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SOLUTION OF A COMBINATORIALLY FORMULATED MONODROMY PROBLEM OF EISENBUD AND HARRIS

BY RONALD D. BERCOV AND ROBERT A. PROCTOR (1)

1. Introduction

In this paper we will (mostly) solve a finite permutation group problem using combinatorial reasoning and a century old result from the theory of permutation groups. This problem arose when Eisenbud and Harris [1] introduced a combinatorial construction into the study of the following geometric problem: Let X be a generic compact Riemann surface of genus g, and consider linear systems of degree d and dimension r on X. If $g=(r+1)(g-d+r)$, then only a finite number $N=N(g, d, r)$ of such linear systems can exist on X. By varying X within a suitable Zariski-open subset of the moduli space of compact Riemann surfaces of genus g, one can obtain monodromy actions on (i. e. permutations of) the set of the N linear systems on some fixed generic X_0. It is natural to ask whether the group of all such actions (called the *monodromy group*) is the entire symmetric group S_N. When combined with the work of Eisenbud and Harris, our main result will imply that these monodromy groups are always at least the alternating groups A_N.

Eisenbud and Harris explicity constructed certain monodromy actions and described these actions in combinatorial terms. Let $m=r+1$ and $n=g-d+r$. Then $m \geqq 1$ and $n \geqq 1$, and m and n determine g, d, and r since we are assuming $g=(r+1)(g-d+r)=mn$. Let $G(m, n)$ denote the monodromy group for the geometric situation indexed by g, d, and r, and let $H(m, n)$ denote the subgroup generated by the actions of Eisenbud and Harris. Hence $H(m, n) \subseteq G(m, n) \subseteq S_N$. We interpret the combinatorial description of [1] in terms of permutations of the set of all $m \times n$ standard Young tableaux. So then $N=N(g, d, r)=N(m, n)$ is the number of standard Young tableaux of $m \times n$ rectangular shape.

When $m=2$ Eisenbud and Harris showed that $H(2, n)=S_N$. Therefore $G(2, n)=S_N$. They then asked whether this approach might work in general: Does $H(m, n)=S_N$ when

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$m \geqq 2$? [It is trivial that $H(m, 1) = S_N$ and $H(1, n) = S_N$.] Our results are as follows:

THEOREM 1. — *For m, $n \geqq 1$, the group $H(m, n)$ is always either the symmetric group S_N or the alternating group A_N.*

THEOREM 2. — *The group $H(m, 2)$ is the symmetric group S_N if and only if $m = 2^i + 2^j - 1$ for some $i \geqq j \geqq 0$.*

THEOREM 3. — *For m, $n \geqq 3$ and $mn \leqq 108$ the group $H(m, n)$ is the symmetric group S_N or the alternating group A_N according to the Table. (There "S" denotes S_N and "-" denotes A_N.)*

TABLE

m\n	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
3	S	S	S	S	S	S	S	S	-	-	-	S	S	S	S	S	S	S	S	-	-	-	-	-	-	-	-	S	S	S	S	S	S	S
4	S	-	S	-	S	-	-	-	-	-	-	-	S	-	-	-	-	-	-	-	-	-	-	-	-									
5	S	S	-	S	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-															
6	S	-	S	-	-	-	-	-	-	-	-	-	-	-	-	-																		
7	S	S	-	-	-	-	-	-	-	-	-	-	-																					
8	S	-	-	-	-	-	-	-	-	-	-																							
9	S	-	-	-	-	-	-	-	-	-																								
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Eisenbud and Harris conjectured that the monodromy group $G(m, n)$ is always the entire symmetric group. The results above imply that $G(m, n)$ is always at least the alternating group A_N. Geometers tell us that it is certainly true that $G(m, n)=G(n, m)$ by use of Serre duality. (A few routine but onerous details need to be checked for this.) We would then have $G(m, n)=S_N$ whenever $H(m, n)=S_N$ or $H(n, m)=S_N$. Since $H(2, n)=S_N$ for all n, but $H(m, 2)=A_N$ for infinitely many certain m, we would then have an infinite sequence of examples (in addition to some others provided by the asymmetry of the Table) where the Eisenbud-Harris group $H(m, n)$ would be known to fall short of the actual monodromy group.

Most or all algebraic geometers would expect that the monodromy group $G(m, n)$ is always the entire symmetric group. In Section 6 we show that the strong tendency for $H(m, n)$ to be the alternating group can be viewed as an expected byproduct of the nature of the actions defined by Eisenbud and Harris. Theorem 1 reduces the original problem to showing that there is always at least one odd permutation in the monodromy group. It is probably quite difficult to determine whether a geometrically constructed permutation is odd or even. The most obvious odd permutations are the transpositions. But these are probably the hardest to construct, since one must pointwise fix $N-2$ objects. Determining which $H(m, n)$ for $m, n \geqq 3$ are equal to S_N seems irrelevant to solving the original monodromy problem in general. Hence we believe that this paper brings things to a high state of completion until some completely new ideas are introduced.

We now describe the combinatorial context of the problem. Let

$$L(m, n)=\{(a_1, a_2, \ldots, a_m): n \geqq a_1 \geqq a_2 \geqq \ldots \geqq a_m \geqq 0, a_i \in \mathbf{Z}\},$$

and consider the set of all paths of mn steps from $(0, 0, \ldots, 0)$ to $(n, n, \ldots, n)$ which stay within $L(m, n)$. Let $N=N(m, n)$ be the cardinality of this set. The monodromy actions of Eisenbud and Harris permute the N paths, We describe these actions in terms of standard Young tableaux in Section 2. In the special case $n=2$, after interchanging the roles of m and n above, the paths at hand are just the "Catalan paths" in the plane. Here

$$N=C_m=\frac{1}{m+1}\binom{2m}{m}$$

is the m-th Catalan number. The Catalan paths which have a corner at a point lying on the line $x+y=2k$ can be grouped into pairs in an obvious way. The following result, which is closely related to Theorem 2, is a combinatorial consequence of our algebraic methods.

THEOREM 4. — *There are an even number of pairs of Catalan paths from* $(0, 0)$ *to* (m, m) *with corners on the line* $x+y=2k$, $0<k<m$, *unless* $m=2^i$; *then there are an odd number of such pairs.*

The result from the theory of permutation groups which we use is Theorem 15.1 of [4].

PROPOSITION 1 (Bochert, 1889). — *Let* H *be a* 2*-transitive permutation group acting on* N *objects. If there is some non-identity element of* H *which moves fewer than* $(1/3)\,N-(2/3)\sqrt{N}$ *objects, then* H *is either the alternating group* A_N *or the symmetric group* S_N.

This proposition is used to obtain Theorem 1: We show that $H(m, n)$ is 2-transitive in Section 3 and then exhibit elements satisfying the degree bound in Section 4. In Sections 5 and 6 the computations which determine the parity of the Eisenbud-Harris generators are described, thereby proving Theorem 2, 3, and 4. In Section 6 we also explain the rarity of symmetric groups in the Table. This numerological phenomenon is closely related to the numerological fact that representations of the symmetric group are rarely odd dimensional.

2. Definitions and actions

The set $L(m, n)$ of n-tuples can be made into a partially ordered set by componentwise comparison. Then the paths described in Section 1 become maximal chains. This poset has been studied by Richard Stanley and others for purely combinatorial reasons. Here it occurs as the poset of Schubert subvarieties of the Grassmannian $G_{m-1,\,m+n-1}$ of $\mathbf{P}^r$'s in $\mathbf{P}^{g-d+2r}$ [or m-dimensional subspaces of an $(m+n)$-dimensional vector space]. The maximal chains of Schubert cycles occuring in [1] can be indexed by standard Young tableaux of $m\times n$ rectangular shape as follows. In the notation of [1] (which is inessential here), if we are given a chain

$$\sigma_{0,0,\ldots,0}\supset\ldots\supset\sigma_{\alpha^{(c-1)}}\supset\sigma_{\alpha^{(c)}}\supset\ldots\supset\sigma_{n,n,\ldots,n},$$

let $T_{ij}=c$ if $\alpha_k^{(c)}=\alpha_k^{(c-1)}$ for all k such that $0\leqq k\leqq m-1=r$ except for $k=i-1$, where $j=\alpha_k^{(c)}=\alpha_k^{(c-1)}+1$. Doing this for the mn values of c from 1 to mn yields a standard Young tableaux T, viz.

$$\{T_{ij},\ 1\leqq i\leqq m,\ 1\leqq j\leqq n\}=\{1,2,3,\ldots,mn\},$$

with $T_{i,j}<T_{i+1,j}$ and $T_{i,j}<T_{i,j+1}$. Let $X(m, n)$ be the set of all such rectangular tableaux and let $N=N(m, n)=|X(m, n)|$.

Here we replace the notation $x_{c,a}$ of [1] with $\pi_{b,c}$, where $2\leqq b\leqq mn-1$ and $1\leqq c\leqq n-1$. Then in the language of tableaux the monodromy actions of Eisenbud and Harris act from the right on the set $X(m, n)$ as follows: If $T\pi_{b,c}=U$, then U is obtained from T by interchanging the entries b and $b+1$ if and only if they were in different rows and columns in T and they were exactly c columns apart. In symbols: If $T_{ij}=b$, $T_{kl}=b+1$, $i\neq k$, and $|l-j|=c$, then $U_{ij}=b+1$, $U_{kl}=b$, and $U_{pq}=T_{pq}$ elsewhere. If $i=k$ or $|l-j|\neq c$ then $U=T$.

Let S_N be the symmetric group on $X(m, n)$, and let $H(m, n)$ be the subgroup of S_N generated by the $\pi_{b,c}$. Then the problem posed by Eisenbud and Harris is:

Problem. — Does $H(m, n)=S_N$?

3. Two-transitivity

Order the elements of $X(m, n)$ by reading the entries of a tableau down the columns from left to right, and then ordering the resulting mn-tuples lexicographically. Then A is the minimum element of $X(m, n)$, where $A_{ij}=(j-1)m+i$; and Z is the maximum element, where $Z_{ij}=(i-1)n+j$. Suppose $T_{ij}=b$ and $T_{kl}=b+1$ with $j-l=c>0$ and $i\neq k$. Then $U=T\pi_{b,c}$ precedes T in the order, viz. $U<T$. The tableau A is the only tableau for which no b lies to the right of $b+1$; all other tableaux can be moved toward A with some $\pi_{b,c}$. So $H(m, n)$ acts transitively on $X(m, n)$. (This was known to Eisenbud and Harris.)

LEMMA 1. — *The action of* $H(m, n)$ *on* $X(m, n)$ *is* 2-*transitive.*

Proof. – Given any T such that $A<T<Z$ we will move the ordered pair (T, Z) to (T′, Z) with $T'<T$. This implies 2-transitivity since the action is transitive. Note that the only $\pi_{b,c}$ which move Z are those with $c=n-1$ and $b\equiv 0 \bmod n$).

Let k be the largest i for which $T_{i,1}=i$. Consider two cases.

(1)$k=m$. The first column of T is minimal. The first entry of T which differs from that for A will be an entry $b+1$ for which b occurs c columns to the right in T where $c<n-1$. Then $T\pi_{b,c}=T'<T$ and $Z\pi_{b,c}=Z$.

(2) $k<m$. Let $T_{k+1,1}=r$. Define p and q by $T_{p,q}=r-1$. In each case below we define $\sigma\in H(m, n)$ and $T'=T\sigma$ such that $Z\sigma=Z$, $T'_{k+1,1}<T_{k+1,1}$, and $T'_{i,1}=T_{i,1}=i$ for $1\leqq i\leqq k$.

If $r-1\not\equiv 0 \pmod n$ or if $q\neq n$, then let $\sigma=\pi_{r-1,q-1}$. So $T'_{k+1,1}=r-1$. Otherwise $r-1\equiv 0 \pmod n$ and $q=n$, i. e. $T_{p,n}=r-1$. Note that p must be $\leqq k$. Consider three cases.

(*a*) $T_{p,n+1-i}=r-i$ for $n\geqq i\geqq 1$.

Note that $p\leqq k$ implies $T_{p,1}=p$ which implies that $T_{p,n}=p+n-1$ which is impossible unless $p=1$ since T is a standard Young tableau. Then $T_{1,2}=2$ implies $T_{2,1}\neq 2$ which implies $k=1$ and $T_{2,1}=n+1$. If $m=2$, then $T=Z$ contrary to assumption. So assume $m\geqq 3$. Now T and Z have identical first rows, so we can use induction on m to assume $T_{3,1}=n+2$. Now set $\sigma=\pi_{n,n-1}\pi_{n+1,n-1}\pi_{n,n-1}$. Then $T'_{2,1}=n<T_{2,1}=n+1$.

(*b*) $T_{p,n+1-i}=r-i$ for $n-1\geqq i\geqq 1$ and $T_{p,1}\neq r-n$.

If $T_{p-1,n}=r-n$ then proceed as in (*c*) below. Otherwise $T_{u,1}=r-n$ with $u>p$. But then $p=1$ as in (*a*) above. Since all of the numbers from 1 to r occur in the first column or first row and $T_{1,n}=r-1$, there are two possible locations for $r+1$ in T. If $T_{k+2,1}=r+1$, then set $\sigma=\pi_{r-1,n-1}\pi_{r,n-1}\pi_{r-1,n-1}$. If $T_{2,2}=r+1$, set $\sigma=\pi_{r,1}\pi_{r-1,1}\pi_{r,1}\pi_{r-1,1}$ if $n=2$, otherwise for $n\geqq 3$ set $\sigma=\pi_{r,1}\pi_{r-1,n-2}\pi_{r,n-1}\pi_{r-1,1}$. In all cases $T'_{k+1,1}=r-1$.

(*c*) $T_{p,n+1-i}=r-i$ for $s\geqq i\geqq 1$ where $n-2\geqq s\geqq 1$ and $T_{p,n+1-s-1}\neq r-s-1$. If $s\geqq 2$ then either $T_{p-1,n}=r-s-1$ or $T_{u,v}=r-s-1$ with $u>p$ and $1<v<n+1-s$. (If $v=1$ then $r-s-1<r$ implies $u\leqq k$ implying that $1, 2, \ldots, r-s-1$ are all used in the first column, leaving no numbers for $T_{p,2}, \ldots, TT_{p,n-s}$.) Act with either $\pi_{r-s-1,s-1}$ or $\pi_{r-s-1,n+1-s-v}$ to replace T with a new tableau T such that $T_{p,n+1-s+1}=r-s+1$ but

$T_{p,n+1-s} \neq r-s$. Repeat this until s has been decreased to 1, i. e. until $T_{p,n-1} \neq r-2$. Consider two cases.

(i) $T_{p-1,n}=r-2$. Set $\sigma=\pi_{r-1,n-1}\pi_{r-2,n-1}\pi_{r-1,n-1}$. Then $T'_{k+1,1}=r-2$.

(ii) $T_{u,v}=r-2$ with $u>p$ and $v<n$. If $v=1$ then $T_{k,1}=r-2=k$ and therefore we must really be in case (b) above with $n=2$. If $v>1$ then set $\sigma=\pi_{r-2,n-v}\pi_{r-1,v-1}$. Then $T'_{k+1,1}=r-1$.

This concludes case (2) and the proof of Lemma 1. ■

4. Degree bound

Let $\lambda=(\lambda_1, \lambda_2, \ldots, \lambda_p)$ be a partition of t; i. e. $\lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_p \geqq 0$ and $\lambda_1+\lambda_2+\ldots+\lambda_p=t$. Consider the (Ferrers or Young) diagram for λ, which has λ_i boxes in the i-th row. The conjugate λ' of λ is the partition of t obtained by reading off the column lengths $\lambda'_1 \geqq \lambda'_2 \geqq \ldots \geqq \lambda'_q > 0$, where $q=\lambda_1$. Let f_λ be the number of standard Young tableaux on the diagram for λ. Let $h_{i,j}$ be the "hook length" at the (i, j) square in the diagram, viz. $h_{i,j}=\lambda_i+\lambda'_j-i-j+1$. Then (Ex. I.5.2 of [2])

$$f_\lambda=\frac{t!}{\prod_{i,j} h_{i,j}}.$$

In particular, if λ is a perfect $m \times n$ rectangle then $N(m, n)=f_\lambda$; explicitly

$$N(m, n)=\frac{(mn)!}{\prod_{1 \leqq i \leqq m} (n+i-1)_n},$$

where $(a)_b=a(a-1)(a-2)\ldots(a-b+1)$.

If $m=1$ or $n=1$ then $N=1$ and $H(m, n)$ is trivially S_1. If $m=2$ or $m=3$ and $n=2$, then $\pi_{n,n-1}$ or $\pi_{3,1}$ respectively are transpositions and we need not appeal to Proposition 1: Lemma 1 can be used to create any transposition. (This is how Eisenbud and Harris handled the $m=2$ case.) If $m=4$ and $n=2$, then $N=14$ and $(1/3)N-(2/3)\sqrt{N}$ is approximately 2.17, forcing us to find a transposition. Random computer search yielded the transposition 3434323434342 (where 3 denotes $\pi_{3,1}$, etc.) of length 13 in the Eisenbud-Harris generators. We believe this is the shortest length for a transposition in $H(4, 2)$.

If $N \geqq 36$ then $(2/9)N$ may be used as the bound in Proposition 1. Clearly $N(m, n) \geqq 42 > 36$ when $m \geqq 5$ and $n=2$ or when $m \geqq 3$ and $n \geqq 3$ because $N(5, 2)=N(3, 3)=42$.

LEMMA 2a. — *If* $m \geqq 3$ *and* $n \geqq 3$, *then the generator* $\pi_{4,2}$ *moves fewer than* $(2/9)N$ *tableaux.*

Proof. — This generator affects only those tableaux T where the entries 4 and 5 are two colums apart. This happens exactly when $T_{1,3}=4$ and $T_{3,1}=5$ or vice versa. Hence

$q=4\,f_\lambda$ paths are moved by $\pi_{4,\,2}$ where λ is the m-tuple $(n, n, \ldots, n, n-1, n-1, n-3)$. The hook lengths $3, 4, 5, 6, \ldots, (n+2)$ and $3, 4, 6, 7, \ldots, (m+2)$ (5 should not be included twice) occur as hooks in the expression for N but not in f_λ, and conversely for the hooks $1, 2, \ldots, (n-3)$ and $1, 2, \ldots, (m-3)$. So then

$$\frac{q}{\mathrm{N}}=\frac{1}{5}\,\frac{(n+2)_5\,(m+2)_5}{(mn)_5}.$$

Then showing

$$(mn)_5 \geqq (n+2)_5\,(m+2)_5$$

will imply $q/\mathrm{N} \leqq 1/5 < 2/9$. But the inequality for $(mn)_5$ is equivalent to

$$5\,m^2\,n^2\,(m-n)^2+20\,(m-n)^2+2\,m^2\,n^2+8 \geqq 4\,(m^2-n^2)^2+10\,mn,$$

which is easily verified for m, $n \geqq 3$. ■

Lemma $2\,b$. — *If* $m=5$ *and* $n=2$, *then* $[\pi_{5,\,1}\,\pi_{6,\,1}]^3$ *moves fewer than* $(2/9)\,\mathrm{N}$ *tableaux. If* $m \geqq 6$ *and* $n \geqq 2$, *then* $[\pi_{2,1}\,\pi_{3,\,1}]^3$ *moves fewer than* $(2/9)\,\mathrm{N}$ *tableaux.*

Proof. – The tableaux (or paths) in the case $n=2$ are treated in detail in the next section. In the language we will use there, by Lemma 3 we will see that $[\pi_5\,\pi_6]^3$ and $[\pi_2\,\pi_3]^3$ consist of $\mathrm{C}_2\,\mathrm{C}_2$ and $\mathrm{C}_1\,\mathrm{C}_{m-2}$ tanspositions respectively. Hence $2\,\mathrm{C}_2\,\mathrm{C}_2=8$ and $2\,\mathrm{C}_1\,\mathrm{C}_{m-2}=2\,\mathrm{C}_{m-2}$ paths are moved. But $\mathrm{N}=42$ and $\mathrm{N}=\mathrm{C}_m$ in the two cases. But $8<(2/9)\,\mathrm{N}$, and it is also easy to see that $2\,\mathrm{C}_{m-2}<(2/9)\,\mathrm{C}_m$ when $m \geqq 6$. ■

Once the above proof has been understood by reading Section 5, then Proposition 1, Lemmas 1, $2\,a$, $2\,b$, and the remarks preceding Lemma $2\,a$ can be combined to provide a proof of Theorem 1.

5. Parity of generators when $n=2$

When $n=2$ it is more convenient to use $\mathrm{L}(2, m)$ to represent the paths in $\mathrm{X}(m, 2)$ than $\mathrm{L}(m, 2)$: Given an element $(2, 2, \ldots, 2, 1, 1, \ldots, 1, 0, 0, \ldots, 0)$ of $\mathrm{L}(m, 2)$, define a corresponding element (x, y), $m \geqq x \geqq y \geqq 0$, of $\mathrm{L}(2, m)$ by letting x be the number of 2's and 1's and y be the number of 2's. Then $\mathrm{X}(m, 2)$ is represented by the set of "Catalan paths" (never rising above the diagonal $x=y$) in the plane from $(0, 0)$ to (m, m). Since $n=2$, we have $\pi_{b,\,c}$ only for $c=1$. We will denote $\pi_{b,\,1}$ by π_b. If a path has an East-then-North corner at a point lying on the line $x+y=b$ and this corner does not have coordinates $(k+1, k)$, then it is interchanged by π_b with its sister path which has a North-then-East corner $\sqrt{2}$ to the northwest. Any path not having such a corner corresponds to a tableau with b and $b+1$ in the same row [when the corner is at $(k, k+1)$], or in the same column [when the path passes straight through a point $(x, b-x)$]. Therefore π_b is a product of $r(m, b)$ transpositions, where $r(m, b)$ is the number of Catalan paths with corners on the line $x+y=b$ not of the form $(k+1, k)$ when $b=2\,k+1$.

At this point it is essential for the reader to draw a picture. It is easy to see that π_2 fixes all paths except those passing through $(2, 1)$. Also note that $\pi_b \pi_{b+1}$ cycles the three paths contained in any 2×1 rectangle lying between $x+y=b-1$ and $x+y=b+2$. Furthermore $\pi_b \pi_{b+1}$ fixes any paths which are straight through these levels. Thus $[\pi_b \pi_{b+1}]^3$ affects only paths near the $x=y$ border. This element interchanges paths passing through $(b/2, (b/2)-1)$ and $((b/2)+1, (b/2)+1)$ when b is even and $((b-1)/2, (b-1)/2)$ and $((b+1)/2+1, (b-1)/2+1)$ *when* b is odd. Let

$$C_n = \frac{1}{n+1}\binom{2n}{n},$$

the n-th Catalan number, which is the number of Catalan paths from $(0, 0)$ to (n, n). We have just proved.

LEMMA 3. — *The generators* π_2 *and* π_{2m-2} *are products of* C_{m-1} *transpositions. The element* $[\pi_b \pi_{b+1}]^3$ *is a product of* $C_k C_{m-k-1}$ *transpositions when* $b=2k$ *or* $b=2k+1$.

When $n!$ is expressed as a product of powers of primes, the exponent of 2 is $n-\text{bin}(n)$, where $\text{bin}(n)$ is the number of 1's occurring in the binary expansion of n. Hence C_m is odd exactly when $m=2^i-1$ for some $i \geqq 0$.

THEOREM 2'. — *The generators* π_b *for* $H(m, 2)$ *are always even unless* $m=2^i+2^j-1$ *for some* $i \geqq j \geqq 0$. *If* $m=2^i$ (*i. e.* $j=0$), *then all* π_b *are odd. If* $m=2^i+2^j-1$ *with* $j>0$, *then only* π_b *with* $b=2^{j+1}-1$ *and* $b=2^{i+1}-1$ *are odd.*

Proof. — The quantity $C_k C_{m-k-1}$ is odd only when $k=2^i-1$ and $m-k-1=2^j-1$ for some $i, j \geqq 0$. Then $m=2^i+2^j-1)$. If $m \neq 2^i+2^j-1$, then $m \neq 2^i$, so C_{m-1} is even and π_2 is even. Then $C_k C_{m-k-1}$ being even implies that $[\pi_b \pi_{b+1}]^3$ is even for $b \geqq 2$. Therefore all π_b are even. Suppose $m=2^i$. Then π_2 is odd. Since $C_k C_{m-k-1}$ is odd only for $k=0$ or $m-1$, this means that $[\pi_b \pi_{b+1}]^3$ is even for $2 \leqq b \leqq 2m-2$, and so all π_b are odd. If $m=2^i+2^j-1$ with $j \geqq 1$, then π_2 and π_{2m-2} are even. And $[\pi_b \pi_{b+1}]^3$ is odd exactly when $k=2^i-1$ or 2^j-1, i. e. when $b=2^{i+1}-2$, $2^{i+1}-1$, $2^{j+1}-2$, $2^{j+1}-1$. This forces π_b to be odd when $b=2^{i+1}-1$ or $2^{j+1}-1$ and even otherwise. ■

The combinatorial version of Theorem 2' is:

THEOREM 4' *The number of pairs of Catalan paths from* $(0, 0)$ *to* (m, m) *with corners on the line* $x+y=b$ *not of the form* $(k+1, k)$ *is even unless* $m=2^i+2^j-1$ *for some* $i \geqq j \geqq 0$. *If* $m=2^i$ (*i. e.* $j=0$), *then there are always an odd number of pairs. If* $m=2^i+2^j-1$ *with* $j>0$, *then there are an odd number of pairs exactly when* $b=2^{i+1}-1$ *or* $b=2^{j+1}-1$.

6. Parity of generators when $n \geqq 3$

The generator $\pi_{b, c}$ interchanges the entries b and $b+1$ in all tableaux wherein these entries are in different rows and $c \geqq 1$ columns apart. Given such locations for b and $b+1$, let λ and μ be the regions occupied by the entries $1, 2, 3, \ldots, b-1$ and $b+2$, $b+3, \ldots, mn$ respectively. Here $\lambda=(\lambda_1, \lambda_2, \ldots, \lambda_m)$ and $\mu=(\mu_1, \ldots, \mu_m)$ are partitions of $b-1$ and $mn-b-1$ respectively, where μ_i is the number of entries in the $(m-i+1)$th row which are larger than $b+1$. Let b and $b+1$ be in the i-th and j-th rows, and let f_λ and f_μ be the numbers of standard Young tableaux on the diagrams λ and μ. Then $f_\lambda f_\mu$ is the number of pairs of paths passing through the points λ and $\lambda'=(\lambda_1, \ldots, \lambda_i+1, \ldots, \lambda_j+1, \ldots, \lambda_m)$. Thus $\pi_{b,c}$ is the product of

$$\sum_{\lambda, i, j} f_\lambda f_\mu$$

disjoint transpositions, where the sum runs over all partitions λ of b and choices of two squares just outside λ which lie in different rows and c columns apart. Computing this quantity (mod 2) for all $\pi_{b,c}$ for each $H(m, n)$ yielded the Table.

The prevalence of the alternating group in the Table is not at all surprising when one considers the question of when the number of transpositions in $\pi_{b,c}$ is odd. In order for this to happen, there must be an odd number of terms in the above sum where f_λ and f_μ are simultaneously odd. We will see that it is rare for just one f_λ to be odd. Furthermore, one of λ or μ must have t squares, where $t \geqq (mn-2)/2$.

Let λ be a partition of t with p parts, viz. $t=\lambda_1+\lambda_2+\ldots+\lambda_p$. According to Example I.1.1 of [2],

$$f_\lambda=\frac{t!\prod_{i<j}(\theta_i-\theta_j)}{\prod_{i=1}^{p}(\theta_i)!},$$

where $\theta_i=\lambda_i+p-i$. Then f_λ is odd only when

$$\binom{p}{2}+\operatorname{bin}(t)=\sum_{i<j}\operatorname{two}(\theta_i-\theta_j)+\sum_{i=1}^{p}\operatorname{bin}(\theta_i),$$

where $\operatorname{bin}(t)$ is as in Section 5, and $\operatorname{two}(k)$ is the exponent of 2 in the prime factorization of k. This condition seems to be especially hard to satisfy when the number of parts p is not small, say $p \geqq 5$, and when the partition is "fat". By this we mean $t \geqq pq/2-1$, where $q=\lambda_1$. Of the 891,042 partitions λ with p, $q \geqq 5$ and $11 \leqq t \leqq 63$ and $t \geqq pq/2-1$, only 108 have odd f_λ. (And all of these had $t \leqq pq/2+3$, supporting the belief that a fat partition is very unlikely to have odd f_λ.) With only 2 exceptions which can be treated by hand, every pair of partitions λ, μ which arises in the transposition count calculation for m, $n \geqq 8$ has at least λ or μ satisfying p, $q \geqq 5$ and $t \geqq (pq-1)/2$. Therefore only on the basis of the preliminary data above concerning the parity of f_λ for λ meeting such conditions, we would expect to find few odd Eisenbud-Harris generators after the full $\Sigma f_\lambda f_\mu$ computations are made for cases with m, $n \geqq 8$. This is because any λ with f_λ odd must be paired with a μ with f_μ odd in order to have any effect, and such a pairing is very unlikely. These considerations lead us to believe that the rarity of S_N in the middle region of the Table is a byproduct of the particular construction of Eisenbud and Harris, and that this rarity does not reflect any geometric phenomenon.

We note in passing that f_λ is the dimension of the λ-th irreducible representation of the symmetric group S_t, where $|\lambda|=t$. John McKay and Ian Macdonald [3] have shown that the number of partitions λ of t for which f_λ is odd is $2^{k_1+k_2+\cdots}$ if $t=2^{k_1}+2^{k_2}+\ldots$ with $k_1<k_2\ldots$ This can be compared with the Hardy-Ramanujan estimate $(4\sqrt{3}\,t)^{-1}e^{\pi\sqrt{2/3}\sqrt{t}}$ for the number of all partitions of t to see how rare odd f_λ are for a given value of t.

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