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THE SOCLE FILTRATION OF A VERMA MODULE

BY RONALD S. IRVING

1. Introduction

1.1. In this paper two fundamental results on socle filtrations of Verma modules are proved. Given a finite length module M, call a filtration of M a *Loewy filtration* if the successive quotients, or *layers*, are semisimple and there is no such filtration of shorter length. The length of such a filtration is called the *Loewy length* of M. Among such filtrations is one which contains any other term-by-term, the socle filtration, and one which is contained in any other, the radical filtration. The first main result of this paper is that the socle filtration of a Verma module is the unique Loewy filtration; in other words, the socle and radical filtrations on a Verma module coincide. From this the second main result easily follows: a description of the simple modules in the layers of the socle filtration as coefficients of Kazhdan-Lusztig polynomials. This second result may be viewed as a strong form of the Kazhdan-Lusztig conjecture, which follows as a special case by setting $q=1$ in the Kazhdan-Lusztig polynomials. In fact, as I will explain further below, the proofs of the two results assume the validity of the Kazhdan-Lusztig conjecture in the equivalent form known as Vogan's conjecture. At the moment, published proofs of the Kazhdan-Lusztig conjecture are available in the integral case, due to Brylinski-Kashiwara and Beilinson-Bernstein ([3], [6]). In the non-integral case, the proof depends on unpublished results of Beilinson and Bernstein and results of Lusztig ([2], [19]). (Vogan's conjecture is discussed in [20], [21], the equivalence to the Kazhdan-Lusztig conjecture being stated and proved in part in [21]. A proof of the equivalence, applicable more generally to generalized Verma modules, is in [14]. It also follows from the Decomposition Theorem of [4], in the integral case.)

Versions of both results have been proved through the use of auxiliary filtrations and deep geometric results. Jantzen defined a filtration on a Verma module called the Jantzen filtration and asked if the filtrations on two different Verma modules are compatible in a suitable sense. Assuming this question (the Jantzen conjecture) has a positive answer, Gabber and Joseph proved that Jantzen filtrations are Loewy filtrations and that coefficients of Kazhdan-Lusztig polynomials count multiplicities of composition factors in layers of the Jantzen filtration ([10], 4.9). Also Barbasch and I independently used this work to deduce that the socle and Jantzen filtrations coincide ([1], [13]). (The Gabber-Joseph result was also conjectured by Deodhar and Gelfand-MacPherson

([9], [11]).) However, no published proof of Jantzen's conjecture has appeared yet, and the proof announced by A. Beilinson and J. Bernstein [2] requires geometric results beyond those necessary to prove the Kazhdan-Lusztig conjecture. In an earlier version of this paper, I constructed another auxiliary Loewy filtration, using antidominant projectives in the category $\mathcal{O}$; using Vogan's conjecture, I proved that this filtration is the socle filtration and that the analogue of the second result holds for it.

Fundamental to the geometric approach which now underlies much of representation theory is the construction due to Gabber of weight filtrations on Verma modules. These filtrations turn out to be Loewy filtrations, for which the analogue of the second main result holds ([2], [4] *see* also [8]). A step in the proof by Beilinson and Bernstein of the Jantzen conjecture is the result that the weight and Jantzen filtrations on a Verma module coincide [3]. From this follows both the Gabber-Joseph result on the Jantzen filtration and the coincidence of socle and weight filtrations. In the earlier version of this paper, I also proved that the socle and weight filtrations coincide. Since the earlier version was written, I learned from L. Casian that he has proved a general result about weight filtrations which implies that the weight filtration on a Verma module coincides with both its radical and socle filtrations [7]. In addition, Beilinson and Ginsburg have announced results which have as a corollary that the weight and radical filtrations coincide [5]. Thus, the first main result of this paper, the uniqueness of Loewy filtrations, is also a consequence of the construction and analysis, by geometric methods, of weight filtrations.

The proof in this paper that Loewy filtrations on a Verma module are unique, in contrast, is an elementary induction argument, and the second main result also follows in an elementary manner. As mentioned, the proofs still depend on geometry in their dependence on the validity of the Kazhdan-Lusztig conjecture, but the actual proofs make no use of geometric methods or of auxiliary objects whose definition depends on geometry. Thus, if a more elementary proof of the Kazhdan-Lusztig conjecture were to become available, the results of the paper would still follow as a consequence. Put another way, the main theorems should be regarded as saying not that there is a unique Loewy filtration on a Verma module, with multiplicities counted by Kazhdan-Lusztig polynomials, but rather that this detailed structural information on a Verma module follows merely from the knowledge of its composition factor multiplicities embodied in the Kazhdan-Lusztig conjecture. It may not be surprising that such additional information should be available, since the Kazhdan-Lusztig conjecture provides not just a list of multiplicities in Verma modules but an algorithm for calculating them. Thus, one might anticipate that buried within the algorithm lies stronger information, and the main results may be regarded as a partial uncovering of this information.

1.2. In order to state some of the results more precisely, let me introduce the notation to be used in this paper. Given a module M of finite length, its Loewy length will be denoted by $ll(\mathrm{M})$. Its socle filtration will be denoted by $0 \subset \mathrm{soc}^1 \mathrm{M} \subset \mathrm{soc}^2 \mathrm{M} \subset \ldots \subset \mathrm{soc}^{ll(\mathrm{M})} \mathrm{M} = \mathrm{M}$. The quotient $\mathrm{soc}^i \mathrm{M}/\mathrm{soc}^{i-1} \mathrm{M}$ will be denoted $\mathrm{soc}_i \mathrm{M}$ and called the i-th layer. The module $\mathrm{soc}^1 \mathrm{M}$ is usually called the socle of M and denoted $\mathrm{soc}\,\mathrm{M}$. Similarly, we will denote the radical filtration by $0 = \mathrm{rad}^{ll(\mathrm{M})}$

$M \subset \ldots \subset \mathrm{rad}^1 M \subset \mathrm{rad}^0 M = M$, and its layer $\mathrm{rad}^i M/\mathrm{rad}^{i+1} M$ by $\mathrm{rad}_i M$. The module $\mathrm{rad}^1 M$ is usually called the radical of M and denoted rad M. Given a submodule N of M, it is well-known and easy to see that $\mathrm{soc}^i N = N \cap \mathrm{soc}^i M$, and that $\mathrm{rad}^i(M/N)$ is the image of $\mathrm{rad}^i M$ under the canonical surjection of M onto M/N. A less standard notion, but one which will be convenient for us, is that of the i-th *capital* of M: it is the quotient $M/\mathrm{rad}^i M$, and will be denoted $\mathrm{cap}^i M$. We may regard $\mathrm{cap}^i M$ as the largest homomorphic image of M of Loewy length i and $\mathrm{soc}^i M$ as the largest submodule of M of Loewy length i. The first capital $\mathrm{cap}^1 M$ is sometimes called the top of M or the head. In this paper it will simply be called the *cap* of M. A simple module L will be said to be *rigidly placed* in M if $(\mathrm{soc}^i M : L) = (\mathrm{rad}^{ll(M)-i} M : L)$ for all i. The module M is *rigid* if every simple module occurring as a composition factor is rigidly placed. Thus M is rigid if and only if its socle and radical filtrations coincide.

A complex, semisimple Lie algebra $\mathfrak{g}$ is fixed throughout, with Cartan subalgebra $\mathfrak{h}$ and Borel subalgebra $\mathfrak{b}$ containing $\mathfrak{h}$. The choice of $\mathfrak{h}$ determines a root system R with Weyl group $\mathscr{W}$, and $\mathfrak{b}$ determines a set of simple roots B. The half-sum of the positive roots is ρ, and a dot action of $\mathscr{W}$ on $\mathfrak{h}^*$ is defined as usual by $w.\lambda = w(\lambda+\rho)-\rho$. An element λ of $\mathfrak{h}^*$ is *antidominant* if $s_\alpha.\lambda \nless \lambda$ for all roots α, where s_α is the reflection about α. The root system R_λ is $\{\alpha \in R : 2(\lambda, \alpha)/(\alpha, \alpha) \in \mathbb{Z}\}$ and B_λ is the unique base of R_λ in $R_\lambda \cap R^+$. The subgroup $\mathscr{W}_\lambda$ of $\mathscr{W}$ is generated by $\{s_\alpha : \alpha \in B_\lambda\}$ and w_λ is its longest element. The Bruhat ordering $\leqq$ is defined on $\mathscr{W}_\lambda$ with e the unique minimal element and w_λ the unique maximal element.

Let λ be a fixed antidominant, regular weight throughout the paper; $\mathcal{O}^\lambda$ is the block of the category $\mathcal{O}$ whose simple modules, up to isomorphism, are $\{L(w.\lambda) : w \in \mathscr{W}_\lambda\}$. We denote by $L(\mu)$ the simple top of the Verma module $M(\mu)$ of highest weight μ, and $P(\mu)$ is its projective cover in $\mathcal{O}$. The Kazhdan-Lusztig polynomials $P_{y,w}(q)$ are defined with respect to the Coxeter group $\mathscr{W}_\lambda$, for any $y, w \in \mathscr{W}_\lambda$, by the dual version of formula 2.2 *a* in [16]. Explicitly, if $y \leqq ws_\alpha$ and there is an $\alpha \in B_\lambda$ with $ys_\alpha < y$ and $ws_\alpha > w$, then

$$P_{y,ws_\alpha}(q) = P_{ys_\alpha,w}(q) + q\,P_{y,w}(q) - \sum_{z \in \mathscr{W}_\lambda} \mu(z, w)\, q^{(l(w)-l(z)+1)/2} P_{y,z}(q),$$

where $\mu(z, w)$ is the coefficient of $q^{(l(w)-l(z)-1)/2}$ in $P_{z,w}(q)$.

The basic assumption used throughout this paper is that all the simple modules in $\mathcal{O}^\lambda$ satisfy Vogan's conjecture. The conjecture is formulated in terms of the functor θ_α of translation across the α-wall, which is defined on $\mathcal{O}^\lambda$ for each α in B_λ. For more on the properties of the functor θ_α, some of which are reviewed presently, one may turn to [10], [12], [13], [20]. Let us recall how θ_α acts on a simple module $L(w.\lambda)$, for $w \in \mathscr{W}_\lambda$. If $ws_\alpha < w$, then $\theta_\alpha L(w.\lambda) = 0$. Alternatively, if $ws_\alpha > w$, then $\theta_\alpha L(w.\lambda)$ has a simple socle $L(w.\lambda)$, a distinct simple cap $L(w.\lambda)$, and an intermediate subquotient $U_\alpha L(w.\lambda)$. Thus, to be precise, $U_\alpha L(w.\lambda) = \mathrm{rad}\, \theta_\alpha L(w.\lambda)/\mathrm{soc}\, \theta_\alpha L(w.\lambda)$. The module $U_\alpha L(w.\lambda)$ is annihilated by θ_α. Vogan's conjecture for $\mathcal{O}^\lambda$ states that $U_\alpha L(w.\lambda)$ is semisimple for all $\alpha \in B_\lambda$ and all $w \in \mathscr{W}_\lambda$ for which $ws_\alpha > w$. We extend U_α to arbitrary semisimple modules of $\mathcal{O}^\lambda$ by having it commute with direct sums and setting $U_\alpha L(w.\lambda) = 0$ if $ws_\alpha < w$.

The effect of θ_α on Verma modules is more easily described. Given $w < ws_\alpha$, the module $\theta_\alpha M(w.\lambda)$ is a non-trivial extension of $M(ws_\alpha.\lambda)$ by $M(w.\lambda)$, and it coincides with $\theta_\alpha M(ws_\alpha.\lambda)$. In fact, this extension is the unique non-trivial one.

An equivalent way to state Vogan's conjecture is that for a semisimple module M in $\mathcal{O}^\lambda$, the module $\theta_\alpha M$ has Loewy length at most 3. This can be extended to the statement that for any M in $\mathcal{O}^\lambda$ the Loewy length of $\theta_\alpha M$ is at most $ll(M)+2$ [13]. It follows from this extension that for any w in $\mathcal{W}_\lambda$, the Loewy length of $M(w.\lambda)$ is exactly $l(w)+1$, and the self-dual projective $P(\lambda)$ has Loewy length $2l(w_\lambda)+1$, where w_λ is the longest element of $\mathcal{W}_\lambda$. In fact it was proved in [13] that Vogan's conjecture holds for $\mathcal{O}^\lambda$ if and only if $P(\lambda)$ has Loewy length $2l(w_\lambda)+1$.

Another property of the category $\mathcal{O}^\lambda$ which will be used is the existence of a duality functor D on it which fixes the simple modules. Given an indecomposable projective $P(w.\lambda)$ in $\mathcal{O}^\lambda$, the dual module $DP(w.\lambda)$ is the injective envelope of $L(w.\lambda)$ and will be denoted by $I(w.\lambda)$.

Let us recall one more notion: a module M in $\mathcal{O}^\lambda$ has a *Verma flag* if it has a filtration whose successive quotients are Verma modules. For any w in $\mathcal{W}_\lambda$, the projective module $P(w.\lambda)$ has a Verma flag; the number of times $[P(w.\lambda): M(y.\lambda)]$ that $M(y.\lambda)$ appears as a quotient of successive modules in the flag (up to isomorphism) is independent of the chosen Verma flag. BGG reciprocity states that this number $[P(w.\lambda): M(y.\lambda)]$ equals the composition factor multiplicity $(M(y.\lambda): L(w.\lambda))$. (A proof of BGG reciprocity will be given in 4.2 as part of a more general result.) It follows easily from this that $[P(\lambda): M(y.\lambda)]=1$ for all y in $\mathcal{W}_\lambda$.

Given y and w in $\mathcal{W}_\lambda$, it is easily seen that any extension of $M(w.\lambda)$ by $M(y.\lambda)$ splits unless $y \leqq w$. As a result, for any $z \in \mathcal{W}_\lambda$, the projective module $P(z.\lambda)$ has a Verma flag $0=M_0 \subset M_1 \subset \ldots \subset M_t=P(z.\lambda)$ with the following property: let $M_i/M_{i-1}=M(w_i.\lambda)$; if $w_i < w_j$, then $i > j$.

Aside from basic facts on θ_α, the proofs of the theorems in this paper depend essentially only on the extended version of Vogan's conjecture mentioned above. Thus, in order to make the paper more self-contained, a proof that the extension follows from Vogan's conjecture is provided in an appendix. This proof has the benefit of being much shorter and to the point than the original proof of [13].

1.3. The results of this paper can now be stated more precisely. Again, it is assumed throughout that λ is a fixed antidominant, regular weight and that Vogan's conjecture holds for the simple modules in the block $\mathcal{O}^\lambda$. These hypotheses are re-stated in the two main theorems for clarity.

THEOREM 1. — *Let λ be an antidominant, regular weight for which $\mathcal{O}^\lambda$ satisfies Vogan's conjecture. For any w in $\mathcal{W}_\lambda$, the Verma module $M(w.\lambda)$ is rigid.*

We will be able to derive easily from this a compatibility condition for a pair of Verma modules $M(w.\lambda)$ and $M(ws_\alpha.\lambda)$ which can be stated as follows:

COROLLARY 1. — *Let y and w be elements of $\mathcal{W}_\lambda$ with $y \leqq w$ and let α in B_λ satisfy $ws_\alpha > w$.*

(i) *If* $ys_\alpha > y$, *then* $(\mathrm{soc}_i \mathrm{M}(ws_\alpha.\lambda) : \mathrm{L}(y.\lambda)) = (\mathrm{soc}_i \mathrm{M}(w.\lambda) : \mathrm{L}(y.\lambda))$.

(ii) *If* $ys_\alpha < y$, *then*

$(\mathrm{U}_\alpha \mathrm{soc}_i \mathrm{M}(w.\lambda) : \mathrm{L}(y.\lambda)) = (\mathrm{soc}_{i+1} \mathrm{M}(ws_\alpha.\lambda) : \mathrm{L}(y.\lambda)) + (\mathrm{soc}_{i-1} \mathrm{M}(w.\lambda) : \mathrm{L}(y.\lambda))$.

The recursion relations are precisely the ones obtained by Gabber and Joseph for the Jantzen filtration [10], and one can now follow their proof to obtain the second main result:

THEOREM 2. — *Let* λ *be an antidominant, regular weight for which* $\mathcal{O}^\lambda$ *satisfies Vogan's conjecture. Let* y, w *be elements of* $\mathcal{W}_\lambda$ *with* $y \leqq w$. *Then*

$$\mathrm{P}_{w_\lambda w, w_\lambda y}(q) = \sum_j (\mathrm{soc}_{l(y)+1+2j} \mathrm{M}(w.\lambda) : \mathrm{L}(y.\lambda))\, q^j.$$

In view of Theorem 1, this could be rewritten as

$$\mathrm{P}_{w_\lambda w, w_\lambda y}(q) = \sum_j (\mathrm{rad}_{l(w)-l(y)-2j} \mathrm{M}(w.\lambda) : \mathrm{L}(y.\lambda))\, q^j.$$

Again following Gabber and Joseph, we can obtain as a consequence that the socle filtration of a Verma module satisfies the Jantzen sum formula, which Jantzen proved for the Jantzen filtration:

COROLLARY 2. — *Given* y, w *in* $\mathcal{W}_\lambda$, *we have*

$$\sum_{j=1}^{l(w)} (\mathrm{soc}^j \mathrm{M}(w.\lambda) : \mathrm{L}(y.\lambda)) = \sum_{\substack{\alpha \in \mathrm{R}^+ \\ s_\alpha w < w}} (\mathrm{M}(s_\alpha w.\lambda) : \mathrm{L}(y.\lambda)).$$

Another important consequence of the Jantzen conjecture is the description of Ext^1 between two simple modules. (It is to be understood that all extensions are within the category $\mathcal{O}^\lambda$.) Obtained by Gabber and Joseph in [10] (and discussed also in [1] and [13]), it can now be proved without the Jantzen conjecture.

COROLLARY 3. — *Given* y *and* w *in* W *with* $y < w$,

$$\dim \mathrm{Ext}^1(\mathrm{L}(y.\lambda), \mathrm{L}(w.\lambda)) = \dim \mathrm{Ext}^1(\mathrm{L}(w.\lambda), \mathrm{L}(y.\lambda)) = \mu(y, w).$$

As A. Joseph pointed out to me, the version of Theorem 2 proved by Gabber and Joseph under the assumption of the Jantzen conjecture provides combinatorial information on Kazhdan-Lusztig polynomials, and this information is apparently not known to be accessible directly from the definition of the polynomials. Such information can be deduced as well from the theorems of this paper. For example, we have:

COROLLARY 4. — *Let* x, y, z *be elements of* $\mathcal{W}_\lambda$ *with* $x \leqq y \leqq z$. *Then* $\mathrm{P}_{x,z}(q) - \mathrm{P}_{y,z}(q)$ *has non-negative coefficients.*

As another example, Corollary 2 is easily translated into a statement about Kazhdan-Lusztig polynomials via Theorem 2. A version of the resulting statement with $q=1$ appears in [9].

Since the weight filtration on a Varma module is a Loewy filtration, we also recover:

COROLLARY 5. — *The socle and weight filtrations on a Verma module coincide.*

The next few results deal with the projective indecomposable modules in $\mathcal{O}^{\lambda}$. A description of the antidominant projective $P(\lambda)$ was obtained in [13] under the assumption of Jantzen's conjecture, and more directly in the earlier version of this paper. But Theorem 1 and the more elementary results of [13] have it as an immediate consequence:

COROLLARY 6. — *The indecomposable projective* $P(\lambda)$ *is rigid, and the layers in its socle filtration are given by the formula*

$$\operatorname{soc}_r P(\lambda) = \bigoplus_{w \in \mathcal{W}_\lambda} \operatorname{soc}_{r-2l(w_\lambda w)} M(w.\lambda).$$

The formula of Corollary 6 can be rephrased, in view of rigidity, in terms of the radical filtration. This version can then be extended to a hypothetical description of the radical filtration of any indecomposable projective module $P(w.\lambda)$, which can be proved in general, using Theorem 1:

COROLLARY 7. — *Let* $w \in \mathcal{W}_\lambda$. *Then*

$$(\operatorname{rad}_r P(w.\lambda) : L(z.\lambda)) = \sum_{y \in \mathcal{W}_\lambda} \sum_{i \geqq 0} (\operatorname{rad}_i (M(y.\lambda) : L(w.\lambda)) (\operatorname{rad}_{r-i} M(y.\lambda) : L(z.\lambda)).$$

Moreover, the module $P(w.\lambda)$ *is rigid if and only if* $(M(w_\lambda.\lambda) : L(w.\lambda)) = 1$.

The formula of Corollary 7 has the following natural structural explanation. The module $P(w.\lambda)$ has a Verma flag with each Verma module $M(y.\lambda)$ occurring as a subquotient $(M(y.\lambda) : L(w.\lambda))$ times by BGG reciprocity. For each occurrence of $L(w.\lambda)$ as a composition factor of $\operatorname{rad}_i M(y.\lambda)$, we may form a quotient Q of $M(y.\lambda)$ of Loewy length $i+1$ with $L(w.\lambda)$ as socle. The dual module DQ has simple cap $L(w.\lambda)$ and simple socle $L(y.\lambda)$, so is a homomorphic image of $P(w.\lambda)$ of Loewy length $i+1$. There is a subquotient $L(y.\lambda)$ of $P(w.\lambda)$ in $\operatorname{rad}_i P(w.\lambda)$ which corresponds under this map of $P(w.\lambda)$ to DQ to the socle of DQ and which may be viewed as the cap of a copy of $M(y.\lambda)$ in a Verma flag for $P(w.\lambda)$. Thus each copy of $L(y.\lambda)$ in $\operatorname{rad}_i M(w.\lambda)$ produces a Verma flag factor of $P(w.\lambda)$ isomorphic to $M(y.\lambda)$, with cap in $\operatorname{rad}_i P(w.\lambda)$. It is then natural to expect that for each $L(z.\lambda)$ in $\operatorname{rad}_{r-i} M(y.\lambda)$, this Verma flag factor lays down a copy of $L(z.\lambda)$ in $\operatorname{rad}_r P(w.\lambda)$. This picture of the structure of $P(w.\lambda)$ leads to the numerical formula of Corollary 7.

The formula can also be re-interpreted as a filtered generalization (or q-analogue) of BGG reciprocity. Let us introduce some notation in order to state this. For any polynomial $F(q)$, let $F(q)^\wedge$ denote the polynomial $F(q^{-1})$. Write $Q_{y,w}(q)$ for the Kazhdan-Lusztig polynomial $P_{w_\lambda w, w_\lambda y}(q)$. By the remark after Theorem 2, we have

$$Q_{y,w}(q) = \sum_j (\operatorname{rad}_{l(w)-l(y)-2j} M(w.\lambda) : L(y.\lambda))\, q^j,$$

or

$$q^{(l(w)-l(y))/2}\, Q_{y,w}(q)^{\wedge} = \sum_j (\mathrm{rad}_j\, M(w.\lambda) : L(y.\lambda))\, q^{j/2}.$$

Let's also introduce the polynomial $U_{y,w}(q)$ which is defined to be

$$\sum_i (\mathrm{rad}_i\, P(w.\lambda) : L(y.\lambda))\, q^{i/2}.$$

We may introduce two square matrices $\mathscr{C}(q)$ and $\mathscr{D}(q)$, each with $|\mathscr{W}_\lambda|$ rows and columns and with polynomial entries: $\mathscr{C}(q)$ has as $y-w$ entry the polynomial $U_{y,w}(q)$ and $\mathscr{D}(q)$ has as $y-w$ entry the polynomial $q^{(l(w)-l(y))/2}\, Q_{y,w}(q)^{\wedge}$. Notice that $\mathscr{C}(1)$ is what is usually called the Cartan matrix of the category $\mathscr{O}^\lambda$, with $y-w$ entry the multiplicity $(P(w.\lambda) : L(y.\lambda))$, while $\mathscr{D}(1)$ has $y-w$ entry $(M(w.\lambda) : L(y.\lambda))$. BGG reciprocity states that

$$\mathscr{C}(1) = {}^t\mathscr{D}(1)\, \mathscr{D}(1),$$

where ^{t} denotes transpose. Corollary 7 may be rewritten to yield the following generalization:

COROLLARY 8. — $\mathscr{C}(q) = {}^t\mathscr{D}(q)\, \mathscr{D}(q)$.

The referee pointed out that the theorems of this paper also have an application to the theory of primitive ideals of $U(\mathfrak{g})$. Let us recall some of this theory. Given $w \in \mathscr{W}_\lambda$, let $J(w.\lambda)$ be the annihilator of $L(w.\lambda)$ in the enveloping algebra $U(\mathfrak{g})$. By a theorem of Duflo, the set X of ideals $\{J(w.\lambda) : w \in \mathscr{W}_\lambda\}$ is the complete set of primitive ideals in $U(\mathfrak{g})$ containing the minimal primitive ideal $J(\lambda)$. The surjection π from $\mathscr{W}_\lambda$ to X sending w to $J(w.\lambda)$ generally is not injective, but there is a distinguished subset Σ_λ^0 of involutions of $\mathscr{W}_\lambda$, known as the Duflo set, such that the restriction of π to Σ_λ^0 is a bijection. For σ in Σ_λ^0, let $C_\sigma = \{w \in \mathscr{W}_\lambda : J(w.\lambda) = J(\sigma.\lambda)\}$. An important consequence of the Kazhdan-Lusztig conjecture is that the sets C_σ are determined by the Kazhdan-Lusztig polynomials [16]. An argument of Joseph in section 4.9 of [15] essentially proves the following statement: given a Loewy filtration $0 \subset M_1 \subset \ldots \subset M_t = M(w_\lambda.\lambda)$ of the dominant Verma module $M(w_\lambda.\lambda)$, there is an i such that $(M(w_\lambda.\lambda)/M_i : L(\sigma.\lambda)) = 1$ but $(M(w_\lambda.\lambda)/M_i : L(w.\lambda)) = 0$ for $w \in C_\sigma$ with $w \neq \sigma$. Thus, the elements of $\mathscr{W}_\lambda$ which lie in Σ_λ^0 are determined by the multiplicities of composition factors in layers of a Loewy filtration of $M(w_\lambda.\lambda)$. By Theorems 1 and 2, these multiplicities are determined by the coefficients of the Kazhdan-Lusztig polynomials. Thus we obtain our final application: Σ_λ^0 can be described in terms of Kazhdan-Lusztig polynomials. Joseph had proved this in [15] with respect to the Jantzen filtration, under the assumption of the Jantzen conjecture.

1.4. Some of the results and arguments of this paper are discussed in a more general, axiomatic setting in [14]. Once one has an appropriate analogue of Corollary 1 for a suitable Loewy filtration, analogues of Theorem 2 and Corollary 2 can be obtained. In particular, this axiomatic setting is applicable to generalized Verma modules, for which the argument for Theorem 1 can be extended to show that the weight filtrations coincide with radical filtrations. (This is also a consequence of Casian's more general result.)

Corollary 1 and Theorem 2 were announced at an Oberwolfach meeting on rings and modules in May 1986, and contained in a preprint circulated in September 1986. As noted, the proof there of Corollary 1 depended on an auxiliary filtration constructed from $P(\lambda)$ and was more circuitous.

I would like to thank David Collingwood for helpful discussions on the results of this paper and the NSF for partial support during the preparation of this paper. In addition, I thank the referee for his considerable patience in reading with care the several versions of this paper. I have incorporated a number of his suggestions into this final version.

2. The proof of rigidity

2.1. For the proof of Theorem 1, it is convenient to rephrase rigidity in terms of the notion of capitals.

LEMMA 1. — *Let* M *be a finite length module of Loewy length* r *and let* L *be a composition factor of* M. *Then*

(i) $(\mathrm{cap}^i M : L) + (\mathrm{soc}^{r-i} M : L) \geqq (M : L)$ *for any* i *between* 0 *and* r, *and*

(ii) L *is rigidly placed in* M *if and only if* $(\mathrm{cap}^i M : L) + (\mathrm{soc}^{r-i} M : L) = (M : L)$ *for all* i *between* 0 *and* r.

Proof. — By definition of capitals, we have

$$(\mathrm{cap}^i M : L) + (\mathrm{soc}^{r-i} M : L) = (M : L) - (\mathrm{rad}^i M : L) + (\mathrm{soc}^{r-i} M : L).$$

But since $\mathrm{soc}^{r-i} M \supseteqq \mathrm{rad}^i M$, we have $(\mathrm{soc}^{r-i} M : L) - (\mathrm{rad}^i M : L) \geqq 0$, and L is rigidly placed in M by definition if equality holds for all i. The result follows immediately.

2.2. The key to the proof of Theorem 1 is to relate rigidity of the three modules $M(w.\lambda)$, $M(ws_\alpha.\lambda)$, and $\theta_\alpha M(w.\lambda)$, for w in $\mathscr{W}_\lambda$. This is done in the following two lemmas.

LEMMA 1. — *Let* $w \in \mathscr{W}_\lambda$ *and* $\alpha \in B_\lambda$ *satisfy* $ws_\alpha > w$. *Let* y *be an element of* $\mathscr{W}_\lambda$. *If* $L(y.\lambda)$ *is rigidly placed in* $M(w.\lambda)$ *and* $M(ws_\alpha.\lambda)$, *then it is rigidly placed in* $\theta_\alpha M(w.\lambda)$.

Proof. — By the Loewy length results reviewed in 1.2, the Loewy length of $\theta_\alpha M(w.\lambda)$ is $l(w)+3$, while $M(ws_\alpha.\lambda)$ is a submodule of Loewy length $l(w)+2$ and $M(w.\lambda)$ is a homomorphic image of Loewy length $l(w)+1$ [13]. One can describe the extension $\theta_\alpha M(w.\lambda)$ of $M(ws_\alpha.\lambda)$ by $M(w.\lambda)$ a little more precisely. The socle of any Verma module $M(z.\lambda)$ is $L(\lambda)$ and $(M(z.\lambda) : L(\lambda)) = 1$. Thus $(\theta_\alpha M(w.\lambda) : L(\lambda)) = 2$ and, since $\theta_\alpha L(\lambda)$ is a submodule of $\theta_\alpha M(w.\lambda)$, the two appearances of $L(\lambda)$ in $\theta_\alpha M(w.\lambda)$ are as its socle and in $\mathrm{soc}_3\, \theta_\alpha M(w.\lambda)$. We may deduce that $\mathrm{soc}^2\, \theta_\alpha M(w.\lambda)$ is in the kernel of the map of $\theta_\alpha M(w.\lambda)$ onto $M(w.\lambda)$.

Let $r = l(w)+3$ and choose i between 0 and r. As an elementary consequence of the definitions of socle and capital, $\mathrm{soc}^{r-i}\, \theta_\alpha M(w.\lambda) \cap M(ws_\alpha.\lambda) = \mathrm{soc}^{r-i} M(ws_\alpha.\lambda)$ and

$\operatorname{cap}^i \theta_\alpha M(w.\lambda)$ maps onto $\operatorname{cap}^i M(w.\lambda)$. The observation at the end of the last paragraph implies that $\operatorname{soc}^{r-i} \theta_\alpha M(w.\lambda)/(\operatorname{soc}^{r-i} \theta_\alpha M(\dot{w}.\lambda) \cap M(ws_\alpha.\lambda))$ has Loewy length at most $r-i-2$, which means that it lies in the submodule $\operatorname{soc}^{r-i-2} M(w.\lambda)$ of $M(w.\lambda)$. Also, since $M(ws_\alpha.\lambda)$ lies in $\operatorname{rad} \theta_\alpha M(w.\lambda)$, the image of $M(ws_\alpha.\lambda)$ in $\operatorname{cap}^i \theta_\alpha M(w.\lambda)$ has Loewy length at most $i-1$. Thus, $\operatorname{cap}^{i-1} M(ws_\alpha.\lambda)$ maps onto the image of $M(ws_\alpha.\lambda)$ in $\operatorname{cap}^i \theta_\alpha M(w.\lambda)$. This yields the two inequalities

$$(1)\quad (\operatorname{soc}^{r-i} \theta_\alpha M(w.\lambda) : L(y.\lambda)) \leqq (\operatorname{soc}^{r-i} M(ws_\alpha.\lambda) : L(y.\lambda)) + (\operatorname{soc}^{r-i-2} M(w.\lambda) : L(y.\lambda)),$$

and

$$(2)\quad (\operatorname{cap}^i \theta_\alpha M(w.\lambda) : L(y.\lambda)) \leqq (\operatorname{cap}^{i-1} M(ws_\alpha.\lambda) : L(y.\lambda)) + (\operatorname{cap}^i M(w.\lambda) : L(y.\lambda)).$$

Let us assume that $L(y.\lambda)$ is not rigidly placed in $\theta_\alpha M(w.\lambda)$. If it is not rigidly placed in $M(ws_\alpha.\lambda)$, the lemma is proved. Assuming instead that it is rigidly placed in $M(ws_\alpha.\lambda)$, what we must show is that it is not rigidly placed in $M(w.\lambda)$. We have the trivial equality

$$(3)\qquad (\theta_\alpha M(w.\lambda) : L(y.\lambda)) = (M(ws_\alpha.\lambda) : L(y.\lambda)) + (M(w.\lambda) : L(y.\lambda)).$$

By Lemma 2.1 there is an i such that

$$(4)\quad (\theta_\alpha M(w.\lambda) : L(y.\lambda)) < (\operatorname{cap}^i \theta_\alpha M(w.\lambda) : L(y.\lambda)) + (\operatorname{soc}^{r-i} \theta_\alpha M(w.\lambda) : L(y.\lambda)).$$

Inequalities (1), (2), and (4) then yield:

$$\begin{aligned}(5)\quad (\theta_\alpha M(w.\lambda) : L(y.\lambda)) &< (\operatorname{cap}^{i-1} M(ws_\alpha.\lambda) : L(y.\lambda)) + (\operatorname{soc}^{r-i}(M(ws_\alpha.\lambda) : L(y.\lambda)) \\ &\quad + (\operatorname{cap}^i M(w.\lambda) : L(y.\lambda)) + (\operatorname{soc}^{r-i-2} M(w.\lambda) : L(y.\lambda)).\end{aligned}$$

By the rigidity of $M(ws_\alpha.\lambda)$, we may rewrite the right side of the inequality as

$$(6)\quad (M(ws_\alpha.\lambda) : L(y.\lambda)) + (\operatorname{cap}^i M(w.\lambda) : L(y.\lambda)) + (\operatorname{soc}^{r-i-2} M(w.\lambda) : L(y.\lambda)).$$

Subtracting $(M(ws_\alpha.\lambda) : L(y.\lambda))$ from both sides of (5), taking (3) and (6) into account, we obtain the inequality

$$(7)\quad (M(w.\lambda) : L(y.\lambda)) < (\operatorname{cap}^i M(w.\lambda) : L(y.\lambda)) + (\operatorname{soc}^{r-i-2} M(w.\lambda) : L(y.\lambda)).$$

By Lemma 2.1, we conclude that $L(y.\lambda)$ is not rigidly placed in $M(w.\lambda)$, proving the lemma.

Remark. — Suppose, as in the preceding Lemma, that $L(y.\lambda)$ is rigidly placed in $M(w.\lambda)$ and $M(ws_\alpha.\lambda)$ [and therefore in $\theta_\alpha M(w.\lambda)$ as well]. Then Lemma 1 of 2.1 implies that the terms on each side of the inequality symbols in (1) and (2) have as their sums the respective terms on either side of the equality in (3). This forces the inequalities in (1) and (2) to be equalities.

LEMMA 2. — *Given y, w in $\mathscr{W}_\lambda$ and α in B_λ, with $ys_\alpha > y$ and $ws_\alpha > w$, if* $L(y.\lambda)$ *is rigidly placed in* $M(w.\lambda)$, *then it is rigidly placed in* $M(ws_\alpha.\lambda)$.

Proof. — The unique up to scalars embedding of $M(w.\lambda)$ in $M(ws_\alpha.\lambda)$ has its image in $\operatorname{rad} M(ws_\alpha.\lambda)$. Thus for any i, the image of $M(w.\lambda)$ in $\operatorname{cap}^{i+1} M(ws_\alpha.\lambda)$ has Loewy length at most i, and must be a homomorphic image of $\operatorname{cap}^i M(w.\lambda)$. The multiplicity of $L(y.\lambda)$ in both $M(w.\lambda)$(and $M(ws_\alpha.\lambda)$ is the same, so all the appearances of $L(y.\lambda)$ in $\operatorname{cap}^{i+1} M(ws_\alpha.\lambda)$ actually must be in the image of $\operatorname{cap}^i M(w.\lambda)$. Thus we have the inequality

$$(\operatorname{cap}^i M(w.\lambda) : L(y.\lambda)) \geqq (\operatorname{cap}^{i+1} M(ws_\alpha.\lambda) : L(y.\lambda)).$$

Let $r = l(w) + 1$, the Loewy length of $M(w.\lambda)$. Using Lemma 2.1 (i) and the inequality, we obtain:

$$\begin{aligned}(M(w.\lambda) : L(y.\lambda)) &= (\operatorname{soc}^{r-i} M(w.\lambda) : L(y.\lambda)) + (\operatorname{cap}^i M(w.\lambda) : L(y.\lambda))\\ &\geqq (\operatorname{soc}^{r-i} M(ws_\alpha.\lambda) : L(y.\lambda)) + (\operatorname{cap}^{i+1} M(ws_\alpha.\lambda) : L(y.\lambda))\\ &\geqq (M(ws_\alpha.\lambda) : L(y.\lambda)).\end{aligned}$$

Since the two extreme numbers are equal, we have equality throughout. The last equality proves that $L(y.\lambda)$ is rigidly placed in $M(ws_\alpha.\lambda)$.

2.3. The proof of Theorem 1 now follows easily from the lemmas of 2.2.

Proof of Theorem 1. — If the theorem is not true, there are y and w in $\mathscr{W}_\lambda$ such that $L(y.\lambda)$ is not rigidly placed in $M(w.\lambda)$. Let us suppose that $y < w_\lambda$. Then there is some α in B_λ for which $ys_\alpha > y$. By Lemma 2 of 2.2, we may assume that w satisfies $ws_\alpha > w$. Let $r = l(w) + 1$ and choose i so that

$$(8) \qquad (\operatorname{soc}^{r-i} M(w.\lambda) : L(y.\lambda)) + (\operatorname{cap}^i M(w.\lambda) : L(y.\lambda)) > (M(w.\lambda) : L(y.\lambda)).$$

The module $\theta_\alpha \operatorname{soc}^{r-i} M(w.\lambda)$ is a submodule of $\theta_\alpha M(w.\lambda)$ with Loewy length at most $r-i+2$, and $L(ys_\alpha.\lambda)$ cannot be in the cap of $\theta_\alpha \operatorname{soc}^{r-i} M(w.\lambda)$, since it is annihilated by θ_α. Therefore we obtain the inequality

$$(9) \qquad (\theta_\alpha \operatorname{soc}^{r-i} M(w.\lambda) : L(ys_\alpha.\lambda)) \leqq (\operatorname{soc}^{r-i+1} \theta_\alpha M(w.\lambda) : L(ys_\alpha.\lambda)).$$

Similarly, $\theta_\alpha \operatorname{cap}^i M(w.\lambda)$ has Loewy length at most $i+2$, and $L(ys_\alpha.\lambda)$ cannot be in its socle, yielding the inequality

$$(10) \qquad (\theta_\alpha \operatorname{cap}^i M(w.\lambda) : L(ys_\alpha.\lambda)) \leqq (\operatorname{cap}^{i+1} \theta_\alpha M(w.\lambda) : L(ys_\alpha.\lambda)).$$

Given any module M in $\mathcal{O}^\lambda$, the occurrences of $L(ys_\alpha.\lambda)$ in $\theta_\alpha M$ can be accounted for as follows: there is one for each occurrence of $L(y.\lambda)$ in M; in addition, any other occurrences arise in the subquotients $\theta_\alpha L(z.\lambda)$, for each $L(z.\lambda)$ in M with $z > y$ and $zs_\alpha > z$. Thus we obtain the formula:

$$(\theta_\alpha M : L(ys_\alpha.\lambda)) = (M : L(y.\lambda)) + \sum_{\substack{z \in \mathscr{W}_\lambda \\ z > y}} (M : L(z.\lambda))(\theta_\alpha L(z.\lambda) : L(ys_\alpha.\lambda)).$$

Combining this formula for various M's with (8) to (10), we obtain ($\ddagger$):

$$(\operatorname{cap}^{i+1}\theta_\alpha \mathrm{M}(w.\lambda): \mathrm{L}(ys_\alpha.\lambda))+(\operatorname{soc}^{r-i+1}\theta_\alpha \mathrm{M}(w.\lambda): \mathrm{L}(ys_\alpha.\lambda))$$

$$\geqq (\theta_\alpha \operatorname{cap}^i \mathrm{M}(w.\lambda): \mathrm{L}(ys_\alpha.\lambda))+(\theta_\alpha \operatorname{soc}^{r-i}\mathrm{M}(w.\lambda): \mathrm{L}(ys_\alpha.\lambda))$$

$$=(\operatorname{cap}^i \mathrm{M}(w.\lambda): \mathrm{L}(y.\lambda))+(\operatorname{soc}^{r-i}\mathrm{M}(w.\lambda): \mathrm{L}(y.\lambda))$$

$$+\sum_{\substack{z\in\mathscr{W}_\lambda\\ z>y}}((\operatorname{cap}^i \mathrm{M}(w.\lambda): \mathrm{L}(z.\lambda))+(\operatorname{soc}^{r-i}\mathrm{M}(w.\lambda): \mathrm{L}(z.\lambda)))(\theta_\alpha \mathrm{L}(z.\lambda): \mathrm{L}(ys_\alpha.\lambda))$$

$$\geqq (\operatorname{cap}^i \mathrm{M}(w.\lambda): \mathrm{L}(y.\lambda))+(\operatorname{soc}^{r-i}\mathrm{M}(w.\lambda): \mathrm{L}(y.\lambda))$$

$$+\sum_{\substack{z\in\mathscr{W}_\lambda\\ z>y}}(\mathrm{M}(w.\lambda): \mathrm{L}(z.\lambda))(\theta_\alpha \mathrm{L}(z.\lambda): \mathrm{L}(ys_\alpha.\lambda))$$

$$>(\mathrm{M}(w.\lambda): \mathrm{L}(y.\lambda)) \quad +\sum_{\substack{z\in\mathscr{W}_\lambda\\ z>y}}(\mathrm{M}(w.\lambda): \mathrm{L}(z.\lambda))(\theta_\alpha \mathrm{L}(z.\lambda): \mathrm{L}(ys_\alpha.\lambda))$$

$$=(\theta_\alpha \mathrm{M}(w.\lambda): \mathrm{L}(ys_\alpha.\lambda)).$$

This proves that $\mathrm{L}(ys_\alpha.\lambda)$ is not rigidly placed in $\theta_\alpha \mathrm{M}(w.\lambda)$. It follows by Lemma 1 of 2.2 that $\mathrm{L}(ys_\alpha.\lambda)$ is not rigidly placed in either $\mathrm{M}(w.\lambda)$ or $\mathrm{M}(ws_\alpha.\lambda)$.

Continuing this argument $l(w_\lambda)-l(y)$ times, we eventually obtain that $\mathrm{L}(w_\lambda.\lambda)$ is not rigidly placed in some Verma module. But the only Verma module in which it occurs as a composition factor is $\mathrm{M}(w_\lambda.\lambda)$, and we are forced to conclude that $\mathrm{L}(w_\lambda.\lambda)$ is not rigidly placed in $\mathrm{M}(w_\lambda.\lambda)$. This is a contradiction, since $\mathrm{L}(w_\lambda.\lambda)$ occurs just once in $\mathrm{M}(w_\lambda.\lambda)$, as the simple cap, trivially implying that it is rigidly placed. The contradiction completes the proof of the theorem.

Remark. — We may deduce a little more from the proof of the theorem. Suppose that $\mathrm{M}(w.\lambda)$ and $\mathrm{M}(ws_\alpha.\lambda)$ are rigid. Then (8) is an equality and the one strict inequality in the chain ($\ddagger$) of relations may be replaced by an equality. By Lemma 1, the module $\theta_\alpha \mathrm{M}(w.\lambda)$ is also rigid. Thus the end terms of ($\ddagger$) are equal. This forces all the relations to be equalities. We may conclude, using the first of these equalities, that formulas (9) and (10) are themselves equalities.

2.4. The results of 2.3 yield a short proof of Corollary 1.

Proof of Corollary 1. — Part (i) is automatic, since $(\mathrm{M}(ws_\alpha.\lambda): \mathrm{L}(y.\lambda))=(\mathrm{M}(w.\lambda): \mathrm{L}(y.\lambda))$ and $\mathrm{M}(w.\lambda)$ is a submodule of $\mathrm{M}(ws_\alpha.\lambda)$. Thus, let us assume that $ys_\alpha<y$. Let $r=l(w)+3$. By the remark after the proof of Theorem 1, formula (9) is an equality. Also, the remark after the proof of Lemma 1 in 2.2 yields equality in formula (1). Combining the equalized versions of (9) and (1), with a change of indexing and parameters, we obtain:

$$(11)\quad (\theta_\alpha \operatorname{soc}^i \mathrm{M}(w.\lambda): \mathrm{L}(y.\lambda))=(\operatorname{soc}^{i+1}\theta_\alpha \mathrm{M}(w.\lambda): \mathrm{L}(y.\lambda))$$
$$=(\operatorname{soc}^{i+1}\mathrm{M}(ws_\alpha.\lambda): \mathrm{L}(y.\lambda))+(\operatorname{soc}^{i-1}\mathrm{M}(w.\lambda): \mathrm{L}(y.\lambda)).$$

Proceeding by induction on i, we obtain from this that:

(12) $(\theta_\alpha \operatorname{soc}_i M(w.\lambda) : L(y.\lambda)) = (\operatorname{soc}_{i+1} M(ws_\alpha.\lambda) : L(y.\lambda)) + (\operatorname{soc}_{i-1} M(w.\lambda) : L(y.\lambda))$.

Since for any z all occurrences of $L(y.\lambda)$ in $\theta_\alpha L(z.\lambda)$ occur in the subquotient $U_\alpha L(z.\lambda)$, this proves (ii) and the corollary.

3. Theorem 2 and three corollaries

3.1. We first prove Theorem 2 and Corollary 2:

Proof of Theorem 2 *and Corollary* 2. — Theorem 2 follows from Corollary 1 by the same argument used by Gabber and Joseph ([10], 4.9) to prove the analogous result for the Jantzen filtration. Parts (i) and (ii) of the corollary play the role of formulas 4.3 (v) and (4.8) (iii) of [10] in the proof. Corollary 2 follows from Theorem 2 by differentiation, as in [10] (4.10). For a further discussion of the arguments used for these two results, with applications to generalized Verma modules, *see* my paper [14].

3.2. Corollaries 3 through 5 follow easily from Theorem 2, as we now see.

Proof of Corollary 3. — Because $\mathcal{O}^\lambda$ has a duality functor which fixes simples, it is obvious that $\dim \operatorname{Ext}^1(L(y.\lambda), L(w.\lambda)) = \dim \operatorname{Ext}^1(L(w.\lambda), L(y.\lambda))$. It is a standard fact that $\dim \operatorname{Ext}^1(L(w.\lambda), L(y.\lambda)) = (\operatorname{rad}_1 M(w.\lambda), L(y.\lambda))$. Basically this is because any non-trivial extension of $L(y.\lambda)$ by $L(w.\lambda)$ is a homomorphic image of $M(w.\lambda)$. By Theorem 2, $\mu(y, w) = (\operatorname{rad}_1 M(w.\lambda), L(y.\lambda))$, proving the corollary.

Proof of Corollary 4. — Using the notation introduced in 1.3, we may prove the equivalent statement that $Q_{x,w}(q) - Q_{x,y}(q)$ has non-negative coefficients for $x \leqq y \leqq w$. By Theorem 2, this is the statement that

$$(\operatorname{soc}^j M(w.\lambda) : L(x.\lambda)) - (\operatorname{soc}^j M(y.\lambda) : L(x.\lambda)) \geqq 0.$$

But $M(y.\lambda)$ embeds as a submodule of $M(w.\lambda)$, so

$$\operatorname{soc}^j M(y.\lambda) = M(y.\lambda) \cap \operatorname{soc}^j M(w.\lambda).$$

The inequality follows immediately from this.

Proof of Corollary 5. — The basic property of a weight filtration on a Verma module is that the successive quotients are semisimple. Moreover, the multiplicities of composition factors in these layers are counted by the coefficients of Kazhdan-Lusztig polynomials in exactly the same way as multiplicities in layers of the socle filtration are counted in Theorem 2. (*See* for instance [8], Theorem 4.6 and Proposition 4.15.) In particular, the weight and socle filtrations have the same length. Using Theorems 1 or 2, we may thus conclude that the two filtrations coincide.

4. Radical filtrations on indecomposable projective modules

4.1. In this section we turn to the proofs of Corollaries 6 and 7. Recall from the discussion at the end of 1.3 that Corollary 8 is a refinement of Corollary 7, requiring no additional proof. Corollary 6 follows easily from Theorem 1 using results of [13].

Proof of Corollary 6. — As discussed at the end of section 1.2, the projective module $P(\lambda)$ has a Verma flag with each $M(w.\lambda)$ occurring once as a factor, for w in $\mathscr{W}_\lambda$. Let $0=M_0 \subset M_1 \subset \ldots \subset M_t = P(\lambda)$ be a Verma flag for $P(\lambda)$ of the form described in 1.2. Each Verma module $M(w.\lambda)$, which has Loewy length $l(w)+1$, occupies the $l(w)+1$ layers from $2l(w_\lambda)-2l(w)+1$ to $2l(w_\lambda)-l(w)+1$ in the socle filtration of $P(\lambda)$. To be more precise, let $n(w)=2l(w_\lambda)-2l(w)$ and choose r with $M_{r+1}/M_r \cong M(w.\lambda)$. Then

$$(\mathrm{soc}_{n(w)+1} M_{r+1} : L(\lambda)) = (\mathrm{soc}_{n(w)+1} M_r : L(\lambda)) + 1$$

([13], 5.1, Proposition 2); in effect, this means that the copy of $L(\lambda)$ contributed to M_{r+1} or $P(\lambda)$ by the socle of the Verma flag factor $M(w.\lambda)$ occurs in layer $n(w)+1$ of the socle filtration of M_{r+1} or $P(\lambda)$. It follows, since $ll\, M(w.\lambda)=l(w)+1$, that $L(w.\lambda)$ occurs once as a composition factor in $M_{r+1}/(M_r+\mathrm{soc}^{n(w)+l(w)} M_{r+1})$. But

$$(P(\lambda)/\mathrm{soc}^{n(w)+l(w)} P(\lambda) : L(w.\lambda)) = 1 = (\mathrm{soc}_{n(w)+l(w)+1} P(\lambda) : L(w.\lambda))$$

by [13], 5.1, Proposition 1. Thus the copy of $L(w.\lambda)$ contributed to $P(\lambda)$ by the cap of the Verma flag factor $M(w.\lambda)$ occurs in layer $n(w)+l(w)+1$ of the socle filtration of M_{r+1} or $P(\lambda)$.

We may conclude, since $M(w.\lambda)$ is rigid, that $\mathrm{soc}_i M(w.\lambda)$ appears in layer $n(w)+i$ of the socle filtration of $P(\lambda)$. This proves the formula of Corollary 6. Rigidity can now be proved in one of two ways. First, versions of the above formulas hold for the radical filtrations, as proved in [13], so one may repeat the argument to get a formula for $\mathrm{rad}_{2l(w_\lambda)+1-r} P(\lambda)$. But by rigidity of the Verma modules, the formulas will coincide. Alternatively, one can use the self-duality of $P(\lambda)$ to deduce rigidity as in [13].

4.2. In order to prove Corollary 7, we need a couple of preliminary lemmas, of interest in their own right. All homomorphisms and Hom spaces below are to be understood as consisting of morphisms in the category $\mathcal{O}^\lambda$; equivalently, they are $\mathfrak{g}$-module homomorphisms.

LEMMA 1. — *Let y and w be elements of $\mathscr{W}_\lambda$. For any nonnegative integer r,*

$$(\mathrm{rad}_r P(w.\lambda) : L(y.\lambda)) = (\mathrm{rad}_r P(y.\lambda) : L(w.\lambda)).$$

Proof. — For a module M and a composition factor L it follows from the definitions that $(\mathrm{rad}_r M : L) = (\mathrm{cap}^{r+1} M : L) - (\mathrm{cap}^r M : L)$. Therefore the lemma will follow by induction if we show for all r that

$$(\mathrm{cap}^r P(w.\lambda) : L(y.\lambda)) = (\mathrm{cap}^r P(y.\lambda) : L(w.\lambda)). \tag{13}$$

The duality functor reverses the socles and capitals of a module. More precisely, for a module M, we have $\mathrm{soc}^i \mathrm{DM} = \mathrm{D}\,\mathrm{cap}^i \mathrm{M}$. Since D fixes simples and therefore preserves composition factor multiplicities, it is equivalent to show that

$$(14) \qquad (\mathrm{cap}^r \mathrm{P}(w.\lambda) : \mathrm{L}(y.\lambda)) = (\mathrm{soc}^r \mathrm{I}(y.\lambda) : \mathrm{L}(w.\lambda))$$

for all r.

Given any z in $\mathscr{W}_\lambda$ and any finite length module M in $\mathcal{O}^\lambda$, it is a standard fact easily proved by induction on length that

$$(15) \qquad \dim \mathrm{Hom}(\mathrm{P}(z.\lambda), \mathrm{M}) = (\mathrm{M} : \mathrm{L}(z.\lambda)) = \dim \mathrm{Hom}(\mathrm{M}, \mathrm{I}(z.\lambda)).$$

Using (15), we may rewrite (14) as:

$$(16) \qquad \dim \mathrm{Hom}(\mathrm{cap}^r \mathrm{P}(w.\lambda), \mathrm{I}(y.\lambda)) = \dim \mathrm{Hom}(\mathrm{P}(w.\lambda), \mathrm{soc}^r \mathrm{I}(y.\lambda)).$$

But $\mathrm{cap}^r \mathrm{P}(w.\lambda)$ is the largest homomorphic image of $\mathrm{P}(w.\lambda)$ of Loewy length r and $\mathrm{soc}^r \mathrm{I}(y.\lambda)$ is the largest submodule of $\mathrm{I}(y.\lambda)$ of Loewy length r. Thus both $\mathrm{Hom}(\mathrm{cap}^r \mathrm{P}(w.\lambda), \mathrm{I}(y.\lambda))$ and $\mathrm{Hom}(\mathrm{P}(w.\lambda), \mathrm{soc}^r \mathrm{I}(y.\lambda)$ represent the subspace of $\mathrm{Hom}(\mathrm{P}(w.\lambda), \mathrm{I}(w.\lambda))$ consisting of those homomorphisms whose image has Loewy length at most r. This proves the equality (16) and the Lemma.

Remark. — Such a result was proved by Landrock for projective modules over group algebras of finite groups ([18], 9.10).

Before stating the next lemma, let us introduce some additional notation. Given w and z in $\mathscr{W}_\lambda$, let $\mathrm{P}(w.\lambda, z.\lambda)$ denote the largest homomorphic image of $\mathrm{P}(w.\lambda)$ all of whose composition actors have highest weight $y.\lambda$ with $y \leqq z$. This may also be characterized as the homomorphic image of $\mathrm{P}(w.\lambda)$ modulo the largest possible submodule in a Verma flag for $\mathrm{P}(w.\lambda)$ all of whose Verma module quotients have highest weight $\nleqq z.\lambda$. An argument along the same lines as the previous proof allows us to obtain a variation of BGG reciprocity, which will itself be reproved along the way.

LEMMA 2. — *Given w and z in $\mathscr{W}_\lambda$,*

$$\text{(i)} \qquad [\mathrm{P}(w.\lambda) : \mathrm{M}(z.\lambda)] = (\mathrm{P}(w.\lambda, z.\lambda) : \mathrm{L}(z.\lambda)) = (\mathrm{M}(z.\lambda) : \mathrm{L}(w.\lambda)).$$

More precisely,

$$\text{(ii)} \qquad (\mathrm{rad}_r \mathrm{P}(w.\lambda, z.\lambda) : \mathrm{L}(z.\lambda)) = (\mathrm{rad}_r \mathrm{M}(z.\lambda) : \mathrm{L}(w.\lambda)).$$

Proof. — (i) The first equality follows from the definition of $\mathrm{P}(w.\lambda, z.\lambda)$. It has a Verma flag containing $\mathrm{M}(z.\lambda)$ as a quotient with the same frequency as $\mathrm{P}(w.\lambda)$ does. But $z.\lambda$ is the highest weight occurring in $\mathrm{P}(w.\lambda, z.\lambda)$. Therefore all appearances of $\mathrm{L}(z.\lambda)$ are as caps of submodules isomorphic to $\mathrm{M}(z.\lambda)$ and every subquotient of the form $\mathrm{M}(z.\lambda)$ is actually a submodule, proving the first equality.

For the second equality, the proof of Lemma 1 shows that it is equivalent to prove that

$$\dim \mathrm{Hom}(\mathrm{P}(w.\lambda, z.\lambda), \mathrm{I}(z.\lambda)) = \dim \mathrm{Hom}(\mathrm{P}(w.\lambda), \mathrm{DM}(z.\lambda)).$$

The module $DM(z.\lambda)$ embeds as a submodule of $I(z.\lambda)$, and the quotient has a filtration whose quotients are dual Verma modules $DM(x.\lambda)$ with $x > z$. With respect to this embedding, we may identify Hom $(P(w.\lambda), DM(z.\lambda))$ as the subspace of $\mathrm{Hom}(P(w.\lambda), I(z.\lambda))$ consisting of those homomorphisms whose image has composition factors only of highest weights $y.\lambda$ with $y \leqq z$. By the definition of $P(w.\lambda, z.\lambda)$, we may identify $\mathrm{Hom}(P(w.\lambda, z.\lambda), I(z.\lambda))$ as the same subspace of $\mathrm{Hom}(P(w.\lambda), I(z.\lambda))$. This proves the lemma.

(ii) As in the proof of Lemma 1, we can replace rad_r in (ii) by cap^r. Then we can translate the resulting equality into the statement that

$$\dim \mathrm{Hom}(\mathrm{cap}^r P(w.\lambda, z.\lambda), I(z.\lambda)) = \dim \mathrm{Hom}(P(w.\lambda), \mathrm{soc}^r DM(z.\lambda)).$$

It is then clear from the argument of (i) that both Hom spaces are the same subspace of $\mathrm{Hom}(P(w.\lambda, z.\lambda), I(z.\lambda))$.

4.3. We can now prove Corollary 7.

Proof of Corollary 7. — Fix w in $\mathscr{W}_\lambda$. The discussion in 1.3 following the statement of Corollary 7 shows that the formula is a consequence of the following description of $P(w.\lambda)$: choose z in $\mathscr{W}_\lambda$ and a nonnegative integer i. Then for each copy of $L(w.\lambda)$ in $\mathrm{rad}_i M(z.\lambda)$, there is a Verma flag factor $M(z.\lambda)$ of $P(w.\lambda)$ which contributes to the radical filtration of $P(w.\lambda)$ by having its cap placed in the i-th layer of $P(w.\lambda)$ and any other layer $\mathrm{rad}_j M(z.\lambda)$ placed in layer $i+j$ of $P(w.\lambda)$. Notice that this is exactly what was shown for $P(\lambda)$ in the proof of Corollary 6.

By Lemma 4.2.2, we have

$$(\mathrm{rad}_i P(w.\lambda, z.\lambda) : L(z.\lambda)) = (\mathrm{rad}_i M(z.\lambda) : L(w.\lambda)).$$

Let this number be denoted by t. By the definition of $P(w.\lambda, z.\lambda)$, as noted in the proof of Lemma 4.2.2, every composition factor $L(z.\lambda)$ in $P(w.\lambda, z.\lambda)$ corresponds to the cap of a submodule isomorphic to $M(z.\lambda)$. Thus it is indeed the case that for each copy of $L(w.\lambda)$ in $\mathrm{rad}_i M(z.\lambda)$ the Verma flag of $P(w.\lambda)$ has a quotient $M(z.\lambda)$ with cap occurring in $\mathrm{rad}_i P(w.\lambda)$. More precisely, there is a Verma flag for $P(w.\lambda)$ such that occurring in the flag (not necessarily consecutively) are two submodules $U \subset V$ satisfying: V/U is a direct sum of t copies of $M(z.\lambda)$; the submodule V lies in $\mathrm{rad}^i P(w.\lambda)$ but not in $\mathrm{rad}^{i+1} P(w.\lambda)$; and $V/V \cap \mathrm{rad}^{i+1} P(w.\lambda)$ is a direct sum of t copies of $L(z.\lambda)$.

It follows that the hypothetical description of the radical filtration of $P(w.\lambda)$ given in the first paragraph is an upper bound in an obvious sense: if it fails to be true, this is because some composition factors occur in a layer of the radical filtration of $P(w.\lambda)$ which is lower down, with a higher indexing number than that predicted. But $M(z.\lambda)$ is rigid, so to prove that the hypothetical description is correct, it suffices to show that the t copies of $M(z.\lambda)$ whose caps appear in $\mathrm{rad}_i P(w.\lambda)$ contribute t copies of their socles $L(\lambda)$ to $\mathrm{rad}_{i+l(z)} P(w.\lambda)$. As we allow z to vary in $\mathscr{W}_\lambda$ and i to vary as well, this leads to a formula for $(\mathrm{rad}_r P(w.\lambda) : L(\lambda))$ which is exactly that given by the statement of the corollary with e in place of z and z in place of y. In other words, we have reduced the proof of the corollary to the proof of the case $z=e$. Since $M(y.\lambda)$ has

Loewy length $l(y)+1$ and $L(\lambda)$ occurs only once, as the socle, what we must prove is the formula

$$(\mathrm{rad}_r P(w.\lambda): L(\lambda)) = \sum_{y \in \mathscr{W}_\lambda} (\mathrm{rad}_{r-l(y)} M(y.\lambda): L(w.\lambda)).$$

But by Lemma 1,

$$(\mathrm{rad}_r P(w.\lambda): L(\lambda)) = (\mathrm{rad}_r P(\lambda): L(w.\lambda)),$$

so we must prove that

$$(\mathrm{rad}_r P(\lambda): L(w.\lambda)) = \sum_{y \in \mathscr{W}_\lambda} (\mathrm{rad}_{r-l(y)} M(y.\lambda): L(w.\lambda)).$$

But this is exactly the formula given by Corollary 6, proving the formula.

Regarding the rigidity statement, let $(M(w_\lambda.\lambda): L(w.\lambda)) = r$ and assume that $r > 1$. Then $M(w_\lambda.\lambda)$ occurs r times as a quotient in a Verma flag for $P(w.\lambda)$, and every appearance is as a submodule. Thus, $P(w.\lambda)$ has as a submodule the direct sum of r copies of $M(w_\lambda.\lambda)$, and this accounts for all appearances of $L(w_\lambda.\lambda)$ as a composition factor in $P(w.\lambda)$. Therefore, $(\mathrm{soc}_{l(w_\lambda)+1} P(w.\lambda): L(w_\lambda.\lambda)) = r$. In contrast, the r occurrences of $L(w.\lambda)$ in $M(w_\lambda.\lambda)$ cannot all be in the same layer in its radical filtration. This follows for instance from Theorems 1 and 2, since the constant term of any Kazhdan-Lusztig polynomial is 1. But the Verma module $M(w_\lambda.\lambda)$ coincides with the projective module $P(w_\lambda.\lambda)$, so Lemma 1 of 4.2 yields that the r appearances of $L(w_\lambda.\lambda)$ in the radical filtration of $P(w.\lambda)$ cannot all be in the same layer. This implies that $L(w_\lambda.\lambda)$ is not rigidly placed in $P(w.\lambda)$.

Suppose instead that $r=1$. Recall that any Verma flag of $P(\lambda)$ has each Verma module occurring once as quotient. The discussion at the end of 1.2 shows that we can choose a Verma flag of $P(\lambda)$ with a submodule Q having the following property: the Verma flag Q inherits as a portion of the Verma flag for $P(\lambda)$ involves all and only those $M(z.\lambda)$'s as quotients for which $z \geqq w$. One can show inductively on $l(w_\lambda w)$ that Q has a simple cap. In other words, Q is a homomorphic image of $P(w.\lambda)$. The assumption that $r=1$ and BGG reciprocity imply that $[P(w.\lambda): M(z.\lambda)]=1$ for any $z \geqq w$. Thus both Q and $P(w.\lambda)$ have Verma flags with the same quotients, so they must coincide. We see that $P(w.\lambda)$ is itself part of a Verma flag for $P(\lambda)$. The argument of Corollary 6, using the results of [13], yields a description of the socle filtration of $P(w.\lambda)$ which is the same as the description of the radical filtration provided by the already proven formula of Corollary 7. Therefore, $P(w.\lambda)$ is rigid, completing the proof of the corollary.

A. APPENDIX

Loewy length under translation

In this appendix, a proof is given of the extension of Vogan's conjecture recalled in 1.2, assuming the validity of Vogan's conjecture, in order to make the paper more self-contained. The extension was proved in [13] in an ungainly manner. In the interim, I realized a much shorter proof could be given, and the referee suggested a short one as well. The proof below incorporates some of the referee's suggestions.

THEOREM. — *Let λ be an antidominant, regular wieght for which $\mathcal{O}^\lambda$ satisfies Vogan's conjecture. Let* M *be a module in $\mathcal{O}^\lambda$ and let α be an element of* B_λ. *Then*

$$ll\,\theta_\alpha \mathrm{M} \leqq ll\,\mathrm{M}+2.$$

Proof. — The theorem will be proved after some preliminary observations. There exist natural transformations I from the identity functor on $\mathcal{O}^\lambda$ to θ_α and J from θ_α to the identity functor, arising from the definition of θ_α as a composite $\varphi_\alpha \psi_\alpha$ of certain adjoint functors (*see* [10], 3.12). Thus for Q in $\mathcal{O}^\lambda$, there exist natural homomorphisms I(Q) from Q to θ_αQ and J(Q) from θ_αQ to Q. If Q is a simple module not annihilated by θ_α, by definition I(Q) and J(Q) are non-zero, so they must be an embedding and a surjection respectively. This observation can be extended to prove for arbitrary Q that the kernel of I(Q) is the largest submodule of Q annihilated by θ_α and the cokernel of J(Q) is the largest homomorphic image of Q annihilated by θ_α ([10], 3.12).

Let Q be a module in $\mathcal{O}^\lambda$ of Loewy length r. There is a commutative diagram of short exact sequences:

$$(*)\qquad \begin{array}{ccccccccc} 0 & \to & \mathrm{rad}\,\mathrm{Q} & \to & \mathrm{Q} & \to & \mathrm{cap}\,\mathrm{Q} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \theta_\alpha \mathrm{rad}\,\mathrm{Q} & \to & \theta_\alpha \mathrm{Q} & \to & \theta_\alpha \mathrm{cap}\,\mathrm{Q} & \to & 0, \end{array}$$

in which the vertical maps are given by the natural transformation I. The image of I(Q) lies in $\mathrm{soc}^r \theta_\alpha \mathrm{Q}$, since $ll\,\mathrm{Q}=r$. We may decompose cap Q as a direct sum $\mathrm{K}_1 \oplus \mathrm{K}_2$, such that θ_α annihilates no simple composition factor of K_1 and $\theta_\alpha \mathrm{K}_2=0$. Then I(cap Q) annihilates K_2 and maps K_1 isomorphically to the socle of θ_α cap Q. The commutativity of the diagram implies our first observation: (i) the image of $\mathrm{soc}^r \theta_\alpha \mathrm{Q}$ under the homomorphism from θ_αQ to θ_α cap Q contains soc(θ_α cap Q).

Let us continue with the same Q and suppose the theorem is known for modules of Loewy length $\leqq r$. Let π be the composition of the surjection from θ_αQ to cap θ_αQ and the map cap J(Q) from cap θ_αQ to cap Q. We will also need the following observation: (ii) Ker π lies in $\mathrm{soc}^{r+1} \theta_\alpha \mathrm{Q}$. There is a commutative diagram (**) with the same rows as (*) but with the vertical arrows reversed, arising from the natural transformation J. The right-hand square of (**) provides two other maps from θ_αQ to cap Q, which equal each other by commutativity. They coincide with π, as one can see because the map in (**) from θ_αQ through Q to cap Q is also induced by J(Q). Viewing π as the composition of the surjection of θ_αQ onto θ_α cap Q and J(cap Q) from $\theta_\alpha \operatorname{cap} Q$ to $\operatorname{cap} Q$, we find that the image of π is the image of $J(\operatorname{cap} Q)$: a submodule K of $\operatorname{cap} Q$ such that θ_α annihilates no simple summand of K but $\theta_\alpha(\operatorname{cap} Q/K)=0$. From (**) one sees that $\theta_\alpha \operatorname{rad} Q$ is in $\operatorname{Ker} \pi$, while the commutativity of (*) implies that the canonical image Q' of Q in $\theta_\alpha Q$ lies in $\operatorname{Ker} \pi$. Hence, π factors through a map σ of $\theta_\alpha Q/(\theta_\alpha \operatorname{rad} Q+Q')$ onto K, and counting multiplicities shows that $\operatorname{Ker} \sigma$ is annihilated by θ_α. Both $\theta_\alpha \operatorname{rad} Q$ and Q' have Loewy length $\leqq r+1$, so $\theta_\alpha \operatorname{rad} Q+Q' \subseteqq \operatorname{soc}^{r+1} \theta_\alpha Q$. Thus, σ factors through a map τ from $\theta_\alpha Q/\operatorname{soc}^{r+1} Q$ onto K such that $\operatorname{Ker} \tau$ is annihilated by θ_α. Since $ll\, \theta_\alpha Q \leqq r+2$, the module $\theta_\alpha Q/\operatorname{soc}^{r+1} \theta_\alpha Q$ is a homomorphic image of $\operatorname{cap} \theta_\alpha Q$, none of whose composition factors is annihilated by θ_α. Thus τ is injective and observation (ii) follows.

We can now prove the theorem. Let M be a module in $\mathcal{O}^\lambda$ of Loewy length s and proceed by induction on s, the case $s=1$ being Vogan's conjecture. Let $N=\operatorname{rad} M$. Then $ll\, \theta_\alpha N \leqq s+1$ by the inductive hypothesis. By observation (i), the image of $\operatorname{soc}^s \theta_\alpha M$ under the map of $\theta_\alpha M$ to $\theta_\alpha \operatorname{cap} M$ contains $\operatorname{soc}(\theta_\alpha \operatorname{cap} M)$. Let $P=\theta_\alpha M/\operatorname{soc}^s \theta_\alpha M$. Then P is an extension of $\operatorname{soc}_{s+1} \theta_\alpha N$ (which has Loewy length 0 or 1) by a quotient of $\theta_\alpha \operatorname{cap} M/\operatorname{soc}(\theta_\alpha \operatorname{cap} M)$ (which has Loewy length 1 or 2). We must show that $ll\, P \leqq 2$. Alternatively, we may show that $\operatorname{rad}^2 \theta_\alpha M \cap \theta_\alpha N \subseteqq \operatorname{soc}^s \theta_\alpha N$, since this implies that P is a homomorphic image of $\operatorname{cap}^2 \theta_\alpha M$. Equivalently, the restriction to $\theta_\alpha N$ of the surjection of $\theta_\alpha M$ onto $\operatorname{cap}^2 \theta_\alpha M$ has kernel in $\operatorname{soc}^s \theta_\alpha N$. Thus, it suffices to show that the restriction to $\theta_\alpha N$ of the composition of maps from $\theta_\alpha M$ to $\operatorname{cap}^2 \theta_\alpha M$ to $\operatorname{cap}^2 M$ has kernel in $\operatorname{soc}^s \theta_\alpha N$. The image of $\theta_\alpha N$ under this composition lies in $\operatorname{rad}(\operatorname{cap}^2 M)$, for the commutativity of (*) with M in place of Q shows that $\theta_\alpha N$ goes to 0 in the map of $\theta_\alpha M$ to $\operatorname{cap} M$. But $\operatorname{rad}(\operatorname{cap}^2 M)=\operatorname{rad}_2 M$, and since $N=\operatorname{rad} M$, this is just $\operatorname{rad}_1 N$ or $\operatorname{cap} N$. Thus we are actually considering the composition of maps from $\theta_\alpha N$ to $\operatorname{cap} \theta_\alpha N$ to $\operatorname{cap} N$. With $Q=N$ and $r=s-1$ in the previous paragraph, this map is π and observation (ii) is the required statement. This proves the theorem.

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