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ARTIN L-FUNCTIONS AND NORMALIZATION OF INTERTWINING OPERATORS

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Introduction

In this paper we prove two results on normalized intertwining operators for quasi-split groups. Our first result is a proof of a conjecture of Langlands [14] on normalizing the standard intertwining operators by means of Artin L-functions in the case of unitary principal series of a quasi-split *p*-adic group. As our second result, we prove certain local and global identities satisfied by these operators in this case. They were suggested by a conjecture of Arthur [1] and are useful in the trace formula. Finally, in the last section of this paper, under a certain natural local assumption, we prove the global identity in general.

More precisely, let $\mathbf{G}$ be a connected reductive quasi-split algebraic group over a non-archimedean field F of characteristic zero. Fix a Borel subgroup $\mathbf{B}$ in $\mathbf{G}$. If $\mathbf{T}$ is a maximal torus in $\mathbf{B}$, we let $G=\mathbf{G}(F)$ and $T=\mathbf{T}(F)$. Write $\mathbf{B}=\mathbf{TU}$, where $\mathbf{U}$ is the unipotent radical of $\mathbf{B}$. More generally, let $\mathbf{P}=\mathbf{MN}$ be a standard parabolic subgroup, $\mathbf{P}\supset\mathbf{B}$, $\mathbf{U}\supset\mathbf{N}$, of $\mathbf{G}$. Denote by ${}^L M$ the L-group of $\mathbf{M}$. If ${}^L\mathfrak{n}$ is the Lie algebra of the L-group ${}^L N$ of $\mathbf{N}$, then ${}^L M$ acts on ${}^L\mathfrak{n}$ by the adjoint representation r.

Next, if $\mathbf{A}_0$ is the maximal F-split torus in $\mathbf{T}$, let $W(\mathbf{A}_0)$ denote its Weyl group in $\mathbf{G}$. Fix $\tilde{w}\in W(\mathbf{A}_0)$ such that $\tilde{w}(\mathbf{M})$ is again a standard Levi subgroup of $\mathbf{G}$. Let ${}^L\mathfrak{n}_{\tilde{w}}$ denote the subspace of ${}^L\mathfrak{n}$ consisting of all those root spaces whose roots are sent to the negative roots under $\tilde{w}$. Let $r_{\tilde{w}}$ be the restriction of r to ${}^L\mathfrak{n}_{\tilde{w}}$.

Now, let $\mathfrak{a}_{\mathbb{C}}^*$ be the complex dual of the real Lie algebra of $\mathbf{A}$. For $\nu\in\mathfrak{a}_{\mathbb{C}}^*$ and an irreducible unitary representation σ of M, let $I(\nu, \sigma)$ be the representation of G induced by σ and ν. Denote by $\mathbf{N}^-$ the unipotent group opposed to $\mathbf{N}$ and let $\mathbf{N}_{\tilde{w}}=\mathbf{N}\cap w\mathbf{N}^- w^{-1}$, where $\tilde{w}$ is as above. Then there exists a cone in $\mathfrak{a}_{\mathbb{C}}^*$ such that for

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every ν in that cone and every f in the space of $I(\nu, \sigma)$, the integral

$$\int_{N_{\tilde{w}}} f(w^{-1} ng)\, dn$$

is absolutely convergent. By means of meromorphic continuation this then defines an intertwining operator $A(\nu, \sigma, w)$ between $I(\nu, \sigma)$ and $I(\tilde{w}(\nu), \tilde{w}(\sigma))$, whenever ν is not a pole (*cf.* [20], [25], and Section 1 of the present paper).

Assume first that we are in the minimal parabolic case. Let λ be a unitary character of T. If W_F is the Weil group of F, there exists a homomorphism $\varphi: W_F \to {}^L T$, the L-group of **T** which determines λ (*cf.* [12], [15], and Section 2 here). For every $\tilde{w} \in W(\mathbf{A}_0)$ we choose a representative w of $\tilde{w}$ in G as in Section 2. Let $A(\lambda, w)$ be the standard intertwining operator $A(0, \lambda, w)$ acting on $I(\lambda)$, the representation of G induced from λ. The map $r_{\tilde{w}} . \varphi$ becomes a representation of W_F whose contragredient is $\tilde{r}_{\tilde{w}} . \varphi$.

Given $s \in \mathbb{C}$, the field of complex numbers, and a non-trivial additive character ψ_F of F, let $L(s, \tilde{r}_{\tilde{w}} . \varphi)$ and $\varepsilon(s, \tilde{r}_{\tilde{w}} . \varphi, \psi_F)$ be the Artin L-function and root number attached to $\tilde{r}_{\tilde{w}} . \varphi$ (*cf.* [6], [16], and [28]). As our first result, in Theorem 3.1 we prove:

THEOREM. — *Let*

$$\mathscr{A}(\lambda, w) = \varepsilon(0, \tilde{r}_{\tilde{w}} . \varphi, \psi_F) \frac{L(1, \tilde{r}_{\tilde{w}} . \varphi)}{L(0, \tilde{r}_{\tilde{w}} . \varphi)} A(\lambda, w).$$

Then:

(*a*) $\mathscr{A}(\lambda, w_1 w_2) = \mathscr{A}(\tilde{w}_2(\lambda), w_1) \mathscr{A}(\lambda, w_2)$

(*b*) $\mathscr{A}(\lambda, w)^* = \mathscr{A}(\tilde{w}(\lambda), w^{-1})$, *i. e.* $\mathscr{A}(\lambda, w)$ *is unitary.*

This proves a conjecture of Langlands [14] in this case (see the remark after Corollary 3.5). We should mention that if **G** is a real group, this has been proved in general by Arthur [2]. Our next result, Theorem 4.1, gives an immediate application of this theorem.

Essential to the proof are the formulas obtained for the Plancherel measures in [8] (*cf.* Theorem 3.1 of the present paper).

Next, assume $\tilde{w}(\lambda) = \lambda$. Let R be the R-group of $I(\lambda)$ (*cf.* [9]). It is equal to the group $\mathbf{S}_\varphi$ attached to φ by Langlands (*cf.* [17]). The character ψ_F defines a non-degenerate character χ of $U = \mathbf{U}(F)$. There exists a unique χ-generic subrepresentation $\pi_0 \subset I(\lambda)$. The irreducible constituents of $I(\lambda)$ are parametrized by equivalence classes of irreducible representations of R. We use $\hat{R}$ to denote this set of equivalence classes. Fix a parametrization such that π_0 corresponds to the trivial character of R (*cf.* Section 4). If $\tilde{w} \in R$ and $\pi \subset I(\lambda)$ corresponds to $\rho_\pi \in \hat{R}$, we let $\langle \rho_\pi, \tilde{w} \rangle = \text{trace}\ \rho_\pi(\tilde{w})$. Otherwise, let $\langle \rho_\pi, \tilde{w} \rangle = 1$. Define the character λ_{ψ_F} of R_0, the group generated by the representatives w of $\tilde{w}$, $\tilde{w} \in R$, as in Section 4. Then, using Theorem 2.6 of [9], in part (*a*) of Theorem 4.1, we prove the local identity:

$$\text{trace}(\mathscr{A}(\lambda, w) I(f)) = \lambda_{\psi_F}(w) \sum_\pi \langle \rho_\pi, \tilde{w} \rangle \chi_\pi(f),$$

where $f \in C_c^\infty(G)$,

$$I(f)=\int_G f(g)\,\pi(g)\,dg,$$

and for $\pi \subset I(\lambda)$, χ_π denotes its character.

To state the global identity in this case, assume F is a number field and $\mathbf{G}$ is again a connected reductive quasi-split algebraic group over F. Let $G=\mathbf{G}(\mathbb{A}_F)$ and $T=\mathbf{T}(\mathbb{A}_F)$, where $\mathbf{T}$ is a maximal torus in $\mathbf{G}$ and $\mathbb{A}_F$ is the ring of adeles of F. Let $\lambda=\bigotimes_v \lambda_v$ be a character of T. Fix $\tilde{w}\in W(\mathbf{A}_0)$. Let $M(\lambda, \tilde{w})$ be the corresponding global intertwining operator [14]. Fix a homomorphism $\varphi: W_F \to {}^L T$ which corresponds to λ (*cf.* [15]). It is not unique, but if $\theta_v: W_{F_v} \to W_F$ is the canonical map, then every local representation $r_{\tilde{w}}.\varphi.\theta_v$ is uniquely determined. The same is true for the Artin L-function $L(s, r_{\tilde{w}}.\varphi)$. Assume $\tilde{w}(\lambda)=\lambda$. Let n be the multiplicity of the identity representation in $r_{\tilde{w}}.\varphi$. Then putting together all the local identities (*cf.* [22] for $v=\infty$), our Theorem 5.1 implies

$$\operatorname{trace}(M(\lambda, \tilde{w})\,I(f))=(-1)^n \prod_v \sum_{\pi_v} \langle \rho_{\pi_v}, \tilde{w}\rangle\,\chi_{\pi_v}(f_v),$$

$f \in C_c^\infty(G)$, $f=\bigotimes_v f_v$, where at each place v of F, π_v runs over all the equivalence classes of irreducible components of $I(\lambda_v)$ (an L-packet). Moreover, given f, almost all the sums in the product reduce to 1.

The significance of such an identity can be explained as follows. Suppose $\tilde{w}$ is an element of $W(\mathbf{A}_0)$ which only fixes the split torus in the center of $\mathbf{G}$ (an element of $W(\mathbf{A}_0)_{\text{reg}}$ in the notation of [1]). Then, up to a constant depending on $\tilde{w}$, the term $\operatorname{trace}(M(\lambda, \tilde{w})\,I(f))$ is among the terms appearing in the discrete part $I_+^G(f)$ of the trace formula coming from the Eisenstein series [*cf.* equation (3.2.2) of [1], for example]. Suppose $n>0$. Then $\tilde{w}$ becomes the trivial element of each local R-group and consequently the distribution $f \mapsto \operatorname{trace}(M(\lambda, \tilde{w})\,I(f))$ becomes stable as expected (*cf.* [1] and [17]). In fact if $n>0$, there must exist a positive root α with $\tilde{w}(\alpha)<0$ for which the Plancherel function ($\mu_\alpha(\lambda_v, s)$ in the notation of [9]) vanishes at zero. Therefore $\alpha \in \Delta'$, the root system for the subgroup W′ of $W(\lambda_v)$ for which normalized intertwining operators act as scalars (*cf.* [9]). Now, if $1\neq\tilde{w}$ is in the R-group, then $\tilde{w}(\beta)>0$ for all the positive roots $\beta\in\Delta'$. But $\tilde{w}(\alpha)<0$, which implies that $\tilde{w}$ is trivial in each local R-group. On the other hand, if $n=0$, then $\tilde{w}(\alpha)>0$ for every positive root $\alpha\in\Delta'$ and therefore $\tilde{w}$ must be a non-trivial element of every local R-group. The distribution $\operatorname{trace}(M(\lambda, \tilde{w})\,I(f))$ is then no longer stable, and is therefore expected to be cancelled off by an unstable distribution coming from a stable distribution attached to some proper cuspidal endoscopic data for $\mathbf{G}$ (*cf.* [1], [17], [24]). These endoscopic data are easy to find. The connected component of the corresponding L-group must contain the connected component of the centralizer of $\varphi(W_F)$ in ${}^L G^0$, the connected component of the L-group of $\mathbf{G}$. Since $\tilde{w}(\lambda)=\lambda$, it then contains a representative for $\tilde{w}$. Consequently, it cannot be contained in any proper Levi subgroup of ${}^L G^0$, $\tilde{w}$

being a member of $W(\mathbf{A}_0)_{\mathrm{reg}}$. This then implies the cuspidality of the data. It is also easily seen that the corresponding group is not equal to **G**. Moreover φ factors through the L-group ${}^{\mathrm{L}}\mathrm{H}$ of the corresponding endoscopic group **H**. In fact, if $\eta: {}^{\mathrm{L}}\mathrm{H} \to {}^{\mathrm{L}}\mathrm{G}$ is the embedding, then $\varphi = \eta . \varphi'$, where $\varphi': \mathrm{W}_{\mathrm{F}} \to {}^{\mathrm{L}}\mathrm{H}$. Thus φ' must correspond to a certain cusp form on H which cannot be lifted to any representation in the discrete spectrum of G (in particular any cusp form). Taking into account all such unstable distributions (for all parabolics) will then determine all those cuspidal representations on all the cuspidal endoscopic groups attached to **G** (and different from **G**) which do not lift to a representation in the discrete spectrum (*cf.* [1]). We refer the reader to Theorem 14.5.2 of [19] for an example of this in the case of $\mathbf{U}(3)$. Finally, we should mention that these identities can also be used to stabilize the trace formula [13], as well as to prove certain local character identities [19].

The characters $\lambda_{\psi_{\mathrm{F}_v}}(w)$ play the role of the functions $c_v(s_v)$ explained in [1]. As it is explained there (p. 27), it is expected that

$$\prod_v c_v(s_v) = 1.$$

This is automatic in our identities since $\lambda_{\psi_{\mathrm{F}_v}}(w)$ is a product of Langlands' λ-functions whose global values are always one (*cf.* [16]). Finally, the reader must observe that an identity as in Theorem 2.6 of [9] can not be used by itself to prove the global identity. In fact, it is important to know how the local normalizing factors relate to each other globally and how a base representation $\pi_{0,v}$ can be chosen in a consistent way at every place. It is exactly for these reasons and the behavior of the intertwining operators at the unramified places that we have chosen Artin factors to normalize them.

We refer to [13] (Lemma 3.6) and [21] for similar types of results.

Finally in Section 6, under an assumption about local L-functions, we prove the global identity in general for a representation induced from a generic cusp form. The proof is based on the functional equation proved in [20]. More precisely, if $\sigma = \bigotimes_v \sigma_v$ is a χ-generic cusp form on $\mathrm{M} = \mathbf{M}(\mathbb{A}_{\mathrm{F}})$, let $\mathrm{I}(\sigma) = \bigotimes_v \mathrm{I}(\sigma_v)$ be the corresponding induced representation. For a fixed $\tilde{w} \in W(\mathbf{A}_0)$, we again denote the global intertwining operator by $\mathrm{M}(\sigma, \tilde{w})$. Assume $\tilde{w}(\sigma) = \sigma$. Let S be a finite set of places including the archimedean ones such that for $v \notin \mathrm{S}$, everything is unramified (*cf.* Section 5). For every $v \notin \mathrm{S}$ and $s \in \mathbb{C}$, let $\mathrm{L}(s, \sigma_v, r_{\tilde{w}} . \eta_v)$ be the local Langlands L-function attached to σ_v and $r_{\tilde{w}} . \eta_v$, where $\eta_v: {}^{\mathrm{L}}\mathrm{M}_v \to {}^{\mathrm{L}}\mathrm{M}$ is the natural map. Set

$$\mathrm{L}_{\mathrm{S}}(s, \sigma, r_{\tilde{w}}) = \prod_{v \notin \mathrm{S}} \mathrm{L}(s, \sigma_v, r_{\tilde{w}} . \eta_v).$$

Let n be the order of the pole of $\mathrm{L}_{\mathrm{S}}(s, \sigma, r_{\tilde{w}})$ at $s = 1$. Under our Assumption 1 it is independent of S. Our global identity is then (Theorem 6.4)

$$\operatorname{trace}(\mathrm{M}(\lambda, \tilde{w}) \mathrm{I}(f)) = (-1)^n \prod_v \sum_{\pi_v} c_{\pi_v}(w) \chi_{\pi_v}(f_v),$$

$f \in C_c^\infty(G)$, $f = \underset{v}{\otimes} f_v$, where for each constituent $\pi_v \subset I(\sigma_v)$, c_{π_v} is a class function and each sum is taken over equivalence classes of constituents of $I(\sigma_v)$. We finally observe (Proposition 6.4) that when σ_v is in the discrete series, the class functions c_{π_v} are in fact characters of irreducible representations of the R-group of $I(\sigma_v)$.

In the special case of a unitary group in three variables these results were first discussed in a lecture with the same title as this paper, given by the second author in the "Seminar on the Analytical Aspects of the Trace Formula II" (*cf.* [19]) at the Institute for Advanced Study, during its special year on Automorphic Forms and L-functions (1983-84). The second author would like to thank the Institute for Advanced Study for its warm hospitality during his stay there.

1. Preliminaries

Let F be a non-archimedean local field of characteristic zero. Denote by O its ring of integers and let R be the unique maximal ideal of O. As usual ϖ denotes a generator of R which we shall fix throughout. Let q denote the number of elements in the residue field O/R. If $|\ |_F$ denote the absolute value on F, then $|\varpi|_F = q^{-1}$.

Let $\mathbf{G}$ be a connected reductive quasi-split algebraic group over F. Fix a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ and write $\mathbf{B} = \mathbf{TU}$, where $\mathbf{T}$ and $\mathbf{U}$ are a maximal torus and the unipotent radical of $\mathbf{B}$, respectively. Let $\mathbf{A}_0$ be the maximal F-split torus inside $\mathbf{T}$.

If we use ψ to denote the set of F-roots of $\mathbf{A}_0$, then $\mathbf{B}$ determines a partition $\psi = \psi^+ \cup \psi^-$ of these roots to the positive roots ψ^+ and the negative roots ψ^-. Let $\Delta \subset \psi^+$ be the set of simple roots.

By a standard parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$, we shall mean a parabolic subgroup $\mathbf{P}$ containing $\mathbf{B}$. Write $\mathbf{P} = \mathbf{MN}$, where $\mathbf{M}$ is a Levi factor, $\mathbf{M} \supset \mathbf{T}$, and $\mathbf{N} \subseteqq \mathbf{U}$ is its unipotent radical. If $\mathbf{A}$ denotes the split component in the center of $\mathbf{M}$, then $\mathbf{A} \subseteqq \mathbf{A}_0$. Let $\theta \subset \Delta$ generate $\mathbf{M}$.

We shall use G to denote $\mathbf{G}(F)$. Similarly we have B, T, U, P, M, N, A, and A_0.

Let $W(\mathbf{A}_0)$ and $W(\mathbf{A})$ be the Weyl groups of $\mathbf{A}_0$ and $\mathbf{A}$ in $\mathbf{G}$, respectively. Then every element of $W(\mathbf{A})$ can be extended to one of $W(\mathbf{A}_0)$.

We fix a special maximal compact subgroup K for G. Then $G = BK$ and $G = PK$.

If $X(\mathbf{M})_F$ denotes the group of F-rational characters of $\mathbf{M}$, we let

$$\mathfrak{a} = \mathrm{Hom}(X(\mathbf{M})_F, \mathbb{R})$$

denote the real Lie algebra of $\mathbf{A}$. Then

$$\mathfrak{a}^* = X(\mathbf{M})_F \otimes_{\mathbb{Z}} \mathbb{R} = X(\mathbf{A})_F \otimes_{\mathbb{Z}} \mathbb{R}$$

and $\mathfrak{a}_{\mathbb{C}}^* = \mathfrak{a}^* \underset{\mathbb{R}}{\otimes} \mathbb{C}$ is the complex dual of $\mathfrak{a}$. Similarly we have $\mathfrak{a}_0$, $\mathfrak{a}_0^*$, and $(\mathfrak{a}_0)_{\mathbb{C}}^*$ for $\mathbf{A}_0$.

Given an irreducible unitary representation σ of M and $\nu \in \mathfrak{a}_{\mathbb{C}}^*$, let $I(\nu, \sigma)$ be the unitarily induced representation

$$I(\nu, \sigma) = \operatorname*{Ind}_{MN \uparrow G} \sigma \otimes q^{\langle \nu, H_P(\ ) \rangle} \otimes 1,$$

where the homomorphism $H_P : M \to \mathfrak{a}$ is defined by

$$q^{\langle \chi, H_P(m) \rangle} = |\chi(m)|_F$$

for all $\chi \in X(\mathbf{M})_F$ and $m \in M$. We use $V(\nu, \sigma)$ to denote the space of $I(\nu, \sigma)$. Finally we let $I(\sigma) = I(0, \sigma)$ and $V(\sigma) = V(0, \sigma)$. Moreover let

$$W(\sigma) = \{ \tilde{w} \in W(\mathbf{A}) \mid \tilde{w}(\sigma) \cong \sigma \}.$$

Fix a $\tilde{w} \in W(\mathbf{A}_0)$ such that $\tilde{w}(\theta) \subset \Delta$ and let $w \in G$ be a representative for $\tilde{w}$. Let $N_{\tilde{w}} = N \cap w N^- w^{-1}$, where N^- is the unipotent subgroup opposed to N. Given $f \in V(\nu, \sigma)$, let

$$(1.1) \qquad A(\nu, \sigma, w) f(g) = \int_{N_{\tilde{w}}} f(w^{-1} n g)\, dn.$$

The integral converges absolutely if $\operatorname{Re} \langle \nu, H_\alpha \rangle \gg 0$ for all $\alpha \in \Delta - \theta$, where $H_\alpha \in \mathfrak{a}$ is the standard coroot attached to α. Moreover, it extends to a meromorphic function of ν on all of $\mathfrak{a}_{\mathbb{C}}^*$ (*cf.* [20], [25]), and away from its poles, it defines an intertwining map between $I(\nu, \sigma)$ and $I(\tilde{w}(\nu), \tilde{w}(\sigma))$, where $\tilde{w}(\sigma)(m') = \sigma(w^{-1} m' w)$ with $m' \in M' = w M w^{-1}$. Finally let $A(\sigma, w) = A(0, \sigma, w)$.

Next, let ${}^L M$ be the L-group of $\mathbf{M}$ and if ${}^L N$ is the L-group of N, we let ${}^L\mathfrak{n}$ be its Lie algebra (*cf.* [4]). The group ${}^L M$ acts by adjoint action on ${}^L\mathfrak{n}$. It sends ${}^L\mathfrak{n}_{\tilde{w}}$, the Lie algebra of the L-group of $N_{\tilde{w}}$, into itself. We use $r_{\tilde{w}}$ to denote the restriction of this adjoint action to ${}^L\mathfrak{n}_{\tilde{w}}$. Let ρ be half the sum of the roots whose root spaces generate N. Then for each α with $X_{\alpha^\vee} \in {}^L\mathfrak{n}$, $\langle 2\rho, \alpha \rangle$ is a positive integer. Let $a_1 < a_2 < \ldots < a_m$ be their distinct values. Set

$$V_i = \{ X_{\alpha^\vee} \in {}^L\mathfrak{n}_{\tilde{w}} \mid \langle 2\rho, \alpha \rangle = a_i \}.$$

Each V_i is invariant under $r_{\tilde{w}}$. We let $r_{\tilde{w}, i}$ be the restriction of $r_{\tilde{w}}$ to V_i.

Let W'_F be the Deligne-Weil group of F (*cf.* [4]). Recall that according to Langlands' conjectures (*cf.* [4]), the representation σ is attached to a homomorphism $\varphi : W'_F \to {}^L M$. Then $r_{\tilde{w}} . \varphi$ is a complex representation of W'_F.

Given a complex numbers $s \in \mathbb{C}$ and a non-trivial additive character $\psi \in \hat{F}$, let $L(s, r_{\tilde{w}} . \varphi)$ and $\varepsilon(s, r_{\tilde{w}} . \varphi, \psi)$ be the Artin L-function and root number attached to $r_{\tilde{w}} . \varphi$, respectively (*cf.* [16]). As it was suggested by Langlands [14], we shall now normalize our intertwining operator $A(\sigma, w)$ by

$$(1.2) \qquad \mathcal{A}(\sigma, w) = \varepsilon(0, \tilde{r}_{\tilde{w}} . \varphi, \psi) \frac{L(1, \tilde{r}_{\tilde{w}} . \varphi)}{L(0, \tilde{r}_{\tilde{w}} . \varphi)} A(\sigma, w),$$

where $\tilde{r}_{\tilde{w}}$ is the contragredient of $r_{\tilde{w}}$. The question is then whether this is a valid normalization, i. e. if it satisfies the following conditions:

$$\mathscr{A}(\sigma, w_1 w_2) = \mathscr{A}(\tilde{w}_2(\sigma), w_1)\mathscr{A}(\sigma, w_2) \tag{1.3}$$

$$\mathscr{A}(\sigma, w)^* = \mathscr{A}(\tilde{w}(\sigma), w^{-1}) \quad \text{i. e. } \mathscr{A}(\sigma, w) \text{ is unitary.} \tag{1.4}$$

As our first result we shall prove this when $\mathbf{P}=\mathbf{B}$ and therefore σ is a character. The induced representation $I(\sigma)$ is then in the principal series. We recall that for real groups this has been verified in general by Arthur in [2].

We shall conclude this section with some necessary facts about generic representations.

Let χ be a generic character of U (*cf.* Section 3 of [23] for its exact definition). Then by restriction χ is also a generic character of $U^0 = U^0(F)$, where $\mathbf{U}^0 = \mathbf{U} \cap \mathbf{M}$. An irreducible admissible representation σ of M is called χ-generic if it can be realized in a space of smooth functions W^0 satisfying

$$W^0(um) = \chi(u) W^0(m),$$

where $m \in M$ and $u = U^0$. We shall call this realization (which is unique), the χ-Whittaker model $W(\sigma)$ for σ.

There exists a canonical functional (*cf.* [20]) $\lambda_\chi(\nu, \sigma)$ on the space $V(\nu, \sigma)$ such that

$$\lambda_\chi(\nu, \sigma)(I(\nu, \sigma)(u) f) = \chi(u) \lambda_\chi(\nu, \sigma)(f),$$

where $u \in U$ and $f \in V(\nu, \sigma)$. If $\lambda_\chi(\tilde{w}(\nu), \tilde{w}(\sigma))$ is the canonical functional attached to $I(\tilde{w}(\nu), \tilde{w}(\sigma))$ (*cf.* [20]), then there exists a complex number $C_\chi(\nu, \sigma, w)$ such that

$$\lambda_\chi(\nu, \sigma) = C_\chi(\nu, \sigma, w) \lambda_\chi(\tilde{w}(\nu), \tilde{w}(\sigma)) A(\nu, \sigma, w). \tag{1.5}$$

This is what we call the local coefficient attached to χ, ν, σ, and w. The reader must observe that we have supressed its dependence on the defining measures. We let $C_\chi(\sigma, w) = C_\chi(0, \sigma, w)$.

We also recall (Lemma 3.1 of [23]) that if σ is a unitary χ-generic representation, then its contragredient $\tilde{\sigma}$ is $\bar{\chi}$-generic and moreover

$$C_{\bar{\chi}}(\tilde{\sigma}, w) = \overline{C_\chi(\sigma, w)}. \tag{1.6}$$

Finally, it was proved in [20] that

$$C_\chi(\tilde{w}(\nu), \tilde{w}(\sigma), w^{-1}) = \overline{C_\chi(-\bar{\nu}, \sigma, w)} \tag{1.7}$$

and

$$A(\nu, \sigma, w) A(\tilde{w}(\nu), \tilde{w}(\sigma), w^{-1}) = C_\chi(\nu, \sigma, w)^{-1} C_\chi(\tilde{w}(\nu), \tilde{w}(\sigma), w^{-1})^{-1}. \tag{1.8}$$

2. Artin factors for the principal series

We shall now concentrate on the case where $\mathbf{P}=\mathbf{B}$. Let λ be a unitary character of T.

For a positive reduced root $\alpha \in \psi^+$, let $\mathbf{G}_\alpha$ be the corresponding rank one subgroup of $\mathbf{G}$. If $\tilde{\mathbf{G}}_{\alpha, \mathrm{D}}$ is the simply connected covering of the derived group of $\mathbf{G}_\alpha$, then there are only two possibilities. Either $\tilde{\mathbf{G}}_{\alpha, \mathrm{D}} = \mathrm{Res}_{\mathrm{F}_\alpha/\mathrm{F}} \mathbf{SL}_2$ or $\tilde{\mathbf{G}}_{\alpha, \mathrm{D}} = \mathrm{Res}_{\mathrm{F}_\alpha/\mathrm{F}} \mathbf{SU}(2, 1)$, where F_α is a finite (separable) extension of F.

Choose $\tilde{w} \in \mathrm{W}(\mathbf{A}_0)$ and let $\tilde{w} = \tilde{w}_{\alpha_1} \ldots \tilde{w}_{\alpha_l}$ be a reduced decomposition (not uniquely) of $\tilde{w}$ where l is the length of $\tilde{w}$ and $\alpha_1, \ldots, \alpha_l \in \Delta$. If α_i is such that $2\alpha_i$ is not a root, we choose

$$w_{\alpha_i} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

as a representative for $\tilde{w}_{\alpha_i}$. Otherwise we let

$$w_{\alpha_i} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let $w = w_{\alpha_1} \ldots w_{\alpha_l}$. Then w is independent of the decomposition of $\tilde{w}$ (*cf.* Part *b* of Lemma 83 of [27]).

Let $\varphi : \mathrm{W}_\mathrm{F} \to {}^L\mathrm{T}$ be the homomorphism attached to λ. Choose a finite Galois extension L of F such that φ factors through $\mathrm{W}_{\mathrm{L/F}}$. Define a homomorphism $\varphi_0 : \mathrm{L}^* \to {}^L\mathrm{T}^0$ by $\varphi(a \times 1) = (\varphi_0(a), a \times 1)$. Unwinding the isomorphism in Theorem 2 of [15] (*cf.* [12] for a new proof) we have

$$\lambda . (\alpha^\vee)^{-1} = \alpha^\vee . \varphi_0, \tag{2.1}$$

where on the left hand side $\alpha^\vee$ denotes a coroot of $\mathbf{T}$, while on the right it is a root of ${}^L\mathrm{T}^0$. More precisely, we first identify ${}^L\mathrm{T}^0$ by $\mathrm{X}(\mathbf{T})_\mathrm{F} \otimes \mathbb{C}^*$. Next if $\langle \ , \ \rangle$ denotes the duality between $\mathrm{X}(\mathbf{T})_\mathrm{F} \otimes \mathbb{C}^*$ and $\mathrm{X}_*(\mathbf{T})_\mathrm{F} \otimes \mathbb{C}^*$, where $\mathrm{X}_*(\mathbf{T})_\mathrm{F}$ is the group of F-rational morphisms from $\mathbf{GL}_1$ into $\mathbf{T}$, then we can identify $\mathrm{X}_*(\mathbf{T})_\mathrm{F} \otimes \mathbb{C}^*$ as the group of complex characters of ${}^L\mathrm{T}^0$ through (*cf.* [18])

$$\eta^\vee(t^\vee) = q^{\langle t^\vee, \eta^\vee \rangle} \quad (t^\vee \in {}^L\mathrm{T}^0, \ \eta^\vee \in \mathrm{X}_*(\mathbf{T})_\mathrm{F} \otimes \mathbb{C}^*). \tag{2.2}$$

In this way $\alpha^\vee$ becomes a root of ${}^L\mathrm{T}^0$.

Now, if $\alpha^\vee$ is considered as a coroot of $\mathbf{T}$, then

$$|\chi(\alpha^\vee(\varpi))|_\mathrm{F} = |\varpi|_\mathrm{F}^{\langle \chi, \alpha^\vee \rangle} \tag{2.3}$$

for every $\chi \in \mathrm{X}(\mathbf{T})_\mathrm{F}$. Comparing (2.2) with (2.3) then justifies the inverse sign for $\alpha^\vee$ in (2.1). Observe that (2.3) implies $\mathrm{H}_\theta(\alpha^\vee(\varpi)) = -\alpha^\vee$.

Fix a positive reduced root α such that $\mathrm{X}_{\alpha^\vee} \in {}^L\mathfrak{n}_{\tilde{w}}$. Let ${}^L\mathfrak{n}_{\alpha^\vee} \subset {}^L\mathfrak{n}_{\tilde{w}}$ be the Lie algebra of ${}^L\mathrm{N}_\alpha$, where $\mathbf{N}_\alpha = \mathbf{G}_\alpha \cap \mathbf{U}$. The adjoint action $r_{\tilde{w}}$ of ${}^L\mathrm{T}$ leaves ${}^L\mathfrak{n}_{\alpha^\vee}$ invariant. If $r^0_{\tilde{w}} = r_{\tilde{w}} | {}^L\mathrm{T}^0$, then equation (2.1) implies that $r^0_{\tilde{w}} . \varphi_0$ acts on $\mathrm{X}_{\alpha^\vee}$ by λ_α^{-1}, where λ_α denotes $\lambda . \alpha^\vee$.

We shall now compute the Artin factors attached to $\tilde{r}_{\tilde{w}}.\varphi$, where $\tilde{r}_{\tilde{w}}$ is the contragredient of $r_{\tilde{w}}$.

First suppose that α is such that $\tilde{\mathbf{G}}_{\alpha,\,\mathrm{D}} \cong \mathrm{Res}_{\mathrm{F}_\alpha/\mathrm{F}}\,\mathbf{SL}_2$ for some finite (separable) extension F_α of F. The representation $\tilde{r}_{\tilde{w}}.\varphi$ when restricted to ${}^{\mathrm{L}}\mathfrak{n}_{\alpha^\vee}$ becomes equivalent to the representation $\operatorname{Ind}_{\mathrm{W}_{\mathrm{F}_\alpha}\uparrow \mathrm{W}_\mathrm{F}} \lambda_\alpha$, where λ_α, which is a character of F_α^* by means of $\lambda_\alpha(a)=\lambda(\alpha^\vee(a))$, $a\in \mathrm{F}_\alpha^*$, is identified by a character of $\mathrm{W}_{\mathrm{F}_\alpha}$ through the isomorphism $\mathrm{W}_{\mathrm{F}_\alpha}^{ab}\cong \mathrm{F}_\alpha^*$ (*cf.* [3], [28]). This is a consequence of Frobenius reciprocity law since $\tilde{r}_{\tilde{w}}^0.\varphi_0$ acts on $\mathrm{X}_{\alpha^\vee}$ by λ_α and $\dim_{\mathbb{C}}^{\mathrm{L}} \mathfrak{n}_{\alpha^\vee}=[\mathrm{F}_\alpha : \mathrm{F}]$.

Throughout this paper we shall fix a non-trivial additive character ψ_F of F.

Fix a complex number $s\in\mathbb{C}$ and for a finite-dimensional representation ρ of W_F, let $\mathrm{L}(s,\rho)$ and $\varepsilon(s,\rho,\psi_\mathrm{F})$ be the corresponding Artin L-function and root number, respectively (*cf.* [16] and [6]; beware of the differences, *cf.* [28], since we are using [16]).

For every finite separable extension E of F and every non-trivial additive character ψ_F, in [16] Langlands has attached a complex number $\lambda(\mathrm{E}/\mathrm{F},\psi_\mathrm{F})$ such that if σ is a finite-dimensional representation of W_E and $\omega=\operatorname{Ind}_{\mathrm{W}_\mathrm{E}\uparrow \mathrm{W}_\mathrm{F}} \sigma$, then

$$\varepsilon(s,\omega,\psi_\mathrm{F})=\lambda(\mathrm{E}/\mathrm{F},\psi_\mathrm{F})^{\dim\sigma}\,\varepsilon(s,\sigma,\psi_{\mathrm{E}/\mathrm{F}}), \tag{2.4}$$

where $\psi_{\mathrm{E}/\mathrm{F}}=\psi_\mathrm{F}.\mathrm{Tr}_{\mathrm{E}/\mathrm{F}}$. It satisfies

$$\lambda(\mathrm{E}/\mathrm{F},\psi_\mathrm{F})\,\lambda(\mathrm{E}/\mathrm{F},\bar{\psi}_\mathrm{F})=1. \tag{2.5}$$

Moreover, if $[\mathrm{E}:\mathrm{F}]=2$, then $\lambda(\mathrm{E}/\mathrm{F},\psi_\mathrm{F})$ is given by the formula

$$\lambda(\mathrm{E}/\mathrm{F},\psi_\mathrm{F})=\frac{\int_{\mathrm{F}^*}\eta(\alpha)\,\psi_\mathrm{F}(\alpha)\,d\alpha}{\left|\int_{\mathrm{F}^*}\eta(\alpha)\,\psi_\mathrm{F}(\alpha)\,d\alpha\right|}, \tag{2.6}$$

where η is the quadratic character of F^* attached to E/F by local class field theory, i. e. $\mathrm{Ker}(\eta)=\mathrm{N}_{\mathrm{E}/\mathrm{F}}(\mathrm{E}^*)$. It satisfies

$$\lambda(\mathrm{E}/\mathrm{F},\psi_\mathrm{F})^2=\eta(-1). \tag{2.7}$$

For any local field E, a non-trivial additive character ψ_E of E, and a character (not necessarily unitary) χ of E^*, let $\mathrm{L}_\mathrm{E}(s,\chi)$ and $\varepsilon_\mathrm{E}(s,\chi,\psi_\mathrm{E})$ be the corresponding Hecke-Tate L-function and root number, respectively (*cf.* [28]). Finally let

$$\gamma_\mathrm{E}(s,\chi,\psi_\mathrm{E})=\varepsilon_\mathrm{E}(s,\chi,\psi_\mathrm{E})\frac{\mathrm{L}_\mathrm{E}(1-s,\chi^{-1})}{\mathrm{L}_\mathrm{E}(s,\chi)}. \tag{2.8}$$

In the case in hand, we then easily see that

$$\mathrm{L}(s,\tilde{r}_{\tilde{w}}.\varphi\,|\,{}^{\mathrm{L}}\mathfrak{n}_{\alpha^\vee})=\mathrm{L}_{\mathrm{F}_\alpha}(s,\lambda_\alpha) \tag{2.9}$$

and

$$\varepsilon(s, \tilde{r}_{\tilde{w}}.\varphi \mid {}^L\mathfrak{n}_{\alpha^\vee}, \psi_F) = \lambda(F_\alpha/F, \psi_F)\, \varepsilon_{F_\alpha}(s, \lambda_\alpha, \psi_{F_\alpha/F}). \tag{2.10}$$

Next, suppose that α is such that $\tilde{\mathbf{G}}_{\alpha,\, D} \cong \mathrm{Res}_{F_\alpha/F}\, \tilde{\mathbf{G}}_\alpha$, where $\tilde{\mathbf{G}}_\alpha$ is the group **SU** (2, 1) as a group over F_α, defined by a quadratic extension E_α of F_α.

Let $\{\alpha_i^\vee, \beta_i^\vee, \alpha_i^\vee + \beta_i^\vee \mid 1 \leqq i \leqq n\}$ denote the set of roots in ${}^L\mathfrak{n}_{\alpha^\vee}$, $n = [F_\alpha : F]$. Then again $\tilde{r}_{\tilde{w}}.\varphi$ leaves the subspace of ${}^L\mathfrak{n}_{\alpha^\vee}$ generated by $\{X_{\alpha_i^\vee}, X_{\beta_i^\vee} \mid 1 \leqq i \leqq n\}$ invariant and for the same reason its restriction to this subspace becomes equivalent to $\operatorname{Ind}_{W_{E_\alpha} \uparrow W_F} \lambda_\alpha$.

Next, let $\gamma_i^\vee = \alpha_i^\vee + \beta_i^\vee$, $1 \leqq i \leqq n$. Let $\mathrm{Gal}(E_\alpha/F_\alpha) = \{1, \sigma_0\}$. Then $X_{\gamma_i^\vee} = [X_{\alpha_i^\vee}, X_{\beta_i^\vee}]$, $\beta_i^\vee = \sigma_0(\alpha_i^\vee)$, and therefore

$$\sigma_0 . X_{\gamma_i^\vee} = -X_{\gamma_i^\vee}. \tag{2.11}$$

Then clearly W_{F_α} acts on each $X_{\gamma_i^\vee}$ by a character of $W_{F_\alpha}^{ab} \cong F_\alpha^*$. More precisely, we have the following commutative diagram (*cf.* [3], [28])

$$\begin{array}{ccc} E_\alpha^* & \xrightarrow{\cong} & W_{E_\alpha}^{ab} \\ {\scriptstyle N_{E_\alpha/F_\alpha}}\downarrow & & \downarrow {\scriptstyle \tilde{i}} \\ F_\alpha^* & \xrightarrow{\cong} & W_{F_\alpha}^{ab} \end{array} \tag{2.12}$$

where the vertical arrow on the right is the homomorphism $\tilde{i}$ induced by the inclusion $i : W_{E_\alpha} \subsetneq W_{F_\alpha}$. Let η_α be the quadratic character of F_α^* attached by class field theory to E_α/F_α. Then $\mathrm{Ker}(\eta_\alpha) = N_{E_\alpha/F_\alpha}(E_\alpha^*)$ can be identified by $\tilde{i}(W_{E_\alpha}^{ab})$. The quotient $W_{F_\alpha}^{ab}/\tilde{i}(W_{E_\alpha}^{ab})$ is then isomorphic to $\mathrm{Gal}(E_\alpha/F_\alpha)$. By (2.11), it acts on each $X_{\gamma_i^\vee}$ by the character η_α. On the other hand by (2.1) the representation $\tilde{r}_{\tilde{w}}.\varphi_0$ acts on each $X_{\gamma_i^\vee}$ by $\lambda.\gamma_i^\vee$. But

$$\lambda.\gamma_i^\vee = \lambda_\alpha . N_{E_\alpha/F_\alpha}.$$

If we now identify $W_{F_\alpha}^{ab}$ as a semi-direct product of $\tilde{i}(W_{E_\alpha}^{ab}) \cong N_{E_\alpha/F_\alpha}(E_\alpha^*)$ and $\mathrm{Gal}(E_\alpha/F_\alpha)$, then $W_{F_\alpha}^{ab} \cong F_\alpha^*$ acts on each $X_{\gamma_i^\vee}$ by the character $\lambda_\alpha \otimes \eta_\alpha$. Note that here λ_α denotes the restriction of λ_α to F_α^*. Consequently, the restriction of $\tilde{r}_{\tilde{w}}.\varphi$ to $\{X_{\gamma_i^\vee} \mid 1 \leqq i \leqq n\}$ is equivalent to $\operatorname{Ind}_{W_{F_\alpha} \uparrow W_F} \eta_\alpha \lambda_\alpha$. We repeat that here λ_α denotes $\lambda_\alpha \mid F_\alpha^*$.

Now the corresponding Artin L-function is given by (*cf.* [16])

$$L(s, \tilde{r}_{\tilde{w}}.\varphi \mid {}^L\mathfrak{n}_{\alpha^\vee}) = L_{E_\alpha}(s, \lambda_\alpha)\, L_{F_\alpha}(s, \eta_\alpha \lambda_\alpha), \tag{2.13}$$

while the corresponding root number is

$$\varepsilon(s, \tilde{r}_{\tilde{w}}.\varphi \mid {}^L\mathfrak{n}_{\alpha^\vee}, \psi_F) = \lambda(F_\alpha/F, \psi_F)\, \lambda(E_\alpha/F, \psi_F)\, \varepsilon_{E_\alpha}(s, \lambda_\alpha, \psi_{E_\alpha/F})\, \varepsilon_{F_\alpha}(s, \eta_\alpha \lambda_\alpha, \psi_{F_\alpha/F}). \tag{2.14}$$

3. Normalization of interwining operators

In this section we prove one of our main results:

THEOREM 3.1. — *Let*

$$\mathscr{A}(\lambda, w)=\varepsilon(0, \tilde{r}_{\tilde{w}}.\varphi, \psi_F)\frac{L(1, \tilde{r}_{\tilde{w}}.\varphi)}{L(0, \tilde{r}_{\tilde{w}}.\varphi)}A(\lambda, w).$$

Then

(*a*) $\mathscr{A}(\lambda, w_1 w_2)=\mathscr{A}(\tilde{w}_2(\lambda), w_1)\mathscr{A}(\lambda, w_2)$

(*b*) $\mathscr{A}(\lambda, w)^*=\mathscr{A}(\tilde{w}(\lambda), w^{-1})$, *i. e.* $\mathscr{A}(\lambda, w)$ *is unitary.*

To prove the theorem we need to compute $A(\lambda, w)A(\tilde{w}(\lambda), w^{-1})$. We shall use (1.8) to do this.

Let for a moment $\mathbf{G}$ be equal to $\mathbf{SU}(2, 1)$, defined by a quadratic extension E/F. More precisely, set

$$Q(x_1, x_2, x_3)=x_1\bar{x}_3-x_2\bar{x}_2+x_3\bar{x}_1,$$

where for $x\in E$, $\bar{x}=\sigma(x)$. Here σ denotes the non-trivial element of Gal(E/F). Then $G=\mathbf{G}(F)$ is the stabilizer of Q in $\mathbf{SL}_3(E)$, i. e.

$$G=\{g\in \mathbf{SL}_3(E)\,|\,{}^t g w\bar{g}=w\},$$

where

$$w=\begin{pmatrix}0&0&1\\0&-1&0\\1&0&0\end{pmatrix}.$$

We may choose $T=\mathbf{T}(F)$ to be

$$T=\left\{\begin{pmatrix}a&0&0\\0&\bar{a}a^{-1}&0\\0&0&(\bar{a})^{-1}\end{pmatrix}\;a\in E^*\right\}.$$

Let λ be a unitary character of E^*. Then it is one of T.

Let ψ_F be a non-trivial additive character of F and denote by $\psi_{E/F}$ the corresponding character of E. We take U to be

$$U=\left\{u=\begin{pmatrix}1&x&y\\0&1&\bar{x}\\0&0&1\end{pmatrix}\;x, y\in E,\; N_{E/F}(x)=\mathrm{Tr}_{E/F}(y)\right\}.$$

We then let $\chi(u)=\psi_{E/F}(x)$. Finally let w represent the non-trivial element of the Weyl group.

Suppose $E=F(\tau)$ and write $x=u+v\tau$, $u, v\in F$. Then $y=(1/2)x\bar{x}+z\tau$ with $z\in F$. The measure dx on $N=U$ is then equal to $dxdz$, where dx and dz are measures on E and F, respectively.

Next, fix a uniformizing parameter ϖ_E of E. Assume for a moment that the largest ideal on which ψ_F is trivial is O, i.e. ψ_F is unramified. Given a positive integer m, define a new additive character ψ_m of E by

$$\psi_m(x)=\psi_{E/F}(N_{E/F}(\varpi_E^{-m})x).$$

Define a function $f_{m,\lambda}\in V(\lambda)$ according to equation (3.3) of [8]. Then there exists a complex number $\gamma_{m,w}(\lambda)$ such that

$$A(\lambda, w)f_{m,\lambda}=\gamma_{m,w}(\lambda)f_{m,\widetilde{w(\lambda)}}. \tag{3.1}$$

Applying an appropriate Whittaker functional on either side of (3.1) and formal manipulations now show that if

$$\gamma_{0,w}(\lambda)=\lim_{m\to 0}\gamma_{m,w}(\lambda),$$

then

$$\gamma_{0,w}(\lambda)=C_\chi(\lambda, w)^{-1}. \tag{3.2}$$

We now borrow the following result from [8]. It follows from equation (3.2) and a careful analysis of the formulas given for $\gamma_{m,w}(\lambda)$ in page 124 of that paper. (ψ_F is no longer assumed to be unramified.)

PROPOSITION 3.2. — *Let dn be the measure defining* $A(\lambda, w)$. *Write* $dn=dxdz$, *where dx and dz are measures on* E *and* F, *respectively. Let each* γ*-function be defined as in equation* (2.8). *Then*

$$C_\chi(\lambda, w)=\lambda(-1)\gamma_E(0, \lambda.N_{E/F}, \psi_{E/F})\gamma_E(0, \lambda, \psi_{E/F})\gamma_F(1, \lambda^{-1}, \psi_F),$$

where the measures defining γ_E *and* γ_F *are dx and dz, respectively.*

COROLLARY 3.3. — *Let* η *be the quadratic character of* F^* *attached to* E/F *by local class field theory. Then*

$$C_\chi(\lambda, w)=\lambda(E/F, \psi_F)^{-1}\gamma_E(0, \lambda, \psi_{E/F})\gamma_F(0, \lambda\eta, \psi_F),$$

where λ *in* $\gamma_E(0, \lambda\eta, \psi_F)$ *denotes* $\lambda|F^*$.

Proof. — This is a consequence of the relations

$$\gamma_F(1-s, \lambda^{-1}, \psi_F)\gamma_F(s, \lambda, \psi_F)=\lambda(-1)$$

and

$$\lambda(E/F, \psi_F)\gamma_E(s, \lambda.N_{E/F}, \psi_{E/F})=\gamma_F(s, \lambda, \psi_F)\gamma_F(s, \lambda\eta, \psi_F),$$

$s \in \mathbb{C}$, and Proposition 3.2.

We shall now go back to the general **G**. Assume that the restriction of χ to every simple root space is given by $\psi_{F_\alpha/F} = \psi_F . Tr_{F_\alpha/F}$, where α is the corresponding simple root. Let $\Delta_1(\tilde{w})$ and $\Delta_2(\tilde{w})$ denote the sets of reduced roots in ψ^+ with $\tilde{w}(\alpha) \in \psi^-$ for which $\tilde{\mathbf{G}}_{\alpha, D} = Res_{F_\alpha/F}\, \mathbf{SL}(2)$ or $Res_{F_\alpha/F}\, \mathbf{SU}(2,1)$, respectively. The product formula for $C_\chi(\lambda, w)$, together with equations (2.9), (2.10), (2.13), (2.14), and

$$(3.3) \qquad \lambda(E_\alpha/F, \psi_F) = \lambda(F_\alpha/F\ \psi_F)^2\, \lambda(E_\alpha/F_\alpha, \psi_{F_\alpha/F}),$$

and finally Corollary 3.3 will now imply:

PROPOSITION 3.4. — *Choose the representative w of $\tilde{w}$ as in Section 2. Then*

$$C_\chi(\lambda, w) = \prod_{\alpha \in \Delta_1(\tilde{w})} \lambda(F_\alpha/F, \psi_F)^{-1} . \prod_{\alpha \in \Delta_2(\tilde{w})} \lambda(E_\alpha/F, \psi_F)^{-2} . \lambda(F_\alpha/F, \psi_F)$$
$$\times \varepsilon(0, \tilde{r}_{\tilde{w}} . \varphi, \psi_F)\, L(1, r_{\tilde{w}} . \varphi)/L(0, \tilde{r}_{\tilde{w}} . \varphi).$$

COROLLARY 3.5. — *Choose the representative w of $\tilde{w}$ as in Section 2. Then*

$$A(\lambda, w)\, A(\tilde{w}(\lambda), w^{-1}) = \varepsilon(0, r_{\tilde{w}} . \varphi, \psi_F)^{-1}\, \varepsilon(0, \tilde{r}_{\tilde{w}} . \varphi, \psi_F)^{-1} \frac{L(0, r_{\tilde{w}} . \varphi)\, L(0, \tilde{r}_{\tilde{w}} . \varphi)}{L(1, r_{\tilde{w}} . \varphi)\, L(1, \tilde{r}_{\tilde{w}} . \varphi)}.$$

Proof. — This is a consequence of equations (1.7) and (1.8), the identity

$$\overline{L(s, r_{\tilde{w}} . \varphi)} = L(\bar{s}, \tilde{r}_{\tilde{w}} . \varphi),$$

and the fact the λ-functions are of absolute value one.

Theorem 3.1 is now immediate

Remark. — We should remark that the normalization of Thorem 3.1 is slightly different from the one suggested by Langlands in [14]. In fact $\varepsilon(0, \tilde{r}_{\tilde{w}} . \varphi, \psi_F)$ in our normalization must be changed to $\varepsilon(0, r_{\tilde{w}} . \varphi, \psi_F)$. By Corollary 3.5 this new normalization is also valid. But we rather use the normalization in Theorem 3.1 since it is more appropriate for the results of the next several sections.

4. Identities for intertwining operators

Throughout this section we shall assume $\tilde{w}\lambda = \lambda$. Then $A(\lambda, w)$ is a self intertwining operator on $V(\lambda) = V(0, \lambda)$.

There exists a finite group $R = R(\lambda) \subset W(\lambda)$, the R-group of $I(\lambda)$, which determines the reducibility of $I(\lambda)$. More precisely, if $\hat{R}$ is the set of classes of finite-dimensional irreducible complex representations of R, then every irreducible component of $I(\lambda)$ is attached to an element of $\hat{R}$, and moreover every such component appears in $I(\lambda)$ with multiplicity equal to the dimension of the corresponding finite-dimensional representation of R (*cf.* [9], [17]). Given $\pi \subset I(\lambda)$, let $\rho_\pi \in \hat{R}$ be the corresponding equivalence class of finite-dimensional representations of R.

Assume λ corresponds to φ. Define a function λ_{ψ_F} by

$$(4.1)\quad \lambda_{\psi_F}(w) = \varepsilon(0, \tilde{r}_{\tilde{w}}.\varphi, \psi_F)\frac{L(1, r_{\tilde{w}}.\varphi)}{L(0, \tilde{r}_{\tilde{w}}.\varphi)}.C_\chi(\lambda, w)^{-1}$$
$$= \prod_{\alpha \in \Delta_1(\tilde{w})} \lambda(F_\alpha/F, \psi_F) \prod_{\alpha \in \Delta_2(\tilde{w})} \lambda(E_\alpha/F, \psi_F)^2 \lambda(F_\alpha/F, \psi_F)^{-1}.$$

Then by (1.7)

$$\lambda_{\psi_F}(w^{-1}) = \overline{\lambda_{\psi_F}(w)} = \lambda_{\psi_F}(w)^{-1}.$$

It is now easy to check that λ_{ψ_F} is a character of the group R_0 generated by

$$\{w \mid \tilde{w} \in R\},$$

where each representative w is fixed as before. We finally observe that there exists a central homomorphism of R_0 onto R and therefore every representation of R can be considered as one of R_0.

Fix $r \in R$ and $\rho \in \hat{R}$, let $\langle \rho, r \rangle = \theta_\rho(r)$, where θ_ρ is simply the character of ρ (its trace). The parametrization of the components of $I(\lambda)$ with the elements of $\hat{R}$ depends on the normalization $\mathscr{A}(\lambda, r)$, $\tilde{r} \in R$, of our intertwining operator. More precisely, let P_ρ be the operator (*cf.* [9])

$$(4.2)\qquad P_\rho = |R|^{-1} \dim \rho \sum_{\tilde{r} \in R} \overline{\theta_\rho(\tilde{r})} \mathscr{A}(\lambda, r).$$

Then P_ρ projects the space $V(\lambda)$ of $I(\lambda)$ onto its π_ρ-isotypic component. Now, suppose $\mathscr{A}(\lambda, r)$ is changed to $\mathscr{A}'(\lambda, r) = \omega(r)\mathscr{A}(\lambda, r)$, $\tilde{r} \in R$, where ω is a one-dimensional unitary character of R (In general they differ by a character of the group R_0 introduced before). Then under $\mathscr{A}'(\lambda, r)$, the representation π_ρ corresponds to $\rho \otimes \omega$. This follows immediately from (4.2).

There exists a unique χ-generic subrepresentation π_0 of $I(\lambda)$.

Given $\pi \subset I(\lambda)$, let χ_π denote its character. Moreover given $f \in C_c^\infty(G)$, let

$$I(f) = \int_G I(\lambda)(g) f(g)\, dg.$$

A character ψ_F of F is called unramified if the ring of integers O of F is the largest ideal of F on which ψ_F is trivial.

We shall say **G** is unramified if it splits over a (finite) unramified extension of F. We then let $K = \mathbf{G}(O)$. Finally a character λ of T is called unramified if $\lambda \mid \mathbf{T}(O)$ is trivial.

We now prove the following theorem.

THEOREM 4.1. — *Fix $\tilde{w} \in W(\lambda)$ and choose a representative w for $\tilde{w}$ as before.*

(*a*) *Let* $\mathscr{A}(\lambda, w)$ *be the operator normalized as in Theorem* 3.1. *Fix a parametrization between* $\pi \subset I(\lambda)$ *and* $\rho_\pi \in \hat{R}$ *such that* π_0 *corresponds to the trivial character of* R. *Then*

$$\text{trace}(\mathscr{A}(\lambda, w) I(f)) = \lambda_{\psi_F}(w) \sum_{\pi} \langle \rho_\pi, \tilde{w} \rangle \chi_\pi(f) \tag{4.11}$$

for $\forall f \in C_c^\infty(G)$, *where* $\langle \rho_\pi, \tilde{w} \rangle = 1$ *unless* $\tilde{w} \in R$ *in which case* $\langle \rho_\pi, \tilde{w} \rangle = \text{trace } \rho_\pi(\tilde{w})$.

(*b*) *Suppose* ψ_F, **G**, *and* λ *are all unramified. Let* $K = \mathbf{G}(O)$. *Then the unique* K-*fixed function of* $I(\lambda)$ *belongs to* π_0 *and* λ_{ψ_F} *is trivial. Consequently*

$$\text{trace}(\mathscr{A}(\lambda, w) I(f)) = \sum_{\pi} \langle \rho_\pi, \tilde{w} \rangle \chi_\pi(f),$$

$f \in C_c^\infty(G)$, *where the coefficient of* $\chi_{\pi_0}(f)$ *in the sum is equal to* 1.

Proof. — With Theorem 2.6 of [9] in hand, to prove part (*a*), we only need the following lemma.

LEMMA 4.2. — *Fix* $\tilde{w} \in R$ *and choose a representative w as before. Let* $\mathscr{A}'(\lambda, w)$ *be the normalization of* $A(\lambda, w)$ *which attaches* π_0 *to the trivial character of* R. *If* $\mathscr{A}(\lambda, w)$ *is the normalization fixed in Theorem* 3.1, *then*

$$\mathscr{A}(\lambda, w) = \lambda_{\psi_F}(w) \mathscr{A}'(\lambda, w),$$

where λ_{ψ_F} *is the character of* R_0 *defined by* (4.1).

Proof. — We first observe that under the assumption $\lambda = \tilde{w}\lambda$, the representations $r_{\tilde{w}} . \varphi$ and $\tilde{r}_{\tilde{w}} . \varphi$ are equivalent and since λ is unitary $L(1, r_{\tilde{w}} . \varphi)$ and $L(1, \tilde{r}_{\tilde{w}} . \varphi)$ are both finite and equal. It then follows from Proposition 3.4 that $\mathscr{A}(\lambda, w)$ acts on the space of π_0 by $\lambda_{\psi_F}(w)$. Finally if π_0 corresponds to the trivial character of R by $\mathscr{A}'(\lambda, w)$, then the orthogonality of characters when applied to (4.2) immediately implies that $\mathscr{A}'(\lambda, w)$ acts on the space of π_0 by 1. The lemma is now complete.

To complete the theorem we shall now prove part (*b*).

Let f_0 be the unique K-fixed function of $I(\lambda)$ normalized by $f_0(e) = 1$. If $\lambda_\chi(0, \lambda)$ is the Whittaker functional attached to $I(\lambda)$ (*cf.* Section 1), then by Theorem 5.4 of [5]

$$\lambda_\chi(0, \lambda)(f_0) = L(1, \tilde{r}_{\tilde{w}} . \varphi)^{-1}$$

which is non-zero since λ is unitary. This completes the theorem.

5. The global identity

Suppose now that F is a number field and for every place v of F, let F_v be the corresponding completion of F. Denote by O_v its ring of integers. Let $\mathbb{A}_F$ denote the ring of adeles of F.

Let **G** be a quasi-split connected reductive algebraic group over F. We use **B**, **T**, **U**, **P**, **M**, **N**, $\mathbf{A}_0$, and **A** to denote the same objects as those in Section 1. Finally for every group $\mathbf{H}$ over F, let $H=\mathbf{H}(\mathbb{A}_F)$ and $H_v=\mathbf{H}(F_v)$ denote its groups of adelic and F_v-rational points, respectively.

If $\mathbf{G}$ is unramified over v, let $K_v=\mathbf{G}(O_v)$. Otherwise K_v denotes a special maximal compact subgroup of G_v. Let $K=\otimes_v K_v$.

Let λ be a unitary character of $\mathbf{T}(F)\backslash T$. Then $\lambda=\otimes_v \lambda_v$, $\lambda_v \in \hat{T}_v$. Define $I(\lambda)$ to be

$$I(\lambda)=\operatorname*{Ind}_{TU \uparrow G} \lambda \otimes 1=\otimes_v I(\lambda_v).$$

Fix $\tilde{w} \in W(\mathbf{A}_0)$ such that $\tilde{w}\lambda=\lambda$. We choose a representative w of $\tilde{w}$ in $\mathbf{G}(F) \cap K$. Moreover we assume that each component of w, which we still denote by w, is as in Section 2. If we let

$$M(\lambda, \tilde{w})=\lim_{\nu \to 0} A(\nu, \lambda, w),$$

then $M(\lambda, \tilde{w})$ becomes a global intertwining operator, sending $I(\lambda)$ onto itself.

Fix $f \in C_c^\infty(G)$. Then $f=\otimes_v f_v$, where for almost all v over which $\mathbf{G}$ is unramified, f_v is the characteristic function of $\mathbf{G}(O_v)$. Next let ψ_F be a non-trivial unitary character of $F\backslash\mathbb{A}_F$. Write $\psi_F=\otimes_v \psi_{F_v}$. Then each ψ_{F_v} is non-trivial and moreover for almost all v, ψ_{F_v} is unramified. At each place v, we then define χ_v as before and let $\pi_{0,v}$ be the unique χ_v-generic component of $I(\lambda_v)$.

By Part (*b*) of Theorem 4.1 for almost all v, the unique $\mathbf{G}(O_v)$-fixed function in the space of $I(\lambda_v)$ belongs to the space of $\pi_{0,v}$ and therefore the coefficient of $\chi_{\pi_{0,v}}$ in the local identity (4.1.1) is equal to 1. Moreover, if at an unramified place v, $f_{0,v}$ denotes the characteristic function of $\mathbf{G}(O_v)$, then $\chi_{\pi_{0,v}}(f_{0,v})=1$, while $\chi_{\pi_v}(f_{0,v})=0$ if $\pi_v \not\cong \pi_{0,v}$.

Let φ be a homomorphism of W_F into ${}^LT$ attached to λ. It is not unique. But if $\theta_v : W_{F_v} \to W_F$ is the canonical homomorphism and φ' is another such homomorphism, then by part (*b*) of Theorem 2 of [15] (also *cf.* Theorem 6.4 of [12]):

$$(\varphi-\varphi').\theta_v=0$$

and therefore at each place v, $\varphi_v=\varphi.\theta_v$ is uniquely determined. It is the homomorphism attached to λ_v. Moreover at each v, the representation $r_{\tilde{w}}.\varphi.\theta_v=r_{\tilde{w}}.\varphi_v$, as well as each $r_{\tilde{w},i}.\varphi.\theta_v=r_{\tilde{w},i}.\varphi_v$, are also uniquely determined. Let n be the multiplicity of the trivial representation in $r_{\tilde{w}}.\varphi$. We then have:

THEOREM 5.1. — *Fix* $\tilde{w} \in W(\lambda)$ *and* $f=\otimes_v f_v$ *in* $C_c^\infty(G)$. *Then*

$$\operatorname{trace}(M(\lambda, \tilde{w}) I(f))=(-1)^n \prod_v \sum_{\pi_v} \langle \rho_{\pi_v}, \tilde{w} \rangle \chi_{\pi_v}(f_v), \tag{5.1.1}$$

where at each v, π_v *runs over all the equivalence classes of irreducible components of* $I(\lambda_v)$. *Moreover at each place* v, *the coefficient of* $\chi_{\pi_{0,v}}(f_v)$ *is equal to one and furthermore for a given* f *almost all the sums in the product are equal to* 1.

Proof. — We first observe that $\prod_v \lambda_{\psi_{F_v}}(w)=1$, using the properties of λ-functions. Next for every representation ρ of W_F and every $s \in \mathbb{C}$ with $\mathrm{Re}(s) \gg 0$, let

$$L(s, \rho)=\prod_v L(s, \rho.\theta_v)$$

and

$$\varepsilon(s, \rho)=\prod_v \varepsilon(s, \rho.\theta_v, \psi_{F_v}).$$

Then

$$L(s, \rho)=\varepsilon(s, \rho) L(1-s, \tilde{\rho}), \tag{5.12}$$

where $\tilde{\rho}$ is the contragredient of ρ (*cf.* [6], [16], [28]).

Using (5.1.2), we then have

$$M(\lambda, \tilde{w})=\lim_{s \to 0^+} \prod_{i=1}^m \varepsilon(is, \tilde{r}_{\tilde{w}, i}.\varphi)^{-1} \frac{L(is, \tilde{r}_{\tilde{w}, i}.\varphi)}{L(1+is, \tilde{r}_{\tilde{w}, i}.\varphi)} \bigotimes_v \mathscr{A}(\lambda_v, w)$$

$$= \lim_{s \to 0^+} \prod_{i=1}^m \frac{L(1-is, r_{\tilde{w}, i}.\varphi)}{L(1+is, \tilde{r}_{\tilde{w}, i}.\varphi)} \bigotimes_v \mathscr{A}(\lambda_v, w).$$

We must now evaluate the following limit

$$\lim_{s \to 0^+} \prod_{i=1}^m L(1-is, r_{\tilde{w}, i}.\varphi)/L(1+is, \tilde{r}_{\tilde{w}, i}.\varphi).$$

This is then clearly equal to

$$(-1)^n \lim_{s \to 0^+} \prod_{i=1}^m L^{(n)}(1-is, r_{\tilde{w}, i}.\varphi)/L^{(n)}(1+is, \tilde{r}_{\tilde{w}, i}.\varphi),$$

where for each ρ, $L^{(n)}(s, \rho)$ denotes the n-th derivative of $L(s, \rho)$ with respect to s. But now for $\mathrm{Re}(s)>0$

$$L^{(n)}(s, r_{\tilde{w}, i}.\varphi)=L^{(n)}(s, \tilde{r}_{\tilde{w}, i}.\varphi),$$

$i=1, \ldots, m$, using the equality $r_{\tilde{w}, i}.\varphi_v=\tilde{r}_{\tilde{w}, i}.\varphi_v$ at each v. Moreover

$$\lim_{s \to 1^-} L^{(n)}(s, r_{\tilde{w}, i}.\varphi)= \lim_{s \to 1^+} L^{(n)}(s, \tilde{r}_{\tilde{w}, i}.\varphi)$$

and they are both finite. Thus

$$\lim_{s \to 0^+} L^{(n)}(1-is, r_{\tilde{w}, i} \cdot \varphi)/L^{(n)}(1+is, \tilde{r}_{\tilde{w}, i} \cdot \varphi) = 1.$$

Now the theorem is a consequence of Theorem 4.1 of the present paper when $v < \infty$ and Theorem 4.1 of [22] otherwise.

6. The global identity in general

In this section, under a certain natural assumption, we shall prove the global identity in general. Thus we resume the notation from the previous section and therefore **G** is a quasi-split connected reductive algebraic group over a number field F. As before, let $\mathbf{P} = \mathbf{MN}$ be a standard ($\mathbf{P} \supset \mathbf{B}$) parabolic subgroup of **G**. Let $\sigma = \otimes \sigma_v$ be a cusp form on $M = \mathbf{M}(\mathbb{A}_F)$.

Fix a unitary character $\chi = \bigotimes_v \chi_v$ of $U = \mathbf{U}(\mathbb{A}_F)$, and let χ also denote its restriction to $U \cap M$. We shall assume σ is χ-generic. This means that each σ_v is χ_v-generic. We shall say σ is generic, if it is generic with respect to some generic character χ.

Let S be a finite set of places, including the archimedean ones, such that for $v \notin S$, **G**, σ_v, and χ_v are all unramified (*cf.* [23] for more detail).

Let $W(\mathbf{A}_0)$ be the Weyl group of $\mathbf{A}_0$ in **G**. Fix $\tilde{w} \in W(\mathbf{A}_0)$ such that $\tilde{w}(\sigma) = \sigma$. Choose a representative $w \in \mathbf{G}(F) \cap K$ for $\tilde{w}$ and let w also denote each of its components.

Fix $\nu \in \mathfrak{a}^*$, where $\mathfrak{a}^* = X(\mathbf{A})_F \otimes \mathbb{R}$. Given $f = \bigotimes_v f_v$, $f_v \in V(\nu, \sigma_v)$, define

$$M(\nu, \sigma, \tilde{w}) f = \bigotimes_v A(\nu, \sigma_v, w) f_v$$

for ν in some appropriate Weyl chamber. Then

$$M(\sigma, \tilde{w}) = \lim_{\nu \to 0} M(\nu, \sigma, \tilde{w})$$

is a self intertwining operator on $I(\sigma) = \bigotimes_v I(\sigma_v)$.

At every place v, let $\pi_{0, v}$ be the unique χ_v-generic component of $I(\sigma_v)$.

The L-group ${}^L M$ of **M** acts on the Lie algebra ${}^L \mathfrak{n}$ of the L-group of **N** by adjoint representation. It sends ${}^L \mathfrak{n}_{\tilde{w}}$, the Lie algebra of the L-group of $\mathbf{N}_{\tilde{w}}$, into itself. We also define $r_{\tilde{w}, i}$ and V_i, $1 \leqq i \leqq m$, as in Section 1.

For each i and $v \notin S$, let $L(s, \sigma_v, r_{\tilde{w}, i} \cdot \eta_v)$ be the corresponding unramified Langlands L-function (*cf.* [4], [23]), where $\eta_v : {}^L M_v \to {}^L M$ is the natural map. Finally, set

$$L_S(s, \sigma, r_{\tilde{w}, i}) = \prod_{v \notin S} L(s, \sigma_v, r_{\tilde{w}, i} \cdot \eta_v).$$

We start with the following lemma.

LEMMA 6.1. — *For each $v \notin S$, let $f_{0,v}$ be the unique K_v-fixed function in $V(\sigma_v)$, normalized by $f_{0,v}(e_v)=1$.*

(*a*) *Assume $m=1$. Then for each $v \notin S$, $f_{0,v}$ belongs to the space of $\pi_{0,v}$.*

(*b*) *Suppose $m>1$. Assume further that the restriction of σ to the center of M is trivial. Then for each $v \notin S$, $f_{0,v}$ belongs to the space of $\pi_{0,v}$.*

Proof. — By Theorem 5.4 of [5] this is equivalent to $L(1, \sigma_v, r_{\tilde{w}} \cdot \eta_v)^{-1} \neq 0$. But this is just Lemma 5.8 of [23], completing the lemma.

Next, let

$$a_\sigma^S(s)=\prod_{i=1}^{m} L_S(1-is, \sigma, r_{\tilde{w}, i})/L_S(1+is, \sigma, \tilde{r}_{\tilde{w}, i}).$$

LEMMA 6.2. — *Assume $\tilde{w}(\sigma)=\sigma$ and furthermore if $m>1$, suppose that the restriction of σ to the center of M is trivial. Then $\lim_{s \to 0} a_\sigma^S(s)$ is independent of S. The same is true for the order n of the pole of $L_S(s, \sigma, r_{\tilde{w}})$ at $s=1$ (It is non-zero by Theorem 5.1 of [20]), and moreover if $a_\sigma=\lim_{s \to 0} a_\sigma^S(s)$, then $a_\sigma=(-1)^n$.*

Proof. — It is easy to see that under the assumption $\tilde{w}(\sigma_v)=\sigma_v$ for each unramified v,

$$L(s, \sigma_v, r_{\tilde{w}, i} \cdot \eta_v)=L(s, \sigma_v, \tilde{r}_{\tilde{w}, i} \cdot \eta_v),$$

$1 \leqq i \leqq m$. Moreover, still resuming the assumption in part (*b*) of Lemma 6.1, they are both finite at $s=1$ and therefore $\lim_{s \to 0} a_\pi^S(s)$ is independent of S. The fact that n is independent of S follows for the same reason. To show $a_\sigma=(-1)^n$, one only has to imitate the same part in the proof of Theorem 5.1.

We are now led to make our assumption.

ASSUMPTION 1. — *For every $v \notin S$, the local L-function $L(s, \sigma_v, r_{\tilde{w}} \cdot \eta_v)$ is holomorphic at $s=1$.*

As we just observed (*cf.* Lemma 5.8 of [23]), its truth can be presently verified in many cases. It is always true if σ_v is tempered [in fact for $\mathrm{Re}(s)>0$].

Next we prove:

PROPOSITION 6.3. — *We resume Assumption 1 and that $\tilde{w}(\sigma)=\sigma$. Let $f=\bigotimes_v f_v$ be a function in $V(\sigma)$. Assume that for every v, f_v belongs to the space of $\pi_{0,v}$ which is possible in view of Lemma 6.1. Let n be the order of the pole of $L_S(s, \sigma, r_{\tilde{w}})$ at $s=1$. Then*

$$M(\sigma, \tilde{w})f=(-1)^n f. \tag{6.3.1}$$

Proof. — Enlarge S, if necessary, so that for every $v \notin S$, $f_v=f_{0,v}$, the unique K_v-fixed function in $V(\sigma_v)$, satisfying $f_{0,v}(e_v)=1$. In view of the equations (2.7) and (3.6) of [23], for $\mathrm{Re}(s) \gg 0$, the operator $M(2s\rho, \sigma, \tilde{w})f'$ can be written as $(f' \in V(2s\rho, \sigma)$,

$f'_v = f'_{0,v}$ for $v \notin S$)

(6.3.2) $$\mathrm{M}(2s\rho, \sigma, \tilde{w})f' = \prod_{v \in S} \mathrm{C}_{\chi_v}(2s\rho_v, \sigma_v, w)$$

$$\times \prod_{i=1}^{m} \frac{\mathrm{L}_S(1-a_i s, \sigma, r_{\tilde{w}, i})}{\mathrm{L}_S(a_i s, \sigma, \tilde{r}_{\tilde{w}, i})} \otimes_v \mathrm{A}(2s\rho_v, \sigma_v, w)f'_v$$

$$= a_\sigma^S(s) \bigotimes_{v \in S} \mathrm{C}_{\chi_v}(2s\rho_v, \sigma_v, w)\mathrm{A}(2s\rho_v, \sigma_v, w)f'_v \otimes \bigotimes_{v \notin S} \tilde{f}'_{0,v},$$

where $\tilde{f}'_{0,v}$ is a similar K_v-fixed function in $\mathrm{V}(2s\tilde{w}(\rho_v), \sigma_v)$. Choose f' such that $f = \lim_{s \to 0^+} f'$.

Now, since each f_v, $v \in S$, belongs to the space of $\pi_{0,v}$ it follows immediately that for $v \in S$

$$(\lim_{s \to 0^+} \mathrm{C}_{\chi_v}(2s\rho_v, \sigma_v, w)\mathrm{A}(2s\rho_v, \sigma_v, w))f_v = f_v$$

[*cf.* equation (1.5)]. Now (6.3.1) is a consequence of taking limits as $s \to 0^+$ in (6.3.2) and Lemma 6.2.

Fix a place v of F. Let $\pi_v \subsetneq \mathrm{I}(\sigma_v)$ and denote by $\mathrm{V}(\pi_v)$ the π_v-isotypic subspace of $\mathrm{V}(\sigma_v)$, the space of $\mathrm{I}(\sigma_v)$. Fix a normalization $\mathscr{A}(\sigma_v, w)$ of $\mathrm{A}(\sigma_v, w)$. It can be easily seen that given an irreducible constituent $\pi_v \subsetneq \mathrm{I}(\sigma_v)$, there exists a function $c_{\pi_v}(w)$, such that

(6.1) $$\mathrm{trace}_{\mathrm{V}(\pi_v)}(\mathscr{A}(\sigma_v, w)\mathrm{I}(f)) = c_{\pi_v}(w)\chi_{\pi_v}(f),$$

where for every $f \in \mathrm{C}_c^\infty(\mathrm{G}_v)$, $\chi_{\pi_v}(f) = \mathrm{trace}\,\pi_v(f)$ with

$$\pi_v(f) = \int_{\mathrm{G}_v} \pi_v(g)f(g)\,dg.$$

In fact $\mathrm{trace}_{\mathrm{V}(\pi_v)}(\mathscr{A}(\sigma_v, w)\mathrm{I}(f))$ must only be computed on those components of $\mathrm{V}(\pi_v)$ which are sent to themselves by $\mathscr{A}(\sigma_v, w)$. It then acts on each of them by a scalar and (6.1) follows. From the properties of the trace it is clear that the functions $c_{\pi_v}(w)$, $\pi_v \subsetneq \mathrm{I}(\sigma_v)$, are all class functions on the group generated by $\{w \mid \tilde{w} \in \mathrm{W}(\sigma_v)\}$, where the representatives w are chosen as in Section 2. We therefore have

(6.2) $$\mathrm{trace}(\mathscr{A}(\sigma_v, w)\mathrm{I}(f)) = \sum_{\pi_v} c_{\pi_v}(w)\chi_{\pi_v}(f),$$

$f \in \mathrm{C}_c^\infty(\mathrm{G}_v)$, where each c_{π_v} is a class function on the group generated by $\{w \mid \tilde{w} \in \mathrm{W}(\sigma_v)\}$.

Observe that if π_v is multiplicity free, then c_{π_v} is in fact a character. We shall now assume that at each v, $\mathscr{A}(\sigma_v, w)$ is so normalized that $c_{\pi_{0,v}}$ is the trivial character. Here $\pi_{0,v}$ is the unique χ_v-generic component of $\mathrm{I}(\sigma_v)$. If v is unramified, we attain this by taking the standard normalization of $\mathscr{A}(\sigma_v, w)$ suggested by equation (2.7) of [23].

Before stating our final result, we shall elaborate more on the functions c_{π_v}.

By the theory of L-packets (*cf.* [17], [24], and Section 4 here), it is desirable to interpret the class functions c_{π_v} as characters of irreducible representations of a certain finite group. In what follows, we shall show that this is the case if σ_v is in the discrete series. We shall also discuss the case of non-discrete series tempered representations.

Assume σ_v is in the discrete series (also generic). Let R_v be the R-group of σ_v which is simply the quotient of $W(\sigma_v)$ by the subgroup $W''(\sigma_v)$ (in the notation of [26]) consisting of those $\tilde{w} \in W(\sigma_v)$ for which $\mathscr{A}(\sigma_v, w)$ are scalars. It can also be considered as a subgroup of $W(\sigma_v)$. Harish-Chandra's Commuting Algebra Theorem (Theorem 5.5.3.2 of [25]) for $I(\sigma_v)$ implies that the normalized operators $\mathscr{A}(\sigma_v, w)$, $\tilde{w} \in R_v$, span the commuting algebra of $I(\sigma_v)$. Moreover by Silberger's Dimension Theorem [26] they form a basis.

Now, given an equivalence class ρ of irreducible (finite-dimensional) representations of R_v, let

$$P_\rho = |R_v|^{-1} \dim \rho \sum_{\tilde{r} \in R_v} \overline{\langle \rho, \tilde{r} \rangle} \mathscr{A}(\sigma_v, r),$$

where $\langle \rho, \tilde{r} \rangle = \operatorname{trace} \rho(\tilde{r})$. Moreover, if V denotes the space of $I(\sigma_v)$, let $V_\rho = P_\rho V$. Then a proof, word by word similar to the one given in Theorem 2.4 of [9], implies that P_ρ is the projection of V onto one of its isotypic components and that every isotypic component of V is obtained in this way. Moreover, if $\pi_v \subset I(\sigma_v)$ corresponds to ρ, then its multiplicity in $I(\sigma_v)$ is equal to $\dim \rho$. (The triviality of the cocycle η of [9] is a consequence of σ_v being generic which results in $I(\sigma_v)$ having a component which appears with multiplicity one; *cf.* the discussion in Section 6 of [9].)

Now, let R_v^0 be the group generated by $\{r \mid \tilde{r} \in R_v\}$. There is a central homomorphism from R_v^0 onto R_v which sends $r \in R_v^0$ to $\tilde{r} \in R_v$. Extend $\mathscr{A}(\sigma_v, r)$ to all of R_v^0 by $\mathscr{A}(\sigma_v, r_1) = \mathscr{A}(\sigma_v, r_2)$ if $\tilde{r}_1 = \tilde{r}_2$. If ρ is an irreducible representation of R_v, then it becomes one of R_v^0, and it can be easily seen that P_ρ is equal to

$$|R_v^0|^{-1} \dim \rho \sum_{r \in R_v^0} \overline{\langle \rho, \tilde{r} \rangle} \mathscr{A}(\sigma_v, r).$$

Then

$$|R_v^0|^{-1} \dim \rho \sum_{r \in R_v^0} \overline{\langle \rho, \tilde{r} \rangle} \operatorname{trace}_{V(\pi_v)} (\mathscr{A}(\sigma_v, r) I(f)) = \dim \rho . \chi_{\pi_v}(f).$$

Since $\operatorname{trace}_{V(\pi_v)} (\mathscr{A}(\sigma_v, r) I(f))$ is a class function on R_v^0, it then follows from the orthogonality of characters that

$$\operatorname{trace}_{V(\pi_v)} (\mathscr{A}(\sigma_v, r) I(f)) = \langle \rho, \tilde{r} \rangle$$

Equation (6.2) can now be stated as follows.

PROPOSITION 6.4. — *Assume σ_v is a generic discrete series representation and let R_v denote its R-group. For $\pi_v \subset I(\sigma_v)$, let ρ_{π_v} be the equivalence class of irreducible (finite-dimensional) representations of R_v which is attached to π_v. (It depends on the choice* *of r.)* *Fix* $f \in C_c^\infty(G_v)$. *Then for every* $\tilde{r} \in R_v$

$$\mathrm{trace}(\mathscr{A}(\sigma_v, r)\,I(f)) = \sum_{\pi_v} \langle \rho_{\pi_v}, \tilde{r} \rangle \chi_{\pi_v}(f), \tag{6.4.1}$$

where the sum is taken over equivalence classes of irreducible constituents of $I(\sigma_v)$. *In particular* $c_{\pi_v}(r) = \langle \rho_{\pi_v}, \tilde{r} \rangle$.

Now suppose σ_v is only tempered (i. e. not necessarily in the discrete series). It is then again expected that the constituents of $I(\sigma_v)$ are parametrized by equivalence classes of irreducible representations of a certain finite group. When $F_v = \mathbb{R}$, this has been accomplished in [11]. In fact, we may take σ_v be a basic representation (i. e. induced from discrete series or limits of discrete series) with non-degenerate data [11]. Then by Theorem 12.6 of [11] the constituents of $I(\sigma_v)$ are parametrized by the equivalence classes of irreducible representations of the R-group of $I(\sigma_v)$ given by a non-degenerate data. The R-group then agrees with the one in [24]. For a non-archimedean field the problem is wide open.

We now state our global identity.

THEOREM 6.5. — *Let* σ *be a generic cusp form on* M *and let* $\tilde{w} \in W(A_0)$ *be such that* $\tilde{w}(\sigma) = \sigma$. *Suppose that for every* $v \notin S$, *the component* σ_v *of* σ *satisfies Assumptions* 1 *(In particular if* σ *is tempered—also see Lemmas* 6.1 *and* 6.2*).* *Let n be the order of the pole of* $L_S(s, \sigma, r_{\tilde{w}})$ *at* $s=1$. *Fix* $f = \bigotimes_v f_v$ *in* $C_c^\infty(G)$. *Then*

$$\mathrm{trace}(M(\sigma, \tilde{w})\,I(f)) = (-1)^n \prod_v \sum_{\pi_v} c_{\pi_v}(w)\,\chi_{\pi_v}(f_v), \tag{6.4.1}$$

where at each v, $c_{\pi_0, v} \equiv 1$, *and the sum is taken over equivalence classes of constituents of* $I(\sigma_v)$.

Proof. — There is a constant $c \neq 0$ such that

$$M(\sigma, \tilde{w}) = c \bigotimes_v \mathscr{A}(\sigma_v, w)$$

Since each normalized operator $\mathscr{A}(\sigma_v, w)$ is such that $c_{\pi_0, v}$ is the trivial character, it follows immediately that the coefficient of $\chi_{\pi_0, v}(f_v)$ is equal to 1 for all v. Now proposition 6.3 implies that $c = (-1)^n$ and the theorem follows from equation (6.2).

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