ANNALES SCIENTIFIQUES DE L'É.N.S.

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Annales scientifiques de l'É.N.S. 4e série, tome 21, nº 2 (1988), p. 287-305
<http://www.numdam.org/item?id=ASENS_1988_4_21_2_287_0>

Ann. scient. Éc. Norm. Sup.,
4^e série, t. 21, 1988, p. 287 à 305.

ON THE EXISTENCE OF MINIMAL HYPERSPHERES IN COMPACT SYMMETRIC SPACES

BY WU-TEH HSIANG, WU-YI HSIANG AND PER TOMTER

1. Introduction

In the study of Riemannian geometry, symmetric spaces constitute a remarkable family of natural generalizations of the classical spaces of constant curvatures; the compact ones are generalizations of the spherical spaces, and the non-compact ones are generalizations of the hyperbolic spaces. Therefore, it is rather natural to seek generalizations of various fundamental results of the spherical (resp. hyperbolic) geometry in the realm of symmetric spaces of compact (resp. non-compact) type. Among all the hypersurfaces of $S^n(1)$, the equator $S^{n-1}(1)$ is certainly one of the simplest, nice global objects. Thus, one is naturally led to the following question of "generalized equator" in a given compact symmetric space M^n [5].

Question. — Among all the hypersurfaces of a given compact symmetric space M^n, what kind of simple, nice hypersurfaces deserve the title of the "generalized equators" of M^n?

In [5], it was proposed that *imbedded, minimal hyperspheres* (i. e., hypersurfaces of the diffeomorphic type of a sphere) should be among the reasonable candidates for generalized equators and the method of equivariant geometry was used to establish the existence of such nice objects in the four compact symmetric spaces of A_2-type. Hence it is quite natural to study the following more specific problem:

PROBLEM 1. — Is it true that every simply connected, compact symmetric space contains some imbedded, minimal hyperspheres?

Furthermore, many new examples of (non-equatorial) minimal hyperspheres in $S^n(1)$ have been constructed in some recent papers ([4], [7], [9], [10], [12]). Therefore, as a generalization of the "opposite" spherical Bernstein problem [7], it is also quite natural to pose the following

PROBLEM 2. — Is it true that every simply connected, compact symmetric space of dimension $\geqq 4$ contains infinitely many congruence classes of imbedded (or immersed) minimal hyperspheres?

Although the whole family of symmetric spaces can be neatly characterized by a single condition that they are *centrally symmetric* with respect to every given point, a survey of the classification list of É. Cartan ([1], [2]) indicates that it consists of a remarkable collection of natural geometric models with fascinating individualities. Therefore, for global geometric problems of the above type, even though there may, eventually, emerge some kind of uniform final results, the technical routes that lead to such answers will, most likely, inevitably, involve quite a lot of case studies. It is in this spirit that we shall begin the study of the above *existence problem of minimal hyperspheres* in some special cases of compact symmetric spaces.

In this paper, we shall mainly treat those special cases which happen to accommodate some particularly suitable geometric structures that enable one to establish the existence of infinitely many distinct minimal hyperspheres with a rather small amount of technicalities. We state the main results of this paper as the following theorems.

THEOREM 1. — *There exist infinitely many congruence classes of imbedded, minimal hyperspheres in the complex projective n-space* $\mathbb{C}\mathrm{P}(n)$, *for each* $n \geqq 2$.

Remark. — In fact, even in the case of spheres, $S^n(1)$, $n \geqq 4$, the existence of infinitely many congruence classes of *imbedded, minimal* hyperspheres have, so far, been proved only for the dimensions 4, 5, 6, 7, 8, 10, 12 and 14. Although one expects that the same type of existence results should also hold for all higher dimensional spheres, it is technically rather difficult to prove. Therefore, it is quite remarkable that there exists a *uniform* construction which works for *all* $\mathbb{C}\mathrm{P}(n)$, $n \geqq 2$. The above result was known to us back in 1983 and was announced in [6].

THEOREM 2. — *There exist infinitely many congruence classes of imbedded, minimal hyperspheres in the real Grassmannian manifold,* $\mathrm{SO}(m+2)/\mathrm{SO}(m)\times\mathrm{SO}(2)$, *for each* $m \geqq 3$.

THEOREM 3. — *Let* M *be one of the following list of compact symmetric spaces of rank* 2, *namely,*

$$S^2(1)\times S^2(1),\qquad S^3(1)\times S^3(1),\qquad \mathrm{SU}(3)/\mathrm{SO}(3),\qquad \mathrm{SU}(3),\qquad \mathrm{Sp}(2).$$

Then there exist infinitely many congruence classes of imbedded minimal hyperspheres in M.

THEOREM 4. — *There exist infinitely many congruence classes of immersed, minimal hyperspheres in the product of two copies of isometric complex projective n-space,* $\mathbb{C}\mathrm{P}(n)\times\mathbb{C}\mathrm{P}(n)$, *for each* $n \geqq 2$.

So far, the only method that enables one to prove theorems of the above type is the method of equivariant differential geometry which, of course, relies heavily on the existence of some particularly suitable equivariant geometric structures of the ambient space. In paragraph 2, we shall exhibit the orbital geometries of those specific transformation groups which enable us to reduce the proofs of the above theorems to the existence of infinitely many global solution curves of specific geometric type of the reduced ODE. The analytical techniques involved in the proof of existence of those global solution curves are essentially modifications of that of ([7], [9]). We believe that the existence of infinitely many congruence classes of *imbedded, minimal, hyperspheres* is probably true for *all* simply connected compact symmetric spaces of dimensions $\geqq 4$. However, such a uniform result would be extremely difficult to prove even if it holds in general for all cases of simply connected, compact symmetric spaces of dimension $\geqq 4$.

2. On the orbital geometries of some specific transformation groups

In this section, we shall exhibit one specific, workable transformation group for each of those compact symmetric spaces mentioned in the above theorems. Various specific features of their orbital geometries are actually the basic ingredients that make the analytic proofs of later sections applicable.

2.1. THE CASE OF $\mathbb{C}P(n)$. — Let $\tilde{G}=U(1)\times U(1)\times U(n-1)$ be the subgroup of a unitary transformation group on $\mathbb{C}^{n+1}=\mathbb{C}^1\oplus\mathbb{C}^1\oplus\mathbb{C}^{n-1}$ and S^1 be the subgroup of scalar multiplications of unit complex numbers. Then the $\tilde{G}$-action on $S^{2n+1}(1)$ induces an action of $G=\tilde{G}/S^1$ on $\mathbb{C}P(n)=S^{2n+1}(1)/S^1$ such that

$$\mathbb{C}P(n)/G \cong S^{2n+1}/\tilde{G}\,(=\Delta). \tag{1}$$

Let (X, Y, Z) be a generic point in $\mathbb{C}^1\oplus\mathbb{C}^1\oplus\mathbb{C}^{n-1}$ and $x=|X|$, $y=|Y|$, $z=|Z|$. Then the above orbit space, Δ, can be parametrized by

$$\Delta \cong \{(x, y, z);\ x, y, z \geqq 0 \text{ and } x^2+y^2+z^2=1\} \tag{2}$$

which is geometrically an octant of the unit sphere. The generic $\tilde{G}$-orbits are of the type $S^1\times S^1\times S^{2n-3}$, namely, in the case $x, y, z>0$, the $\tilde{G}$-orbit with coordinate (x, y, z) is isometric to $S^1(x)\times S^1(y)\times S^{2n-3}(z)$. Therefore, the volume function, $\tilde{v}$, which records the volume of generic $\tilde{G}$-orbits, is as follows

$$\tilde{v}(x, y, z)=(2\pi)^2\, C\, xyz^{2n-3} \tag{3}$$

where C = the $(2n-3)$-dimensional volume of $S^{2n-3}(1)$. Since each $\tilde{G}$-orbit is a Riemannian fibration over its corresponding G-orbit with totally geodesic fibres isometric to $S^1(1)$, it is easy to see that

$$v(x, y, z)=2\pi\, C\, xyz^{2n-3} \tag{4}$$

records the volume of generic G-orbits in $\mathbb{C}P(n)$.

2.2. THE CASE OF $S^n(1)\times S^n(1)$. — Let $G=O(n)\times O(n)$. Then the space of G-orbits

$$\Delta=M/G=[S^n(1)/O(n)]\times[S^n(1)/O(n)]\simeq[0, \pi]\times[0, \pi] \tag{5}$$

is isometric to a flat square of size π, namely,

$$\Delta=\{(x, y);\ 0\leqq x, y\leqq \pi\},\qquad ds^2=dx^2+dy^2. \tag{5'}$$

For the point (x, y) with $0 < x, y < \pi$, the corresponding orbit is isometric to $S^{n-1}(\sin x) \times S^{n-1}(\sin y)$. Hence, the volume function is given by

$$v(x, y) = C^2 . [\sin x . \sin y]^{n-1} \tag{6}$$

where C = the $(n-1)$-dimensional volume of $S^{n-1}(1)$.

[In this paper, we shall only need the special case of $n = 2, 3$].

2.3. THE CASE OF $\mathbb{C}P(n) \times \mathbb{C}P(n)$. — Let $\tilde{G} = U(1) \times U(n) \times U(1) \times U(n)$ be the subgroup of $U(n+1) \times U(n+1)$ acting on $S^{2n+1}(1) \times S^{2n+1}(1)$ and $S^1 \times S^1$ be the subgroup of scalar multiples. Then

$$\mathbb{C}P(n) \times \mathbb{C}P(n) = [S^{2n+1}(1) \times S^{2n+1}(1)]/[S^1 \times S^1] \tag{7}$$

and the above $\tilde{G}$-action induces an action of $G = \tilde{G}/[S^1 \times S^1]$ on $\mathbb{C}P(n) \times \mathbb{C}P(n)$ such that

$$[\mathbb{C}P(n) \times \mathbb{C}P(n)]/G \cong [S^{2n+1}(1) \times S^{2n+1}(1)]/\tilde{G} \quad (=\Delta). \tag{8}$$

It is easy to see that $S^{2n+1}(1)/[U(1) \times U(n)]$ is isometric to an interval of length $\pi/2$ and the generic $U(1) \times U(n)$-orbit corresponding to $0 < x < \pi/2$ is isometric to $S^1(\sin x) \times S^{2n-1}(\cos x)$. Hence

$$\Delta = \left\{(x, y);\ 0 \leqq x, y \leqq \frac{\pi}{2}\right\}, \qquad ds^2 = dx^2 + dy^2 \tag{9}$$

and the volume function which records the volume of G-orbits in $\mathbb{C}P(n) \times \mathbb{C}P(n)$ is given by

$$v(x, y) = C^2 . \sin x \sin y . [\cos x \cos y]^{2n-1} \tag{10}$$

where C = the $(2n-1)$-dimensional volume of $S^{2n-1}(1)$.

2.4. THE CASES OF $SU(3)/SO(3)$, $SU(3)$ AND $Sp(2)$. — Let $K = SO(3)$ (resp. $SU(3)$, $Sp(2)$) which acts on $M = SU(3)/SO(3)$ [resp. $SU(3)$, $Sp(2)$] via left translations (resp. conjugations). Then the principal isotropy subgroup type is the maximal $\mathbb{Z}_2$-tori (resp. tori) of K. Let H be an arbitrarily chosen principal isotropy subgroup and $W = N(H)/H$. Then, the fixed point set of H in M, F(H, M), is a flat, totally geodesic torus of rank 2 with a natural induced action of W. Moreover, the above flat torus, T^2, intersects every K-orbit perpendicularly and

$$M/K \cong T^2/W \quad (=\Delta). \tag{11}$$

Therefore, the orbit space Δ is isometric to a flat triangle in $\mathbb{R}^2 = \{(x, y);\ x, y \in \mathbb{R}\}$ defined by the following inequalities, namely,

$$\alpha_1 \geqq 0, \qquad \alpha_2 \geqq 0 \quad \text{and} \quad \beta \leqq \pi \tag{12}$$

where α_1, α_2 are the simple roots and β is the highest root (with respect to a chosen ordering). Therefore, in the cases of SU(3)/SO(3) and SU(3), Δ can be represented by

$$\Delta=\{(x, y);\ y \geqq 0,\ y-\sqrt{3}\,x \leqq 0 \text{ and } y+\sqrt{3}\,x \leqq 2\pi\}, \tag{12'}$$

and in the case of Sp(2), Δ can be represented by

$$\Delta=\{(x, y);\ y \geqq 0,\ x-y \geqq 0 \text{ and } x+y \leqq \pi\}. \tag{12''}$$

The generic K-orbits are represented by interior points of the above triangle and the volume functions that record the volumes of generic K-orbits are, respectively, as follows:

$$v(x, y)=\begin{cases} 8\pi^3 . \sin y . \sin 1/2(\sqrt{3}\,x-y) . \sin 1/2(\sqrt{3}\,x+y), \\ 64\pi^3 . [\sin y . \sin 1/2(\sqrt{3}\,x-y) . \sin 1/2(\sqrt{3}\,x+y)]^2, \\ (4\pi)^4 . [\sin x \sin y \sin(x+y) \sin(x-y)]^2. \end{cases} \tag{13}$$

2.5. The Case of $SO(2+m)/SO(2)\times SO(m)$. — In the general setting of a simply connected symmetric space $M=G/K$, let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be a decomposition of the Lie algebra of G into the (± 1)-eigenspace of the involution and $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{g}$ contained in $\mathfrak{p}$. Then $A=\operatorname{Exp}(\mathfrak{a})$ is a maximal, flat, totally geodesic submanifold of M which intersects perpendicularly with all K-orbits in M. Let $\mathfrak{h}=\mathfrak{a}\oplus(\mathfrak{k}\cap\mathfrak{h})$ be a Cartan subalgebra of $\mathfrak{g}$ and $\Delta_{\mathfrak{h}}(\mathfrak{g}, \mathfrak{k})$ be the set of complementary roots of the pair $(\mathfrak{g}, \mathfrak{k})$ with respect to $\mathfrak{h}$ and $\Delta(M)$ be the projection of $\Delta_{\mathfrak{h}}(\mathfrak{g}, \mathfrak{k})$ onto $\mathfrak{a}$. Then, it follows from É. Cartan's generalization of the maximal tori theorem in the realm of symmetric spaces that

$$M/K \cong A/W \tag{14}$$

where W is the group generated by reflections with respect to the perpendicular hyperplanes of the roots in $\Delta(M)$. In the special case of $G=SO(2+m)$, $K=SO(2)\times SO(m)$, the Lie algebra $\mathfrak{g}$ is equal to the set of all anti-symmetric real matrices of rank $(2+m)$. It is convenient to choose the following basis for $\mathfrak{g}$, namely

$$\{A_{ij}=E_{ij}-E_{ji};\ 1 \leqq i<j \leqq m+2\} \tag{15}$$

where E_{ij} is the matrix with only one non-zero entry, 1, at the (i, j)-place.

Then

$$\begin{cases} \{A_{ij};\ (i, j)=(1, 2) \text{ or } 3 \leqq i<j \leqq m+2\} \\ \{A_{ij};\ i \leqq 2,\ j \geqq 3 \text{ and } i<j \leqq m+2\} \end{cases} \tag{16}$$

are, respectively, bases of $\mathfrak{k}$ and $\mathfrak{p}$. Moreover, it is convenient to choose $\mathfrak{a}$ and $\mathfrak{h}$ as follows:

$$\begin{aligned} &\mathfrak{a}=\{x.\mathrm{A}_{1,3}+y.\mathrm{A}_{2,4};\ x,\ y\in\mathbb{R}\}, \\ &\mathfrak{h}=\mathfrak{a}\oplus\left\{\sum_{i=1}^{l}\lambda_i\mathrm{A}_{2i+3,\,2i+4}\right\}, \qquad l=\left[\frac{m}{2}\right]-1. \end{aligned} \tag{17}$$

Geometrically speaking, $\mathrm{A}=\exp\mathfrak{a}$ is a flat totally geodesic subtorus of M, which is the product of two great circles passing the base point 0 contained in $\mathrm{S}^2\times\mathrm{S}^2=$the SO(4) orbit of 0. Straightforward computation will show that the W-action on $\mathfrak{a}$ is generated by the reflections with respect to the following four lines, namely

$$x=0, \qquad y=0, \qquad x+y=0 \quad \text{and} \quad x-y=0. \tag{18}$$

Therefore $\mathrm{M}/\mathrm{K}\cong\mathrm{A}/\mathrm{W}$ is isometric to the following flat triangle

$$\Delta=\{(x,\ y);\ y\geqq 0,\ x-y\geqq 0 \text{ and } x+y\leqq\pi\}. \tag{19}$$

Moreover, the volume function that records the volumes of generic K-orbits is given by

$$v(x,\ y)=\mathrm{C}.(\sin x.\sin y)^{m-2}\sin(x-y)\sin(x+y). \tag{20}$$

3. The reduced differential equations and some analytical lemmas

Let (K, M) be an equivariant geometric system of cohomogeneity 2, i. e., $\dim(\mathrm{M}/\mathrm{K})=2$, and let Σ be a K-invariant hypersurface. Then Σ/K is called the generating curve of Σ and the mean curvature of Σ at a generic point $p\in\Sigma$ can be computed as follows. It is quite natural to equip the orbit space, M/K, with the *orbital distance metric* which measures the distance between orbits. Let $v:\mathrm{M}/\mathrm{K}\to\mathbb{R}$ be the volume function whose value at a *generic* orbit ξ is exactly the volume of ξ in M, namely,

$$v(\mathrm{K}(p))=\text{the volume of } \mathrm{K}(p)\subset\mathrm{M} \tag{21}$$

if $\mathrm{K}(p)$ is an orbit of the principal type. Then the mean curvature of Σ at a point, p, on a principal K-orbit $\xi=\mathrm{K}(p)$ can be neatly computed by the following formula

$$\mathrm{H}(\Sigma,\ p)=\kappa(\Sigma/\mathrm{K},\ \xi)-\frac{d}{d\mathfrak{n}_\xi}\ln\mathfrak{v} \tag{22}$$

where $\kappa(\Sigma/\mathrm{K},\ \xi)$ is the curvature of the curve, Σ/K, and $d/d\mathfrak{n}_\xi$ is the directional differentiation in the direction of the unit normal $\mathfrak{n}_\xi$.

3.1. Reduced ODE of invariant minimal hypersurfaces. — In the specific cases of the equivariant geometric systems of paragraph 2, one can easily apply the above formula to compute the following explicit forms of ODE which characterize the generating curves of *invariant*, *minimal* hypersurfaces in the respective systems. We state the final form of each case separately as follows.

(i) *The case of* $\mathbb{C}P(n)$. — In this case, it is convenient to parametrize the orbit space, Δ, by the following spherical polar coordinates, namely, set

$$x=\sin r\cos\theta,\qquad y=\sin r\sin\theta,\qquad z=\cos r,$$
$$0\leqq\theta\leqq\frac{\pi}{2},\qquad 0\leqq r\leqq\frac{\pi}{2}. \tag{23}$$

Thus it follows from (2) that

$$\Delta=\left\{(r,\ \theta);\ 0\leqq r,\ \theta\leqq\frac{\pi}{2}\right\},\qquad ds^2=dr^2+\sin^2 r\,d\theta^2 \tag{2'}$$

and the volume function becomes

$$v(r,\ \theta)=2\pi c\sin^2 r\cos^{2n-3} r\sin\theta\cos\theta. \tag{4'}$$

Hence, the generating curve, γ, of a G-invariant *minimal* hypersurface, Γ, in $\mathbb{C}P(n)$ is characterized by the following ODE:

$$\frac{dr}{ds}=\cos\alpha,\qquad \frac{d\theta}{ds}=\frac{\sin\alpha}{\sin r}$$
$$\frac{d\alpha}{ds}+3\cos r\frac{d\theta}{ds}-\frac{2}{\sin r}\cot 2\theta\frac{dr}{ds}-(2n-3)\sin r\tan r\frac{d\theta}{ds}=0 \tag{24}$$

where α is the angle between the radial direction $\partial/\partial r$ and the tangential direction of γ.

(ii) *The case of* $S^n(1)\times S^n(1)$. — In this case, the orbit space is isometric to a flat square of size π, namely,

$$\Delta=\{(x,\ y);\ 0\leqq x,\ y\leqq\pi\},\qquad ds^2=dx^2+dy^2 \tag{5'}$$

and $v(x,\ y)=c^2.(\sin x\sin y)^{n-1}$. Hence, the generating curve, γ, of a G-invariant *minimal* hypersurface, Γ, in $S^n(1)\times S^n(1)$ is characterized by the following ODE:

$$\frac{dx}{ds}=\cos\sigma,\qquad \frac{dy}{ds}=\sin\sigma$$
$$\frac{d\sigma}{ds}+(n-1)\sin\sigma\cot x-(n-1)\cos\sigma\cot y=0 \tag{25}$$

where σ is the angle between the direction $\partial/\partial x$ and the tangential direction of γ.

(iii) *The case of* $\mathbb{C}P(n)\times\mathbb{C}P(n)$. — According to paragraph 2.3, the orbit space is a square of size $\pi/2$, namely

$$\Delta=\left\{(x,\ y);\ 0\leqq x,\ y\leqq\frac{\pi}{2}\right\},\qquad ds^2=dx^2+dy^2 \tag{9}$$

and $v(x,\ y)=c^2.\sin x\sin y[\cos x\cos y]^{2n-1}$. Therefore, it follows from (22) that the generating curve, γ, of a G-invariant *minimal* hypersurface Γ in $\mathbb{C}P(n)\times\mathbb{C}P(n)$ is characterized by the following ODE:

$$\begin{gathered}\frac{dx}{ds}=\cos\sigma,\qquad \frac{dy}{ds}=\sin\sigma,\\ \frac{d\sigma}{ds}+\sin\sigma[\cot x-(2n-1)\tan x]+\cos\sigma.[(2n-1)\tan y-\cot y]=0\end{gathered} \tag{26}$$

where σ is, again, the angle between $\partial/\partial x$ and the tangential direction of γ

(iv) *The cases of* SU(3)/SO(3) *and* SU(3). — In these two cases, the orbit space is a flat, regular triangle of height π. Hence, direct application of (13) and (22) will show that the characterizing ODE for the generating curves of minimal K-invariant hypersurfaces are, respectively, as follows.

$$\begin{aligned}&\frac{dx}{ds}=\cos\sigma,\qquad \frac{dy}{ds}=\sin\sigma\\ &\frac{d\sigma}{ds}=k.\left\{\cos\sigma\cot y-\sin\left(\sigma+\frac{\pi}{6}\right)\cot 1/2(\sqrt{3}\,x-y)\right.\\ &\qquad\left.-\sin\left(\sigma-\frac{\pi}{6}\right)\cot 1/2(\sqrt{3}\,x+y)\right\}\end{aligned} \tag{27}$$

where $k=1$ and 2 and σ is the angle between $\partial/\partial x$ and the tangential direction of γ.

(v) *The cases of* Sp(2) *and* $SO(2+m)/SO(2)\times SO(m)$. — In the above two cases, the orbit space is a flat, rectangular, isosceles triangle, namely,

$$\Delta=\{(x,\ y);\ y\geqq 0,\ x-y\geqq 0 \text{ and } x+y\leqq\pi\}$$

and, moreover, the volume function

$$v(x,\ y)=\begin{cases}(4\pi)^4.[\sin x\sin y\sin(x+y)\sin(x-y)]^2\\ c.(\sin x.\sin y)^{m-2}\sin(x+y)\sin(x-y)\end{cases}$$

for Sp(2) and $SO(2+m)/SO(2)\times SO(m)$ respectively. Hence it follows from (22) that the characterizing ODE for the generating curves of minimal K-invariant hypersurfaces in the above two cases is as follows.

$$\frac{dx}{ds}=\cos\sigma,\qquad \frac{dy}{ds}=\sin\sigma$$

$$\frac{d\sigma}{ds}=2\left\{\cos\sigma\cot y-\sin\sigma\cot x-\sqrt{2}\sin\left(\sigma+\frac{\pi}{4}\right)\cot(x-y)\right.$$
$$\left.-\sqrt{2}\sin\left(\sigma-\frac{\pi}{4}\right)\cot(x+y)\right\}\tag{28}$$

$$\left[\text{resp. }\frac{d\sigma}{ds}=(m-2).\{\cos\sigma\cot y-\sin\sigma\cot x\}\right.$$
$$\left.-\sqrt{2}\left\{\sin\left(\sigma+\frac{\pi}{4}\right)\cot(x-y)+\sin\left(\sigma-\frac{\pi}{4}\right)\cot(x+y)\right\}\right].$$

3.2. Symmetries of orbital geometry and explicit simple solutions. — In the study of existence of global solution curves with certain specific geometric characteristics, the existence of symmetries as well as certain explicit, simple solution curves are often very useful. We list below such useful "assets" of the above analytical systems.

(i) *The case of* $\mathbb{C}P(n)$. — The ODE (24) is reflectionally symmetric with respect to the line $\theta=\pi/4$ and it has the following four explicit simple solution curves, namely,

(1) the line $\theta=\pi/4$;

(2) the line $r=\tan^{-1}\sqrt{\dfrac{3}{2n-3}}$;

(3) $\sin r\cos\theta=\sqrt{\dfrac{1}{2n}}$;

(4) $\sin r\sin\theta=\sqrt{\dfrac{1}{2n}}$.

(ii) *The case of* $S^n(1)\times S^n(1)$. — In this case, one has the following four straight-line solutions of the ODE (25), namely

(1) $x=\pi/2$;

(2) $y=\pi/2$;

(3) $x-y=0$;

(4) $x+y=\pi$,

and, moreover, the ODE (25) is reflectionally symmetric with respect to the above four lines.

(iii) *The case of* $\mathbb{C}P(n) \times \mathbb{C}P(n)$. — The ODE (26) is reflectionally symmetric with respect to the line $x-y=0$ and there are the following three straight-line solutions to the ODE (26), namely,

(1) $x-y=0$,

(2) $x=\tan^{-1}\sqrt{\dfrac{1}{2n-1}}$;

(3) $y=\tan^{-1}\sqrt{\dfrac{1}{2n-1}}$.

(iv) *The cases of* $SU(3)/SO(3)$ *and* $SU(3)$. — The ODE (27) is reflectionally symmetric with respect to the three bisectors, namely,

(1) $x=\pi/\sqrt{3}$,

(2) $x-\sqrt{3}\,y=0$,

(3) $\sqrt{3}\,x+3y=2\pi$,

and they are solutions of the ODE (27).

(v) *The cases of* $Sp(2)$ *and* $SO(2+m)/SO(2)\times SO(m)$. — The line $x=\pi/2$ is a solution of the ODE (28) and the system is reflectionally symmetric with respect to the line $x=\pi/2$.

3.3. THE BEHAVIOR OF THE SINGULAR BOUNDARY. — Geometrically, the boundary points of the orbit spaces (of § 2) represent singular orbits of those specific transformation groups. Correspondingly, the volume function vanishes at the boundary and hence the $d/dn \ln v$ term of the reduced ODE becomes singular. Since minimal submanifolds of a given Riemannian manifold M are clearly invariant under homothetic magnifications, the ODE of paragraph 3.1 is also naturally invariant under homothetic magnifications. Therefore, it is advantageous to exploit the above *homothetic invariance* to reduce the study of local analysis in the neighborhood of a singular point to that of a limiting ODE which is geometrically associated to the equivariant geometry of the slice representation of the given singular orbit (*cf.* p. 66 of [7]-[I]).

In the specific cases considered in this paper, the orbit spaces are either a flat square or a flat (or spherical) triangle. Moreover, for the interior points of each given side, the singularity of the ODE is of a regular type studied in [5] and one has the following lemma on the unique existence as well as the analytical dependence of solution curves originating at such a singular point.

LEMMA 1. — *Let* Δ *be one of the orbit spaces described in paragraph* 2 *and* $\overline{AB}$ *be one of its sides. Then, to each interior point* b *of* $\overline{AB}$, *there exists a unique solution curve*, γ_b, *of the corresponding* ODE *with* b *as its initial point and, moreover, the family of such solution curves* $\{\gamma_b;\ b\in\overline{AB}^0\}$ *depends analytically on the coordinate of* b.

Proof. — The above lemma is a direct consequence of Proposition 1 of [5]. Due to the particularly simple type of singularity, a solution curve originating at such a boundary point b is necessarily *perpendicular* to $\overline{AB}$ and locally *analytic*. Therefore, the unique existence follows from the method of power series substitution and majoration. We refer to [5] for the details of such a proof.

Q. E. D.

Next, let us consider the singularities of some corner points. One of the important special features of the family of specific transformation groups of paragraph 2 is that every one of them contains an isolated singular orbit whose slice representation is of the *focal type* (*cf.* p. 361 of [7]-[II]). In terms of the coordinate system of Δ, used in paragraph 3.1 in exhibiting the reduced ODE, they are respectively the following.

(i) the origin $r=0$ for the case of $\mathbb{C}\,P(n)$;

(ii) all four corners, namely, $(0, 0)$, $(\pi, 0)$, $(0, \pi)$, and (π, π), for the case of $S^n(1)\times S^n(1)$, $n=2, 3$;

(iii) the origin, $(0, 0)$, for the case $\mathbb{C}\,P(n)\times\mathbb{C}\,P(n)$;

(iv) all three corners for the cases of $SU(3)/SO(3)$, $SU(3)$ and $Sp(2)$;

(v) the corner $(\pi/2, \pi/2)$ for the case $SO(2+m)/SO(2)\times SO(m)$.

Straightforward computation will show that the slice representations of the singular orbits corresponding to the above corner points are as follows

(*a*) $(SO(2)\times SO(2), \mathbb{R}^2\oplus\mathbb{R}^2)$ for the cases of $\mathbb{C}\,P(n)$, $S^2(1)\times S^2(1)$, $\mathbb{C}\,P(n)\times\mathbb{C}\,P(n)$ and $SO(2+m)/SO(2)\times SO(m)$;

(*b*) $(SO(3)\times SO(3), \mathbb{R}^3\oplus\mathbb{R}^3)$ for the cases of $S^3(1)\times S^3(1)$ and the corner $(\pi/2, \pi/2)$ in the $Sp(2)$ case;

(*c*) $(SO(3), \mathbb{R}^5)$ for the case of $SU(3)/SO(3)$;

(*d*) $(SU(3), \mathbb{R}^8)$ for the case of $SU(3)$;

(*e*) $(SO(5), \mathbb{R}^{10})$ for the corners $(0, 0)$ and $(\pi, 0)$ in the case of $Sp(2)$.

Therefore, the proofs of [7]-[I], p. 66 and [7]-[II], p. 358-361 can easily be adapted to obtain the following lemma.

LEMMA 2. — *Let Δ be one of the orbit spaces described in paragraph* 2 *and let* A *be one of its corners, which is of the focal type, and moreover, with an explicit solution curve l* (cf. § 3.2) *originating at* A. *Let b be a nearby boundary point and γ_b be the unique solution curve originating at b. Then as $b\to A$ along the boundary of Δ, the number of intersection points of l and the portion of γ_b within a fixed neighborhood,* U, *of* A *tends to infinity, namely,*

$$\#(\gamma_b\cap l\cap U)\to\infty \quad \text{as } b\to A.$$

LEMMA 2′. — *Let u be a non-trivial solution of the Jacobi equation along l, namely, the linearization of the* ODE *of paragraph* 3.1 *along l. Then u has infinitely many zeros on $U\cap l$.*

(We refer to page 66 of [7]-[I], pages 358-361 of [7]-[II], and pages 227-229 of [9] for a proof of the above two closely related lemmas.)

4. The proof of theorem 1

In this section we shall construct infinitely many mutually non-congruent examples of *imbeded, minimal hyperspheres* in $\mathbb{C}P(n)$, for each $n \geqq 2$, which are G-*invariant* with respect to the specific G-action given in paragraph 2.1. Analytically, this amounts to proving the existence of *infinitely many* global solution curves of the ODE (24) which originate at the side $r=\pi/2$ and terminate at the sides $\theta=0$ or $\pi/2$.

The orbit space $\Delta=\mathbb{C}P(n)/G$ is a spherical triangle which can be conveniently parametrized by polar coordinate (r, θ). Then the generating curves of G-invariant minimal hypersurfaces in $\mathbb{C}P(n)$ are characterized by the ODE (24). It has four explicit, simple, solution curves, given by $\theta=\pi/4$, $r=\tan^{-1}\sqrt{3/(2n-3)}$, $\sin r\cos\theta=\sqrt{1/2n}$ and $\sin r\sin\theta=\sqrt{1/2n}$, and the corner A is a singularity of the focal type. By Lemma 1, to each boundary point $b(\pi/2, t)\in\overline{BC}$, $0 < t < \pi/2$, there exist a unique solution curve, γ_b, that originates at $b(\pi/2, t)$. Let γ_t be the arc of γ_b from b up to its first minimum in r. Then $\{\gamma_t;\ 0 < t < \pi/2\}$ forms an *analytical* family of solution curves perpendicular to the side $\overline{BC}$, e. g., γ_t are exactly the following explicit simple solution curves, namely

$$\theta=\frac{\theta}{4},\qquad \sin r\sin\theta=\sqrt{\frac{1}{2n}}\quad\text{and}\quad \sin r\cos\theta=\sqrt{\frac{1}{2n}}$$

when

$$t=\frac{\pi}{4},\qquad \sin^{-1}\sqrt{\frac{1}{2n}}\quad\text{and}\quad \cos^{-1}\sqrt{\frac{1}{2n}}\quad\text{(cf. \textit{Fig.} 1).}$$

Set $U=\{(r, \theta);\ r < \tan^{-1}\sqrt{3/(2n-3)}\}$. It follows easily from the ODE (24) that the terminating point (i. e., the first r-minimum point) of γ_t must be in U. Let $N(t)$ be the number of intersection points of γ_t and $\gamma_{\pi/4}$, namely

$$N(t)=\#\{\gamma_t\cap\gamma_{\pi/4}\},\qquad t\neq\frac{\pi}{4}. \tag{29}$$

Then it follows from Lemma 2′ that

$$N(t)\to\infty\quad\text{as } t\to\frac{\pi}{4}. \tag{30}$$

Based on the above facts, we shall prove the following existence result, namely

"To each positive integer i, there exists a suitable value t_i such that $N(t_i)=i$ and γ_{t_i} terminates at a boundary point different from A."

Let i be an arbitrary given positive integer. It follows from (30) that there exists a sufficient small $\delta > 0$ such that $N((\pi/4)-\delta)\geqq(i+1)$. Observe that $N(\sin^{-1}\sqrt{1/2n})=1$ and, as t continuously varies from $(\pi/4)-\delta$ to $\sin^{-1}\sqrt{1/2n}$, γ_t can never become tangential to $\gamma_{\pi/4}$. Therefore, the deformation must go through the following two stages, namely, there exists u_i and u_{i+1} such that

$$N(u_i)=i,\qquad N(u_{i+1})=i+1,\qquad \sin^{-1}\sqrt{\frac{1}{2n}}\leqq u_i < u_{i+1}\leqq\frac{\pi}{4}-\delta, \tag{31}$$

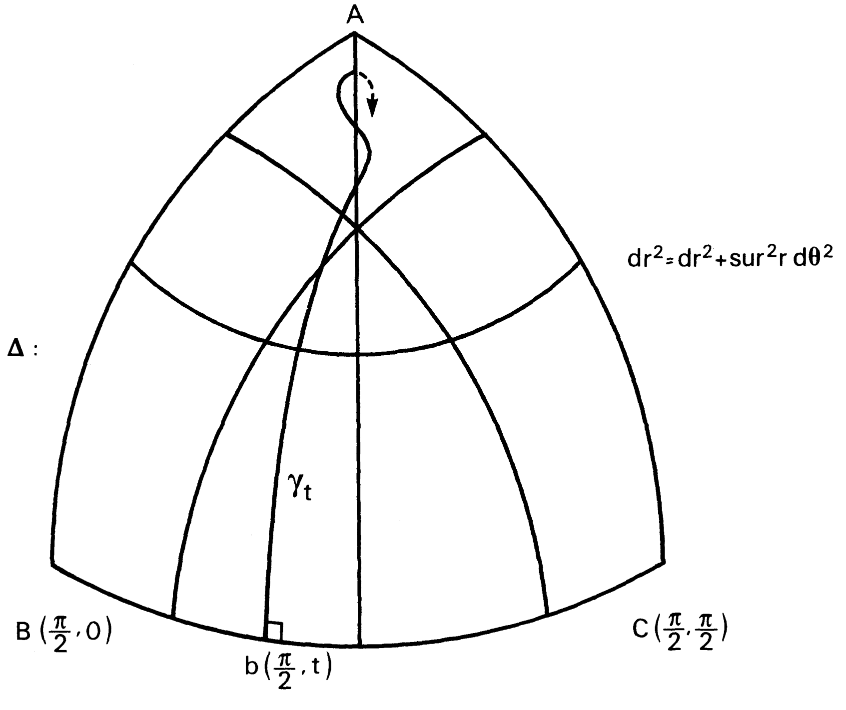

Fig. 1.

and γ_{u_i}, $\gamma_{u_{i+1}}$ intersect $\gamma_{\pi/4}$ perpendicularly, i. e., exactly at the points of r-minima. It is easy to see that the values of $d\theta/ds$ at the above two points of r-minima must be of opposite signs. Therefore, there must exist a value $u_i < t_i < u_{i+1}$ such that $\mathrm{N}(t_i) = i$ and γ_{t_i} terminates at a boundary point. For otherwise, the family of solution curves $\{\gamma_t;\ t \in [u_i, u_{i+1}]\}$ forms a C^1-continuous family and hence *cannot have* a sudden reversal of signs of $d\theta/ds$ at the end points! Finally, one may, of course, assume that u_{i+1} is the smallest such value satisfying (31). Then $\mathrm{N}(t) < i+1$ for all $t < u_{i+1}$ and it follows from Lemma 2 that the terminal point of γ_{t_i} cannot be A. Hence, it is easy to see that the inverse images of $\{\gamma_{t_i},\ i \leqq i < \infty\}$ forms an infinite family of imbedded, minimal hyperspheres which are mutually non-congruent. This completes the proof of Theorem 1.

Remark. — Among the above infinite family of examples, the first one is generated by the curve γ_{t_1} with $t_1 = \sin^{-1}\sqrt{1/2n}$. It is clearly the simplest, nicest one which deserves the title of the equator. Hence, it is a natural problem to work out various characterizations of this simplest, hypersphere of $\mathbb{C}\,\mathrm{P}(n)$ in the same fashion as J. Simons did for the equator of $S^n(1)$ in [11].

5. The proof of theorem 2

Let $M = SO(2+m)/SO(2)\times SO(m)$, $m \geqq 3$, and $K = SO(2)\times SO(m)$. Then the orbit space $\Delta = M/K$ is a flat, rectangular, isosceles triangle and the generating curves of K-invariant minimal hypersurfaces of M are characterized by the ODE (28). Moreover, it is symmetric with respect to the line $x = \pi/2$; the corner $A(\pi/2, \pi/2)$ is a singularity of the focal type and the line $x = \pi/2$ is a solution curve of the ODE (28).

Let us first consider the family of solution curves

$$\mathscr{I} = \left\{\gamma_b;\ \gamma_b(0) = b = (t, t),\ 0 < t < \frac{\pi}{2}\right\}. \tag{32}$$

Set γ_t to be the arc of γ_b between its initial point $b = (t, t)$ and its first y-minimum. Let $N(t)$ be the number of intersection points of γ_t with the line l: $x = \pi/2$, namely

$$N(t) = \#\{\gamma_t \cap l\}. \tag{33}$$

Then for sufficiently small $\delta > 0$, one has

$$N(\delta) = 0 \qquad \text{and} \qquad N\left(\frac{\pi}{2} - \delta\right) \text{ is rather large.} \tag{34}$$

Hence, it follows from the continuity of $\mathscr{I}$ that there exists a smallest value t_0 such that $N(t_0) > 0$. Then γ_{t_0} must intersect l perpendicularly at its first y-minimum and, hence, by the reflectional symmetry, the γ_{b_0} with $b_0 = (t_0, t_0)$ is a global solution curve which starts at (t_0, t_0), terminates at $(\pi - t_0, t_0)$, and intersects l perpendicularly at its unique y-minimum point (as indicated in the following Figure 2).

Next let us consider the following family of solution curves

$$\mathscr{I}' = \{\gamma'_b;\ \gamma'_b(0) = b = (u, 0),\ 0 < u < \pi\}. \tag{35}$$

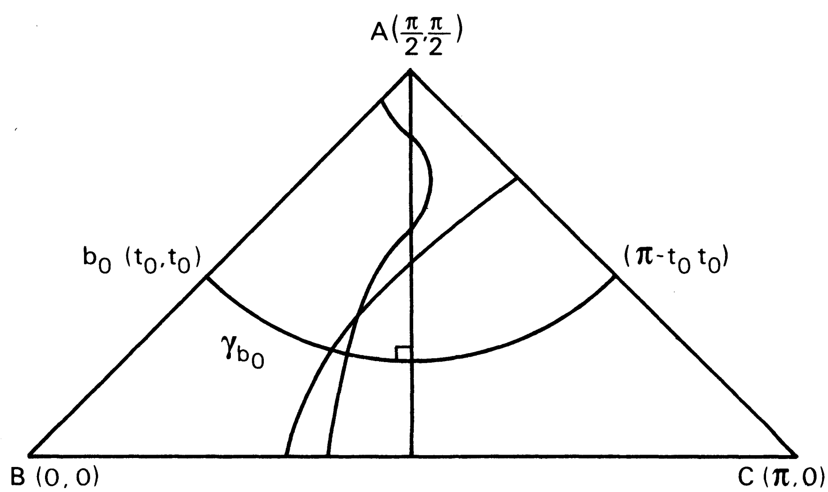

Fig. 2.

Set γ'_u to be the arc of γ'_b between its initial point $b=(u, 0)$ and its first y-maximum. Let $\nu(u)$ be the number of intersection points of γ'_u and l; U be the region of Δ above the above solution curve γ_0. Then it is not difficult to show that all the terminal points of γ'_u must be inside of U.

Again, it follows from Lemma 2′ that

$$\nu(u) \to \infty \quad \text{as } u \to \frac{\pi}{2}. \tag{36}$$

Let a be the minimal value of $\nu(u)$, $0 < u < \pi/2$. Then, essentially the same kind of proof as that of paragraph 4 will show that

"to each integer $i \geqq a+1$, there exists a suitable value u_i such that $\nu(u_i)=i$ and γ'_{u_i} terminates at a boundary point different from A".

Hence, the inverse images of $\{\gamma'_{u_i};\ i \geqq a+1\}$ provide infinitely many mutually non-congruent examples of imbedded, minimal hyperspheres in $SO(2+m)/SO(2) \times SO(m)$. This completes the proof of Theorem 2.

Remarks. — (i) It is not difficult to use the method of numerical estimation to establish the existence of a γ'_u such that $\nu(u)=1$ and γ'_u terminates at $\overline{AC}$. Therefore, actually, there exists a γ'_{u_i} for all positive integer i.

(ii) Again, among all the examples of imbedded, minimal hyperspheres in $SO(2+m)/SO(2) \times SO(m)$, the one with $\nu(u)=1$ is clearly the simplest minimal hypersphere and should be crowned with the title of the equator in M.

6. The proofs of theorems 3 and 4

6.1. THE PROOF OF THEOREM 3. — In view of the orbital geometries of various cases listed in Theorem 3, it is quite natural to divide the proof of Theorem 3 into the following three cases.

(i) *The case of* Sp (2). — The proof for this case is almost identical to that of Theorem 2 in paragraph 5 and hence omitted.

(ii) *The cases of* SU (3)/SO (3) *and* SU (3). — In this case, the orbit space is a flat regular triangle and the ODE (27) is symmetric with respect to each of the three bisectors which are themselves solution curves of the ODE (27). Moreover, a global solution curve of the same geometric type as γ_{b_0} in the proof of Theorem 2 had already been established in [5]. Therefore, it is, again, quite straightforward to adapt the proof of Theorem 2 to prove the existence of global solution curves of the desired geometric type with $\nu(u)$ = an arbitrary positive integer i.

(iii) *The cases of* $S^n(1) \times S^n(1)$, $n=2, 3$. — In this case, the orbit space is a flat square of size π. The ODE (25) has four straight-line solutions, namely

$$x=\frac{\pi}{2}, \qquad y=\frac{\pi}{2}, \qquad x-y=0 \quad \text{and} \quad x+y=\pi$$

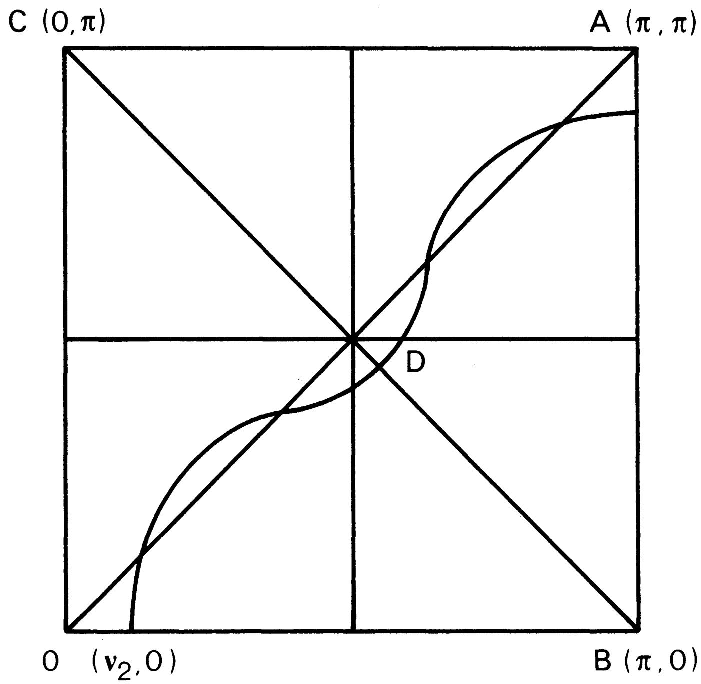

Fig. 3.

and the equation is symmetric with respect to the above four lines.

Let γ be an arbitrary solution curve of the ODE (25) other than $x=\pi/2$ and $y=\pi/2$. Then it is easy to deduce from the ODE (25) that the x-minima (resp. x-maxima, y-minima, y-maxima) of γ can occur only in the region $x<\pi/2$ (resp., $x>\pi/2$, $y<\pi/2$, $y>\pi/2$). Moreover, in the case $n=2$, 3, the four corner points are singularities of the focal type.

Let us consider the following family of solution curves

$$\mathscr{I}=\left\{\gamma_b;\ \gamma_b(0)=b=(t,\ 0),\ 0<t\leqq\frac{\pi}{2}\right\} \tag{37}$$

and set γ_t to be the arc of γ_b between its initial point b and its first x-maximum. Let D be the middle point of the diagonals $\overline{OA}$ and $\overline{BC}$. Set

$$N(t)=\#\{\gamma_t\cap\overline{OD}\}. \tag{38}$$

Then, it follows from Lemma 2 that

$$N(t)\to\infty\quad\text{as } t\to 0. \tag{39}$$

Therefore, it follows from the continuity of $\mathscr{I}$ that, for each given positive integer i, there exists a suitable value t_i such that $N(t_i)=i$ and D is exactly the *last* intersection point, e. g., $t_1=\pi/2$. It is easy to see that the angle between γ_{t_i} and the diagonal BC is less than $\pi/2$ if i is an odd integer; but is greater than $\pi/2$ if i is an even integer. Hence, again by the continuity of $\mathscr{I}$ there exists a suitable value, $t_{i+1} < v_i < t_i$, such that

$$\text{(40)} \qquad \mathrm{N}(v_i)=i \quad \text{and} \quad \gamma_{v_i} \perp \overline{\mathrm{BC}}.$$

Therefore, by the reflectional symmetry of the ODE (25) with respect to $\overline{\mathrm{BC}}$, γv_i is a global solution curve which starts at $(v_i, 0)$, terminates at $(\pi, \pi-v_i)$ and has exactly $2i$ intersection points with $\overline{\mathrm{OA}}$. The inverse images of $\{\gamma_{v_i};\ i=1, 2, \ldots\}$ provide infinitely many mutually non-congruent examples of imbedded, minimal hyperspheres in $\mathrm{S}^n(1)\times\mathrm{S}^n(1)$, $n=2$ or 3. This completes the proof of Theorem 3.

6.2. THE PROOF OF THEOREM 4. – Let (G, M) be the equivariant geometric system described in paragraph 2.3. Then the orbit space, $\Delta=\mathrm{M}/\mathrm{G}$, is a flat square of size $\pi/2$ and the generating curves of G-invariant minimal hypersurfaces in M are characterized by the ODE (26). It is reflectionally symmetric with respect to the line $x-y=0$ and there are the following three straight line solutions, namely,

$$x-y=0, \qquad x=\tan^{-1}\sqrt{\frac{1}{2n-1}}, \qquad y=\tan^{-1}\sqrt{\frac{1}{2n-1}}.$$

(See *Fig.* 4)

Let

$$\text{(41)} \qquad \mathrm{F}=\left\{\gamma_b;\ \gamma_b(0)=b=(t, 0),\ 0<t\leqq \tan^{-1}\sqrt{\frac{1}{2n-1}}\right\}$$

and let γ_t be the arc of γ_b between its initial point b and its first x-maximum. Set

$$\text{(42)} \qquad \mathrm{N}(t)=\#\{\gamma_t\cap\overline{\mathrm{OA}}\}.$$

Since A is a corner singularity of the focal type, it follows from Lemma 2′ that

$$\text{(43)} \qquad \mathrm{N}(t)\to\infty \quad \text{as } t\to 0.$$

Therefore, it follows from the continuity of F and the fact that $\mathrm{N}(\tan^{-1}\sqrt{1/(2n-1)})=1$ that, for every positive integer i, there exists a suitable value t_i such that

$$\text{(44)} \qquad \mathrm{N}(t_i)=i+1 \quad \text{and} \quad \gamma_t\perp\overline{\mathrm{OA}} \text{ at the last intersection.}$$

Hence, by reflection symmetry, one can continue γ_t to obtain a symmetric global solution which starts at $(t_i, 0)$, terminates at $(0, t_i)$, and has exactly i double points on OA. The inverse images of the above family of global solution curves provide infinitely many mutually non-congruent examples of *immersed* minimal hyperspheres in $\mathbb{C}\mathrm{P}(n)\times\mathbb{C}\mathrm{P}(n)$. This completes the proof of Theorem 4.

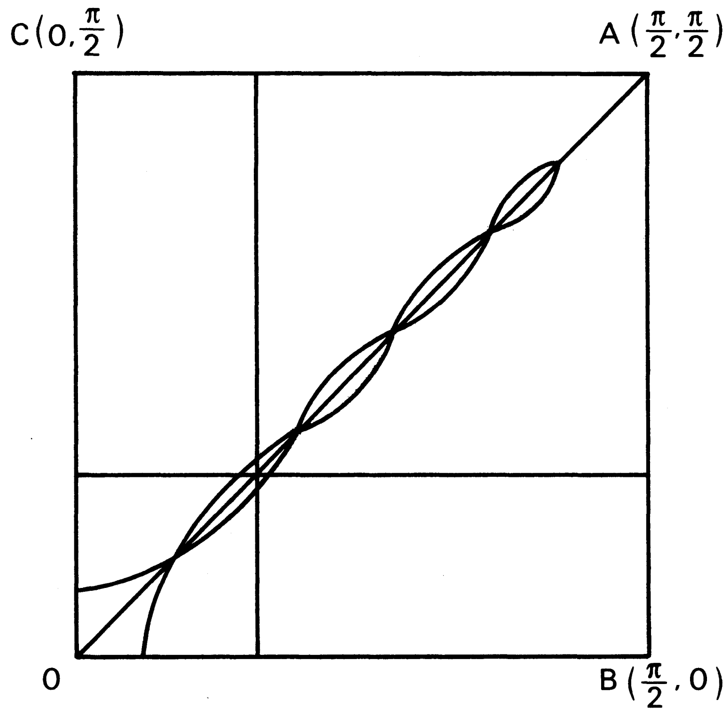

Fig. 4.

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(Manuscrit reçu le 19 août 1987).

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