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IRREDUCIBILITY AND MONODROMY OF SOME FAMILIES OF LINEAR SERIES

BY DAVID EISENBUD AND JOE HARRIS (1)

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ABSTRACT. — Let g, r, and d be positive integers such that $g=(r+1)(g-d+r)$, so that the general curve of genus g has only finitely many g^r_d's. We will show in this paper that for suitable families of curves $\mathscr{C} \rightarrow B$, the family of all g^r_d's on all fibers of $\mathscr{C} \rightarrow B$ is irreducible. We do this by analyzing the monodromy action on the set of g^r_d's on a fibre, using a degeneration to reducible curves and our technique of limit series [198? *a*].

In the case $r=1$ we prove the sharper statement that the monodromy is the full symmetric group, a result motivated by a problem posed by Verdier, and applied by him in the study of harmonic maps from 2^2 to S^4 (Verdier [198?]). If we take $\mathscr{C}$ to be the universal curve over a suitable open set B of the moduli space $\mathscr{M}_g$, then the family of g^r_d's is a finite cover of B, and the branch locus of this cover (in the case $r=1$), analyzed through the tools developed in this paper, plays a fundamental role in the even-genus case in our proof [198? *b*] that $\mathscr{M}_g$ has general type for all $g \geqq 24$.

Introduction

In this paper *curves* will be complex algebraic, reduced, connected, and projective.

A g^r_d on a smooth curve C is by definition a linear series of degree d and dimension r; that is, a pair $(\mathscr{L}, V)$ where $\mathscr{L}$ is a line bundle of degree d on C and $V \subset H^0(C, \mathscr{L})$ is an $r+1$-dimensional space of sections. It is known (*see* Gieseker [1982], Eisenbud-Harris [1983 *b*] and Fulton-Lazarsfeld [1981]) that if C is a smooth curve of genus g with general

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moduli then the space $G^r_d(C)$ of all g^r_d's on C is naturally a smooth variety of dimension $\rho := g-(r+1)(g-d+r)$, and is irreducible if $\rho>0$.

We will fix numbers g, r, and d for the remainder of this paper so that $\rho=0$. In this case $G^r_d(C)$ is reducible and consists of

$$N(g,r,d) := g! \prod_{i=0}^{r} \frac{i!}{(g-d+r+i)!}$$

reduced points. The natural analogue of the irreducibility statement is that there exist smooth *families*

$$\mathscr{C} \xrightarrow{\pi} B$$

of smooth curves such that the natural map

$$G^r_d(\mathscr{C}/B) \xrightarrow{\pi} B$$

from the family of g^r_d's on fibers of π to B is finite, and so that $G^r_d(\mathscr{C}/B)$ is irreducible:

THEOREM 1. — *There is a family of smooth curves* $\mathscr{C}/B$ *such that the family* $G^r_d(\mathscr{C}/B) \to B$ *has fibers consisting of* $N(g,r,d)$ *reduced points, and such that the monodromy of the family acts transitively, so that* $G^r_d(\mathscr{C}/B)$ *is smooth and irreducible of dimension* $\rho+\dim B=\dim B$. *Further, if* $r=1$, *then the monodromy acts as the full symmetric group.*

These statements are true for any sufficiently small irreducible smooth family $\mathscr{C}/B$ containing, as stable limits, curves of the form given in Figure 1, for all relative positions of the points $p_1, \ldots, p_g$ on $\mathbb{P}^1$.

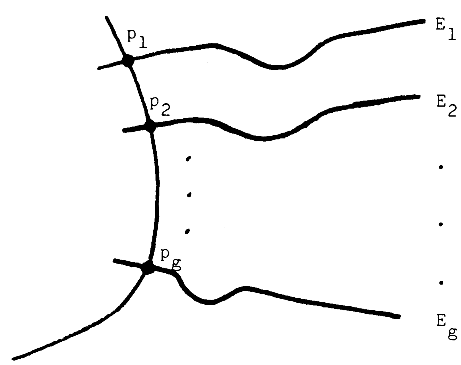


Fig. 1. — $E_1, \ldots, E_g$ curves of genus 1.

Remark. — It seems reasonable to conjecture that the monodromy acts as the full symmetric group in all cases, and even possible that the monodromy to be constructed here actually generates the full symmetric group. We will make this combinatorial problem explicit in section 3.

To prove the theorem we make use of the theory of *limit linear series* developed in our paper [198? *a*], which allows us to work directly with "limit $\mathfrak{g}^r_d$'s" on reducible curves like that of Figure 1 rather than with ordinary $\mathfrak{g}^r_d$'s on nearby smooth curves. It turns out that a limit $\mathfrak{g}^r_d$ on the curve of Figure 1, in the case $\rho=0$, is completely determined by its "$\mathbb{P}^1$-aspect"-that is, by an ordinary $\mathfrak{g}^r_d$ on $\mathbb{P}^1$ having *cusps* at the points $p_1, \ldots, p_g$. [Recall that if $(\mathscr{L}, V)$ is the series, then it is said to have a *cusp at p* if the subspace of V of sections vanishing to order $\geqq 2$ at p has codimension $\leqq 1$ in V.] We may therefore work interchangeably with limit $\mathfrak{g}^r_d$'s on the curve of Figure 1 and its degenerations or with linear series on $\mathbb{P}^1$ having cusps at $p_1, \ldots, p_g$, and with limit series on g-pointed stable degenerations of this situation. We thus get the following equivalent reformulation of Theorem 1:

THEOREM 1′. — *If $p_1, \ldots, p_g \in \mathbb{P}^1$ are points in general position, then the variety of $\mathfrak{g}^r_d$'s on $\mathbb{P}^1$ having cusps at $p_1, \ldots, p_g$ consists of* N (g, r, d) *reduced points. The monodromy induced by motions of $p_1, \ldots, p_g \in \mathbb{P}^1$ acts transitively on these $\mathfrak{g}^r_d$'s. If $r=1$, then the monodromy acts as the full symmetric group.*

Since $\mathcal{O}_{\mathbb{P}^1}(d)$ is the only line bundle of degree d on $\mathbb{P}^1$, the variety $G^r_d(\mathbb{P}^1)$ is just a Grassmannian; and the condition of having a cusp at a point p is a Schubert condition with respect to the flag defined by vanishing orders at p, so Theorem 1′ may be reformulated in terms of these Schubert cycles. In one case this is particularly interesting: Every ramification point of a $\mathfrak{g}^1_d$ is a cusp and every $\mathfrak{g}^1_d$ on $\mathbb{P}^1$ thus has $2d-2$ cusps (with multiplicity). Thus if we take $r=1$ in Theorem 1′, so that $g=2d-2$, the variety of $\mathfrak{g}^r_d$'s having cusps at *some* distinct points $p_1, \ldots, p_g$ may be identified with an open subset of the Grassmannian (of course smooth and irreducible!) and we get a result used by Verdier [1986] to study harmonic maps from S^2 to S^4:

COROLLARY 2. — *Let* $C \subset \mathbb{P}^d$ *be the rational normal curve, and let* G *be the variety of* $(d-2)$*-planes not meeting* C. *The map*

$$G \to (\mathbb{P}^1)^{(2d-2)} \cong \mathbb{P}^{2d-2}$$

associating to each plane Λ the ramification points of the projection of C from Λ onto $\mathbb{P}^1$ is generically finite and has monodromy equal to the full symmetric group on the

$$\frac{(2d-2)!}{d!\,(d-1)!}$$

points of the general fiber.

We next recall the central definitions from our [198? *a*] so that we can explain the proofs of these results:

A (possibly) reducible curve is of *compact type* if its irreducible components are smooth and meet transversely two at a time, and its dual graph (a vertex for each component, an edge for each node) has no loops.

A *limit* $\mathfrak{g}^r_d$ on a curve C of compact type is a collection

$$L = \{ L_Y = (\mathscr{L}_Y, V_Y)\ a\ \mathfrak{g}^r_d \text{ on } Y \mid Y \text{ an irreducible component of } C \}$$

of g^r_d's on the irreducible components Y of C satisfying the *compatibility condition* that if Y and Z are components meeting in a point p and V_Y contains a section vanishing to order a at p, then V_Z contains a section vanishing to order $d-a$ at p.

According to the theory of our [198? *a*], specialized to our case $\rho=0$, if C is a curve of compact type with precisely $N(g, r, d)$ limit g^r_d's on it, then the g^r_d's on each curve in a 1-parameter family of curves can be indexed uniquely by the g^r_d's on C, by associating each g^r_d on the nearby curve to its limit on C.

Our plan of attack is the following: We will give a particular curve C_∞ of compact type on which there are precisely $N(g, r, d)\, g^r_d$'s, and we will show how to label these by the facets (=maximal dimensional faces) of a certain simplicial complex Σ, actually a triangulation of a high-dimensional ball.

We will construct a number of 1-parameter families $C_{i,p}$ specializing to C_∞ as $p \to \infty$, and we will compute generators for the monodromy permutation groups that these families induce on the set of g^r_d's. We will show that if Δ_1 and Δ_2 are facets of Σ meeting in a face of codimension 1, then we obtain from one of the families $C_{i,p}$ a permutation interchanging Δ_1 and Δ_2. Since Σ is equi-dimensional and connected in codimension 1, any facet can be connected to any other by a path which crosses only codimension 1 faces of Σ, so that the monodromy group acts transitively on the g^r_d's, as claimed.

In the case $r=1$ the combinatorics simplify, and we are able to show that the monodromy group is the full symmetric group.

Since the curves $C_{i,p}$ and C are all stable limits of curves of the form given in Figure 1, this finishes the argument.

The fact that Σ is a triangulation of a ball was proved by Richard Stanley; the fact that Σ is equidimensional and connected in codimension 1 are far more elementary, and we give a simple direct proof.

To be more specific, C_∞ will be a genus g curve of compact type of the form exhibited in Figure 2.

Thus C consists of a chain of $g+1$ smooth rational curves $Y_1, \ldots, T_{g+1}$, with "elliptic tails" $E_1, \ldots, E_g$ attached to $Y_1, \ldots, Y_g$, and an extra smooth point p_1 marked on Y_1. (The curve Y_{g+1} is of course just a "place-holder"; which will simplify our subsequent notation.)

The families we consider consist of curves of the form exhibited in Figure 3.

Note that $C_{i,p}$ is similar to C_∞ except that in place of the two rational components Y_i, Y_{i+1} and their elliptic tails ($i=1, \ldots, g-1$) we have one component Y_i, with two elliptic tails E_i and E_{i+1} hanging from it, and Y_{i-1}, Y_{i+2}, E_i, and E_{i+1} are attached to Y_i at points which in a suitable coordinate system are 0, ∞, 1, and $p \neq 0, 1, \infty$, respectively.

Now in the family of stable 4-pointed rational curves, the limit of a family of $\mathbb{P}^1$'s with marked points 0, 1, p, ∞, as p approaches ∞, is obtained by blowing up the obvious family, and consists of two copies of $\mathbb{P}^1$, with the limit of 0 and 1 on one copy and the limit of p and ∞ distinct points on the other copy.

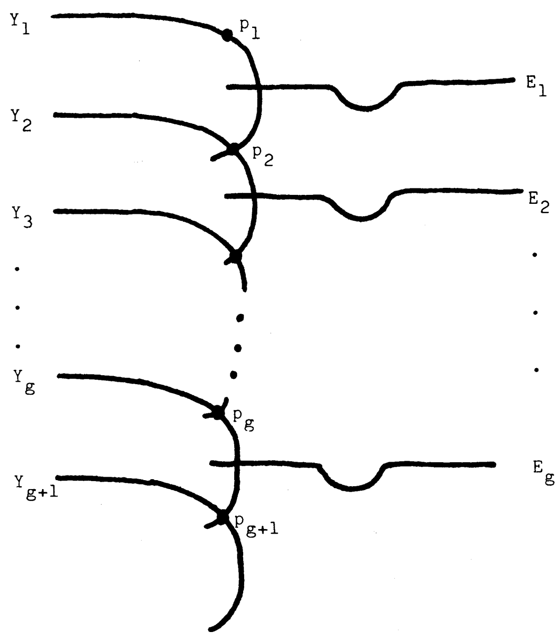


Fig. 2. – $Y_1, \ldots, Y_{g+1} \cong \mathbb{P}^1$; $E_1, \ldots, E_g$ elliptic.

Thus as p approaches ∞, *the limit of* $C_{i,p}$, *in the family of stable curves, is* C_∞. In a similar way, one sees that each of the $C_{i,p}$ is a limit of the curves of Figure 1 as the points p_i come together properly.

The rest of this paper will be concerned with an analysis of the families of $\mathfrak{g}_d^r$'s on $C_{i,p}$ and C_∞, and the monodromy, as p varies, of these families.

The first step, which occupies section 1, is to investigate the limit $\mathfrak{g}_d^r$'s on the curves D and D′ exhibited in Figure 5.

Here D consists of an elliptic component meeting a smooth rational component Y, while in D′ two elliptic components meet the smooth rational components. We have marked two points, p_1, p_2 on D and on D′, and we shall be especially interested in the vanishing behavior of sections in the Y-aspects of the $\mathfrak{g}_d^r$'s at these two points.

Limit $\mathfrak{g}_d^r$'s on C_∞ and $C_{i,p}$ are built out of limit $\mathfrak{g}_d^r$'s on curves like D and D′, which can be patched together at p_1 and p_2 if the vanishing behavior is suitable, and this explains our need for this material; however, D and D′ are the most interesting reducible curves of compact type having genus 1 and 2, so the subject has some independent interest.

In section 2 we study the combinatorics involved in putting together the $\mathfrak{g}_d^r$'s constructed on D and D′ to get refined limit series on C_∞ and $C_{i,p}$; here the simplicial complex Σ plays the central role.

In section 3 we complete the proof of Theorem 1.

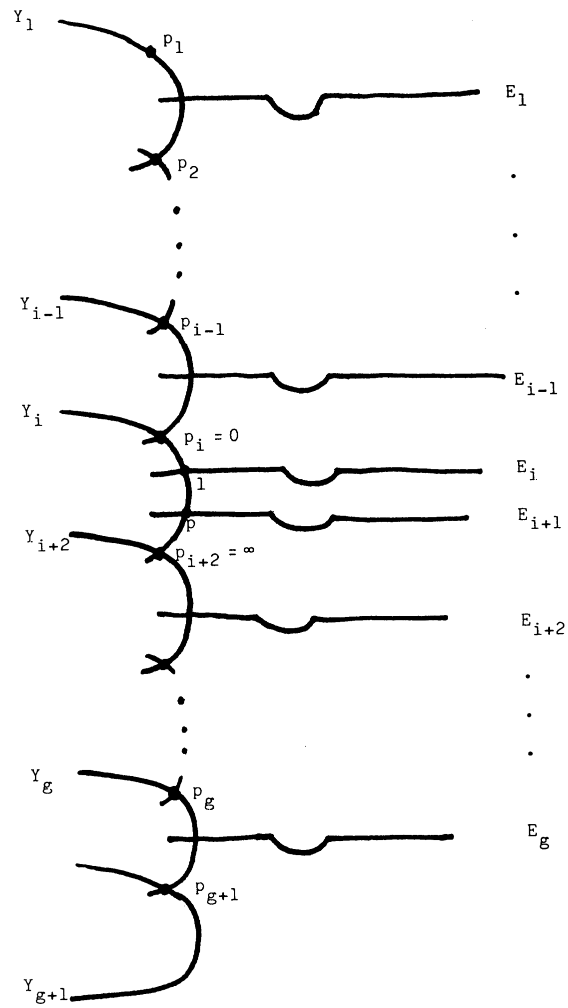


Fig. 3. — On Y_i, in suitable coordinates, $p_i=0$, E_i is attached at 1, E_{i+1} at $p\neq 0, 1$, and $p_{i+2}=\infty$.

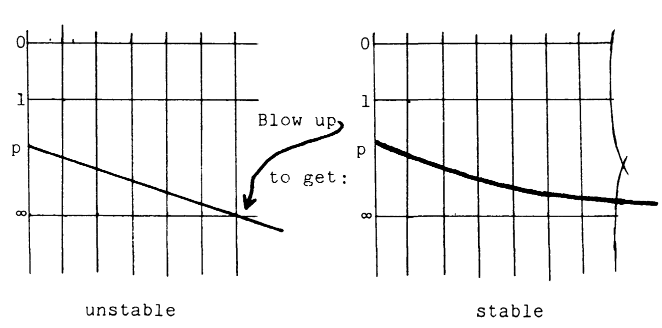


Fig. 4

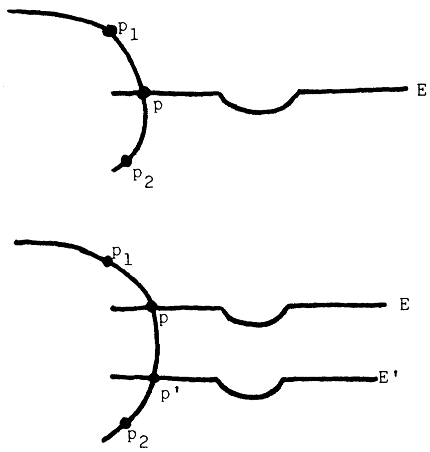


Fig. 5. — $Y \cong \mathbb{P}^1$; E, E′ elliptic.

1. g_d^r's on some reducible curves of genus 1 and 2

Recall that if $L=(\mathscr{L}, V)$ is a g_d^r on an irreducible curve Y, and $q \in Y$ is any point, then the *vanishing sequence*

$$a_0 = a_0^L(p), \ldots, a_r = a_r^L(p)$$

is the set of $r+1$ distinct orders of vanishing of sections in V at p, arranged so that $a_0 < \ldots < a_r$.

The *weight* of p with respect to $(\mathscr{L}, V)$ is by definition

$$w^V(p) = w^L(p) = \sum_{i=0}^{r} (a_i^L(p) - i).$$

The "Plücker formula" exploited in our [1983 *a*] says that

$$\sum_{p \in C} w^V(p) = (r+1)\,d + \binom{r+1}{2}(2g-2),$$

where g is the genus of Y.

If C is a curve of compact type and, for each irreducible component Y of C, $L_Y = (\mathscr{L}_Y, V_Y)$ is a g_d^r on Y, then the collection

$$L = \{ L_Y \mid Y \text{ a component of } C \}$$

is a *crude limit* g_d^r on C if, for each intersection $p = Y \cap Z$ of components of C we have

$$a_j^{L_Y}(p) + a_{r-j}^{L_Z}(p) \geqq d, \qquad j = 0, \ldots, r.$$

If the inequalities are all equalities, we say that L is a *refined limit* g_d^r or simply a *limit* g_d^r on C. This definition is easily seen to be equivalent to the one given in the introduction.

Suppose that L is a limit g_d^r on one of the curves D or D′ described in the introduction (*Fig.* 3). We will write

$$\underline{a}=(a_0, \ldots, a_r) \text{ for the vanishing sequence of } L_Y \text{ at } p_1$$

and

$$\underline{b}=(b_0, \ldots, b_r) \text{ for the vanishing sequence of } L_Y \text{ at } p_2.$$

It will be convenient also to have standard notations for the associated "Schubert indices"

$$\alpha=(\alpha_0, \ldots, \alpha_r), \qquad \alpha_i := a_{r-i}-(r-i)$$
$$\beta=(\beta_0, \ldots, \beta_r), \qquad \beta_i := b_{r-i}-(r-i)$$
$$\alpha'=(\alpha'_0, \ldots, \alpha'_r), \qquad \alpha'_i := d-r-\beta_{r-i};$$

here α' is the so-called dual to β, and would correspond, in the sense that α corresponds to $(a_0, \ldots, a_r)$, to the vanishing sequence

$$(d-b_r, \ldots, d-b_0),$$

which will appear when we consider refined limit series on a curve like D or D′, but with another curve attached at p_2. The following lemma is the first step in our analysis.

LEMMA 1.1. — *If* L *is a crude limit* g_d^r *on* D *then* L_Y *has a cusp at* p (*that is,* $a_1^{L_Y}(p)\geqq 2$), *and if* $\mathscr{L}_E \neq \mathscr{O}_E(dp)$, *then* L_Y has a base-point at p (that is, $a_0^{L_Y}(p)\geqq 1$). *If* L *is a limit series on* D′, *then similar conclusions hold for both p and p′.*

Proof. — The first statement is a special case of our [1983 *b*, Prop. 1.5]; the second follows immediately from the definitions, since $\mathscr{L}_E \neq \mathscr{O}_E(dp)$ implies $a_r^{L_E}(p)<d$, and $a_r^{L_E}(p)+a_0^{L_Y}(p)\geqq d$ by hypothesis. □

To go further, it is convenient to introduce the Schubert cycles defined in terms of orders of vanishing, as in [E−H−2]. Let G(r, d) be the Grassmannian of projective r-planes in the d-dimensional projective space $\mathbb{P}^d$ of lines in $H^0(Y, \mathscr{O}_Y(d))$ (remember that $Y\cong\mathbb{P}^1$!). For any point $q\in Y$, the spaces

$$f^i(q)=\{\sigma\in H^0(Y, \mathscr{O}_Y(d)) \mid \operatorname{ord}_q(\sigma)\geqq d-i\}$$

form a complete flag of subspaces of $\mathbb{P}^d$. For any *Schubert index* $\alpha=(\alpha_0, \ldots, \alpha_r)$ with $d-r\geqq\alpha_0\geqq\ldots\geqq\alpha_r\geqq 0$ we define the *Schubert Cycle*

$$\sigma_\alpha(q)=\{V\in G(r, d) \mid \dim(V\cap f^{d-r+i-\alpha_i}(q))>i\},$$

which is a codimension $|\alpha|=\sum\alpha_i$ subvariety of G(c, d). It is easy to see that the vanishing sequence of a linear series $(\mathscr{L}=\mathscr{O}_V(d), V)$ on Y at a point q is termwise $\geqq(a_0, \ldots, a_r)$ iff $V\in\sigma_{a_r-r,\, a_{r-1}-(r-1), \ldots, a_0}(q)$.

It follows from the Plücker formula, as explained in our [1983 a] that the intersection of any collection of Schubert cycles

$$\sigma_{\alpha(i)}(q_i)$$

(q_i distincts point of $Y=\mathbb{P}^1$) has the expected codimension, $\sum_i |\alpha^{(i)}|$; in particular, if we write $[\sigma_{\alpha(i)}]$ for the homology class of $\sigma_{\alpha^{(i)}}$, then $\bigcap_i \sigma_{\alpha^{(i)}}(q_i) \neq \varnothing$ iff the intersection product of the $\sigma_{\alpha^{(i)}}$ is nonzero in the homology ring of the Grassmanian.

In terms of Schubert cycles, Lemma 1.1 says that if V_Y has vanishing sequences $\underline{a}$ and $\underline{b}$ at p_1 and p_2 then

$$V_Y \in \sigma_\alpha(p_1) \cap \sigma_{1,\ldots,1,0}(p) \cap \sigma_\beta(p_2)$$

in case of the curve D, or

$$V_Y \in \sigma_\alpha(p_1) \cap \sigma_{1,\ldots,1,0}(p) \cap \sigma_{1,\ldots,1,0}(p') \cap \sigma_\beta(p_2)$$

in case of the curve D′. Using some Schubert calculus, we can now derive most of the combinatorics we need:

COROLLARY 1.2. — *With notations as above.*

1. *On the curve* D, *we have* $|\alpha'| \geqq |\alpha| + r$. *If equality holds then* $\mathscr{L}_E = \mathcal{O}_E(dp)$, V_E *is the image of* $H^0(\mathscr{L}_E(-(r+1)p)$ *in* $H^0(\mathscr{L}_E)$, *and there is a unique i such that*

$$b_{r-i} = d - a_i,$$

while

$$b_{r-j} = d - a_j - 1 \quad \textit{for all } j \neq i.$$

Given vanishing sequences a and b satisfying this condition, there is a unique limit g^r_d *with these vanishing sequences at* p_1 *and* p_2.

2. *On the curve* D′ *we have* $|\alpha'| \geqq |\alpha| + 2r$. *If equality holds then* $(\mathscr{L}_E, V_E)$ *and* $(\mathscr{L}_{E'}, V_{E'})$ *are determined as in case* 1. *Given vanishing sequences a and b such that the associated ramification sequences* α, α' *satisfy* $|\alpha'| = |\alpha| + 2r$, *there is at most one limit* g^r_d *on* D′ *with these sequences except in the following case, where there are either* 1 *or* 2 *such series:*

There exist integers $i<j$ *such that* $a_{i-1} < a_i - 1$ *if* $i>0$, $a_{j-1} < a_j - 1$, *and*

$$b_{r-i} = d - a_i - 1$$
$$b_{r-j} = d - a_j - 1$$
$$b_{r-k} = d - a_k - 2 \quad \textit{for all } k \neq i, j.$$

Sketch of proofs. — The Schubert calculus is applicable because of the dimensional transversality of the Schubert cycles $\sigma_{\alpha^{(i)}}(q_i)$ in $G(r, d)$, mentioned above. The pertinent facts are

1. For any two Schubert cycles σ_α and σ_β, we have

$$\sigma_\alpha \cdot \sigma_\beta = \sum_{|\gamma| = |\alpha| + |\beta|} \delta(\alpha, \beta, \gamma)\, \sigma_\gamma$$

for suitable integers $\delta(\alpha, \beta, \gamma)$, and if we write

$$\bigstar \gamma = ((d-r)-\gamma_r, \ldots, (d-r)-\gamma_0),$$

then $\sigma_{*\gamma}$ is dual to σ_γ so $\delta(\alpha, \beta, \gamma) = \sigma_\alpha \cdot \sigma_\beta \cdot \sigma_{*\gamma}$.

2. For any Schubert cycle σ_α,

$$\sigma_\alpha \cdot \sigma_{1, \ldots, 1, 0}$$

is the sum with multiplicities 1 of all σ_γ such that

$$\gamma_i = \alpha_i \quad \text{for some } i$$
$$\gamma_j = \alpha_j + 1 \quad \text{for all } j \neq i.$$

These facts may be deduced from what is found, for example, in Griffiths-Harris [1978], p. 197-204; in particular, 2. is the dual of "Pieri's formula", given on p. 203. To calculate $(\mathscr{L}_E, V_E)$ in both cases of Corollary 1.2, note that $\mathscr{L}_E = \mathscr{O}_E(dp)$, and by the compatibility condition, the vanishing sequence at p is $(d-(r+1), \ldots, d-2, d)$. Thus V_E is the complete series associated to $\mathscr{O}_E((d-(r+1))p)$, with an $(r+1)$-fold base point at p added. The same remarks apply to E′. Thus the only way in which more than one refined limit linear series can appear on D or D′ is for there to be more than one choice of $V \in \sigma_\alpha \cdot \sigma_{1, \ldots, 1, 0} \cdot \sigma_\beta$ (or, in the case of D′, in $\sigma_\alpha \cdot (\sigma_{1, \ldots, 1, 0})^2 \cdot \sigma_\beta$), and this corresponds to the conclusion of the Corollary. □

It is perhaps amusing to see directly the unique g_d^r on D with vanishing sequences as above in the case

$$b_i = d - a_{r-i}$$
$$b_j = d - a_{r-j} - 1 \qquad (j \neq i).$$

(Note that for $b_0, \ldots, b_r$ to be increasing we must have $a_{i-1} < a_i - 1$) The techniques to be introduced in a moment would allow us to analyze what it musk look like; however, since there is only one, we may simply exhibit it. We have already computed $(\mathscr{L}_E, V_E)$. It remains to specify

$$V_Y = \langle x_0 t^{a_0} + t^{a_0+1}, \ldots, x_{i-1} t^{a_{i-1}} + t^{a_{i-1}+1}, \ldots, t^{a_i}, x_{i+1} t^{a_{i+1}} + t^{a_{i+1}+1}, \ldots \rangle$$

where $x_j = (1 - a_i + a_j)/(a_i - a_j)$.

The case of the curves D′ is substantially more complex. For the purposes of the next section, we wish to know, in the last case mentioned in Corollary 1.2, the cross-ratios p for which there is only one refined limit g_d^r with the given vanishing sequence on D′; of course the uniqueness of $(\mathscr{L}_E, V_E)$ and $(\mathscr{L}_{E'}, V_{E'})$ has already been demonstrated so it suffices to examine $V = V_Y \subset H^0(Y, \mathscr{O}_Y(d))$. We give a treatment which actually avoids the Schubert calculus involved above by appealing directly to the Plücker formula. To simply, we introduce coordinates on Y so that $p_1=0$, $p_2=\infty$, $p'.1$, and consider p as number $\neq 0, 1, \infty$.

THEOREM 1.3. — *Let $0 \leqq i < j \leqq r$, and d, and $0 \leqq a_0 < \ldots < a_r$ be integers, and suppose that*

$$a_{i-1} < a_i - 1 \quad (\textit{if } i > 0) \qquad \textit{and} \qquad a_{j-1} < a_j - 1.$$

Set

$$\begin{aligned} b_{r-i} &= d - a_i - 1 \\ b_{r-j} &= d - a_j - 1 \\ b_{r-k} &= d - a_k - 2 \qquad \textit{for} \quad k \neq i, j. \end{aligned}$$

The variety G of g_d^r's on $\mathbb{P}^1$ having vanishing sequences $\geqq \underline{a}$ at 0, $\geqq \underline{b}$ at ∞ and with at least a cusp at 1 and at some further point p in $\mathbb{P}^1 - \{0, 1, \infty\}$ is an irreducible rational curve. The mapping $G \to \mathbb{P}^1$ that associates to each such series its further cusp point p is a finite double covering, branched over two points of $\mathbb{P}^1 - \{0, 1, \infty\}$. These branch points are determined by the number $a_j - a_i$, and either branch point determines the value of $a_j - a_i$.

Remark. — Setting $\varepsilon = 1/(a_j - a_i)$ one sees from the proof below that the branch points are $1 - 2\varepsilon^2 \pm 2\varepsilon\sqrt{\varepsilon^2 - 1}$.

We will make use of the following lemma, which clarifies the structure of the linear series described in the theorem:

LEMMA 1.4. — *If $V \subset H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$ is a g_d^r satisfying the conditions of the theorem, then the space*

$$V \cap f^{d-a_k}(0) \cap f^{d-b_{r-k}}(\infty) \qquad (k = 0, \ldots, r)$$

is 1-dimensional, and if φ_k is a nonzero element of this space, then the φ_k form a basis of V such that

$$\operatorname{ord}_0 \varphi_k = a_k \tag{1}$$

and

$$\operatorname{ord}_\infty \varphi_k = b_{r-k}. \tag{2}$$

Remark. — A similar fact was true for the g_d^r on D exhibited above; the techniques below would prove it in that case even if we didn't know the g_d^r explicitly. How generally does such an assertion hold?

Proof. — We will repeatedly use the following observation: If a g_d^r having cusps at 1 and $p \neq 0, 1, \infty$ on $\mathbb{P}^1$ has vanishing sequences $(a'_0, \ldots, a'_r)$ at 0 and $(b'_0, \ldots, b'_r)$ at ∞, then since the cusps each have weight $\geqq r$, the "Plücker Formula" described above becomes

$$(r+1)(d-2) + 2 \geqq \sum a'_i + \sum b'_i.$$

Now in the case of the lemma,

$$\dim(V\cap f^{d-a_k}(0))+\dim(V\cap f^{d-b_{r-k}}(\infty))=r-k+1+k+1=\dim V+1,$$

there is a nonzero $\varphi_k\in V$ vanishing to order $\geqq a_k$ at 0 and $\geqq b_{r-k}$ at ∞. It is enough for the lemma to prove that $\operatorname{ord}_0\varphi_k=a_k$ and $\operatorname{ord}_\infty\varphi_k=b_{r-k}$.

Consider first the cases $k=i$ and $k=j$. Since $a_j+b_{r-i}>d$, the sections φ_i and φ_j are independent. In these cases $b_{r-k}=d-a_k-1$, so if either φ_k vanished to orders $>a_k$ or $>b_{r-k}$, then for suitable a and b we would have: a section vanishing to orders a at 0 and $d-a$ at ∞ and a section vanishing to orders b at 0 and $\geqq d-b-1$ at ∞. The existence of such a pencil would contradict the first observation above, so the result is established for $k=i, j$.

Now suppose $k\neq i, j$. We will show that φ_i, φ_j and φ_k are linearly independent; then we can apply the first observation to the g_d^r that they span, and get

$$\operatorname{ord}_0\varphi_k+\operatorname{ord}_\infty\varphi_k\leqq d-2,$$

whence $\operatorname{ord}_0\varphi_k=a_k$ and $\operatorname{ord}_\infty\varphi_k=d-a_k-2=b_{r-k}$ as desired.

It remains to show that φ_k is not in the space $\langle\varphi_i,\varphi_j\rangle$.

If $k<i$, then φ_k vanishes to order $b_{r-k}>b_{r-i}$ at ∞, so φ_k is not in the span of φ_i, φ_j. The case $k>j$ is similar. Finally, if $i<k<j$ and if φ_k were in the span of φ_i, φ_j, then we would have

$$\operatorname{ord}_0\varphi_k=a_j,\qquad \operatorname{ord}_\infty\varphi_k=d-a_i-1,$$

a contradiction since

$$a_j+(d-a_i-1)>d.\quad\square$$

Proof of Theorem 1.3. — Write $\varphi_0,\ldots,\varphi_r$ for the basis whose existence is guaranteed by Lemma 1.4.

We deal first with the case $r=1$, where $i=0$, $j=1$; we will see that the general case reduces to this one. If $\varphi(t)$, $\psi(t)$ are any rational functions of degree d, then the pencil $\langle\varphi,\psi\rangle$ is ramified at a point p if

$$\det\begin{vmatrix}\varphi(p) & \psi(p)\\ \varphi'(p) & \psi'(p)\end{vmatrix}=0.$$

By Lemma 1.4 we may take in our case

$$\varphi_i=xt^{a_i}+t^{a_i+1},\qquad \varphi_j=yt^{a_j}+t^{a_j+1}$$

with $x, y\neq 0$. If we set $\varepsilon=1/(a_j-a_i)$ then the equation above becomes (using $p\neq 0,\infty$)

$$(\bigstar)\qquad p^2+[x(1+\varepsilon)+y(1-\varepsilon)]p+xy=0.$$

Thus the equation of the locus of x, y such that the pencil $\langle\varphi_i,\varphi_j\rangle$ has a cusp at 1 is

$$1+x(1+\varepsilon)+y(1-\varepsilon)+xy=0,$$

which is a nonsingular conic if, as in our case, $\varepsilon \neq 0, \infty$.

Further, the pencil $\langle \varphi_i, \varphi_j \rangle$ has cusps at both $p=1$ and $p=p_0$, a further point, if and only if

$$xy = 1 . p_0 = p_0$$

and

$$-[x(1+\varepsilon)+y(1-\varepsilon)] = 1+p_0,$$

or

$$y = p_0/x$$
$$(1+\varepsilon)x^2 + (1+p_0)x + (1-\varepsilon)p_0 = 0.$$

The discriminant of this last equation is

$$(1+p_0)^2 - 4p_0(1-\varepsilon^2).$$

Thus $\varepsilon = 1/(a_j - a_i)$ determines the two points p_0 in $\mathbb{P}^1 - \{0, 1, \infty\}$ over which G ramifies, and either of these, since they are $\neq 0$, determines ε up to sign; since $\varepsilon > 0$, we are done with the case $r=1$.

Turning to the case $r>1$, we see from the lemma that V contains a distinguished pencil $\langle \varphi_i, \varphi_j \rangle$, also with cusps at 1 and p. We will complete the proof by showing that there is one and only one g_d^r V with the given properties containing given pencil V_0 with these properties.

By the lemma, we may assume that φ_k $(k \neq i, j)$ will have the form

$$(\bigstar) \qquad \varphi_k = z_0 t^{a_k} + z_1 t^{a_k+1} + z_2 t^{a_k+2} \neq 0.$$

By the Plücker formula, the pencil V_0 cannot have a base point at either 1 or p. Thus the conditions on (z_0, z_1, z_2) that $\langle V_0, \varphi_k \rangle$ have a cusp at 1 and at p are linear, so the set of such φ_k of the form ($\bigstar$) is the set of nonzero elements of a vectorspace. If its dimension were $\geqq 2$, then such a φ_k could be found with $Z_0 = 0$, or $Z_2 = 0$, and then $\langle V_0, \varphi_k \rangle$ would be a g_d^r contradicting the observation at the beginning of the lemma.

Since $V_0 = \langle \varphi_i, \varphi_j \rangle$ has a cusp but does not have a base point at 1 or at p, the g_d^r $V = \langle \varphi_0, \ldots, \varphi_r \rangle$ has cusp at 1 and p if and only if each of the series $\langle \varphi_i, \varphi_j, \varphi_k \rangle$ do. This proves that there is a unique g_d^r extending $\langle \varphi_i, \varphi_j \rangle$. □

2. g_d^r's and chains of Schubert cycles

In this section we will study the g_d^r's on the stable curves of genus g, as described in the introduction. We first note that in the cases of interest to us the problem reduces to a problem involving only certain g_d^r's with cusps on curves of (arithmetic) genus 0. Recall that g, r, d are fixed so that $\rho = 0$.

PROPOSITION 2.1. — *Let* F *be a curve of arithmetic genus* 0 *whose components meet transversely two at a time, and let* $p_1, \ldots, p_g$ *be* g *smooth points of* F. *Let* C *be the curve obtained from* F *by attaching elliptic curves* E_i *at marked points* $p_1, \ldots, p_g$ *of* F (Fig. 6).

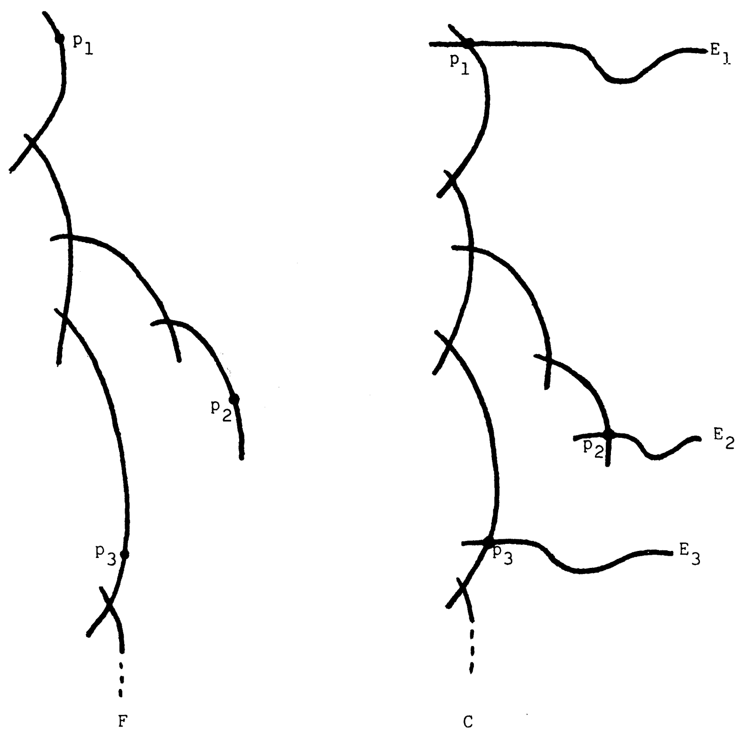


Fig. 6. — Arithmetic genus of F=0, branches meet two at a time. E_i all smooth, elliptic.

If $L=\{L_Y=(\mathscr{L}_Y, V_Y) \mid Y$ is a component of $C\}$ *is a crude limit* g_d^r *on* C, *then* L *is refined and for each* E_i *the* E_i-aspect *of* L *is given by*

$$\mathscr{L}_{E_i}=\mathcal{O}_{E_i}(dp_i)$$

$$V_{E_i}=\text{Image } H^0 \mathscr{L}_{E_i}(-(d-r-1)p) \subsetneq H^0 \mathscr{L}_{E_i}.$$

The addition of these aspects gives a 1-1 *correspondence between the set of limit* g_d^r's *on* C *and the set of limit* g_d^r's *on* F *having at cusp at each of the* p_i.

Remark. — One may check that this correspondence is an isomorphism of schemes; we shall not need this.

Proof. — The linear series $(\mathscr{L}, V)$ on the elliptic curve E_i given by

$$\mathscr{L}=\mathcal{O}_E(dp_i)$$

$$V=\text{Image}(H^0 \mathscr{L}((d-r-1)p_i) \to H^0 \mathscr{L}$$

has vanishing sequence

$$d-r-1, \ldots, d-2, d$$

at p_i, so the weight of p_i as a ramification point of this series is $(r+1)(d-r)-r$. It is easy to see that $(\mathscr{L}, V)$ is the unique g_d^r to achieve such a high weight at p_i.

If $L=\{(\mathscr{L}_Y, V_Y)\}$ is a crude limit g_d^r on C then by the compatibility conditions

$$w^{V_{Y_i}}(p_i)+w^{V_{E_i}}(p_i)\geqq(r+1)(d-r).$$

Thus by the remarks above we have

$$w^{V_{Y_i}}(p_i)\geqq r$$

and if equality holds then L satisfies the compatibility condition for a refined limit g_d^r at p_i and $(\mathscr{L}_{Y_i}, V_{Y_i})$ *has* a cusp at p_i.

On the other hand, by induction on the number of components we see that for any crude limit g_d^r, say $L'=\{(\mathscr{L}'_Y, V'_Y)\}$ on F we have

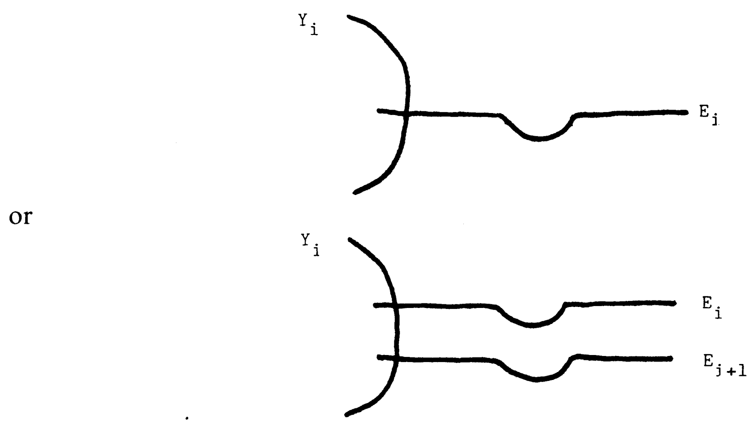
$$\sum_{Y} \sum_{\substack{p \in Y \\ \text{smooth on } F}} w^{V'_Y}(p)\leqq(r+1)(d-r)=rg,$$

with equality if and only if L' is refined. Putting these facts together, the theorem follows. □

We now wish to classify the limit g_d^r's on curves of the form $C=C_\infty$ or $C=C_{i,p}$ of Figures 2, 3. By Proposition 2.1, the aspects $(\mathscr{L}_{E_i}, V_{E_i})$ are the same for all the g_d^r's, so we may ignore them.

Let L be such a limit g_d^r.

Of course the restriction of L to each of the curves



contained in C will be a limit g_d^r of the type considered in the previous section. Write

$$(a_0^{(i)}, \ldots, a_r^{(i)})$$

for the vanishing sequence of $(\mathscr{L}_{Y_i}, V_{Y_i})$ at p_i, and $\alpha^{(i)}=(a_r^{(i)}-r, \ldots, a_0^{(i)})$ for the corresponding Schubert index. By Corollary 1.2, we have

$$|\alpha^{(j+1)}| \geqq |\alpha^{(j)}| + r$$

for all j where $\alpha^{(j+1)}$ makes sense and

$$|\alpha^{(i+2)}| \geqq |\alpha^{(i)}| + 2r \quad \text{on } C_{i,\,p}.$$

On the other hand, $|\alpha^{(j)}|$ is always less than $\dim G(r, d) = (r+1)(d-r)$, and since under our hypothesis $(r+1)(d-r) = rg$, we see that we must have $|\sigma^{(j)}| = r(j-1)$ for every j.

We associate to L the chain of Schubert cycles

$$\Delta(L) = (\sigma_{\alpha^{(1)}}, \ldots, \sigma_{\alpha^{(g+1)}}),$$

where $\sigma_{\alpha^{(i+1)}}$ is omitted if $C = C_{i,\,p}$.

By Corollary 1.2, L is classified by $\Delta(L)$ completely if $C = C_\infty$, and up to a choice of at most two limit series if $C = C_{i,\,p}$.

The data in $\Delta(L)$ may be conveniently organized as follows:

Recall that a (combinatorial) *simplicial complex* is a collection of subsets, called *faces* or *implices*) of a given set, called the *vertex set*, such that a subset of a face is a face, and each one-element subset is a face. The maximal faces are called *facets.* The simplicial complex may be specified by giving the facets.

Let Σ be the simplicial complex whose facets are the sets of $g+1$ Schubert cycles in $G(r, d)$

$$(s_1, \ldots, s_{g+1})$$

such that, writing $s_{i+1}^\vee$ for the Schubert cycle Poincaré-dual to s_{i+1},

$$s_i \cdot \sigma_{1, \ldots, 1, 0} \cdot s_{i+1}^\vee \neq 0.$$

As above, it follows that the codimension of s_i is ri.

We can now summarize Corollary 1.2 and Theorem 1.3 as follows:

THEOREM 2.2. — *There is a* 1-1 *correspondence between the limit* g_d^r's *on* C_∞ *and the facets of* Σ *established by*

$$L \mapsto \Delta(L).$$

(ii) *If* L *is a limit* g_d^r *on a curve* $C_{i,\,p}$, *then* $\Delta(L)$ *is a codimension* 1 *face of* Σ. *Every codimension* 1 *face of* Σ *occurs in this way. For general* p, *the number of distinct* L *with a given image* $\Delta(L)$ *is the number of facets containing* $\Delta(L)$, *which is either* 1 *or* 2. *As* p *varies, the family formed by those* L *with a given* $\Delta(L)$ *is irreducible.*

COROLLARY 2.2. — *Let* L_1 *and* L_2 *be limit* g_d^r's *on* C *and suppose that* $\Delta(L_1)$ *and* $\Delta(L_2)$ *meet along a codimension* 1 *face* Δ *of* Σ. *If* i *is such that there is a limit series* L *on* $C_{i,\,p}$ *with* $\Delta(L) = \Delta$, *then* L_1 *and* L_2 *are interchanged by the monodromy of the family of limit* g_d^r's *over* $C_{i,\,p}$.

Proof of the Corollary. — $C_{i,\infty}=C_\infty$, so the Corollary makes sense. Further, the smoothing theorem of [E-H-3] may easily be adapted to show that both L_1 and L_2 are the limits of refined limit g_d^r's on $C_{i,p}$ as $p\to\infty$. The corollary now follows from the last statement of the theorem. □

We can now easily count the number of refined limit g_d^r's on our curves

PROPOSITION 2.4. — *The number of refined limit g_d^r's on C_∞ or, for general p, on $C_{i,p}$ is the same as the number of g_d^r's on a general curve of genus g.*

Proof. — The number of facets of Σ is the number of chains of Schubert cycles $s_1, \ldots, s_{g+1}$ with codim $s_j=r(j-1)$ and $s_{j+1}\subset s_j.\sigma_{1,\ldots,1,0}$. It follows that $s_1=\sigma_{0\ldots 0}$, $s_{g+1}=\sigma_{d-r,\ldots,d-r}=$ one point, and the number of facets is simply the intersection number $(\sigma_{1,\ldots,1,0})^g$. On the other hand, by Griffiths-Harris [1980] (or our papers [1983 *a* or *b*]) the number of g_d^r's on a general smooth curve is the same as the number on a general nodal or cuspidal curve, and this is $(\sigma_{1,1,\ldots,1,0})^g$ as required. □

We will need to know that Σ is equi-dimensional and connected in codimension 1, and for this it is convenient to exhibit Σ in a different way. Recall that a *chain* in a partially ordered set is a totally ordered subset. The family of all chains in a partially ordered set S is of course a simplicial complex, which we denote $\Sigma(S)$. We will apply this to the set of Schubert cycles, ordered by inclusion (recall $\sigma_\alpha\subseteqq\sigma_\beta$ iff $\alpha_i\geqq\beta_i$ for each *i*).

PROPOSITION 2.5. — *Σ is isomorphic to the simplicial complex of all chains of Schubert cycles in the Grassmann variety $G(r, g-d+2r)$. Further, Σ is equi-dimensional and connected in codimension* 1.

Proof. — The isomorphism is obtained by sending the vertex $s_i=\sigma_{\alpha^{(i)}}$ of the facet $(s_1, \ldots, s_{g+1})$ of Σ to the Schubert cycle

$$\varphi(s_i)=\sigma_{i-\alpha_r^{(i)},\ldots,i-\alpha_0^{(i)}}\subset G(r, g-d+2r).$$

The condition $s_i.\sigma_{1,\ldots,1,0}.s_{i+1}^\vee$, which implies that one of the indices of s_{i+1} is the same as the corresponding index of s_i while all the others have risen by 1 translates into the statement that precisely one index of $\varphi(s_{i+1})$ is higher than the corresponding index of $\varphi(s_i)$, and that by exactly 1. Thus

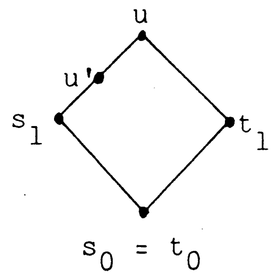
$$\varphi(s_1), \ldots, \varphi(s_{g+1})$$

is a maximal chain of Schubert cycles in $G(r, g-d+2r)$, and every such maximal chain appears as the image under φ of a unique facet of Σ, as required.

To prove the statement about Σ we write S for the partially ordered set of Schubert cycles of $G(r, g-d+2r)$. S is actually a distributive lattice (since $\sigma_{\alpha_0,\ldots,\alpha_r}\cap\sigma_{\beta_0,\ldots,\beta_r}=\sigma_{\min(\alpha_0,\beta_0),\ldots,\min(\alpha_r,\beta_r)}$, S is a sublattice of the product of $r+1$ totally ordered sets) and this suffices.

If $s_0 < \ldots < s_d$ and $t_0 < \ldots < t_d$ are maximal chains of S, and thus facets of Σ, then $s_0 = t_0$ is the infinum of S. Set $u = s_1 \vee t_1$. If there were an element u' between s_1 and u, then

$$s_1 \vee (u' \wedge t_1) = s_1 \neq (s_1 \vee u') \wedge (s_1 \vee t_1)$$



contradicting distributivity. If we choose a maximal chain (u_i) of the form

$$s_0 = u_0 < s_1 = u_1 < u = u_2 < \ldots < u_d,$$

and use the theorem inductively on the sublattices lying above s_1 and above t_1, the desired conclusions follow.

3. The monodromy groups

To complete the program outlined in the introduction, we begin with a remark on monodromy: If $\mathscr{C} \to \mathrm{B}$ is an irreducible family of smooth curves containing the curves of Figure 1 as stable limits, and thus containing the curves $\mathrm{C}_{i,\,p}$ and C_∞ as limits too, then by the theory of our [198? *a*] the family of g^r_d's on the fibers of $\mathscr{C}/\mathrm{B}$ extends to the family of limit g^r_d's on the limiting fibers $\mathrm{C}_{i,\,p}$ and C_∞. at least along 1-parameter families. Since monodromy is a birational invariant (*see* for example Harris [1979]) it is enough to show that the monodromy actions on the limit g^r_d's of C induced by the 1-parameter families $\mathrm{C}_{i,\,p}$ generate permutation groups with the required properties—that is, transitive in general, and the full symmetric group in the case $r = 1$.

As we have seen, the limit g^r_d's on C_∞, and therefore on nearby curves, are indexed by maximal chains of Schubert cycles

$$(\sigma_{0\ldots 0} \supset \ldots \supset \sigma_{g-d+r,\,\ldots,\,g-d+r})$$

in the Grassmanian of $\mathbb{P}^r$'s in $\mathbb{P}^{g-d+2r}$.

If we recast Theorem 1.3 using Proposition 2.5, we obtain a description of the monodromy action of the family $\mathrm{C}_{i,\,p}$ in this language:

THEOREM 3.1. — *With the identifications above, the monodromy actions of the various families* $\mathrm{C}_{i,\,p}$ *are generated by permutations* $x_{c,\,a}$, *for positive integers c and a, where* $x_{c,\,a}$ *is the product of all transpositions of pairs of chains*

$$\sigma_{0,\,\ldots,\,0} \supset \ldots \supset \sigma_{\underline{\alpha}(c-1)} \supset \sigma_{\underline{\alpha}(c)} \supset \sigma_{\underline{\alpha}(c+1)} \supset \ldots$$

and

$$\sigma_{0, \ldots, 0} \supset \ldots \supset \sigma_{\underline{\beta}^{(c-1)}} \supset \sigma_{\underline{\beta}^{(c)}} \supset \sigma_{\underline{\beta}^{(c+1)}} \supset \ldots$$

where the

$$\underline{\alpha}^{(i)} = (\alpha_0^{(i)} \geqq \ldots \geqq \alpha_r^{(i)})$$
$$\underline{\beta}^{(i)} = (\beta_0^{(i)} \geqq \ldots \geqq \beta_r^{(i)})$$

are Schubert indices corresponding to Schubert cycles $\sigma_{\underline{\alpha}^{(i)}}$ *and* $\sigma_{\underline{\beta}^{(i)}}$ *of* $G(r, g-d+2r)$ *such that*

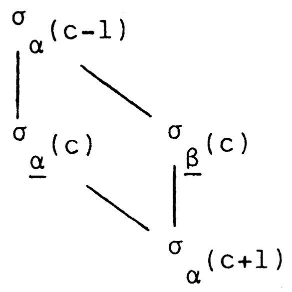
$$\begin{array}{ccc} \underline{\alpha}^{(0)} & =0, \ldots, 0= & \underline{\beta}^{(0)} \\ \vdots & & \vdots \\ \underline{\alpha}^{(c-1)} & = & \underline{\beta}^{(c-1)}, \\ \underline{\alpha}^{(c+1)} & = & \underline{\beta}^{(c+1)} \\ \vdots & \vdots & \vdots \end{array}$$

$\alpha_i^{(c)} = \beta_i^{(c)}$ *for all but precisely two values* $j < k$ *of* i, *and* $a = \alpha_k^{(c-1)} - \alpha_j^{(c-1)}$. □

As already remarked, these transformations are obviously sufficient to interchange any two maximal chains of Schubert cycles that agree in all but one place; and the fact that the associated simplicial complex is connected in codimension 1 shows that the monodromy acts transitively.

We now specialize to the case $r=1$. Here the combinatorics simplify. Note that, setting $n=g-d+2$, the lattice of Schubert cycles $\sigma_{\alpha_0 \alpha_1} \subset G(1, g-d+2)$ may be represented by the diagram (*Fig.* 7).

In this case, the transformations given in Theorem 3.1 are in 1-1 correspondence with the diamonds of this diagram:



Proposition 3.2. — *In the case* $r=1$, *the monodromy group contains, for each diamond as above, an element which is the product of all transpositions of pairs of maximal chains of Schubert cycles in Figure 7 which are obtained by adding to the right-hand or left-hand pairs of sides of the given diamond the same maximal chain leading from* $\sigma_{0,0}$ *to the top of the diamond* $\sigma_{\underline{\alpha}^{(c-1)}}$ *and from the bottom of the diamond* $\sigma_{\underline{\alpha}^{(c+1)}}$ *to* σ_{nn}.

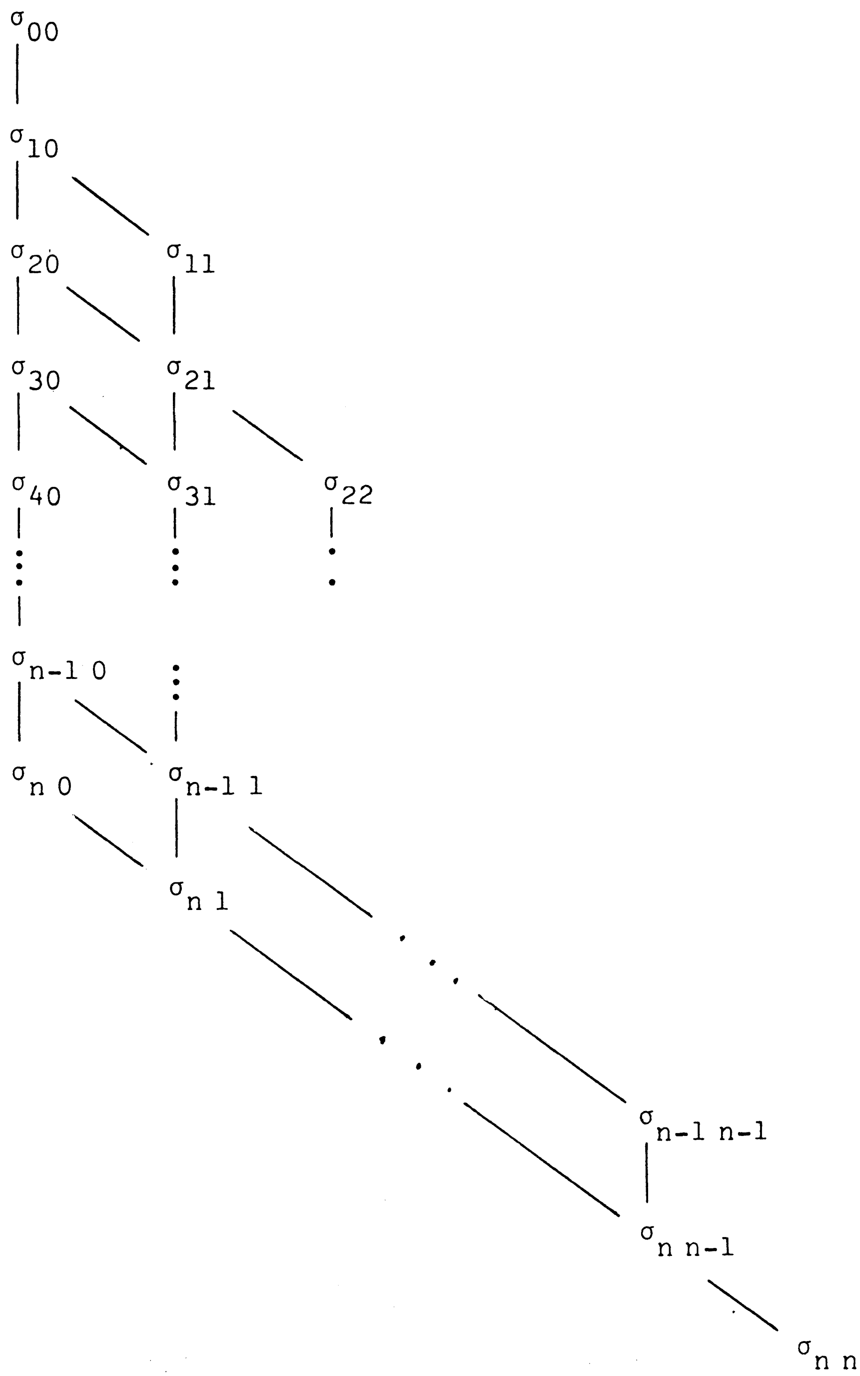
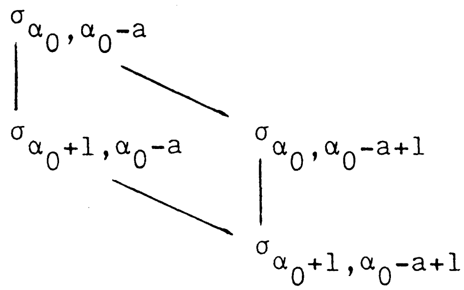

σ00
σ10
σ20
σ11
σ30
σ21
σ40
σ31
σ22
σn-1 0
σn 0
σn-1 1
σn 1
σn-1 n-1
σn n-1
σn n

Fig. 7

Proof. — $x_{c,a}$ corresponds to the diamond

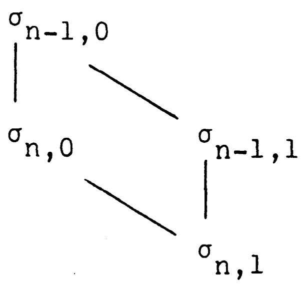


where $2\alpha_0 = a+c-1$. □

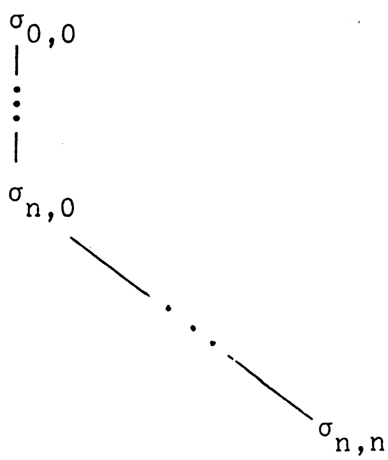
From this we may deduce the desired result:

COROLLARY 3.3. — *If* $r=1$, *the permutation group generated by the monodromy actions of the families* $C_{i,p}$ *on the limit* g_d^r*'s on* C_∞ *is the full symmetric group.*

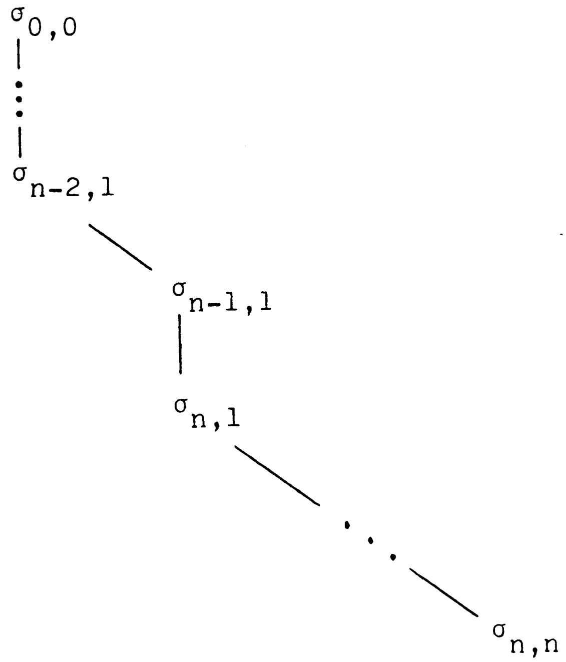
Proof. — The transformation coming from the diamond δ given below:



is a simple transposition since the maximal chains ascending from $\sigma_{n-1,0}$ and descending from $\sigma_{n,1}$ are unique. Thus it is enough to show that the monodromy is doubly transitive. It is transitive because any maximal chain can be pushed across diamonds by the $x_{c,a}$'s until it reaches the extremal chain



(Of course, we already know that the monodromy is transitive even for general r.) Further, this extremal chain is fixed by the permutations associated to all the diamonds except for the diamond δ just exhibited. But it is possible to push any maximal chain other the extremal one above across diamonds $\neq \delta$ until it reaches the chain



so the stabilizer of the extremal chain acts transitively on the rest, and the group generated by the given monodromy is doubly transitive as required. □

R. Proctor and R. D. Bercov [198?] have recently shown that for arbitrary r the monodromy elements constructed above generate either the symmetric or alternating group, and that either case can occur for different values of g, r, d. We conjecture that the monodromy group is never the less the full symmetric group in every case. To prove this it would be enough to construct in each case a family yielding a simple transposition as its monodromy.

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