymbol{\mu}_l)^{\otimes 2})$ just the kernel of the restriction map

$$H^2(R(X), (\boldsymbol{\mu}_l)^{\otimes 2}) \to H^2(\bar{R}(X), (\boldsymbol{\mu}_l)^{\otimes 2})$$

so $\{a, b\} \in K_2(R(X), J(X))$ goes to zero under $c_{2,2}$ if and only if the crossed product algebra $(a, b)_l$ is split. By Corollary 4.10, $\{a, b\}=0$ in $k_2(R(X), J(X))$. We now prove that

$$c_{2,2}: \quad K_2(R(X), J(X))/l \to H^2(R(X), i_!(\boldsymbol{\mu}_l)^{\otimes 2})$$

is an isomorphism by induction on the length of an element η in the kernel

$$\eta=\sum \{a_i, b_i\}.$$

This is done by going up to the Brauer-Severi scheme associated to $\{a_1, b_1\}$ and using the corollary above. □

THEOREM 4.12. — *The Chern class*

$$c_{2,1}: \quad K_3(E, \mathbb{Z}/l^n)^{\text{ind}} \to H^1(E, (\boldsymbol{\mu}_{l^n})^{\otimes 2})$$

is an isomorphism.

Proof. — We reduce as in the proof of Theorem 4.5 to the case $n=1$, and may assume that E contains $\boldsymbol{\mu}_l$. We have the commutative ladder

$$\begin{array}{ccccccccc}
K_3(R, J; \mathbb{Z}/l) & \xrightarrow{\kappa} & K_3(R; \mathbb{Z}/l) & \xrightarrow{\alpha} & K_3(E; \mathbb{Z}/l) & \to & K_2(R, J)/l & \to & K_2(R)/l \\
\downarrow \varepsilon & & \downarrow \gamma & & \downarrow \delta & & \downarrow \wr & & \downarrow \wr \\
H^1(R, i_!(\boldsymbol{\mu}_l^{\otimes 2}))_\iota & \to & H^1(R, \boldsymbol{\mu}_l^{\otimes 2}) & \to & H^1(E, \boldsymbol{\mu}_l^{\otimes 2}) & \to & H^2(R, i_!(\boldsymbol{\mu}_l^{\otimes 2})) & \to & H^2(R, \boldsymbol{\mu}_l^{\otimes 2})
\end{array}$$

We have already shown that δ is surjective. Since β is surjective, α is also surjective. As in the proof of Theorem 4.5, α and β factor through $K_3(R; \mathbb{Z}/l)/K_3(E; \mathbb{Z}/l)$ and $H^1(R, \boldsymbol{\mu}_l^{\otimes 2})/H^1(E, \boldsymbol{\mu}_l^{\otimes 2})$ respectively. We claim that ε maps $K_3(R, J; \mathbb{Z}/l)$ onto $\iota(H^1(R, i_!(\boldsymbol{\mu}_l^{\otimes 2})))$.

Indeed, we have the commutative triangle

$$\begin{array}{ccc}
K_3(R; \mathbb{Z}/l) & \to & {}_lK_2(R) \\
\gamma \downarrow & \nearrow \text{symb}_R & \\
H^1(R, \boldsymbol{\mu}_l) \otimes \boldsymbol{\mu}_l = R^\times \otimes \boldsymbol{\mu}_l & &
\end{array}$$

The image $\iota(H^1(R, i_!(\boldsymbol{\mu}_l^{\otimes 2})))$ is $(1+J)^\times/l$; let f be in $(1+J)^\times$, and let η be a lifting of the element $\{f, \zeta_l\}$ of ${}_lK_2(R, J)$ to $K_3(R, J; \mathbb{Z}/l)$. Then

$$\text{symb}_R \circ \gamma \circ \kappa(\eta) = \text{symb}_R(f \otimes \zeta_l).$$

On the other hand, the kernel of symb_R injects into $H^1(E, \boldsymbol{\mu}_l^{\otimes 2})^2$, hence $\gamma \circ \kappa(K_3(R, J; \mathbb{Z}/l))$ maps isomorphically onto ${}_lK_2(R)$ via symb_R. Thus

$\gamma\circ\kappa(\eta)=(f\otimes\zeta_l)$, proving our claim. Since

$$\bar{\gamma}:\quad K_3(R;\mathbb{Z}/l)/K_3(E;\mathbb{Z}/l)\to H^1(R,\mu_l^{\otimes 2})/H^1(E,\mu_l^{\otimes 2})$$

is an isomorphism, δ is an isomorphism, as claimed. □

Let E/F be a finite Galois extension of fields which are finitely generated over the prime field. Since $H^0_{et}(E,\mathbb{Z}_l(2))=0$, the Hochschild-Serre spectral sequence shows that

$$H^1(F,\mathbb{Z}_l(2))=H^1(E,\mathbb{Z}_l(2))^{\mathrm{Gal}(E/F)}.$$

In addition, using the Bloch-Ogus sequence relating $H^1_{et}(E(t),\mu_{l^v}{}^{\otimes 2})$ and $H^1_{et}(\mathbb{A}^1_E,\mu_{l^v}{}^{\otimes 2})$, we find that $H^1(-,\mathbb{Z}_l(2))$ is invariant under pure transcendental extensions.

THEOREM 4.13. — *Let* E *be a field. Then the map*

$$c_{2,1}:\quad \varprojlim K_3(E)^{\mathrm{ind}}/l^n\to H^1_{et}(E,\mathbb{Z}_l(2))$$

is an isomorphism, so the kernel of $c_{2,1}: K_3(E)^{\mathrm{ind}}\to H^1_{et}(E,\mathbb{Z}_l(2))$ *is the maximal l-divisible subgroup of* $K_3(E)^{\mathrm{ind}}$. *If* E_0 *is the field of constants in* E, *then*

$$K_3(E_0)^{\mathrm{ind}}/l^n\to K_3(E)^{\mathrm{ind}}/l^n$$

is an isomorphism. If $E\to F$ *is an algebraic Galois extension with Group* G, *such that every finite quotient of* G *has order prime to the characteristic, then*

$$K_3(E)^{\mathrm{ind}}=(K_3(F)^{\mathrm{ind}})^G.$$

Proof. — Suslin has shown that

$$\ker(H^1(E_0,(\mu_{l^n})^{\otimes 2})\to {}_{l^n}K_2(E_0))\to\ker(H^1(E,(\mu_{l^n})^{\otimes 2})\to {}_{l^n}K_2(E))$$

is an isomorphism, and that these kernels are the image under $c_{2,1}$ of $K_3(E_0)^{\mathrm{ind}}$ and $K_3(E)^{\mathrm{ind}}$ respectively. In addition, he has shown that the map

$$H^1(E_0,\mathbb{Z}_l(2))\to H^1(E,\mathbb{Z}_l(2))$$

is an isomorphism. The first two results follow from this, Theorem 4.5 and Theorem 4.12. To prove the third, we may assume that F is finite over E, of degree say d. Since $K_3(E)^{\mathrm{ind}}\to K_3(F)^{\mathrm{ind}}$ is injective we have the inclusions

$$d.K_3(E)^{\mathrm{ind}}\subset d.(K_3(F)^{\mathrm{ind}})^G\subset K_3(E)^{\mathrm{ind}}\subset(K_3(F)^{\mathrm{ind}})^G$$

Thus we need only show that

$$K_3(E)^{\mathrm{ind}}/l=(K_3(F)^{\mathrm{ind}}/l)^G.$$

for all $l\mid d$. The result now follows from the isomorphism

$$H^1(E,\mathbb{Z}_l(2))\tilde{\to}H^1(F,\mathbb{Z}_l(2))^G$$

and Theorem 4.12. □

Let F be a field. We recall the definition of Bloch's group B(F). Let D(F) be the free abelian group on $F^*-\{1\}$; P(F) the quotient of D(F) by the subgroup generated by elements of the form

$$[x]-[y]+[y/x]-[(1-y)/(1-x)]+[(1-y^{-1})/(1-x^{-1})].$$

The map $D(F) \to F^* \otimes F^*/\langle a \otimes b + b \otimes a \rangle$ gotten by sending $[x]$ to $x \otimes (1-x)$ descends to P(F). B(F) is defined to be the kernel of

$$T(F) \to F^* \otimes F^*/\langle a \otimes b + b \otimes a \rangle$$

COROLLARY 4.14. — *Let* E *be a field containing an algebraically closed field. Then Bloch's group* B(E) *is uniquely l-divisible for l prime to the characteristic.*

Proof. — We may assume that E is finitely generated over the algebraic closure of the prime field. Suslin has shown that B(E) is just $K_3(E)^{\mathrm{ind}}$ modulo the image of $\mathbb{Q}_l/\mathbb{Z}_l(2)$. By Corollary 4.6 B(E) is torsion free. Since $H^1(E, \mathbb{Z}_l(2))=0$ by Suslin's computation (Cor. 2.7 [S]), it follows from the previous theorem that B(E) is l-divisible. □

5. Relative K_2 and l-adic cohomology

We now proceed to prove an analogue of the theorem of Merkurjev and Suslin for relative K_2 of semi-local PIR's. Since the receptor cohomology group for the relevant Galois symbols are the étale cohomology groups of Spec(R), R a semi-local PIR, we need a good cohomology theory with $\mathbb{Z}_l(i)$ coefficients. Uwe Jannsen [J] has constructed such a theory by viewing $\mathbb{Z}_l(i)$ as an object in the category of inverse systems of étale sheaves. A similar theory has been constructed by Dwyer and Friedlander [D-F], using étale homotopy theory.

5.1. CONTINUOUS COHOMOLOGY. — Let $\mathcal{S}_{et}(X)$ denote the category of sheaves in the small étale site over X, $\mathcal{AB}$ the category of abelian groups. If $\mathcal{A}$ is an abelian category, let $\mathcal{A}^{\mathbb{N}}$ denote the category of inverse systems in $\mathcal{A}$ indexed by the natural numbers. Jannsen defines the continuous cohomology on X of the limit F of an inverse system $(F_n) \in \mathcal{S}_{ét}(X)^{\mathbb{N}}$, $H^*_{cont}(X, \varprojlim(F_n))$, to be the derived functors of the composition

$$(F_n) \to (H^0_{et}(X, F_n)) \to \varprojlim (H^0_{et}(X, F_n))$$

from $\mathcal{S}_{et}(X)^{\mathbb{N}}$ to $\mathcal{AB}$. The functor H^*_{cont} satisfies many of the properties of continuous Galois cohomology; in particular if X is the spectrum of a field, and the (F_n) satisfies the Mittag-Leffler condition (*e.g.* all F_n sheaves of finite groups) then $H^*_{cont}(X, F)$ is the usual continuous Galois cohomology. There is a Hochschild-Serre spectral sequence if

X is over a field, and short exact sequences

$$0 \to \lim^1 (\mathrm{H}^{p-1}(\mathrm{X}, \mathrm{F}_n)) \to \mathrm{H}^p_{\mathrm{cont}}(\mathrm{X}, \varprojlim (\mathrm{F}_n)) \to \varprojlim (\mathrm{H}^p_{\mathrm{cont}}(\mathrm{X}, \mathrm{F}_n)) \to 0.$$

In particular, if X is of finite type over $\mathbb{Z}[1/l]$, the cohomology groups $\mathrm{H}^p(\mathrm{X}, \mathrm{F}_n)$ are finite if the F_n are sheaves of finite groups, hence

$$\mathrm{H}^p_{\mathrm{cont}}(\mathrm{X}, \varprojlim (\mathrm{F}_n)) = \varprojlim \mathrm{H}^p(\mathrm{X}, \mathrm{F}_n).$$

Let X be a scheme essentially of finite type over a field k. Let $\mathrm{Fin}(\mathrm{X}/)$ be the category of pairs (Y, f), where Y is a scheme of finite type over $\mathbb{Z}[1/l]$ and $f: \mathrm{X} \to \mathrm{Y}$ is a morphism. A morphism from (Y, f) to (Z, g) is a commutative diagram

$$\begin{array}{ccc} \mathrm{X} & \to & \mathrm{Z} \\ \downarrow & \nearrow & \\ \mathrm{Y} & & \end{array}$$

Then X is the inverse limit

$$\varprojlim_{\mathrm{Fin}(\mathrm{X}/)} \mathrm{Y}$$

hence $\mathrm{K}_*(\mathrm{X})$ is the direct limit

$$\mathrm{K}_*(\mathrm{X}) = \varinjlim_{\mathrm{Fin}(\mathrm{X}/)^{\mathrm{op}}} \mathrm{K}_*(\mathrm{Y}).$$

We have the Chern classes ([Gillet] or [So]):

$$c_{p,q}: \quad \mathrm{K}_{2p-q}(\mathrm{Y}) \to \varprojlim \mathrm{H}^q_{\mathrm{et}}(\mathrm{Y}, (\mu_{l^n})^{\otimes p}) = \mathrm{H}^q_{\mathrm{cont}}(\mathrm{Y}, \mathbb{Z}_l(p)).$$

This defines the Chern classes $c_{p,q}: \mathrm{K}_{2p-q}(\mathrm{X}) \to \mathrm{H}^q_{\mathrm{cont}}(\mathrm{X}, \mathbb{Z}_l(p))$ via the composition

$$\mathrm{K}_{2p-q}(\mathrm{X}) \xrightarrow{\sim} \varinjlim_{\mathrm{Fin}(\mathrm{X}/)^{\mathrm{op}}} \mathrm{K}_{2p-q}(\mathrm{Y}) \xrightarrow{c_{p,q}} \varinjlim_{\mathrm{Fin}(\mathrm{X}/)^{\mathrm{op}}} \mathrm{H}^q_{\mathrm{cont}}(\mathrm{Y}, \mathbb{Z}_l(p)) \to \mathrm{H}^q_{\mathrm{cont}}(\mathrm{X}, \mathbb{Z}_l(p)).$$

If $\mathrm{X} = \mathrm{Spec}(\mathrm{R})$, where R is a semi-local PIR with Jacobson radical J, and $i: x \to \mathrm{X}$ the inclusion of the generic point, we similarly get Chern classes

$$c_{p,q}: \quad \mathrm{K}_{2p-q}(\mathrm{R}, \mathrm{J}) \to \mathrm{H}^q_{\mathrm{cont}}(\mathrm{X}, i_!(\mathbb{Z}_l(p))),$$

compatible with the relativization sequences for K-theory and continuous cohomology. This is done using the relative Chern classes of paragraph 1.12 and a limit argument as above.

5.2. MERKURJEV-SUSLIN FOR RELATIVE K_2.

THEOREM 5.1. — *Let* R *be a semi-local PIR containing a field* k_0, J *the Jacobson radical. Let* l *be a prime distinct from* $\mathrm{char}(k_0)$. *Then the Chern class*

$$c_{2,2}:\quad K_2(R, J)/l^n \to H^2(R, i_!(\mu_{l^n})^{\otimes 2})$$

is an isomorphism. The map

$$c_{2,2}:\quad K_2(R, J) \to H^2_{\mathrm{cont}}(R, i_!(\mathbb{Z}_l(2)))$$

is injective mod the maximal l-divisible subgroup of $K_2(R, J)$. If all the residue fields of R are finite extensions of the prime field, then $c_{2,2}$ induces a natural isomorphism

$$K_2(R, J)\{l\} \to H^2(R, i_!\mathbb{Z}_l(2))\{l\}.$$

Proof. — To prove the first statement, we reduce to the case $n=1$, and may assume that R contains μ_l. Let $\bar{R}=R/J$. By Theorem 4.13, the map

$$c_{2,1}:\quad K_3(\bar{R};\ \mathbb{Z}/l)^{\mathrm{ind}} \to H^1(\bar{R},\ \mathbb{Z}/l(2))$$

is an isomorphism. Arguing as in Theorem 4.5, the map

$$c_{2,1}:\quad K_3(R;\ \mathbb{Z}/l)^{\mathrm{ind}} \to H^1(R,\ \mathbb{Z}/l(2))$$

is surjective. Since R contains μ_l, the map

$$H^1(R,\ \mathbb{Z}/l(2)) \to H^1(\bar{R},\ \mathbb{Z}/l(2))$$

is surjective, hence

$$K_3(R;\ \mathbb{Z}/l)^{\mathrm{ind}} \to K_3(\bar{R};\ \mathbb{Z}/l)^{\mathrm{ind}}$$

is surjective. Thus

$$K_2(R, J)/l \to K_2(R)$$

and

$$H^2(R, i_!(\mu_l)^{\otimes 2}) \to H^2(R, (\mu_l)^{\otimes 2})$$

are injective. Since ${}_lK_1(R, J)=0$ and ${}_lK_1(R) \to {}_lK_1(\bar{R})$ is injective, the relativization sequence

$$\to K_2(R, J;\ \mathbb{Z}/l) \to K_2(R;\ \mathbb{Z}/l) \to K_2(\bar{R};\ \mathbb{Z}/l) \to$$

yields the commutative ladder

$$\begin{array}{ccccccc} 0 \to & K_2(R, J)/l & \to & K_2(R)/l & \to & K_2(\bar{R})/l \\ & \downarrow & & \downarrow \wr & & \downarrow \wr \\ 0 \to & H^2(R, i_!(\mu_l)^{\otimes 2}) & \to & H^2(R, (\mu_l)^{\otimes 2}) & \to & H^2(\bar{R}, (\mu_l)^{\otimes 2}) \end{array}$$

Thus

$$c_{2,2}:\quad K_2(R, J)/l \to H^2(R, i_!(\mu_l)^{\otimes 2})$$

is an isomorphism as claimed. Passing to the limit, we see that

$$c_{2,2}: \quad \varprojlim \mathrm{K}_2(\mathrm{R}, \mathrm{J})/l^n \to \varprojlim \mathrm{H}^2(\mathrm{R}, i_!(\boldsymbol{\mu}_{l^n})^{\otimes 2})$$

is an isomorphism. We have the commutative diagram

$$\begin{array}{ccc} \mathrm{K}_2(\mathrm{R}, \mathrm{J}) & \xrightarrow{c_{2,2}} & \mathrm{H}^2_{\mathrm{cont}}(\mathrm{R}, i_!(\mathbb{Z}_l(2))) \\ \downarrow & & \downarrow \\ \varprojlim \mathrm{K}_2(\mathrm{R}, \mathrm{J})/l^n & \xrightarrow[c_{2,2}]{} & \varprojlim \mathrm{H}^2(\mathrm{R}, i_!(\boldsymbol{\mu}_{l^n})^{\otimes 2}) \end{array}$$

proving the second statement. To prove the third, we note that our assumptions on R, together with Quillen's finiteness theorem [Q3] for the K-theory of number rings, implies that the maximal l-divisible subgroup of $\mathrm{K}_2(\mathrm{R}, \mathrm{J})$ is just the prime to l torsion. We may assume that k_0 contains $\boldsymbol{\mu}_l$. Then the sequence

$$\mathrm{H}^1(\mathrm{R}, i_!(\boldsymbol{\mu}_l)^{\otimes 2}) \to \mathrm{H}^2_{\mathrm{cont}}(\mathrm{R}, i_!\mathbb{Z}_l(2)) \xrightarrow{\times l} \mathrm{H}^2_{\mathrm{cont}}(\mathrm{R}, i_!\mathbb{Z}_l(2))$$

together with the symbol map

$$\begin{aligned} \mathrm{symb}: \quad & \mathrm{H}^1(\mathrm{R}, i_!(\boldsymbol{\mu}_l)^{\otimes 2}) \to {}_l\mathrm{K}_2(\mathrm{R}, \mathrm{J}) \\ & f \otimes \zeta \to \{f, \zeta\} \end{aligned}$$

shows that $c_{2,2}$ maps ${}_l\mathrm{K}_2(\mathrm{R}, \mathrm{J})$ onto ${}_l\mathrm{H}^2_{\mathrm{cont}}(\mathrm{R}, i_!\mathbb{Z}_l(2))$, completing the proof. □

We have the Chern classes

$$c_{2,1}: \quad \mathrm{K}_3(\mathrm{R}, \mathrm{J}; \mathbb{Z}/l^n) \to \mathrm{H}^1(\mathrm{R}, i_!(\boldsymbol{\mu}_{l^n})^{\otimes 2})$$

and

$$c_{2,2}: \quad \mathrm{K}_2(\mathrm{R}, \mathrm{J}) \to \mathrm{H}^2(\mathrm{R}, i_!(\boldsymbol{\mu}_{l^n})^{\otimes 2}).$$

As in the absolute case, these are compatible with the Bockstein homomorphisms, *i. e.* we have the commutative square

$$\begin{array}{ccc} \mathrm{K}_3(\mathrm{R}, \mathrm{J}; \mathbb{Z}/l^n) \to {}_{l^n}\mathrm{K}_2(\mathrm{R}, \mathrm{J}) & \subset & \mathrm{K}_2(\mathrm{R}, \mathrm{J}) \\ {\scriptstyle c_{2,1}}\downarrow & & {\scriptstyle c_{2,2}}\downarrow \\ \mathrm{H}^1(\mathrm{R}, i_!(\boldsymbol{\mu}_{l^n})^{\otimes 2}) & \xrightarrow{\delta} & \mathrm{H}^2(\mathrm{R}, i_!(\boldsymbol{\mu}_l)^{\otimes 2}) \end{array}$$

In addition, if R contains $\boldsymbol{\mu}_{l^n}$, the map

$$\bar{c}_{2,1}: \quad {}_{l^n}\mathrm{K}_2(\mathrm{R}, \mathrm{J}) \to \mathrm{H}^1(\mathrm{R}, i_!(\boldsymbol{\mu}_{l^n})^{\otimes 2})/c_{2,1}(\mathrm{K}_3(\mathrm{R}, \mathrm{J}))$$

satisfies

$$\bar{c}_{2,1}(\{f, \zeta_{l^n}\}) = \bar{f} \otimes \zeta_{l^n} \bmod c_{2,1}(\mathrm{K}_3(\mathrm{R}, \mathrm{J})); \qquad \bar{f} = f \bmod (1+\mathrm{J})^{*l^n}.$$

We have the relativization sequence:

$$H^0_{\text{cont}}(\bar{R},\ \mathbb{Z}_l(2)) \to H^1_{\text{cont}}(R,\ i_!\,\mathbb{Z}_l(2)) \to H^1_{\text{cont}}(R,\ \mathbb{Z}_l(2)) \to H^0_{\text{cont}}(\bar{R},\ \mathbb{Z}_l(2)).$$

Suppose that R is essentially of finite type over $\mathbb{Z}$. Then the H^0 terms both vanish, and the $\lim^1$ terms for the H^1's also vanish so we get

$$H^1_{\text{cont}}(R,\ \mathbb{Z}_l(2)) = \varprojlim H^1(R,\ (\boldsymbol{\mu}_{l^n})^{\otimes 2}).$$

By the Bloch-Ogus sequence, we get

$$\begin{aligned}\varprojlim H^1(R,\ (\boldsymbol{\mu}_{l^n})^{\otimes 2}) &= \varprojlim H^1(L,\ (\boldsymbol{\mu}_{l^n})^{\otimes 2})\\ &= H^1(L,\ \mathbb{Z}_l(2));\end{aligned}$$

by Suslin's theorem (Cor. 2.7 [S]), the restriction map

$$H^1_{\text{cont}}(R,\ \mathbb{Z}_l(2)) \to H^1_{\text{cont}}(\bar{R},\ \mathbb{Z}_l(2));$$

is injective, hence

$$H^1_{\text{cont}}(R,\ i_!\,\mathbb{Z}_l(2)) = 0.$$

Lemma 5.2. — *Let* R *be a semi-local PIR. Then*

$$c_{2,1}(K_3(R,\ J)) = 0.$$

Proof. — We may suppose the R is essentially of finite type over $\mathbb{Z}$. We have the commutative diagram

$$\begin{array}{ccc} K_3(R,\ J) & \to & K_3(R,\ J,\ \mathbb{Z}/l^n) \\ {\scriptstyle c_{2,1}}\downarrow & & {\scriptstyle c_{2,1}}\downarrow \\ 0 = H^1\,\text{cont}(R,\ i_!\,\mathbb{Z}_l(2)) & \to & H^1(R,\ i_!(\boldsymbol{\mu}_{l^n})^{\otimes 2}) \end{array}$$

which proves the lemma. □

Let L be the quotient field of R, L_0 the field of constants in L, and $R_0 = L_0 \cap R$. R_0 is a semi-local PIR and R is a smooth, faithfully flat extension of R_0. We call R_0 the *ring of constants* in R.

Theorem 5.3. — *Suppose* R *contains* $\boldsymbol{\mu}_{l^n}$. *Let* R_0 *be the ring of constants in* R. *Then the following are equivalent*

(*a*) $\{f,\ \zeta_{l^n}\} = 0$ in $K_2(R,\ J)$

(*b*) $f = f_0\, g^{l^r}$, *with* g *in* $(1+J)^*$, f_0 *in* $(1+J_0)^*$, *and* $\{f_0,\ \zeta_{l^n}\} = 0$ *in* $K_2(R_0,\ J_0)$.

Proof. — This is the same as the proof of Theorem 3.5 in [S]. □

THEOREM 5.4. — *Let* R → S *be a smooth faithfully flat extension of semi-local PIR's with* R *algebraically closed in* S. *Suppose that* $JS \cap R = J$. *Then*

$$K_2(R, J) \to K_2(S, JS)$$

is injective.

Proof. — The same as the proof of Theorem 3.9 of [S]. □

Since $c_{2,1}: K_3(R, J) \to H^1_{\text{cont}}(R, i_!(\mu_{l^n})^{\otimes 2})$ is the zero map, we get a well-defined map $\Phi: K_2(R, J)\{l\} \to H^1_{\text{cont}}(R, i_1 \mathbb{Q}_l/\mathbb{Z}_l(2))$.

COROLLARY 5.5. — *There is a natural surjection*

$$H^1(R, i_!(\mu_{l^n})^{\otimes 2}) \to {}_{l^n}K_2(R, J).$$

If R *has characteristic zero, then*

$$\Phi:\quad K_2(R, J)\{l\} \to H^1_{\text{cont}}(R, i_! \mathbb{Q}_l/\mathbb{Z}_l(2))/\operatorname{Im}(H^1_{\text{cont}}(R, i_! \mathbb{Q}_l(2)))$$

is an isomorphism. If R *has characteristic* $p > 0$, $p \neq l$ *then*

$$\Phi:\quad K_2(R, J)\{l\} \to H^1_{\text{cont}}(R, i_! \mathbb{Q}_l/\mathbb{Z}_l(2))$$

is an isomorphism.

Proof. — Make the obvious modifications in the argument Suslin uses to prove Theorems 3.9, 3.10, and Corollary 3.13 in [S]. □

COROLLARY 5.4. — *Let* $\mathscr{D}$ *be an Azumaya algebra over* R *with* $\bar{\mathscr{D}}$ *split,* $\pi: X \to \operatorname{Spec}(R)$ *the associated Brauer-Severi scheme. Then the map*

$$\pi^*:\quad K_2(R, J) \to E_2^{0,-2}(X, \bar{X})$$

is an isomorphism.

Proof. — Same as Theorem 4.9. □

COROLLARY 5.5. — *Let* $\mathscr{D}$ *be an Azumaya algebra over* R *with* $\bar{\mathscr{D}}$ *split. Then there is a unique homomorphism*

$$\mathrm{Nrd}:\quad K_2(\mathscr{D}, \bar{\mathscr{D}}) \to K_2(R, J)$$

such that for every smooth extension R → S *splitting* $\mathscr{D}$, *the diagram*

$$\begin{array}{rccc} \mathrm{Nrd}: & K_2(\mathscr{D}, \bar{\mathscr{D}}) & \to & K_2(R, J) \\ & \downarrow & & \downarrow \\ & K_2(\mathscr{D}_S, \bar{\mathscr{D}}_S) & \xrightarrow{\sim} & K_2(S, JS) \end{array}$$

commutes.

Proof. — Let X/R be the associated Brauer-Severi scheme. Define Nrd to be the composition

$$\begin{tikzcd} K_2(\mathscr{D}, \bar{\mathscr{D}}) \arrow[r] \arrow[drr, "\mathrm{Nrd}"'] & K_2(X, \bar{X}) = K_2(R, J) \oplus K_2(\mathscr{D}, \bar{\mathscr{D}}) \oplus \ldots \arrow[r] & E_2^{0,-2}(X, \bar{X}) \arrow[d, "\wr"] \\ & & K_2(R, J). \end{tikzcd} \quad \square$$

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