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ENTROPY AND INDEX FOR SUBFACTORS

By MIHAI PIMSNER and SORIN POPA

Introduction

Let M be a type II_1 factor with normalized trace τ and $N \subset M$ a subfactor. In a recent paper [13] V. Jones considered the coupling constant of N in his representation on $L^2(M, \tau)$ as an invariant for N up to conjugations by automorphisms of M. He calls this invariant the index of N in M and denotes it $[M:N]$. In the case M and N are the group algebras of some discrete groups $G_0 \subset G$, $[M:N]$ is just the index of G_0 in G. This exemple provides the motivation for the notation and for the name "index". It also suggests that $[M:N]$ may take only integer values. However, even in the hyperfinite II_1 factor R one can construct subfactors of any index $\geqq 4$ just by identifying the reduced algebras of R by the projections f, $1-f \in R$ (*cf.* [13]). The situation is much more complicated when one requires N to have trivial relative commutant in M, $N' \cap M = \mathbb{C}$. As a consequence of the properties of the coupling constant Jones shows that in case $[M:N] < 4$ the condition $N' \cap M = \mathbb{C}$ is automatically fulfilled. He then proves the striking result that if the index is less than 4, then it can only take the values $4 \cos^2 \pi/n$, $n \geqq 3$. Moreover for each

$$k \in \{ 4 \cos^2 \pi/n \mid n \geqq 3 \} \cup [4, \infty)$$

he constructs in a natural way a subfactor in R having index k. We denote it in the sequel by R_λ where $\lambda = k^{-1}$. For index greater than 4 these subfactors have nontrivial relative commutant (*cf.* [13]). It is worth noting that the pairs $R_\lambda \subset R$ are obtained as increasing limits of finite dimensional subalgebras $A_n \subset B_n$, such that the corresponding conditional expectations commute.

There are at least two important problems arising from Jones' work: (1) Find the possible values of the index on the halfline $(4, \infty)$ in the trivial relative commutant case. (2) Classify the subfactors of R having the same index.

This paper originates in our attempt to get more insight on these and other index problems.

We begin by considering M as a module over its subfactor N and get an interpretation of the index as the dimension of M over N. Then we obtain some formulas for [M : N] expressing the flatteness of the positive elements in M when projected on N. Further we introduce the Connes-Störmer relative entropy $H(M|N)$ as an invariant of N up to conjugation. Quite surprinsingly the relative entropy is very closely related to the index and is actually finite whenever the index is finite. In fact we obtain an explicit formula of $H(M|N)$ depending on the index and the relative commutant of N in M. By this formula, if $N' \cap M = \mathbb{C}$ then $H(M|N) = \ln [M:N]$. But more interesting is that for small enough index, e. g. $4 < [M:N] < 3 + 2\sqrt{2}$, the converse is also true: if $H(M|N) = \ln [M:N]$ then $N' \cap M = \mathbb{C}$.

In more detail the main results of the 6-sections are as follows.

In section 1 we prove that the index $[M:N]$ is finite if and only if M is a finitely generated projective module over N and that if this is the case then M has an "orthonormal decomposition" over N. This result was obtained independently by U. Haagerup (paper in preparation). We use this to get some duality type results similar to the case when M is the crossed product of N by a finite group. Moreover, using the "orthonormal decomposition" we show that if the index is finite then M and N have the same type of central sequence algebra.

In Section 2 we prove the formulas for the index. We show that if $\lambda = [M:N]^{-1}$ (with the convention $\infty^{-1} = 0$) then $E_N(x) \geqq \lambda x$ for all $x \in M_+$ and that λ is the best constant for which this inequality holds (E_N denotes the trace preserving conditional expectation onto N). Along the line we also prove that λ is the infimum of all the norms $\| E_N(f) \|$, f running over the nonzero projections in M. We then define for von Neumann subalgebras $B_1 \subset B_2 \subset M$ the constant $\lambda(B_2, B_1) = \max \{ \lambda \geqq 0 \mid E_{B_1}(x) \geqq \lambda E_{B_2}(x) \text{ for all } x \in B_{2+} \}$ as a remplacement of the index when B_1, B_2 are not necessary factors (the definition actually works for arbitrary von Neumann algebras, whenever there exists a normal conditional expectation from B_2 onto B_1). The consideration of the constant λ makes possible the computation of the index whenever the pair $N \subset M$ is the inductive limit of pairs of finite dimensional algebras, under certain compatibility conditions.

In Section 3 we recall the definitions and basic properties of the Connes-Störmer relative entropy $H(\ |\ )$. We also prove some technical results and note the important relation between H and the constant $\lambda(\ ,\ )$ namely $H \leqq -\ln \lambda$.

Section 4 contains the computation of $H(M|N)$ for II_1 factors: if $N' \cap M$ has a diffuse part then $H(M|N) = \infty$; if $N' \cap M$ is atomic and $\{ f_n \}$ are minimal projections in $N' \cap M$ such that $\sum f_n = 1$ then $H(M|N) = \sum \tau(f_n) \ln ([M_{f_n} : N_{f_n}]/\tau(f_n)^2)$. As a consequence we characterize the pairs of minimal and maximal entropy (for a given index). In particular we obtain the earlier mentioned condition for N to have trivial relative commutant in M.

Section 5 deals with applications. We show that Jones' pairs of subfactors $R_\lambda \subset R$ are of minimal entropy. Then we consider a family of automorphisms Θ_λ of the hyperfinite factor R, related with the construction of the subfactors R_λ, $\lambda^{-1} \in \{ 4 \cos^2 \pi/n \mid n \geqq 3 \} \cup [4, \infty)$, and compute their entropy. For $\lambda^{-1} > 4$ we show that in fact Θ_λ are noncommutative Bernoulli shifts.

In section 6 we compute λ and H in the finite dimensional case thus providing a tool for computing the index and the entropy for pairs of hyperfinite factors $N \subset R$ that can be obtained as inductive limits of appropriate pairs of finite dimensional algebras. It is an open problem whether any pair $N \subset R$ of finite index can be realized in this way. But anyway it seems reasonable to believe that, if $\mathscr{C}_R$ denotes the set of all possible values of the index of subfactors with trivial relative commutant, then, as in the case <4, for any $k \in \mathscr{C}_R$, $k>4$, there exists a subfactor of R with index k and trivial relative commutant, such that the corresponding pair of factors is the inductive limit of pairs of finite dimensional algebras as before. Thus to characterize $\mathscr{C}_R$ at least for small values of the index k (e. g. $k<3+2\sqrt{2}$) it would be enough to construct sequences of finite dimensional algebras such that the limit of the entropies equals the limit of the logarithm of the index. By our results this would avoid the computation of the relative commutant (which is usually very difficult).

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0. Preliminaries

In this section we fix the notations and, for the convenience of the reader, recall some of Jones' terminology and results in [13], to be more frequently used in the paper.

Throughout M will be a finite von Neumann algebra with a fixed normal faithful trace τ, $\tau(1)=1$. We denote by $\|x\|_2=\tau(x^*x)^{1/2}$ the Hilbert norm on M given by τ and $L^2(M, \tau)$ the completion of M in the norm $\| \ \|_2$. Thus $L^2(M, \tau)$ is the Hilbert space of the GNS representation of M, given by τ, and M acts on $L^2(M, \tau)$ by left multiplication. This representation of M is called the standard representation. The canonical conjugation on $L^2(M, \tau)$ is denoted by J. It acts on the dense subspace $M \subset L^2(M, \tau)$ by $Jx=x^*$. Then J satisfies $JMJ=M'$ and in fact JxJ is the operator of multiplication on the right with x^* : $JxJ(y)=yx^*$, $y \in M \subset L^2(M, \tau)$.

If $N \subset M$ is a von Neuamnn subalgebra ($1_N=1_M$) then E_N denotes the unique τ-preserving conditional expectation of M onto N [24]. E_N is in fact the restriction to M of the orthogonal projection of $L^2(M, \tau)$ onto $L^2(N, \tau)$ (which is the closure in $L^2(M, \tau)$ of N). We shall denote this orthogonal projection by e_N, or simply by e, if no confusion is possible. The following properties of $e=e_N$ are easy consequences of the definition (*see* [13], 3.1.4, and also [5], [12]).

0.1 $exe=E_N(x)e$, $x \in M$;

0.2 If $x \in M$ then $x \in N$ iff $ex=xe$;

0.3 $N'=(M' \cup \{e\})''$;

0.4 J commutes with e.

By 0.3, 0.4 it follows that if M_1 denotes the von Neumann algebra on $L^2(M, \tau)$ generated by M and e then $M_1=JN'J$. This is called the *basic construction* for $N \subset M$

(*cf.* [13]). We now list some of its properties ([13], 3.1.5):

0.5. Operators of the form $a+\sum_{i=1}^{n} a_i e b_i$ with a, a_i, $b_i \in \mathrm{M}$, give a dense $*$-subalgebra in M_1;

0.6. $\mathrm{N} \ni x \mapsto xe \in e\,\mathrm{M}_1\, e$ is an isomorphism;

0.7. The central support of e in M_1 is 1;

0.8. M_1 is a factor iff N is;

0.9. M_1 is finite iff N′ is.

If M_1 satisfies 0.9 and if there exists a trace τ_1 on M_1 such that $\tau_{1\,|\,\mathrm{M}}=\tau$ and $\mathrm{E}_{\mathrm{M}}(e)=\lambda\, 1_{\mathrm{M}}$, where E_{M} is the τ_1-preserving conditional expectation of M_1 onto M and $\lambda>0$ is a scalar, then we say that (M_1, τ_1) is a *λ-extension of* M *by* N. By 0.5 it follows that if such a trace τ_1 exists then it is unique.

If M is a finite factor and acts on the Hilbert space $\mathscr{H}$, the Murray and von Neumann coupling constant $\dim_{\mathrm{M}} \mathscr{H}$ is defined as $\tau([\mathrm{M}'\xi])/\tau'([\mathrm{M}\xi])$, where $0 \neq \xi \in \mathscr{H}$ and for $\mathrm{A} \subset \mathscr{B}(\mathrm{H})$ a von Neumann algebra $[\mathrm{A}\,\xi]$ is the orthogonal projection onto $\overline{\mathrm{A}\,\xi}$. It is shown in [18] that this definition is independent of $\xi \neq 0$. For a pair of finite factors $\mathrm{N} \subset \mathrm{M}$ V. Jones defined in [13] the *index of* N *in* M, [M:N], to be the number $\dim_{\mathrm{N}}(\mathscr{H})/\dim_{\mathrm{M}}(\mathscr{H})$ or equivalently $\dim_{\mathrm{N}} \mathrm{L}^2(\mathrm{M}, \tau)$. In particular, [M:N] is a conjugacy invariant for N as a subfactor of M. In the case $\mathrm{N} \subset \mathrm{M}$ comes from the group construction in [18], for some I.C.C. discrete groups $\mathrm{G}_0 \subset \mathrm{G}$, then [M:N] coincide with the index of G_0 in G. Another important example to be noted is when M is the crossed product of N by some outer action of a discrete group K, $\mathrm{M}=\mathrm{N} \rtimes \mathrm{K}$, when [M:N] is just the cardinal of K.

The index [M:N] has all the nice properties of the index for subgroups (*see* [13], 2.1.8):

0.10. $[\mathrm{M}:\mathrm{M}]=1$;

0.11. $[\mathrm{M}:\mathrm{N}] \geqq 1$;

0.12. If $\mathrm{N} \subset \mathrm{P} \subset \mathrm{M}$ then $[\mathrm{M}:\mathrm{P}]\,[\mathrm{P}:\mathrm{N}]=[\mathrm{M}:\mathrm{N}]$, $[\mathrm{M}:\mathrm{P}] \leqq [\mathrm{M}:\mathrm{N}]$ with equality iff $\mathrm{N}=\mathrm{P}$.

Note also that $[\mathrm{M}:\mathrm{N}]=\infty$ iff N′ (or equivalently M_1) is of type II_∞.

If $[\mathrm{M}:\mathrm{N}]<\infty$ then M_1 is a finite factor and so it has a unique normalized trace, to be denoted also by τ (as the trace of M). Moreover if $\lambda=[\mathrm{M}:\mathrm{N}]^{-1}$ then it follows that $\mathrm{M}_1=(\mathrm{M} \cup \{e_{\mathrm{N}}\})''$ is a λ-extension of M by N (*cf.* [13], 3.1.7). In this case we simply call M_1 the extension of M by N. Then the pair of finite factors $\mathrm{M} \subset \mathrm{M}_1$ satisfies the important relation $[\mathrm{M}:\mathrm{N}]=[\mathrm{M}_1:\mathrm{M}]$ ([13], 3.1.7). So, by using the basic construction, from a pair of factors $\mathrm{N} \subset \mathrm{M}$ of finite index $k=[\mathrm{M}:\mathrm{N}]$ one can get a new pair of facors $\mathrm{M} \subset \mathrm{M}_1$ with the same index, $[\mathrm{M}_1:\mathrm{M}]=k$. It turns our that in fact the basic construction is generic for subfactors of finite index. More precisely Jones showed in [13], 3.1.9 that if $\mathrm{N} \subset \mathrm{M}$ are type II_1 factors with $[\mathrm{M}:\mathrm{N}]<\infty$ then there is a subfactor $\mathrm{N}_1 \subset \mathrm{N}$ such that M is the extension of N by N_1, i. e. there is a projection $e \in \mathrm{M}$ with $\mathrm{E}_{\mathrm{N}}(e)=[\mathrm{M}:\mathrm{N}]^{-1}$, $[e, \mathrm{N}_1]=0$ and $exe=\mathrm{E}_{\mathrm{N}_1}(x)\, e$ for $x \in \mathrm{N}$, such that M is generated as a von Neumann algebra by N and e. We shall sometimes refer to the construction of N_1 as the *downward* *basic construction.* This construction is no more canonical as it is the usual basic construction, because the sub factor $N_1 \subset N$ and the projection e with the above properties are not unique. However we shall prove in Section 1 that any two such subfactors N_1 of N are conjugated by a unitary element in N.

Now we mention two useful formulas relating the index [M:N] with the index of the induced algebras M_p, $\bar{N}_p$, where $p \in N' \cap M$ ([13], 2.2).

0.13 If $[M:N] < \infty$ and $p \in N' \cap M$ is a projection then

$$[M_p : N_p] = [M:N]\,\tau(p)\,\tau'(p),$$

where τ' is the unique normalized trace on N'.

0.14. If $p_i \in N' \cap M$ are projections with $\sum p_i = 1$ then

$$[M:N] = \sum [M_{p_i} : N_{p_i}]/\tau(p_i).$$

We note that if $[M:N] < \infty$ then 0.14 follows by 0.13 and it is easy to see that if $[M:N] = \infty$ then $\sum [M_{p_i} : N_{p_i}]/\tau(p_i) = \infty$.

These formulas allow to compute the index of the following example of factors $N \subset M$ ([13], 2.2.5): Let M be a type II_1 factor and $\alpha > 0$ in the fundamental group $\mathscr{F}(M)$ of M (so that M is isomorphic to its amplification by α, M_α). Let $f \in M$ be a projection such that $\tau(f)/\tau(1-f) = \alpha$ and denote by ϑ an isomorphism of M_f onto M_{1-f} [it exists by the assumption $\alpha \in \mathscr{F}(M)$]. Denote $N = \{ x \oplus \vartheta(x) \mid x \in M_f \}$. Then 0.14 applies to get $[M:N] = \tau(f)^{-1} + \tau(1-f)^{-1}$.

Another important consequence of 0.14 is that if $[M:N] < 4$ then $N' \cap M = \mathbb{C}$. Moreover Jones proves in [13] the remarkable result that if $[M:N] < 4$ then the only possible values for [M:N] are $\{ 4\cos^2 \pi/n \mid n \geqq 3 \}$.

1. Factors as modules over their subfactors

If $N \subset M$ are type II_1 factors then in particular M may be regarded as a right Hilbert N-module [22] with N valued inner product $E_N(m_1^* m_2)$. We shall prove in this section that M is a finitely generated projective module over N iff the index [M:N] is finite. Since projective modules over II_1 factors are of a simple form, this will make possible to chose an "orthonormal basis" of M over N. Such a basis yields a decomposition of $L^2(M, \tau)$ into n copies of L^2 and a "remainder" and it is a useful tool for proving several duality type results.

For the next two lemmas we only assume M to be a finite von Neumann algebra (not necessary a factor). The notations are those of Section 0.

1.1. Lemma. — *Operators of the form* $\sum_{i=1}^{n} a_i e_N b_i$, $a_i, b_i \in M$ *give a dense* $*$*-subalgebra in* $M_1 = (M \cup \{ e_N \})''$.

Proof. — The relation $e_N x e_N = E_N(x) e_N$ shows that these operators form a $*$-subalgebra. To see the density it suffices to prove that the projection onto the closure of $M e_N L^2(M, \tau)$ is the identity on $L^2(M, \tau)$. But this projection is the central support of e_N in M_1 so that 0.7 yields the conclusion.

Q.E.D.

1.2. LEMMA. — *Suppose* $M_1 = (M \cup \{e_N\})''$ *is a* λ*-extension of* M *by* N. *For any* $x \in M_1$ *there exists a unique* $m \in M$ *such that* $x e_N = m e_N$.

Proof. — Let E_M be the τ_1-preserving conditional expectation of M_1 onto M (where the trace τ_1 on M_1 make it into a λ-extension of M by N). Of course if m exists then it must equal $\lambda^{-1} E_M(x e_N)$. The weak continuity of E_M implies that we only have to prove the existence part for x in a dense subset of M_1. But for $x = \sum_{i=1}^{n} a_i e_N b_i$, a_i, $b_i \in M$, $x e_N = (\sum_i a_i E_N(b_i)) e_N$.

Q.E.D.

In the rest of this section $N \subset M$ are assumed to be factors such that $[M:N] < \infty$.

If $\alpha > 0$ we shall identify the elements in the amplification N_α of N with $(n+1) \times (n+1)$ matrices $(a_{ij})_{i,\, j}$ where n is the integer part of α and the entries a_{ij} *satisfy* $a_{ij} \in N$, $a_{i,\, n+1} \in N p$, $a_{n+1,\, i} \in p N$, $a_{n+1,\, n+1} \in p N p$, where $p \in N$ is a fixed projection of trace $\alpha - n$.

1.3. PROPOSITION. — *There exists a family* $\{m_j\}_{1 \leq j \leq n+1}$ *of elements in* M, *with n equal to the integer part of* $[M:N]$, *satisfying the properties:*

(*a*) $E_N(m_j^* m_k) = 0$, $j \neq k$;

(*b*) $E_N(m_j^* m_j) = 1$, $1 \leq j \leq n$;

(*c*) $E_N(m_{n+1}^* m_{n+1})$ *is a projection of trace* $[M:N] - n$. *Moreover any such family satisfies:*

(1) $m_j e_N$ *are partial isometries,* $1 \leq j \leq n+1$;

(2) $\sum_{j=1}^{n+1} m_j e_N m_j^* = 1$;

(3) $\sum_{j=1}^{n+1} m_j m_j^* = [M:N]$;

(4) Every $m \in M$ has a unique decomposition $m = \sum_{j=1}^{n+1} m_j y_j$ with $y_j \in N$, $y_{n+1} \in E_N(m_{n+1}^* m_{n+1}) N$.

If $\{m'_j\}_{1 \leq j \leq n+1}$ *is another family with the properties* (*a*), (*b*), (*c*) *then* $(E_N(m_i^* m'_j))_{i,\, j} = (a_{ij})_{1 \leq i,\, j \leq n+1}$ *is a unitary element in* N_α (*where* $\alpha = [M:N]$ *and* N_α *is the* α*-amplification of* N) *such that*

$$m'_k = \sum_{l=1}^{n+1} m_l a_{lk}.$$

Proof. — Let $g_1, \ldots, g_{n+1}$ be a family of pairwise orthogonal projections in the extension of M by N, M_1, satisfying $\tau(g_i)=[M:N]^{-1}$, $1\leq i\leq n$, $\tau(g_{n+1})=1-n[M:N]^{-1}$, $\sum_{i=1}^{n+1} g_i=1$. Choose partial isometries $v_j\in M_1$ such that $v_j v_j^*=g_j$, $1\leq j\leq n+1$, $v_j^* v_j=e_N$, $1\leq j\leq n$, $v_{n+1}^* v_{n+1}\leq e_N$. Lemma 1.2 then implies the existence of $\{m_j\}_{j=1}^{n+1}$ such that $v_j=m_j e_N$. Since $v_j^* v_k=0$ for $j\neq k$ it follows that $0=e_N m_j^* m_k e_N=E_N(m_j^* m_k) e_N$, so that $E_N(m_j^* m_k)=0$. Further $E_N(m_j^* m_j) e_N=e_N m_j^* m_j e_N=e_N$, $1\leq j\leq n$, so that $E_N(m_j^* m_j)=1$ and similary $E_N(m_{n+1}^* m_{n+1}) e_N=f e_N$ where $f\in N$ is a projection such that $\tau(g_{n+1})=\tau(f e_N)=\tau(f)\tau(e_N)=\tau(f)[M:N]^{-1}$. Thus $E_N(m_{n+1}^* m_{n+1})=f$ and $\tau(f)=\tau(g_{n+1})[M:N]=[M:N]-n$.

Let $\{m_j'\}_{1\leq j\leq n+1}$ be another family satisfying (*a*), (*b*), (*c*). If $1\leq j\leq n$ then $e_N m_j'^* m_j' e_N=E_N(m_j'^* m_j') e_N=e_N$ so that $m_j' e_N$ are partial isometries and similary $m_{n+1}' e_N$ is a partial isometry. Moreover by (*a*), $0=E_N(m_j'^* m_k') e_N=e_N m_j'^* m_k' e_N$, for $j\neq k$, so that $m_j' e_N$ have mutually orthogonal left supports and by (*b*), (*c*), $m_j' e_N m_j'^*$ fill up the identity in M_1. Applying the conditional expectation E_M of M_1 on M we also get $1=E_M(\sum m_j' e_N m_j'^*)=[M:N]^{-1}\sum m_j' m_j'^*$. This shows that m_j' satisfy (1), (2), (3). Finally if $m\in M$, by (2) we obtain that $\sum m_j' E_N(m_j'^* m) e_M=\sum m_j' e_N m_j'^* m e_N=m e_N$ which shows that $m=\sum m_j' y_j$, where $y_j=E_N(m_j'^* m)$. This decomposition is easily seen to be unique by (*a*).

Let now $m_j'=\sum_s m_s b_{sj}$ be the decomposition of m_j' in the given basis $\{m_i\}$. Then $m_i^* m_j'=\sum_s m_i^* m_s b_{sj}$ so that $E_N(m_i^* m_j')=\sum E_N(m_i^* m_s) b_{sj}$ which by (*a*), (b), (c) equals b_{ij}. Thus $b_{ij}=a_{ij}$. Also by (2) we have

$$\sum_k a_{ki}^* a_{kj} e_N=\sum_k e_N m_i'^* m_k e_N m_k^* m_j' e_N=e_N m_i'^* m_j' e_N=E_N(m_i'^* m_j') e_N$$

and since $\{m_j'\}$ satisfy (*a*), (b), (c), $(a_{ij})_{i,\,j}$ is a unitary element in N_α.

Q.E.D.

1.4. *Remarks.* — 1° By (4) in 1.3, the family $\{m_j\}_{1\leq j\leq n+1}$ forms an "orthonormal basis" in M with respect to the N valued inner product $E_N(m_1^* m_2)$, $m_1, m_2\in M$.

2° Property (3) in 1.3 shows that if [M : N] is not an integer then one cannot find an "orthonormal basis" of n unitaries $m_1, m_2, \ldots, m_n$ plus a remainder.

3° At the $L^2(M, \tau)$ level one gets the decomposition $L^2(M, \tau)=\bigoplus_{j=1}^{n+1}\overline{m_j N}$ [the closure is in the $\|\ \|_2$ topology on $L^2(M, \tau)$]. The orthogonal projections g_j on $\overline{m_j N}$ are equivalent in M_1 with $e_N=\overline{1 N}$, $1\leq j\leq n$, while $\overline{m_{n+1} N}$ is equivalent in M_1 with the projection on $E_N(m_{n+1}^* m_{n+1}) L^2(N, \tau)$.

4° From 1° it follows that M is isomorphic as a right N-module with $N^n\oplus E_N(m_{n+1}^* m_{n+1}) N$ so that M is projective and finitely generated.

Conversely if M is a finitely generated N-module then there exist $m_1, \ldots, m_k\in M$ such that $M=\sum m_i N$. Thus $M=\sum N m_i^*$. This means that the unit in N′ can be filled up with k cyclic projections. Thus the coupling constant of N is $\leq k$, so that $[M:N]\leq k$.

Note that since $N^n \oplus E_N(m^*_{n+1} m_{n+1}) N$ is of the form $p N^{n+1}$, where p is a projection in the $n+1$-amplification of N with $\tau(p)=[M:N]/n+1$ it follows that the class of M in $K_0(N)$ is equal to $[M:N]$ via the usual isomorphism $K_0(N) \simeq \mathbb{R}$.

For the next proposition we shall denote by σ_α the amplification of the $*$-morphism $\sigma : N \to M$, i. e. $\sigma_\alpha : N_\alpha \to M_\alpha$ acts on the entires of the matrix.

1.5. Proposition. — *Let* $N \subset M$ *be type* II_1 *factors with finite index* $\alpha=[M:N]$. *Let* $M \subset M_1 \subset M_2$ *be the factors obtained by iterating the basic construction, i. e.* M_1 *is the extension of* M *by* N *and* M_2 *is the extension of* M_1 *by* M. *If* $i : N \to M$, $j : M_1 \to M_2$ *are the inclusion maps then there exist isomorphisms* $\rho : N_\alpha \to M_1$, $\sigma : M_\alpha \to M_2$ *such that the diagram*

$$\begin{array}{ccc} N_\alpha & \xrightarrow{i_\alpha} & M_\alpha \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\sigma} \\ M_1 & \xrightarrow{j} & M_2 \end{array}$$

commutes.

Proof. — Let us first show that any family $\{m_i\}_{1 \leqq j \leqq n+1} \subset M$ as described in the preceding proposition defines a $*$-isomorphism $\rho : N_\alpha \to M_1$ by $\rho((y_{ij})_{1 \leqq i,\, j \leqq n+1}) = \sum_{i,\, j} m_i y_{ij} e_N m^*_j$. Since $e_N m e_N = E_N(m) e_N$, ρ is easily seen to be a $*$-homomorphism.

Since the algebras involved are simple, ρ is injective. To show that it is surjective, let $x \in M_1 (= (M \cup \{e_N\})'')$ and write

$$x = (\sum_i m_i e_N m^*_i) x (\sum_j m_j e_N m^*_j) = \sum_{i,\, j} m_i e_N m^*_i x m_j e_N m^*_j.$$

By 1.2 there exist elements $a_{ij} \in M$ such that $m^*_i x m_j e_N = a_{ij} e_N$ so that $x = \sum_{i,\, j} m_i e_N a_{ij} e_N m^*_j = \sum_{i,\, j} m_i E_N(a_{ij}) e_N m^*_j$. This concludes the proof that ρ is an isomorphism.

To prove the commutativity of the diagram note that the family $\{\alpha^{-1/2} m_j e_N\}_{1 \leqq j \leqq n+1} \subset M_1$ satisfies the properties (*a*), (b), (*c*) of 1.3 for the pair $M \subset M_1$. By the first part of the proof this family implements an isomorphism σ of M_α onto M_2. So all we have to prove is that the map

$$M_\alpha \ni (x_{ij})_{i,\, j} \overset{\sigma}{\mapsto} \sum_{i,\, j} (\alpha^{-1/2} m_i e_N) x_{ij} e_M \cdot (\alpha^{-1/2} e_N m^*_j) \in M_2$$

takes values in M_1 when restricted to N_α. This follows from the fact that $e_N e_M e_N = [M:N]^{-1} e_N = \alpha^{-1} e_N$ (*cf.* [13]).

Q.E.D.

1.6. *Remark.* — The isomorphism between N_α and the extension of M by N, M_1, in the preceding proposition is just the fact that N and M_1 are Morita equivalent via M. The second statement may be viewed as an analogue of the duality for crossed products by finite groups, and express the fact that M_2 is obtained by inducing the module M.

The next proposition gives another type of duality results, connecting unitaries in M to projections in M_1 (the extension of M by N).

1.7. PROPOSITION. — *Let* $\mathscr{U}(M)$ *denote the unitary group of* M *and* $\mathscr{N}(N)$ *the normalizer of* N *in* M. *For* $u \in \mathscr{U}(M)$ *let* $\varphi(u) = u e_N u^* \in M_1$.

(i) *The map* φ *induces a one to one correspondence between* $\mathscr{U}(M)/\mathscr{U}(N)$ *and the projections* p *in* M_1 *with* $E_M(p) = [M:N]^{-1} . 1_M$. *Moreover* $E_N(u) = 0$ *iff* $e_N \varphi(u) = 0$.

Q.E.D.

(ii) *The map* φ *induces a one to one correspondence between* $\mathscr{N}(N)/\mathscr{U}(N)$ *and the projections* p *in* $N' \cap M_1$ *with* $E_M(p) = [M:N]^{-1} . 1_M$.

Proof. — (i) Since e_N commutes with $\mathscr{U}(N)$ and $E_M(u e_N u^*) = \lambda u u^* = \lambda$ (where $\lambda = [M:N]^{-1}$), φ induces the desired map. If $\varphi(u_1) = \varphi(u_2)$ then $u_2^* u_1 e_N = e_N u_2^* u_1$, so that $u_2^* u_1 \in N$ (*cf.* 0.2). To see that φ is onto let $p \in M_1$ be a projection with $E_M(p) = \lambda$. In particular $\tau(p) = \lambda$ so that p is equivalent in M_1 with e_N. By 1.2 there is $m \in M$ such that $m e_N m^* = p$ and applying E_M on both sides we see that $m m^* = 1$, i.e. $m \in \mathscr{U}(M)$.

Moreover if $u \in \mathscr{U}(M)$ then $E_N(u) = 0$ iff $e_N u e_N = 0$ iff $e_N u e_N u^* = 0$.

(ii) If $u \in \mathscr{N}(N)$ and $y \in N$ then

$$u e_N u^* y = u e_N (u^* y u) u^* = u (u^* y u) e_N u^* = y u e_N u^*.$$

Conversely if $u e_N u^* y = y u e_N u^*$ then $u^* y u$ commutes with e_N so that $u^* y u \in N$. So we have to prove only the surjectivity, which follows by (i).

Q.E.D.

We show now that the downward basic construction ([13], 3.1.9; *see* Section 0) is unique up to unitary conjugacy.

1.8. COROLLARY. — *Let* $N \subset M$ *be type* II_1 *factors with* $[M:N] < \infty$.

(i) *If* $e \in M$ *is a projection such that* $E_N(e) = [M:N]^{-1} . 1_M$ *then* $P = \{e\}' \cap N$ *is a type* II_1 *factor,* $[N:P] = [M:N]$ *and* $e y e = E_P(y) e$ *for all* $y \in N$. *Thus* M *is the extension of* N *by* P.

(ii) *If* $e_1, e_2 \in M$ *are projections such that* $E_N(e_i) = [M:N]^{-1} . 1_N$, $i = 1, 2$, *then there exists a unitary element* $u \in N$ *such that* $u e_1 u^* = e_2$. *Moreover if* $P_i = \{e_i\}' \cap N$ *are as in* (i) *then* $u P_1 u^* = P_2$.

Proof. — By [13], 3.1.9 there exists a projection $e_0 \in M$ and a subfactor $P_0 \subset N$ such that $E_N(e_0) = [M:N]^{-1} . 1_N$, $[e_0, P_0] = 0$, $e_0 y e_0 = E_{P_0}(y) e_0$ for all $y \in N$ and $[N:P_0] = [M:N]$. If $e \in M$ is another projection with $E_N(e) = [M:N]^{-1} . 1_N$ then applying 1.7 (i) to the pair $P_0 \subset N$ it follows that there exists a unitary $u \in N$ such that $u e_0 u^* = e$. Thus $u P_0 u^* = u(\{e_0\}' \cap N) u^* = \{e\}' \cap N = P$ and the rest of the statement follows now easily.

Q.E.D.

The next proposition is motivated by the following example. If M is a II_1 factor and G is a finite group of outer automorphisms of M, then the relative commutant of the fixed point algebra $M^G = N$ in $M \rtimes G$ is isomorphic to $L(G)$. Moreover the extension of M by N, M_1, is isomorphic to $M \rtimes G$ and $e_N = e_{M^G}$ corresponds to the projection in $L(G)$ determined by the trivial representation of G.

1.9. PROPOSITION. — *Let* $N \subset M$ *be* II_1 *factors with* $[M:N] = \lambda^{-1} < \infty$. *Suppose that* $N' \cap M = \mathbb{C}.1$.

(1) *The trace of every projection p in* $N' \cap M_1$ *is greater than or equal to* λ.

(2) *If the trace of the projection* $p \in N' \cap M_1$ *equals* λ, *then p is central.*

(3) *More generally if* $p \in N' \cap M_1$ *and* $\tau(p) < (k+1)\lambda$ *then p lies in a factor-summand of* $N' \cap M_1$ *of dimension at most k (as usual* M_1 *is the extension of* M *by* N).

Proof. — It is enough to prove that it is impossible to have $k+1$ mutually orthogonal projections $p_j \in N' \cap M_1$ of trace less than $(k+1)\lambda$ which are pairwise equivalent in $N' \cap M_1$.

Suppose the contrary and choose partial isometries

$$v_{i,s} \in M_1, \qquad 1 \leqq i \leqq k+1, \quad 1 \leqq s \leqq l+1$$

where $\alpha := \tau(p_j) = \lambda . l + \nu$, $0 \leqq \nu < \lambda$, such that

$$v_{i,s}^* v_{i,s} = e_N, \qquad 1 \leqq s \leqq l, \qquad 1 \leqq i \leqq k+1$$

$$v_{i,l+1}^* v_{i,l+1} = f \leqq e_N, \qquad 1 \leqq i \leqq k+1$$

$$\sum_{s=1}^{l+1} v_{i,s} v_{i,s}^* = p_i, \qquad 1 \leqq i \leqq k+1$$

and

$$w_i = \sum_s v_{is} e_N v_{k+1,s}^*$$

$1 \leqq i \leqq k$ are partial isometries in $N' \cap M_1$ such that

$$w_i w_i^* = p_i, \qquad 1 \leqq i \leqq k$$

$$w_i^* w_i = p_{k+1}.$$

Put also $w_{k+1} = p_{k+1}$.

Note that $e_{ij} = w_i w_j^*$, $1 \leqq i, j \leqq k+1$, is a system of matrix units in $N' \cap M_1$ such that

$$e_{ii} = p_i, \qquad 1 \leqq i \leqq k+1,$$

$$e_{ij} = \sum_{s=1}^{l+1} v_{i,s} e_N v_{js}^*.$$

Lemma 1.2 implies that each v_{is} is of the form $m_{i,s} e_N$ for some $m_{i,s} \in M$. Since $e_{ij} \in N' \cap M_1$ it follows that $E_M(e_{ij}) \in N' \cap M = \mathbb{C}.1$ so that

$$E_M(e_{ij}) = \delta_{ij} \tau(p_j) = \alpha \delta_{ij}$$

Applying the conditional expectation E_M on both sides of the equality

$$(*) \qquad e_{ij}=\sum_{s=1}^{l+1} v_{is}\, e_N\, v_{js}^* = \sum_{s=1}^{l+1} m_{is}\, e_N\, m_{js}^*$$

we get

$$\alpha\delta_{ij}=\lambda \sum_{s=1}^{l+1} m_{is}\, m_{js}^*, \qquad 1 \leqq i, j \leqq k+1.$$

Since $l \leqq k$ the above relations show that the matrix

$$(\lambda/\alpha)^{1/2}\, m = ((\lambda/\alpha)^{1/2}\, m_{ts})_{t,\,s} \in \mathscr{M}_{k+1}(\mathrm{M})$$

where

$$m_{t,\,s}=\begin{cases} m_{t,\,s} & \text{for } \ t \leqq l, \\ 0 & \text{for } \ t > 1 \end{cases}$$

is a unitary operator, so that $l=k$. Moreover if we denote by $\tilde{e}$ the diagonal matrix diag $(e_N, e_N, \ldots, e_N)$ and by a the matrix $a=(e_{ij})_{i,\,j}$, formula $(*)$ shows that $m\tilde{e}m^*=a$ and since $(\lambda/\alpha)^{1/2}\, m$ is unitary, $(\lambda/\alpha)\; a$ must be a projection. But $a^2=(k+1)\, a$ so that $\alpha=(k+1)\,\lambda$ which contradicts our assumption that $\alpha=\tau(p_j)<(k+1)\,\lambda$.

Q.E.D.

It is natural to expect that if $\mathrm{N} \subset \mathrm{M}$ are II_1 factors with finite index $[\mathrm{M}:\mathrm{N}]$ then M and N share many properties, or even that they are isomorphic. For instance, by Connes' theorem it follows that M is hyperfinite iff N is (*cf.* [13]). It turns out that in general M and N may be nonisomorphic. In fact it is proved in [8] that there exists a type II_1 factor N with a period 2 automorphism such that if $\mathrm{M}=\mathrm{N} \rtimes \mathbb{Z}/2\mathbb{Z}$ then $\chi(\mathrm{M}) \neq \chi(\mathrm{N})$, where χ is the Connes invariant. Thus $[\mathrm{M}:\mathrm{N}]=2$ but $\mathrm{M} \simeq \mathrm{N}$.

Yet there are some important properties that M and N have in common: existence of nontrivial central sequences (i. e. property Γ of Murray and von Neumann [18]) or splitting by R (i. e. McDuff's property [17]). In order to prove these results we first need to relate the index $[\mathrm{M}:\mathrm{N}]$ with the index of the corresponding ultrapower factors. This will be a simple consequence of 1.3:

1.10. Proposition. — *Let* $\mathrm{N} \subset \mathrm{M}$ *be type* II_1 *factors,* ω *a free ultrafilter on* $\mathbb{N}$ *and* $\mathrm{N}^\omega \subset \mathrm{M}^\omega$ *the corresponding ultrapower factors* [17]. *Then* $[\mathrm{M}^\omega:\mathrm{N}^\omega]=[\mathrm{M}:\mathrm{N}]$.

Proof. — If $[\mathrm{M}:\mathrm{N}]<\infty$ then let $\{m_i\}_{1 \leqq i \leqq n+1}$ be as in 1.3 an "orthonormal basis" of M over N. We claim that $\{m_i\}_{1 \leqq i \leqq n+1}$ is also a basis of M^ω as a module over N^ω. Indeed, since $E_{N^\omega}((x_n)_n)=(E_N(x_n))_n$ it follows that $\{m_j \mathrm{N}^\omega\}_{1 \leqq j \leqq n+1}$ are mutually orthogonal subspaces and that $(m_j E_N(m_j^* x_n))_n$ is the orthogonal projection of $x=(x_n)_n \in \mathrm{M}^\omega$ onto $m_j \mathrm{N}^\omega$. Moreover $x^j=(m_j E_N(m_j^* x_n))_n \in m_j \mathrm{N}^\omega$ and $\sum_{j=1}^{n+1} x^j = x$.

This shows in particular that if $[\mathrm{M}:\mathrm{N}]<\infty$ then $[\mathrm{M}:\mathrm{N}]=[\mathrm{M}^\omega:\mathrm{N}^\omega]$.

To end the proof note that $L^2(M, \tau)$ is canonically imbedded in $L^2(M^\omega, \tau_\omega)$ and that if $\xi, \eta \in L^2(M, \tau)$ are such that $N\xi$ and $N\eta$ are mutually orthogonal in $L^2(M, \tau)$ then $N^\omega \xi$ and $N^\omega \eta$ are mutually orthogonal in $L^2(M^\omega, \tau_\omega)$. Moreover if ξ_0 denotes the image of 1 in $L^2(M, \tau)$ then $[N\xi]$ is equivalent to $[N\xi_0]$ in N' iff ξ is separating for N, i.e. $x \in N$, $x\xi = 0$ implies $x = 0$. But it is easily seen that this holds iff $x \in N^\omega$, $x\xi = 0$ implies $x = 0$, which in turn means that $[N^\omega \xi]$ is equivalent to $[N^\omega \xi_0]$ in $(N^\omega)' \subset \mathscr{B}(L^2(M^\omega, \tau_\omega))$. This shows that $[M^\omega : N^\omega] \geqq [M : N]$ and thus if $[M : N] = \infty$ then $[M^\omega : N^\omega] = \infty$.

Q.E.D.

1.11. Proposition. — *Let* M *be a type* II_1 *factor* $N \subset M$ *a subfactor of finite index.*

(i) M *is a full factor iff* N *is.*

(ii) M *is a McDuff factor iff* N *is.*

Proof. — In both (i) and (ii) we only need to prove one implication. This is because by 0.6 N is isomorphic to a reduced algebra of $M_1 (= (M \cup \{e_N\})'')$ and because each of the above properties is invariant to amplification (*cf.* [7]). Fix ω a free ultrafilter on $\mathbb{N}$.

(i) By Connes' results [6] we only need to show that if $N' \cap N^\omega = \mathbb{C}$ then $M' \cap M^\omega$ has atoms. We shall actually prove that $N' \cap M^\omega$ has atoms. Suppose on the contrary that $N' \cap M^\omega$ is completely nonatomic.

But $\mathbb{C} = N' \cap N^\omega \subset N' \cap M^\omega$ and $E_{N^\omega}(N' \cap M^\omega) \subset N' \cap N^\omega = \mathbb{C}$, i'e. $N' \cap M^\omega$ is orthogonal to N^ω (*see* [21]). Since $N' \cap M^\omega$ is completely nonatomic we can find an infinite set of unitaries $\{u_n\}_{n \in \mathbb{N}}$ in $N' \cap M^\omega$ such that $\tau(u_n u_m^*) = 0$ for $n \neq m$. Thus $\{u_n N^\omega\}_n$ are mutually orthogonal in M^ω with respect to the trace so that $[M^\omega : N^\omega] = \infty$, in contradiction with 1.10.

(ii) Suppose M is a McDuff factor and N is not. By (i) it follows that N has property Γ so that $N' \cap N^\omega$ is a diffuse abelian algebra (*cf.* [6]). Since $M' \cap M^\omega$ is a type II_1 von Neumann algebra (*cf.* [6]) it follows that $N' \cap M^\omega \supset M' \cap M^\omega$ is of type II_1. As we pointed out in (i) we have $E_{N^\omega}(N' \cap M^\omega) \subset N' \cap N^\omega$. Thus if $y \in N' \cap M^\omega$ is such that $E_{N' \cap N^\omega}(y) = 0$ then $E_{N^\omega}(y) = 0$. Let $n \geqq 1$ such that $2^n > [M : N]$. Since $N' \cap M^\omega$ is of type II_1 and $N' \cap N^\omega$ is abelian we can find, as in [21], a unitary element $u \in N' \cap M^\omega$ such that $E_{N' \cap N^\omega}(u^k) = 0$ for $1 \leqq k \leqq 2^n - 1$. It follows that $E_{N^\omega}(u^k) = 0$ so that $\{u^k N^\omega\}_{0 \leqq k \leqq 2^n - 1}$ are mutually orthogonal in M^ω. Thus $[M^\omega : N^\omega] \geqq 2^n > [M : N]$, contradicting 1.10.

Q.E.D.

We end this section by mentioning an interesting consequence of [20] and of Jones result that the basic construction is generic for factors with finite index.

1.12. Proposition. — *Let* $N \subset M$ *be separable type* II_1 *factors with* $[M : N] < \infty$. *If any maximal abelian subalgebra of* N *is maximal abelian in* M *then* $M = N$.

Proof. — By [13] there exists a projection $e_1 \in M$ and $N_1 \subset N$ such that $e_1 x e_1 = E_{N_1}(x) e_1$ for $x \in N$, $[e_1, N_1] = 0$, $E_N(e_1) = [M : N]^{-1}$. By the hypothesis it follows that $N' \cap M = \mathbb{C}$ so that $N_1' \cap N = \mathbb{C}$ (*cf.* e.g. 1.5). Thus by [20] N_1 has a maximal abelian subalgebra $A_1 \subset N_1$ which is maximal abelian in N. But e_1 commutes with $A_1 (\subset N_1)$ and $E_{A_1}(e_1) = E_{A_1} E_N(e_1) = [M:N]^{-1}$ so that e_1 is not in A_1, unless $e_1 = 1$. Thus if A_1 is maximal abelian in M then $[M:N] = 1$ and $M = N$.

Q.E.D.

The preceding proposition is related to a well known problem of R. V. Kadison asking whether if $N \subset M$ are type II_1 factors and any maximal abelian subalgebra in N is maximal abelian in M then $M = N$. By a counterexample in [21] this fails to be true if the index $[M:N]$ is infinite. Thus 1.12 seem to be the best positive result that can be obtained in this direction.

2. Some formulas for the index

In the first section we considered the index of N in M only as a module-dimension. We shall now provide a more analytical characterization of $[M:N]$, depending on the behaviour of the conditional expectation E_N on the positive cone of M: the number $[M:N]$ will show how "flat" $E_N(x)$ can be, compared to x, $x \in M_+$.

2.1. Proposition. — *Let* M *be a type* II_1 *factor and* $N \subset M$ *a subfactor of finite index* $k = [M:N]$. *Then* $E_N(x) \geq k^{-1} x$ *for all* $x \in M_+$.

Proof. — Since $k < \infty$, M is the extension of N by some subfactor $N_1 \subset N$ (*cf.* [13], 3.1.9). Denote by $e \in M$ a projection implementing the conditional expectation of N onto N_1, i.e. $E_N(e) = k^{-1}$, $[e, N_1] = 0$, $eye = E_{N_1}(y)e$, for all $y \in N$. If $x \in M_+$ then by the preceding section there exist $a_1, \ldots, a_n, b_1, \ldots, b_n \in N$ such that

$$x = (\sum_j a_j e b_j)^* (\sum_i a_i e b_i) = \sum_{i,j} b_j^* E_{N_1}(a_j^* a_i) e b_i.$$

Since $(a_j^* a_i)_{i,j}$ is a positive matrix and E_{N_1} is completely positive it follows that $(E_{N_1}(a_j^* a_i))_{i,j}$ is also positive. Thus there exists a matrix $(c_{rs})_{r,s}$, $c_{rs} \in N_1$ such that $E_{N_1}(a_j^* a_i) = \sum_k c_{kj}^* c_{ki}$, for all i, j. If we denote by $\tilde{c}$ the matrix $(c_{rs})_{r,s}$, $\tilde{b}$ the column matrix

$$\begin{matrix} b_1 \\ \vdots \\ b_n \end{matrix} \quad \text{and} \quad \tilde{e} = \begin{pmatrix} e & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & e \end{pmatrix},$$

then we get

$$\begin{aligned} x = \sum_{i,j} b_j^* E_{N_1}(a_j^* a_i) e b_i &= \tilde{b}^* \tilde{c}^* \widetilde{e c b} \leq \tilde{b}^* \tilde{c}^* \widetilde{cb} \\ &= \tilde{b}^* (E_{N_1}(a_j^* a_i))_{i,j} \tilde{b} = \sum_{i,j} b_j^* E_{N_1}(a_j^* a_i) b_i \\ &= k E_N(\sum_{i,j} b_j^* E_{N_1}(a_j^* a_i) e b_i) = k E_N(x). \end{aligned}$$

Q.E.D.

In fact the existence of a constant k with the above property caracterizes the finite index. Moreover $k=[\mathrm{M}:\mathrm{N}]$ is best possible in the inequality 2.1. More precisely we have the following:

2.2. THEOREM. — *If* N *is a subfactor of the type* II_1 *factor* M *then*

$$[\mathrm{M}:\mathrm{N}]^{-1}=\max\{\lambda\in\mathbb{R}_+ \mid \mathrm{E_N}(x)\geqq\lambda x, x\in\mathrm{M}_+\}$$
$$=\inf\{\|\mathrm{E_N}(x)\|_2^2/\|x\|_2^2 \mid x\in\mathrm{M}_+, x\neq 0\}$$
$$=\inf\{\|\mathrm{E_N}(x)\|/\|x\| \mid x\in\mathrm{M}_+, x\neq 0\}$$
$$=\inf\{\|\mathrm{E_N}(f)\| \mid f \text{ nonzero projection in } \mathrm{M}\}.$$

Proof. — Denote

$$\lambda_1=\max\{\lambda\in\mathbb{R}_+ \mid \mathrm{E_N}(x)\geqq\lambda x, x\in\mathrm{M}_+\},$$
$$\lambda_2=\inf\{\|\mathrm{E_N}(y)\|_2^2/\|x\|_2^2 \mid x\in\mathrm{M}_+, x\neq 0\},$$
$$\lambda_3=\inf\|\mathrm{E_N}(x)\|/\|x\| \mid x\in\mathrm{M}_+, x\neq 0\},$$
$$\lambda_4=\inf\{\|\mathrm{E_N}(f)\| \mid f\in\mathrm{M} \text{ nonzero projection}\}.$$

Obviously $\lambda_1\leqq\lambda_3\leqq\lambda_4$. Also, if $\mathrm{E_N}(x)\geqq\lambda x$ for some $x\in\mathrm{M}^+$, $\lambda\in\mathbb{R}_+$, then $\tau(\mathrm{E_N}(x)^2)=\tau(x\,\mathrm{E_N}(x))\geqq\lambda\tau(x^2)$, so that $\|\mathrm{E_N}(x)\|_2^2\geqq\lambda\|x\|_2^2$. This shows that $\lambda_1\leqq\lambda_2$. Moreover if $f\in\mathrm{M}$ is a projection, then

$$\|\mathrm{E_N}(f)\|_2^2=\tau(\mathrm{E_N}(f)^2)\leqq\|\mathrm{E_N}(f)\|\,\tau(\mathrm{E_N}(f))=\|\mathrm{E_N}(f)\|\,\tau(f)=\|\mathrm{E_N}(f)\|\,\|f\|_2^2.$$

Thus $\lambda_2\leqq\lambda_4$.

Now if the index $[\mathrm{M}:\mathrm{N}]$ is finite then by 2.1 $[\mathrm{M}:\mathrm{N}]^{-1}\leqq\lambda_1$ and by [13] there exists a projection $e\in\mathrm{M}$ such that $\mathrm{E_N}(e)=[\mathrm{M}:\mathrm{N}]^{-1}$. Thus $\lambda_4\leqq\|\mathrm{E_N}(e)\|=[\mathrm{M}:\mathrm{N}]^{-1}$.

For the proof of the case $[\mathrm{M}:\mathrm{N}]=\infty$ we need a technical result. Its proof is the same as that of 2.4 in [20] so we give here only a sketch.

2.3. LEMMA. — *Let* M_1 *be a type* II_∞ *factor with semifinite trace* φ *and* $\mathrm{M}\subset\mathrm{M}_1$ *a type* II_1 *subfactor. Assume that* $\mathrm{M}'\cap\mathrm{M}_1$ *contains no finite projections of* M_1, *then for any* $\varepsilon>0$ *and* $x\in\mathrm{M}_{1+}$ *with* $\varphi(x)<\infty$ *there exist projections* $e_1,\ldots,e_n\in\mathrm{M}$ *such that* $\sum e_i=1$ *and* $\|\sum c_i x e_i\|_\varphi<\varepsilon\|x\|_\varphi$ ($\|x\|_\varphi=\varphi(x^*x)^{1/2}$ *is the Hilbert norm given by* φ).

Proof. — We may suppose $x\neq 0$ and first prove that there exists a unitary $u\in\mathrm{M}$ such that $\|uxu^*-x\|_\varphi>\|x\|_\varphi$. To do this let $\mathrm{K}_x=\overline{\mathrm{co}}^w\{vxv^* \mid v \text{ unitary element in } \mathrm{M}\}$, so that K_x is a weakly compact convex set in M_1 and $y\geqq 0$, $\varphi(y)\leqq\varphi(x)$ for all $y\in\mathrm{K}_x$. By the weak inferior semicontinuity of the norm $\|\ \|_\varphi$ there exists $y_0\in\mathrm{K}_x$ such that $\|y_0\|_\varphi=\inf\{\|y\|_\varphi \mid y\in\mathrm{K}_x\}$. Since $\|\ \|_\varphi$ is a Hilbert norm and K_x is convex, y_0 is unique with this property. But $vy_0v^*\in\mathrm{K}_x$ for all unitaries $v\in\mathrm{M}$ and $\|vy_0v^*\|_\varphi=\|y_0\|_\varphi$ so that

$vy_0v^*=y_0$ and thus $y_0 \in M' \cap M_1$. As $y_0 \geqq 0$, $\varphi(y_0)<\infty$ it follows by the hypothesis that $y_0=0$. Now if $\|vxv^*-x\|_\varphi^2 \leqq \|x\|_\varphi^2$ for all unitaries $v \in M$, we get $2\operatorname{Re}\varphi(x^* vxv^*) \geqq \|x\|_\varphi^2$ so that $2\operatorname{Re}\varphi(x^* y) \geqq \|x\|_\varphi^2$, for all $y \in K_x$, which for $y=y_0=0$ gives $0 \geqq \|x\|_\varphi$, a contradiction.

By an approximation argument (e. g. using spectral decomposition) we may assume that the unitary $u \in M$ satisfying $\|uxu^*-x\|_\varphi > \|x\|_\varphi$ has finite spectrum, i. e. it is of the form $u=\sum \lambda_i e_i$ for some scalars $\lambda_i \in \mathbb{C}$, $|\lambda_i|=1$ and projections $e_i \in M$, $\sum e_i=1$. Then we have

$$4\|x\|_\varphi^2-4\|\sum e_i x e_i\|_\varphi^2=4\|\sum_{i\neq j} e_i x e_j\|_\varphi^2 \geqq \|\sum_{i\neq j}(\lambda_i\bar{\lambda}_j-1)e_i x e_j\|_\varphi^2=\|uxu^*-x\|_\varphi^2>\|x\|_\varphi^2,$$

so that

$$\|\sum e_i x e_i\|_\varphi^2=\sum\|e_i x e_i\|_\varphi^2 \leqq 3/4\|x\|_\varphi^2.$$

This proves the statement for $\varepsilon=\sqrt{3}/2$. But for any projection $e \in M$ the algebras $M_e \subset (M_1)_e$ still satisfy that $M_e' \cap (M_1)_e=(M' \cap M_1)_e$ contains no finite projections of $(M_1)_e$. Indeed because if $f \in M' \cap M_1$ is such that $\varphi(ef)<\infty$ then letting $w_1, \ldots, w_m \in M$ be partial isometries such that $\sum_i w_i e w_i^*=1$, we get $\sum w_i e f w_i^*=\sum w_i e w_i^* f=f$ and thus $\varphi(f) \leqq m\varphi(ef)<\infty$, which is a contradiction unless $f=0$. This shows that we can apply recursively s times the inequality for $\varepsilon=\sqrt{3}/2$ to get it for $\varepsilon=(\sqrt{3}/2)^s$.

Q.E.D.

End of the proof of theorem 2.2. — If $[M:N]=\infty$ then obviously $[M:N]^{-1}=0 \leqq \lambda_1$ so, with the preceding notations, we only need to show $\lambda_4=0$. There are two possibilities: $N' \cap M$ is infinite of finite dimensional. If $N' \cap M$ is infinite dimensional then for any $\varepsilon>0$ there exists a projection $f \in N' \cap M$ such that $\varepsilon>\tau(f)>0$. Since N is a factor and $E_N(f) \in N' \cap M$, we get $E_N(f)=\tau(f)1_N$, which shows that $\lambda_4<\varepsilon$ and as ε is arbitrary, $\lambda_4=0$. If $N' \cap M$ has finite dimension then by Jones' formula 0.14 there exists a minimal projection $f_0 \in N' \cap M$ such that $[M_{f_0}:N_{f_0}]=\infty$. Since $f \leqq f_0$ implies $\|E_N(f)\| \leqq \|E_{N_{f_0}}(f)\|$, it is sufficient to show that for any $\varepsilon>0$ there exists $f \in M_{f_0}$ such that $\|E_{N_{f_0}}(f)\|<\varepsilon$. Consequently we may assume $N' \cap M=\mathbb{C}$, $[M:N]=\infty$.

Let $M \subset \mathscr{B}(L^2(M,\tau))$ be in standard form with canonical conjugation J and denote $M_1=JN'J$. Since $[M:N]=\infty$, M_1 is a type II_∞ factor. The condition $N' \cap M=\mathbb{C}$ implies $M' \cap M_1=JN'J \cap JMJ=\mathbb{C}$. Let $e_N=e$ be the extension to $L^2(M,\tau)$ of the conditional expectation of M onto N, as in Section 0. Then $e \in M_1$, $M_1=(M \cup \{e\})''$, e is a finite projection in M_1 and the reduced algebra $(M_1)_e$ is isomorphic to N. We denote by φ the semifinite trace on M_1 that satisfies $\varphi(e)=1$ and by $\|x\|_\varphi=\varphi(x^*x)^{1/2}$, $x \in M_1$ the Hilbert norm given by φ. By 2.3 for any $\varepsilon>0$ there exist projections $f_1, \ldots, f_n \in M$ such that

$$\sum f_i=1 \qquad \text{and} \qquad \sum_i\|f_i e f_i\|_\varphi^2=\|\sum f_i e f_i\|_\varphi^2<\varepsilon^2=\varepsilon^2\sum_i\tau(f_i).$$

It follows that there exists $i \in \{1, \ldots, n\}$ such that $\| f_i e f_i \|_\varphi^2 < \varepsilon^2 \tau(f_i)$. But

$$\| f_i e f_i \|_\varphi^2 = \| e f_i e \|_\varphi^2 = \| \mathrm{E_N}(f_i) e \|_\varphi^2 = \| \mathrm{E_N}(f_i) \|_2^2.$$

Thus $\| \mathrm{E_N}(f_i) \|_2 < \varepsilon \| f_i \|_2$.

Let now p be the spectral projection of $\mathrm{E_N}(f_i)$ corresponding to the interval $[0, \varepsilon^{1/2}]$. Then we have

$$\varepsilon \tau(f_i) = \varepsilon \| f_i \|_2^2 > \tau(\mathrm{E_N}(f_i)^2) \geqq \tau((1-p)\,\mathrm{E_N}(f_i)^2) \geqq \varepsilon \tau(1-p)$$

so that $\tau(p) + \tau(f_i) > 1$. If we denote $q = p \wedge f_i \in \mathrm{M}$ then

$$\tau(q) = \tau(p) + \tau(f_i) - \tau(p \vee f_i) \geqq \tau(p) + \tau(f_i) - 1 > 0.$$

Moreover q satisfies

$$\mathrm{E_N}(q) = \mathrm{E_N}(p \wedge f_i) \leqq \mathrm{E_N}(p f_i p) = p\, \mathrm{E_N}(f_i)\, p \leqq \varepsilon^{1/2} p \leqq \varepsilon^{1/2}.$$

Thus given arbitrary $\varepsilon > 0$ we can find a nonzero projection $q \in \mathrm{M}$ such that $\mathrm{E_N}(q) \leqq \varepsilon^{1/2}$. This shows that $\lambda_4 = 0$.

Q.E.D.

2.4. *Remark.* — One can use the preceding theorem and a maximality argument to show that if $[\mathrm{M}:\mathrm{N}] = \infty$ then for any $\varepsilon > 0$ there exists a nonzero projection $e \in \mathrm{M}$ such that: (i). $\mathrm{E_N}(e) \leqq \varepsilon . 1_\mathrm{N}$; (ii) the spectral projection of $\mathrm{E_N}(e)$ corresponding to the set $\{\varepsilon\}$ has trace $\geqq 1 - \varepsilon$. It would be interesting to decide whether one can find the projection e such that $\mathrm{E_N}(e) = \varepsilon\, 1_\mathrm{N}$.

Each of the four constants involved in the formulas of Theorem 2.2 make sense for any pair of finite von Neumann algebras (and even for arbitrary von Neumann algebras in case there is a normal conditional expectation of M onto N). Therefore we can chose any of them as a replacement of the index for the general case when $\mathrm{N} \subset \mathrm{M}$ are not necessary finite factors. We shall use the first constant because of its close relation to the relative entropy that will be considered in the next two sections.

2.5. Notation. — Let M be a finite von Neumann algebra with faithful trace τ, $\tau(1) = 1$, and B_1, $\mathrm{B}_2 \subset \mathrm{M}$ two von Neumann subalgebras of M, with $\mathrm{B}_2 \subset \mathrm{B}_1$. We denote $\lambda(\mathrm{B}_1, \mathrm{B}_2) = \max\{\lambda \geqq 0 \mid \mathrm{E}_{\mathrm{B}_2}(x) \geqq \lambda x, x \in \mathrm{B}_{1+}\}$.

The consideration of the constant λ is particularly useful to reduce the index problems from the II_1 factor case to problems concerning imbeddings of finite dimensional algebras. To be more precise, let us consider the following situation: Let $\{\mathrm{N}_k\}$, $\{\mathrm{M}_k\}$ be increasing sequences of von Neumann subalgebras of the finite factor M, with $\mathrm{N}_k \subset \mathrm{M}_k$, and assume that M_k generate M and N_k generate a subfactor $\mathrm{N} \subset \mathrm{M}$. Then we wish to have $[\mathrm{M}:\mathrm{N}]^{-1} = \lim \lambda(\mathrm{M}_k, \mathrm{N}_k)$. This is obviously false in general, as we can modify the limit of $\lambda(\mathrm{M}_k, \mathrm{N}_k)$, by taking the limit of $\lambda(\mathrm{M}_{k+p}, \mathrm{N}_k)$ instead. The additional hypothesis that has to be made is $\mathrm{E}_{\mathrm{N}_{k+1}} \mathrm{E}_{\mathrm{M}_k} = \mathrm{E}_{\mathrm{N}_k}$, $k \geqq 1$. Note that this condition is equivalent to $E_{N_{k+1}} E_{M_k} = E_{M_k} E_{N_{k+1}} = E_{N_k}$ and it implies $N_{k+1} \cap M_k = N_k$. In fact for this to hold it is sufficient that $E_{N_{k+1}}(M_k) \subset N_k$.

2.6. Proposition. — (i) *If* $\{B_n\}$, $\{A_n\}$ *are increasing sequences of von Neumann subalgebras in the finite von Neumann algebra* M, *such that* $A_n \subset B_n$, $n \geqq 1$, *and if* $B = \overline{\bigcup B_n}^w$, $A = \overline{\bigcup A_n}^w$, *then* $\lambda(B, A) \geqq \limsup \lambda(B_n, A_n)$. (ii) *If in addition* $E_{A_{n+1}} E_{B_n} = E_{A_n}$, $n \geqq 1$, *then* $\lambda(B, A) = \lim \lambda(B_n, A_n)$, *decreasingly*.

Proof. — Let $\lambda_n = \lambda(B_n, A_n)$, $\lambda = \limsup \lambda(B_n, A_n)$ and $\varepsilon > 0$. Then there exists a subsequence $\{\lambda_{k_n}\}_n$ such that $\lambda_{k_n} \geqq \lambda - \varepsilon$, $n \geqq 1$. Since $\bigcup_n B_n = \bigcup_n B_{k_n}$ it follows that if $x \in \overline{\bigcup_n B_n}^w$ then

$$E_{A_{k_n}}(x) \geqq \lambda_{k_n} E_{B_{k_n}}(x) \geqq (\lambda - \varepsilon) E_{B_{k_n}}(x)$$

and letting $n \to \infty$, $E_A(x) \geqq (\lambda - \varepsilon) E_B(x) = (\lambda - \varepsilon) x$. As $\varepsilon > 0$ is arbitrary we get the first part of the proposition.

If $E_{A_{n+1}} E_{B_n} = E_{A_n}$ then by induction $E_{A_{n+k}} E_{B_n} = E_{A_n}$ and letting $k \to \infty$, $E_A E_{B_n} = E_{A_n}$. Thus if $x \in B_{n+}$ then $E_{A_n}(x) = E_A(x) \geqq \lambda(B, A) x$, so that $\lambda(B, A) \leqq \lambda_n$. This shows that $\lambda(B, A) \leqq \liminf \lambda_n$.

Q.E.D.

A similar proof to that used in Theorem 2.2 for factors shows that if $A \subset B$ then

$$\lambda(B, A) \leqq \lambda_4 = \inf\{\| E_A(f) \| \mid f \in B \text{ nonzero Projection}\}$$

and that

$$\lambda_2 = \inf\left\{\| E_A(x) \|_2^2 / \| x \|_2^2 \mid x \in B_+, x \neq 0\right\},$$

$$\lambda_3 = \inf\{\| E_A(x) \| / \| x \| \mid x \in B_+, x \neq 0\}$$

lie between $\lambda(B, A)$ and λ_4. It is easily seen that λ_2, λ_3, λ_4 satisfy a statement similar to 2.6 (i) and that λ_2 satisfies 2.6 (ii) as well, but it is not clear whether λ_3 and λ_4 also satisfy it. In fact even the problem of whether all these constants coincide or not is open. However we shall prove in Section 6 that for finite dimensional algebras they are all equal to λ. Proposition 2.6 then applies to get the equality for more general pairs of approximately finite dimensional algebras. Indeed if A_n, B_n satisfy the hypothesis of 2.6 (ii) and $\lambda(B_n, A_n) = \lambda_4(B_n, A_n)$ for all n, then $\lambda(B, A) \geqq \lambda_4(B, A)$ and since the opposite inequality allways holds we actually have $\lambda(B, A) = \lambda_4(B, A)$.

It seems to be of great interest to prove (or disprove) that any pair of hyperfinite factors can be constructed as an inductive limit of finite dimensional algebras $\{A_n\}$, $\{B_n\}$ satisfying the conditions of proposition 2.5. In other words: if $R_0 \subset R$ are hyperfinite factors, does there exist an increasing sequence of finite dimensional subalgebras B_n in R with $\overline{\bigcup_n B_n}^w = R$ and such that $E_{B_n} E_{R_0} = E_{R_0} E_{B_n}$? Or at least such that $\overline{\bigcup_n (B_n \cap R_0)}^w = R_0$? This last problem is related to a problem of Sakai (*see* [23] p. 241).

3. Relative entropy: some generalities

In [9] A. Connes and E. Störmer extended the notion of entropy from the classical ergodic theory to the nonabelian frame of operator algebras. Their first step was to define the entropy of a finite dimensional subalgebra and more generally the relative entropy between two finite dimensional subalgebras $B_1, B_2 \subset M$, as a substitute of the entropy and relative entopy for partitions: Let S be the set of all finite families $(x_1, \ldots, x_n)$ of positive elements in M with $\Sigma x_i = 1$ (as usual M is a finite von Neumann algebra with fixed trace τ). If $\eta: [0, \infty) \to (-\infty, \infty)$ is defined by $\eta(t) = -t\ln t$, then

$$(*) \qquad H(B_1 | B_2) = \sup_{(x_i) \in S} \sum_i (\tau\eta\, E_{B_2}(x_i) - \tau\eta\, E_{B_1}(x_i))$$

is the entropy of B_1 relative to B_2. If $B_2 = \mathbb{C}$ then $H(B_1 | \mathbb{C})$ is simply the entropy of B_1 and is denoted by $H(B_1)$. In the particular case when M is commutative, B_1, B_2 are generated by some partitions of the unity P_1, P_2 and $H(B_1 | B_2)$ coincides with the classical relative entropy $h(P_1 | P_2)$.

Connes and Störmer use this relative entropy mainly as a technical tool in the proof of their Kolmogorow-Sinai type theorem. They show that for subalgebras of dimension less than some $n \in \mathbb{N}$ the usual « distance »

$$\delta(B_1, B_2) = \sup \{ \| x - E_{B_2}(x) \|_2 \mid x \in B_1, \| x \| \leqq 1 \}$$

is comparable with $H(B_1 | B_2)$. Thus if δ is small then H is small. However it is easy to see that $\delta \leqq 1$ and that in fact $\delta = 1$ whenever B_1 is "far" from B_2, whereas H may take different values even if $\delta = 1$. In particular if $B_1 \supset B_2$ the relative entropy $H(B_1 | B_2)$ is more appropriate to express the relative size of B_1 with respect to B_2. It is this feature of the relative entropy that we shall exploit here.

Note that the definition $(*)$ does not depend on B_1, B_2 being finite dimensional, so that we may consider $H(B_1 | B_2)$ as in $(*)$ for arbitrary von Neumann subalgebras B_1, $B_2 \subset M$ and allow $H(B_1 | B_2) = \infty$.

In fact we shall consider the case when $B_1 = M$ and $B_2 = N$ is a subalgebra of M. Then $H(M | N)$ is clearly invariant to conjugation of N by τ-preserving automorphisms of M. Our aim in the rest of the paper is to relate this invariant to the index.

In this section we first recall the basic properties concerning the relative entropy and then prove some useful technical results.

1° ln is operator increasing on $(0, \infty)$;

2° η is operator concave on $[0, \infty)$ and operator continuous on $[0, 1]$;

3° if $x, y \in M_+$, $xy = yx$, then $\eta(xy) = \eta(x) y + x \eta(y)$;

4° if $x \in M_+$ then $\eta x = 0$ iff x is a projection;

5° if $x, y \in M_+$, then $\tau\eta(x+y) \leqq \tau\eta\, x + \tau\eta\, y$ and if $xy = 0$ then we have equality;

6° if $B \subset M$ is a von Neumann subalgebra and $x \in M_+$ then $\eta\, E_B(x) \geqq E_B(\eta\, x)$, in particular $\eta\tau(x) \geqq \tau\eta\, E_B(x) \geqq \tau\eta\, x$;

7° if $x \in M_+$ is a scalar multiple of a projection then $\tau\eta\, E_B(x) = \tau\eta\, x$ iff $x \in B$;

8° if $B_1, B_2 \subset M$ are von Neumann subalgebras then $H(B_1 | B_2) \geqq 0$ and $H(B_1 | B_2) = 0$ iff $B_1 \subset B_2$;

9° if $B_1, B_2, B_3 \subset M$ are von Neumann subalgebras then $H(B_1 | B_3) \leqq H(B_1 | B_2) + H(B_2 | B_3)$;

10° $H(B_1 | B_2)$ is increasing in B_1 and decreasing in B_2;

11° if $B \subset M$ is finite dimensional and $e_1, \ldots, e_k \in B$ is a set of minimal projections in B with $\sum e_i = 1$ then $H(B | \mathbb{C}) = H(B) = \sum \eta\tau(e_i)$.

For the proof of 1°, 2°, 6°, *see* [15], [16], [11], [4]. A unified exposition of these results can be found in [1].

3° and 4° are easy consequences of the functional calculus.

To prove 5° (*see* [10]) note first that by adding some small positive scalar we may assume x, y, $x+y$ have the spectrum in $(0, \infty)$. As ln is operator increasing on $(0, \infty)$ we get $\ln(x+y) \geqq \ln x$, $\ln(x+y) \geqq \ln y$ so that $\tau(x \ln(x+y)) \geqq \tau(x \ln x)$, $\tau(y \ln(x+y)) \geqq \tau(y \ln y)$. Summing up we get $\tau\eta(x+y) \geqq \tau\eta\, x + \tau\eta\, y$.

If $x = ce$ in 7°, for some projection $e \in M$ and $c \in \mathbb{R}_+$, then $\tau\eta\, E_B(x) = \eta(c)\, \tau(e) + c\, \tau\eta\, E_B(e)$ and $\tau\eta\, x = \eta(c)\, \tau(e)$ so that $\tau\eta\, E_B(x) = \tau\eta\, x$ iff $\eta\, E_B(e) = 0$. As $E_B(e)$ is a projection iff $E_B(e) = e$, 4° applies to get 7°.

8° is now an easy consequence of 6°, 7°: if $B_1 \not\subset B_2$ then there exists a projection $e \in B_1$ such that $e \notin B_2$, thus

$$H(B_1 | B_2) \geqq (\tau\eta\, E_{B_2}(e) - \tau\eta\, e) + (\tau\eta\, E_{B_2}(1-e) - \tau\eta\,(1-e)) \geqq \tau\eta\, E_{B_2}() > 0.$$

9° and 10° are clear by the definition and by 6°.

11° is proved in [9].

From now on a finite family $x_1, \ldots, x_n \in M$ of positive elements with $\sum x_i = 1$ will be called a partition of the unity in M.

3.1. Lemma. — *If* $B_2 \subset B_1$ *are von Neumann subalgebras in* M *and* $S' \subset S$ *is the set of all finite families* $(x'_0, x'_1, x'_2, \ldots, x'_n)$ *in* S *with each* x'_i, $i \geqq 1$, *a scalar multiple of a projection and* $x'_i \in B_1$, *then*

$$H(B_1 | B_2) = \sup_{(x'_i) \in S'} \sum_i (\tau\eta\, E_{B_2}(x'_i) - \tau\eta\, E_{B_1}(x'_i)).$$

Proof. — It is clear from the definition (∗) that to compute $H(B_1 | B_2)$ it is enough to consider partitions (x_i) in B_1. If $(x_i) \in S$ with all $x_i \in B_1$, then by spectral decomposition there exist $x'_i \in B_1$ such that $0 \leqq x'_i \leqq x_i$, $x_i - x'_i \leqq \varepsilon$ and x'_i have finite spectrum, i.e. $x'_i = \sum_j \alpha_{ij} e_{ij}$ for some scalars $\alpha_{ij} \geqq 0$ and some mutually orthogonal projections $(e_{ij})_j$ in B_1. As η is continuous we get

$$\sum (\tau\eta\, E_{B_2}(x_i) - \tau\eta\, x_i) \leqq \sum (\tau\eta\, E_{B_2}(x'_i) - \tau\eta\, x'_i) + \delta(\varepsilon)$$

where $\delta(\varepsilon) \to 0$ when $\varepsilon \to 0$. By 5° we have

$$\tau\eta\, E_{B_2}(x'_i) \leqq \sum_j \tau\eta\, E_{B_2}(\alpha_{ij} e_{ij}) \qquad \text{and} \qquad \tau\eta\,(x'_i) = \sum_j \tau\eta\,(\alpha_{ij} e_{ij})$$

so that

$$\sum_i (\tau\eta\, E_{B_2}(x_i) - \tau\eta\, x_i) \leqq \sum_{i,\, j} (\tau\eta\, E_{B_2}(\alpha_{ij} e_{ij})$$

$$- \tau\eta\, (\alpha_{ij} e_{ij})) + \delta(\varepsilon) \leqq \sup_{(y_i) \in S'} (\tau\eta\, E_{B_2}(y_i) - \tau\eta\, y_i) + \delta(\varepsilon).$$

Q.E.D.

3.2. LEMMA. — *Let* $B \subset M$ *be a von Neumann subalgebra and* $\{f_n\}_{n \geqq 1}$ *a sequence of projections in* M *with* $\| f_n - 1 \|_2 \to 0$.

(i) *If* $f_n \in B$, $n \geqq 1$, *then* $H(M \mid B) = \lim_n H(M_{f_n} \mid B_{f_n})$.

(ii) *If* $f_n \in B' \cap M$ *and* B *is a factor then again*

$$H(M \mid B) = \lim_n H(M_{f_n} \mid B_{f_n}).$$

[In both (i), (ii) the entropy $H(M_{f_n} \mid B_{f_n})$ is considered with respect to the induced trace $\tau_{f_n}(f_n x f_n) = \tau(f_n)^{-1} \tau(f_n x f_n)$.]

Proof. — Consider first the case when $f_n \in B$. Let $(x_1, \ldots, x_n)$ be a partition of the unity in M_{f_n}. If $E_{B_{f_n}}$ denotes the τ_{f_n}-preserving conditional expectation of M_{f_n} onto B_{f_n} then

$$H(M \mid B) \geqq \sum_i (\tau\eta\, E_B(x_i) - \tau\eta\, x_i) + (\tau\eta\, E_B(1 - f_n) - \tau\eta\, (1 - f_n))$$

$$= \tau(f_n) (\sum \tau_{f_n} \eta\, E_{B_{f_n}}(x_i) - \tau_{f_n} \eta\, x_i) + \tau\eta\, E_B(1 - f_n).$$

This shows that $H(M \mid B) \geqq \limsup_n H(M_{f_n} \mid B_{f_n})$.

Now if $(x_1, \ldots, x_n)$ is a partition of the unity in M then $f_n x_i f_n$ tends strongly to x_i so that $E_{B_{f_n}}(f_n x_i f_n)$ tends strongly to $E_B(x_i)$ and as η is operator continuous

$$\sum_i (\tau_{f_n} \eta\, E_{B_{f_n}}(f_n x_i f_n) - \tau_{f_n}(f_n x_i f_n))$$

tends to $\sum (\tau\eta\, E_B(x_i) - \tau\eta\, x_i)$. This shows that $H(M \mid B) \leqq \liminf_n H(M_{f_n} \mid B_{f_n})$.

If $f_n \in B' \cap M$ then note that for $x \in f_n M f_n$ we have $E_B(f_n) \in B' \cap B = Z(B)$ and $E_B(f_n) E_{B_{f_n}}(x) = f_n E_B(x)$. It follows as before (for $f_n \in B$) that $H(M \mid B) \leqq \liminf H(M_{f_n} \mid B_{f_n})$.

Finally let $x_1, \ldots, x_n$ be a partition of f_n for some fixed n. Then

$$H(M \mid B) \geqq \sum_i (\tau\eta\, E_B(x_i) - \tau\eta\, x_i) \geqq \sum_i (\tau(f_n \eta\, E_B(x_i)) - \tau\eta\, x_i)$$

$$= \sum_i (\tau\eta\, (f_n E_B(x_i)) - \tau\eta\, x_i) = \sum_i (\tau\eta\, (E_B(f_n) E_{B_{f_n}}(x_i)) - \tau\eta\, x_i)$$

$$= \sum_i \tau(E_B(f_n) \eta\, E_{B_{f_n}}(x_i)) + \tau(E_{B_{f_n}}(x_i) \eta\, E_B(f_n)) - \tau\eta\, x_i).$$

Now if B is a factor then $E_B(f_n)=\tau(f_n).1_B$ and by the above inequalities we get $H(M|B)\geqq\tau(f_n)^2\sum_i(\tau_{f_n}\eta E_{B_{f_n}}(x_i)-\tau_{f_n}\eta x_i)$ so that

$$H(M|B)\geqq\limsup_n H(M_{f_n}|B_{f_n}).$$

Q.E.D.

3.3. *Remark.* — In (ii) of the preceding lemma the condition that B is a factor is in fact superfluous. However the proof of the general case requires a much more careful analysis.

Like the constant λ considered in Section 2, the relative entropy H behaves well with respect to inductive limits:

3.4. PROPOSITION. — *If* $\{B_n\}$, $\{A_n\}$ *are increasing sequences of von Neumann subalgebras in* M, *such that* $A_n\subset B_n$, $n\geqq 1$, and if $B=\bigcup B_n^w$, $A=\bigcup A_n^w$, *then* $H(B|A)\leqq\liminf H(B_n|A_n)$. *If in addition* $E_{A_{n+1}}E_{B_n}=E_{A_n}$, $n\geqq 1$, *then* $H(B|A)=\lim_n H(B_n|A_n)$, *increasingly.*

Proof. — Let $x_1, \ldots, x_n$ be a partition of the unity in M. As $E_A(x_i)=\lim_n E_{A_n}(x_i)$, $E_B(x_i)=\lim_n E_{B_n}(x_i)$ in the strong operator topology, it follows that

$$\sum_i(\tau\eta E_A(x_i)-\tau\eta E_B(x_i))=\lim_n\sum_i(\tau\eta E_{A_n}(x_i)-\tau\eta E_{B_n}(x_i))\leqq\liminf_n H(B_n|A_n).$$

If $E_{A_{n+1}}E_{B_n}=E_{A_n}$ then by induction $E_{A_{n+k}}E_{B_n}=E_{A_n}$ so that $E_A E_{B_n}=E_{A_n}$ and if $x_1, \ldots, x_n$ is a partition of the unity in B_n then

$$\sum_i(\tau\eta E_{A_n}(x_i)-\tau\eta x_i)=\sum_i(\tau\eta E_A(x_i)-\tau\eta x_i)$$

so that $H(B_n|A_n)\leqq H(B_n|A)\leqq H(B|A)$, which completes the proof.

Q.E.D.

We now establish the basic inequality that relates the constant λ with the relative entropy H.

3.5. PROPOSITION. — *If* $N\subset M$ *is a von Neumann subalgebra then* $H(M|N)\leqq-\ln\lambda(M, N)$.

Proof. — If $\lambda=\lambda(M, N)$ then $E_N(x)\geqq\lambda x$, $x\in M_+$. Since ln is operator increasing we get

$$x^{1/2}(\ln E_N(x))\,x^{1/2}\geqq x^{1/2}(\ln\lambda x)\,x^{1/2}=x\ln\lambda+x\ln x$$

and thus $\tau\eta\, E_N(x) - \tau\eta\, x \leqq \tau(x) \ln \lambda^{-1}$. Summing up over a partition of the unity (x_i) in M and taking the supremum we get $H(M|N) \leqq \ln \lambda^{-1}$.

Q.E.D.

We end this section with a useful technical lemma. Its proof is standard calculus and will be omitted.

3.6. LEMMA. — *The maximum of the expression*

$$-\sum_i v_i \left(\sum_k w_{ki} \ln w_{ki}\right)$$

on the set $\sum_k w_{ki} = 1$, $\sum_i v_i w_{ki} = \beta_k$, $w_{ki} \geqq 0$, $v_i \geqq 0$ *for every* $i \in I$ *and* $k \in K$, *is*

$$-\sum_k \beta_k \ln \left(\beta_k \left(\sum_l \beta_l\right)^{-1}\right)$$

and is attained only when

$$w_{ki} = \beta_k \left(\sum_l \beta_l\right)^{-1} \textit{for every } i,\ v_i \textit{ being arbitrary.}$$

4. Computation of H (MIN) for type II_1 factors

In this section we shall obtain a formula for the relative entropy $H(M|N)$ in the case M and N are factors. It will depend on both the relative commutant and the index of N in M.

First of all we note a straightforward consequence of 2.2 and 3.4; it will provide an estimate of $H(M|N)$ from above.

4.1. COROLLARY. — **If** M *is a type* II_1 *factor and* $N \subset M$ *a subfactor then* $H(M|N) \leqq \ln [M:N]$.

The next lemma is the key result in proving the estimate of $H(M|N)$ from below.

4.2. LEMMA. — *Let* $N \subset M$ *be arbitrary finite von Neumann algebras and* $q \in M$ *a projection such that* $E_{N' \cap M}(q) = cf$ *for some scalar c and some projection* $f \in N' \cap M$. *Then* $H(M|N) \geqq c^{-1} \tau\eta\, E_N(q)$.

Proof. — Since $\tau(q) = \tau(E_{N' \cap M}(q)) = c\,\tau(f)$ we have $c = \tau(q)\,\tau(f)^{-1}$. Let $x = q - cf$ and $K_x = \overline{\mathrm{co}}^w \{vxv^* \mid v \text{ unitary element in } N\}$. As in the proof of 2.3 it follows that there exists $y_0 \in K_x \cap N'$. But $E_{N' \cap M}(vxv^*) = v\, E_{N' \cap M}(x)\, v^* = E_{N' \cap M}(x) = 0$ (by the uniqueness of the conditional expectation), so that $E_{N' \cap M}(y) = 0$ for all $y \in K_x$, in particular $y_0 = E_{N' \cap M}(y_0) = 0$. Since the closures of $\mathrm{co}\{vxv^* \mid v \text{ unitary element in } N\}$ in the weak and $\| \ \|_2$ topologies coincide it follows that for any $\varepsilon > 0$ there exist unitary elements $v_1, \ldots, v_n$ in N such that

$$\left\| \frac{1}{n} \sum_i v_i q v_i^* - cf \right\|_2 = \left\| \frac{1}{n} \sum_i v_i x v_1^* \right\|_2 < \varepsilon c \| f \|_2.$$

Denote by $y=\sum_i (cn)^{-1} v_i q v_i^*$. Then $\|y-f\|_2 \leqq \varepsilon \|f\|_2$, $0 \leqq y \leqq c^{-1} . 1_M$. Moreover, since

$$0=(1-f) E_{N' \cap M}(q)(1-f)=E_{N' \cap M}((1-f) q (1-f)),\ (1-f) q=0$$

so that $q \leqq f$ and $v_i q v_i^* \leqq f$, thus $y \leqq c^{-1} f$. Let $\delta>0$ and denote by p the spectral projection of y corresponding to $[0, 1+\delta]$ in the algebra $f \mathrm{M} f$. Then

$$y(f-p) \geqq (1+\delta)(f-p)$$

so that

$$\|y-f\|_2^2 \geqq \tau((y-f)^2 (f-p)) \geqq \delta^2 \tau(f-p).$$

Thus

$$\|f-p\|_2^2 \leqq \delta^{-2} \|y-f\|_2^2 \leqq (\varepsilon\delta^{-1})^2 \tau(f).$$

Denote

$$x_i=((1+\delta) cn)^{-1} p \wedge v_i q v_i^*$$

and

$$y_i=((1+\delta) cn)^{-1} (v_i q v_i^* - p \wedge v_i q v_i^*).$$

It follows that x_i, $y_i \geqq 0$, $\sum x_i \leqq (1+\delta)^{-1} pyp \leqq f$, $x_i y_i=0$. Applying first 5° then 6° of section 3 we obtain:

$$\begin{aligned}
\sum_i (\tau\eta\, E_N(x_i)-\tau\eta\, x_i) &\geqq \sum_i (\tau\eta\, E_N(((1+\delta) cn)^{-1} v_i q v_i^*) \\
&\quad -\tau\eta\, (((1+\delta) cn)^{-1} v_i q v_i^*)) - \sum_i (\tau\eta\, E_N(y_i)-\tau\eta\, y_i) \\
&\geqq n\,\eta\, ((1+\delta) cn)^{-1} \tau(q) + n ((1+\delta) cn)^{-1} \tau\eta\, E_N(q) \\
&\quad - n\,\eta\, (((1+\delta) cn)^{-1}) \tau(q) - \sum_i (\eta\tau\, y_i - \tau\eta\, y_i) \\
&= ((1+\delta) c)^{-1} \tau\eta\, E_N(q) - \sum_i ((1+\delta) cn)^{-1} \eta\tau (v_i q v_i^* - p \wedge v_i q v_i^*).
\end{aligned}$$

Since $v_i q v_i^*$, $p \leqq f$ we have

$$\tau(p \wedge v_i q v_i^*) \geqq \tau(q)+\tau(p)-\tau(f) \geqq \tau(q)-(\varepsilon\delta^{-1})^2 \tau(f)$$

so that $\tau(v_i q v_i^* - p \wedge v_i q v_i^*) \leqq (\varepsilon\delta^{-1})^2 \tau(f)$. But for ε small enough, $(\varepsilon\delta^{-1})^2 \tau(f) \leqq e^{-1}$ and because η is increasing on $[0, e^{-1}]$ we obtain $\eta\tau (v_i q v_i^* - p \wedge v_i q v_i^*) \leqq \eta((\varepsilon\delta^{-1})^2 \tau(f))$, so that by the preceding inequalities:

$$\begin{aligned}
H(M \mid N) &\geqq (1+\delta)^{-1} c^{-1} \tau\eta\, E_N(q) - n (1+\delta)^{-1} n^{-1} c^{-1} \eta((\varepsilon\delta^{-1})^2 \tau(f)) \\
&= (1+\delta)^{-1} c^{-1} \tau\eta\, E_N(q) - (1+\delta)^{-1} c^{-1} \eta((\varepsilon\delta^{-1})^2 \tau(f)).
\end{aligned}$$

Letting now first $\varepsilon \to 0$ and then $\delta \to 0$ we get the result.

Q.E.D.

4.3. Lemma. — *Let* M *be a finite factor and* $\{e_n\}_n$ *a sequence of projections in* M *with* $\sum e_n = 1$. *If* $B = \{e_n\}'_n \cap M$ *then* $H(M \mid B) = \sum_n \eta\tau(e_n)$.

Proof. — By 3.2 we may assume that $\{e_n\}_n = \{e_1, e_2, \ldots, e_m\}$ is finite. Let $0 \neq q_i \leqq e_i$ be some mutually equivalent projections and choose $\{v_{ij}\}_{1 \leqq i,\, j \leqq m}$ a set of matrix units such that $v_{ii} = q_i$ (this is possible because M is a factor). If $q = \sum_{i,\, j} (\tau(e_i)\tau(e_j))^{1/2} v_{ij}$ then it is easy to see that q is a projection of the same trace as q_i. Moreover if $B_0 = B' \cap M = \{e_i\}''$ then $E_{B_0}(q)$ is a scalar multiple of the identity and since $\tau E_{B_0}(q) = \tau(q)$, $E_{B_0}(q) = \tau(q).1_M$. By the preceding lemma we get

$$H(M \mid B) \geqq \tau(q)^{-1} \tau\eta\, E_B(q) = \tau(q)^{-1} \sum_i \eta(\tau(e_i))\tau(q_i) = \sum_i \eta\tau(e_i).$$

For the opposite inequality let $\{c_j f_j\}_j$ be a finite partition of the unity in M, $c_j \in \mathbb{R}_+$, f_j projections. For any $\varepsilon > 0$ there exists a refinement $\{g_{jk}\}_k$ of f_j and some nonnegative scalars α^i_{jk} such that

$$\sum_k g_{jk} = f_j \text{ and } 0 \leqq g_{jk} e_i g_{jk} - \alpha^i_{jk} g_{jk} \leqq \varepsilon g_{jk}$$

for all i, j, k. Indeed, by spectral decomposition of $f_j e_i f_j$ one can find g^1_{jk} satisfying the conditions for $i = 1$. Then apply the same argument to $g^1_{jk} e_2 g^1_{jk}$ to get projections g^2_{js} satisfying the conditions for $i = 2$. Since g^2_{js} is a refinement of g^1_{jk} it will also satisfy the conditions for $i = 1$. Recursively we finally obtain the projections $g^m_{jr} = g_{jr}$ that satisfy the conditions for all $i = 1, 2, \ldots, m$.

It follows that $\{c_j g_{jk}\}_{j,\, k}$ is also a partition of the unity (since $\sum_k c_j g_{jk} = c_j f_j$) and that

$$\sum_{j,\, k} (\tau\eta\, E_B(c_j g_{jk}) - \tau\eta\, c_j g_{jk}) \geqq \sum_j (\tau\eta\, E_B(\sum_k c_j g_{jk}) - \tau\eta(\sum_k c_j g_{jk})) = \sum_j (\tau\eta\, E_B(c_j f_j) - \tau\eta\, c_j f_j).$$

Moreover using 3°, 5° and that η is increasing on $[0, e^{-1}]$, we get for $\varepsilon \leqq e^{-1}$:

$$\begin{aligned}\sum_{j,\, k} (\tau\eta\, E_B(c_j g_{jk}) - \tau\eta\, c_j g_{jk}) &= \sum_{j,\, k} c_j \tau\eta\, E_B(g_{jk}) \\ &= \sum_{j,\, k} c_j \tau\eta(\sum_i e_i g_{jk} e_i) = \sum_{i,\, j,\, k} c_j \tau\eta(e_i g_{jk} e_i) = \sum_{i,\, j,\, k} c_j \tau\eta(g_{jk} e_i g_{jk}) \\ &= \sum_{i,\, j} c_j \tau\eta(\sum_k g_{jk} e_i g_{jk}) \leqq \sum_{i,\, j} c_j \tau\eta(\sum_k \alpha^i_{jk} g_{jk}) + \sum_{i,\, j} c_j \eta\tau(\varepsilon f_j) \\ &= \sum_{i,\, j,\, k} c_j \tau\eta(\alpha^i_{jk} g_{jk}) + \delta(\varepsilon) = \sum_{i,\, j,\, k} c_j \eta(\alpha^i_{jk})\tau(g_{jk}) + \delta(\varepsilon),\end{aligned}$$

where $\delta(\varepsilon)\to 0$ when $\varepsilon\to 0$. By the above properties of the projections g_{jk} and of the scalars α^i_{jk} it follows that $0\leqq g_{jk}-\sum_i \alpha^i_{jk} g_{jk}\leqq m\,\varepsilon\, g_{jk}$ so that $0\leqq 1-\sum_i \alpha^i_{jk}\leqq m\,\varepsilon=\varepsilon'$. Also,

$$0\leqq \sum_{j,\,k} c_j (g_{jk} e_i g_{jk}-\alpha^i_{jk} g_{jk})\leqq \varepsilon \sum_{j,\,k} c_j g_{jk}$$

so that

$$0\leqq \tau(e_i \sum_{j,\,k} c_j g_{jk})-\sum_{j,\,k} c_j \alpha^i_{jk}\,\tau(g_{jk})=\tau(e_i)-\sum_{j,\,k} c_j\alpha^i_{jk}\,\tau(g_{jk})\leqq\varepsilon.$$

Thus $\tau(e_i)\geqq \sum_{j,\,k} c_j\alpha^i_{jk}\,\tau(g^i_{jk})\geqq \tau(e_i)-\varepsilon$.

We can now apply the calculus lemma 3.6 to get

$$\sum_{i,\,j,\,k} c_j\alpha^i_{jk}\,\tau(g^i_{jk})\leqq \sum_i \eta\tau(e_i)+\delta'(\varepsilon),$$

where $\delta'(\varepsilon)\to 0$ when $\varepsilon\to 0$. This together with the preceding computations show that

$$\sum_j (\tau\eta\, E_B(c_j f_j)-\tau\eta\, c_j f_j)\leqq \sum_i \eta\tau(e_i)+\delta''(\varepsilon),$$

and letting $\varepsilon\to 0$, $H(M\,|\,B)\leqq \sum_i \eta\tau(e_i)$.

Q.E.D.

4.4. THEOREM. — *Let* M *be a type* II_1 *factor and* N *a subfactor of* M. *If* $N'\cap M$ *has a completely nonatomic part then* $H(M\,|\,N)=\infty$. *If* $N'\cap M$ *is atomic and* $\{f_k\}_k$ *are atoms in* $N'\cap M$ *such that* $\sum f_k=1$ *then*

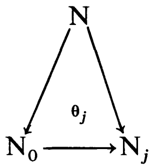
$$H(M\,|\,N)=2\sum_k \eta\tau(f_k)+\sum_k \tau(f_k)\ln[M_{f_k}:N_{f_k}]=\sum_k \tau(f_k)\ln([M_{f_k}:N_{f_k}]/\tau(f_k)^2)).$$

Proof. — If $N'\cap M$ has a completely nonatomic part with support $f\neq 0$ then for each $n\geqq 1$ there exist $f'_1, f'_2, \ldots, f'_n\in N'\cap M$ such that $\tau(f'_i)=n^{-1}\tau(f)$, $\sum_i f'_i=f$. Since N is a factor $E_N(f'_i)$ is a scalar, so that $E_N(f'_i)=n^{-1}\tau(f).1_N$. Thus

$$H(M\,|\,N)\geqq \sum_i (\tau\eta\, E_N(f'_i)-\tau\eta\, f'_i)=n\,\eta\,(n^{-1}\tau(f))=\tau(f)\ln(\tau(f)^{-1}n)\to\infty.$$

Assume now that $N'\cap M$ is atomic and denote $B=\{f_k\}'_k\cap M$. By 3.2 we may suppose $\{f_k\}_k=\{f_0,\ldots,f_m\}$ is a finite set. By the preceding lemma we have $H(M\,|\,N)\leqq H(M\,|\,B)+H(B\,|\,N)=\sum_k \eta\tau(f_k)+H(B\,|\,N)$. Let $M_{f_k}=M_k$, $N_{f_k}=N_k$, $\tau_{f_k}=\tau_k$ and E_{N_k} the τ_k-preserving conditional expectation of M_k onto N_k. Fix the projection

$f_0 \in \{ f_k \}_k$ and denote by $\vartheta_j : \mathrm{N}_0 \rightarrow \mathrm{N}_j$ the isomorphism making commutative the diagram



where the verticals are the induced isomorphisms corresponding to f_0, $f_j (\varepsilon \mathrm{N}')$. Note that if $x_j \varepsilon \mathrm{N}_j$ then $\mathrm{E}_\mathrm{N}(x_j) = \tau(f_i) \sum_i \theta_i \vartheta_j^{-1}(x_j)$ (N_j is also considered as the subalgebra $f_j \mathrm{N} f_j$ in M).

To estimate $\mathrm{H}(\mathrm{B} \mid \mathrm{N})$ consider first a projection $e \in \mathrm{B}$ majorized by f_0, and c a nonnegative scalar. Then we have

$$\begin{aligned}\mathrm{E}_\mathrm{N}(ce) = \mathrm{E}_\mathrm{N}(f_0 (\bigoplus_j \vartheta_j \mathrm{E}_{\mathrm{N}_0}(ce))) = \mathrm{E}_\mathrm{N}(f_0) (\bigoplus_j \vartheta_j (\mathrm{E}_{\mathrm{N}_0}(ce)))\\ = \tau(f_0) (\bigoplus_j \vartheta_j (\mathrm{E}_{\mathrm{N}_0}(ce))).\end{aligned}$$

Since for $x \in \mathrm{N}_j$ we have $\tau_j(x) = \tau(f_j)^{-1} \tau(x)$ and $\tau_j \circ \vartheta_j = \tau_0$, it follows that

$$\begin{aligned}\tau\eta\, \mathrm{E}_\mathrm{N}(ce) - \tau\eta\, ce &= \sum_j \tau\eta\, (\tau(f_0) \vartheta_j (\mathrm{E}_{\mathrm{N}_0}(ce))) - \eta(c) \tau(e)\\ &= \sum_j \tau(f_j) \tau_j \eta\, (\tau(f_0) \vartheta_j (\mathrm{E}_{\mathrm{N}_0}(ce))) - \eta(c) \tau(e)\\ &= \sum_j \tau(f_j) \tau_0\, \eta\, (\tau(f_0) \mathrm{E}_{\mathrm{N}_0}(ce)) - \eta(c) \tau(e)\\ &= \tau_0\, \eta\, (\tau(f_0) \mathrm{E}_{\mathrm{N}_0}(ce)) - \eta(c) \tau(e)\\ &= \eta(\tau(f_0)) \tau_0(ce) + \tau(f_0) \tau_0\, \eta\, \mathrm{E}_{\mathrm{N}_0}(ce) - \tau(f_0) \tau_0 (\eta\, ce).\end{aligned}$$

If we take now a partition $\{ c_i e \}_i$ of f_0, with e_i projections and $c_i \in \mathbb{R}_+$ then by the above computations we get:

$$\begin{aligned}\sum_i (\tau\eta\, \mathrm{E}_\mathrm{N}(c_i e_i) - \tau\eta\, c_i e_i) = \eta(\tau(f_0)) + \tau(f_0) (\sum_i \tau_0\, \eta\, \mathrm{E}_{\mathrm{N}_0}(c_i e_i) - \tau_0\, \eta\, c_i e_i)\\ \leqq \eta\tau(f_0) + \tau(f_0) \mathrm{H}(\mathrm{M}_0 \mid \mathrm{N}_0).\end{aligned}$$

By the second part of property 5° in Section 3 and by 3.1 the relative entropy $\mathrm{H}(\mathrm{B} \mid \mathrm{N})$ can be computed by considering only partitions of the unity $\{ \alpha_i p_i \}$ with p_i projections, $\alpha_i \in \mathbb{R}_+$, and each p_i majorised by some f_j. The preceding inequality and Corollary 4.1 eventually give

$$\sum_i (\tau\eta\, \mathrm{E}_\mathrm{N}(\alpha_i p_i) - \tau\eta\, \alpha_i p_i) = \sum_j (\sum_{p_i \leqq f_j} (\tau\eta\, \mathrm{E}_\mathrm{N}(\alpha_i p_i) - \tau\eta\, \alpha_i p_i)) \leqq \sum_j \eta\tau(f_j) + \tau(f_j) \ln\, [\mathrm{M}_j \mathrm{N}_j].$$

This shows that $H(B|N) \leqq \sum \eta\tau(f_j) + \sum_j \tau(f_j) \ln [M_j : N_j]$ so that

$$H(M|N) \leqq 2\sum \eta\tau(f_j) + \sum_j \tau(f_j) \ln [M_j : N_j].$$

To prove the opposite inequality we shall use Lemma 4.2.

We first assume that $[M_j : N_j] < \infty$ for all $j \geqq 0$, and show that there exists a projection $q \in M$ such that:

(1) $\tau(q) = [M : N]^{-1}$;

(2) $f_j q f_j = \tau(f_j) q_j$, where $q_j \leqq f_j$ are projections with $\tau(q_j) = \tau(q)$, $j \geqq 0$;

(3) $E_{N_j}(q_j) = [M_j : N_j]^{-1} p_j$ for some projections $p_j \in N_j$ with

$$\tau(p_j) = [M_j : N_j]/[M : N], \qquad j \geqq 0;$$

(4) $E_N(p_j) = \tau(f_j) s_j$, where $s_j \in N$ are mutually orthogonal projections, $\sum s_j = 1_N$ and $\tau(s_j) = [M_j : N_j]/(\tau(f_j)[M : N])$.

Then we prove that for such a projection q, $E_{N' \cap M}(q)$ is a scalar multiple of the identity and $\tau(q)^{-1} \tau\eta\, E_N(q) = 2\sum_i \eta\tau(f_i) + \sum_j \tau(f_j) \ln [M_j : N_j]$.

Let r_j be mutually orthogonal projections in N_0 with $\sum r_j = 1_{N_0} (= f_0)$ and with traces proportional to the numbers $\tau(f_j)^{-1}$. $[M_j : N_j]$, $m \geqq j \geqq 0$, i.e. $\tau(r_j) = c\,\tau(f_j)^{-1} [M_j : N_j]$ for some $c \in \mathbb{R}^+$. This is possible because $N_0 \simeq N$ is a type II_1 factor (as $N' \cap M$ is atomic). Since $\sum_j r_j = f_0$ it follows by Jones' formula 0.14 that

$$\tau(f_0) = \sum_j \tau(r_j) = c \sum_i [M_{f_i} : N_{f_i}]/\tau(f_i) = c\,[M : N],$$

so that $\tau(r_j) = (\tau(f_0)/\tau(f_j)) . ([M_j : N_j]/[M : N])$. Denote by $p_j = \vartheta_j(r_j) \in N_j$. Then

$$\tau(p_j) = (\tau(f_j)/\tau(f_0))\,\tau(r_j) = [M_j : N_j]/[M : N]$$

and

$$E_N(p_j) = \tau(f_j) \sum_i \vartheta_i \vartheta_j^{-1}(p_j) = \tau(f_j) \sum_i \vartheta_i(r_j)$$

so that if $s_j = \sum_i \vartheta_i(r_j)$ then condition (4) is fulfilled.

There exist projections $e_j \in M_j$ such that $E_{N_j}(e_j) = [M_j : N_j]^{-1} 1_{N_j}$ and such that $[e_j, p_j] = 0$, $\tau_j(e_j p_j) = \tau_j(e_j)\tau_j(p_j)$. To prove this note that by [13] we may regard M_j as the extension of N_j by some subfactor $N_j^0 \subset N_j$. Thus there exist $e_j^0 \in M_j$ such that $E_{N_j}(e_j^0) = [M_j : N_j]^{-1} 1_{N_j}$ and $[e_j^0, N_j^0] = 0$. Since N_j^0, N_j are type II_1 factors and $p_j \in N_j$, there exists a unitary $u_j \in N_j$ such that $u_j p_j u_j^* \in N_j^0 \subset N_j$. Taking $e_j = u_j^* e_j^0 u_j$ we get that

$$[e_j, p_j] = u_j^* [e_j^0, u_j p_j u_j^*] u_j = 0,$$

$$\tau_j(e_j p_j) = \tau_j(e_j^0 u_j p_j u_j^*) = \tau_j(E_{N_j}(e_j^0) u_j p_j u_j^*) = \tau_j(e_j)\tau_j(p_j)$$

and

$$E_{N_j}(e_j)=u_j^* E_{N_j}(e_j^0)\, u_j=[M_j : N_j]^{-1}.$$

It follows that $E_{N_j}(p_j e_j)=[M_j : N_j]^{-1} p_j$ so that if $q_j=p_j e_j$ then condition (3) is satisfied and

$$\tau(q_j)=\tau(f_j)\,\tau_j(p_j e_j)=\tau(f_j)\,\tau_j(p_j)\,\tau_j(e_j)$$
$$=\tau(p_j)\,\tau_j(e_j)=([M_j : N_j]/[M:N])\,[M_j : N_j]^{-1}=[M:N]^{-1}.$$

In particular we get that q_j are mutually equivalent projections in M so that there exists a set of matrix units in M, $\{v_{ij}\}_{0\leqq i, j\leqq m}$ having q_j as diagonal, $q_j=v_{jj}$. An easy computation shows that $q=\sum_{i,j}(\tau(f_i)\,\tau(f_j))^{1/2}\, v_{ij}$ satisfies (1) and (2).

We now show that $E_{N'\cap M}(q)\in\mathbb{C}$. First we consider some notations. Since $N'\cap M$ is atomic, there exist partial isometries $f_{ij}\in N'\cap M$, $0\leqq i, j\leqq m$, such that:

(*a*) $f_{ii}=f_i$, $f_{ij}=f_{ji}^*$;

(*b*) $f_{ij}\neq 0$ if and only if the isomorphism $\vartheta_j\vartheta_i^{-1} : N_i\to N_j$ is implemented by f_{ij}:

$$\vartheta_j\vartheta_i^{-1}(x)=f_{ji}x f_{ij},\qquad x\in N_i;$$

(*c*) if $f_{ij}\neq 0$, $f_{jk}\neq 0$ then $f_{ij}\, f_{jk}=f_{ik}$;

(*d*) $N'\cap M=\text{span}\,\{f_{ij}\}_{i, j}$.

By (2) and (3) we have $f_i\geqq p_i\geqq q_i$ and $f_j q f_i=q_j q q_i$ so that if $f_{ij}\neq 0$ then

$$\tau(f_{ij}\, q)=\tau(f_{ij}(f_j q f_i))=\tau(f_{ij}\, q_j\, q q_i)=\tau(f_{ij}\, p_j\, q p_i)$$
$$=\tau(\vartheta_i\vartheta_j^{-1}(p_j) f_{ij}\, q p_i)=\tau(f_{ij}\, q p_i\,\vartheta_i\vartheta_j^{-1}(p_j))=0,$$

where the last equality follows by (4). As

$$\tau(f_{ii}\, q)=\tau(f_i\, q)=\tau(f_i\, q f_i)=\tau(f_i)\,\tau(q_i)=\tau(f_{ii})\,\tau(q)$$

it follows by (*d*) that $\tau(qx)=\tau(q)\,\tau(x)$, for $x\in N'\cap M$, i.e. $E_{N'\cap M}(2)=\tau(q).1_M$.

We compute now $E_N(q)$ and $\tau\eta\, E_N(q)$. Since $N\subset\bigoplus_j N_j\subset\bigoplus_j M_j$ we have $E_N=E_N\circ E_{\oplus N_j}\circ E_{\oplus M_j}$ so that by (1)-(4),

$$E_N(q)=\sum_j\tau(f_j)\,[M_j : N_j]^{-1}\, E_N(p_j)=\sum_j(\tau(f_j)^2/[M_j : N_j])\, s_j.$$

Since s_j are mutually orthogonal and $\tau(s_j)=[M_j : N_j]/\tau(f_j)\,[M:N]$ we get

$$\tau\eta\, E_N(q)=\sum\eta(\tau(f_j)^2/[M_j : N_j])\,\tau(s_j)=[M:N]^{-1}\sum_j\tau(f_j)\ln([M_j : N_j]/\tau(f_j)^2).$$

Applying Lemma 4.2 we get that

$$H(M\,|\,N)\geqq\tau(q)^{-1}\,\tau\eta\, E_N(q)=\sum_j\tau(f_j)\ln([M_j : N_j]/\tau(f_j)^2).$$

To complete the proof of the theorem we now consider the case when $[M_f : N_f]=\infty$ for some minimal projection $f\in N'\cap M$. Then by Theorem 2.2 for any $\varepsilon>0$ there exists a projection $0\neq e\in M_f$ such that $E_{N_f}(e)\leqq\varepsilon$. Thus $E_N(e)\leqq\varepsilon$ and $E_{N'\cap M}(e)=\tau(e)\tau(f)^{-1}f$, so that by 4.2 we get $H(M|N)\geqq\tau(f)\tau(e)^{-1}\tau\eta E_N(e)$. As ln is operatorial increasing we have $\ln E_N(e)\leqq\ln\varepsilon$ so that

$$\tau(E_N(e)\ln E_N(e))\leqq\tau(\ln\varepsilon . E_N(e))=(\ln\varepsilon)\tau(e)$$

and thus $\tau\eta E_N(e)\geqq\tau(e)\ln\varepsilon^{-1}$ which yields $H(M|N)\geqq\tau(f)\ln\varepsilon^{-1}\to\infty$.

Q.E.D.

4.5. COROLLARY. — *Let* $N\subset M$ *be type* II_1 *factors with* $[M:N]<\infty$. *Denote by* $f_1,\ldots,f_n$ *a set of minimal projections in* $N'\cap M$ *with* $\sum f_i=1$ *and by* e *a projection in* M *with* $E_N(e)=[M:N]^{-1}.1_M$ (cf. [13]). *Then the following conditions are equivalent:*

(i) $H(M|N)=\ln[M:N]$;

(ii) $[M_{f_i}:N_{f_i}]/\tau(f_i)^2=[M:N]$ *for all* $i=1,\ldots,n$;

(iii) $E_{N'\cap M}(e)=[M:N]^{-1}.1_M$;

(iv) *if* τ' *is the normalized trace on* N' *then* $\tau|N'\cap M=\tau'|N'\cap M$;

(v) *if* M_1 *is the extension of* M *by* N *then the antiisomorphism* $N'\cap M\ni x\mapsto JxJ\in M'\cap M_1$ *is trace preserving.*

Proof. — (i)$\Leftrightarrow$(ii). By the preceding theorem,

$$H(M|N)=\sum\tau(f_i)\ln([M_{f_i}:N_{f_i}]/\tau(f_i)^2).$$

Since ln is strictly concave and $\sum\tau(f_i)=1$,

$$\sum_i\tau(f_i)\ln([M_{f_i}:N_{f_i}]/\tau(f_i)^2)\leqq\ln\sum_i[M_{f_i}:N_{f_i}]/\tau(f_i),$$

with equality iff all the terms $[M_{f_i}:N_{f_i}]/\tau(f_i)^2$ are equal. But by formula 0.14 $\sum[M_{f_i}:N_{f_i}]/\tau(f_i)=[M:N]$.

(iii) $\Rightarrow$(i) If $E_{N'\cap M}(e)=[M:N]^{-1}.1_M$ then taking $q=e$ in Lemma 4.2 it follows that $H(M|N)\geqq\ln[M:N]$, so that by 4.1, $H(M|N)=\ln[M:N]$.

(ii) $\Rightarrow$ (iii). The projection q constructed in the proof of 4.4 satisfies $E_{N'\cap M}(q)=[M:N]^{-1}.1_M$ and $E_N(q)=\sum_i(\tau(f_i)^2/[M_{f_i}:N_{f_i}])s_i$ for some mutually orthogonal projections $s_i\in N$, $\sum s_i=1$. Thus if (ii) holds then $E_N(q)=[M:N]^{-1}.1_M$. By 1.8, e and q are conjugated by a unitary element $u\in N$, $e=uqu^*$. Thus $E_{N'\cap M}(e)=E_{N'\cap M}(u^*eu)=[M:N]^{-1}.1_M$.

(ii) $\Leftrightarrow$ (iv) is an imediate consequence of the formula 0.13 for the index.

(iv) $\Leftrightarrow$ (v) is trivial.

Q.E.D.

4.6. COROLLARY. — *If* $N'\cap M=\mathbb{C}$ *then* $H(M|N)=\ln[M:N]$. *Conversely if* $H(M|N)=\ln[M:N]$ *and* $4<[M:N]<8$, $[M:N]\neq(1+2\cos\pi/n)^2$, $n\geqq3$, *then* $N'\cap M=\mathbb{C}$.

Proof. — The first part is a particular case of 4.4 (or 4.5). For the second part of the statement assume on the contrary that there exists $f \in N' \cap M$, $0 < \tau(f) \leqq 2^{-1}$. Then f and $1-f$ are minimal projections in $N' \cap M$. Indeed, because if $N' \cap M$ would have three mutually orthogonal projections, say f_1, f_2, f_3, then by formula 0.14, $[M:N] \geqq \sum \tau(f_i)^{-1} \geqq 9$.

Denote by $r=[M_f : N_f]$, $s=[M_{1-f} : N_{1-f}]$, $t=\tau(f)$. Applying again formula 0.14 we have $[M:N]=rt^{-1}+s(1-t)^{-1}$ and by 4.5, $rt^{-2}=s(1-t)^{-2}$. Moreover $r, s \in \{4\cos^2(\pi/n) \mid n \geqq 3\} \cup [4, \infty)$ (*cf.* [13]). Now if both $r, s \neq 1$ then $r, s \geqq 2$ so that $[M:N] \geqq 2(t^{-1}+(1-t)^{-1}) \geqq 8$. If $r=s=1$ then $t^{-2}=(1-t)^2$ forces $t=1-t=1/2$, so that $[M:N]=4$. As $t \leqq 1-t$, we have $rs^{-1}=(t/1-t)^2 \leqq 1$ so that $r \leqq s$. Thus the only possible case left is when $r=1$ and $s \geqq 2$. But then $1-t/t=s^{1/2}$ so that $t=1/(1+s^{1/2})$ and $[M:N]=1+s^{1/2}+s\,(s^{1/2}/1+s^{1/2})^{-1}=(1+s^{1/2})^2$.

Q.E.D.

The preceding corollaries deal with the upper extremal case of the entropy $H(M \mid N)$, for given index $[M:N]$. We shall now derive from theorem 4.4 the structure of the pairs $N \subset M$ when $H(M \mid N)$ is minimal. We shall consider only the case $[M:N] \geqq 4$, because otherwise we automatically have $N' \cap M = \mathbb{C}$ and $H(M \mid N) = \ln [M:N]$.

4.7. Lemma. — *(a) If* $0<t<1$ *then* $2\eta t+2\eta(1-t) \leqq -\ln t(1-t)$ *with equality iff* $t=1/2$.

(b) If $t_1, \ldots, t_n \in \mathbb{R}_+$, $n \geqq 2$, $\sum t_i = 1$, $0<t<1$ *and* $\sum_i t_i^{-1} \leqq t^{-1}+(1-t)^{-1}$ *then* $\sum \eta\, t_i \geqq \eta\, t + \eta(1-t)$ *and the equality holds iff* $n=2$ *and* $\{t_1, t_2\}=\{t, 1-t\}$.

Proof. — (*a*) By alementary calculus $2\eta t+2\eta(1-t)+\ln t(1-t)$ is strictly increasing on $(0, 1/2]$, and if $t=1/2$ then

$$2\eta t+2\eta(1-t)+\ln t(1-t)=0.$$

(*b*) Assume $0<t \leqq 1-t$ and $t_1 \leqq t_2 \leqq \ldots \leqq t_n$ so that $t \leqq 1/2$, $t_1 \leqq 1/2$. If $t_1<t$ then $\sum t_i^{-1} > t^{-1}+(1-t)^{-1}$, in contradiction with the hypothesis, so $t \leqq t_1 \leqq 1/2$. Since ln is strictly increasing we have $\sum \eta\, t_i \geqq \eta\, t_1 + \eta(1-t_1)$ with equality only if $n=2$. Moreover, as $\eta\, t+\eta(1-t)$ is strictly increasing on $(0, 1/2]$, we have $\eta\, t_1+\eta(1-t_1) \geqq \eta\, t+\eta(1-t)$ with equality iff $t_1=t$.

Q.E.D.

4.8. Corollary. — *Let* M *be a type* II_1 *factor* $N \subset M$ *a subfactor of finite index* $[M:N]=\lambda^{-1} \geqq 4$ *and* $t>0$ *with* $t(1-t)=[M:N]^{-1}$.

(i) $H(M \mid N) \geqq 2\eta t+2\eta(1-t)$;

(ii) *If* $[M:N]>4$ *then* $H(M \mid N)=2\eta t+2\eta(1-t)$ *if and only if there exist a projection* $f \in N' \cap M$ *with* $\tau(f)=t$ *and an isomorphism* $\vartheta : M_f \to M_{1-f}$ *such that* $N=\{x \oplus \vartheta(x) \mid x \in M_f\}$;

(iii) *If* $[M:N]=4$ *then* $H(M \mid N)=\ln 4$ *and either* $N' \cap M=\mathbb{C}$ *or there exist a projection* $f \in N' \cap M$ *with* $\tau(f)=1/2$ *and an isomorphism* $\vartheta : M_f \to M_{1-f}$ *such that* $N=\{x \oplus \vartheta(x) \mid x \in M_f\}$.

Proof. — (i) is a consequence of 4.4 and 4.7 (*b*). If $[M:N]>4$ and $H(M|N)=2\eta t+2\eta(1-t)$ then by 4.7 (*a*) and 4.4, $N'\cap M\neq\mathbb{C}$. Let $f_1, \ldots, f_n$ be minimal projections in $N'\cap M$, $\sum f_i=1$. Then $[M:N]=(t(1-t))^{-1}\geqq\sum_i \tau(f_i)^{-1}$ so that by 4.7 (*b*), $n=2$, $f_1=f$, $f_2=1-f$, $\tau(f)=t$. Moreover by 4.4 $[M_f : N_f]=[M_{1-f} : N_{1-f}]=1$ so that $M_f=N_f\simeq N\simeq N_{1-f}=M_f$ and if ϑ denotes the resulting isomorphism $\vartheta : M_f\to M_{1-f}$ then $N=\{x\oplus\vartheta(x)\,|\,x\in M_f\}$. This proves (ii).

If $[M:N]=4$ and $N'\cap M=\mathbb{C}$ then by 4.6, $H(M|N)=\ln\,[M:N]=\ln 4$. If $f\in N'\cap M\neq\mathbb{C}$ then $[M:N]\geqq\tau(f)^{-1}+(1-\tau(f))^{-1}$ forces $\tau(f)=1/2$ and then the proof is as for (ii).

Q.E.D.

4.9. *Remark.* — 1° Note that the preceding corollary doesn't solve the problem of classifying the conjugacy classes of subfactors $N\subset M$ of the form $\{x\oplus\vartheta(x)\,|\,x\in M_f\}$, ϑ an isomorphism from M_f onto M_{1-f}. In fact if $N, N_0\subset M$ are given as above by $\vartheta, \vartheta_0 : M_f\to M_{1-f}$ then N, N_0 are conjugated in M iff there exists an automorphism σ of M such that $\sigma(f)=f$ and $\sigma.\vartheta=\vartheta_0\,\sigma$.

2° By 4.8 it follows that if the factor M has a subfactor N of index $[M:N]>4$ and entropy $2\eta t+2\eta(1-t)$ then the fundamental group of M is nontrivial.

5. Some applications

In this section we give two applications of the preceding results. First we compute the entropy $H(R\,|\,R_\lambda)$ for Jones' subfactors R_λ of the hyperfinite factor R (*see* [13]), and use this to show that for $\lambda<1/4$, R_λ is a subalgebra of the form $\{x\oplus\theta(x)\,|\,x\in fRf\}$, where $f\in R$ is a projection, with $\tau(f)(1-\tau(f))=\lambda$, and $\theta : fRf\to(1-f)R(1-f)$ is an isomorphism. Then we compute the entropy of some ergodic automorphisms of R, attached to Jones' construction of subfactors.

Let us briefly recall the definition of R_λ, where $0<\lambda\leqq 1$. Let $e_0, e_1, e_2, \ldots$ be a sequence of projections in R satisfying the axioms:

(*a*) $e_i e_{i\pm 1} e_i=\lambda e_i$;

(*b*) $e_i e_j=e_j e_i$, for $|i-j|\geqq 2$;

(*c*) $\tau(we_i)=\lambda\tau(w)$, if w is a word on $\{1, e_0, e_1, \ldots, e_{i-1}\}$.

In [13] it is shown that if the constant λ exceeds 1/4, then it can only be $(4\cos^2\pi/n)^{-1}$ for $n=3, 4, \ldots$, and that if

$$\lambda\in(0, 1/4]\cup\{(4\cos^2\pi/n)^{-1}\,|\,n\geqq 3\},$$

then there exists such a sequence of projections in R which in addition may be chosen to generate R. Moreover if

$$R_\lambda=\{1, e_1, e_2, \ldots\}''\subset\{1, e_0, e_1, \ldots\}''=R$$

then $[R : R_\lambda]=\lambda^{-1}$. It is also shown that if $A_{0,n}$ is the von Neumann algebra generated by $\{1, e_0, \ldots, e_n\}$ then $A_{0,n}$ is finite dimensional and uniquely determined by the axioms (*a*), (*b*), (*c*), i.e. if $g_0, g_1, \ldots, g_n$ is another set of projections satisfying the above conditions then $e_i \to g_i$ extends to a trace preserving isomorphism of the algebra $A_{0,n}$ onto the von Neumann algebra generated by $1, g_0, \ldots, g_n$.

Let $A_{1,n} = \{1, e_1, \ldots, e_n\}$, so that $R = \overline{\mathrm{U}A_{0,n}}^w$ and $R_\lambda = \overline{\mathrm{U}A_{1,n}}^w$. Using [13] it is easy to see that the algebras $\{A_{0,n}\}_n$ and $\{A_{1,n}\}_n$ satisfy the conditions of Propositions 2.6, 3.4. Moreover these algebras may be associated in a natural way with any pair of type II_1 factors $N \subset M$ with finite index $[M:N] = \lambda^{-1}$: Let $N_1 \subset N = N_0$ be a downward basic construction for $N \subset M$, i.e. N_1 is a subfactor such that M is the extension of N_0 by N_1 and let $e_0 \in M$ be a projection that implements the conditional expectation of N_0 onto N_1, i.e. $E_{N_0}(e_0) = \lambda$, $e_0 x e_0 = E_{N_1}(x) e_0$, for $x \in N_0$ (*cf.* [13], 3.1.9). Iterating the downward basic construction we obtain recursively subfactors $N_0 \supset N_1 \supset N_2 \supset \ldots$ and projections $e_0 \in M$, $e_k \in N_{k-1}$ such that $[N_k : N_{k+1}] = \lambda^{-1}$, $E_{N_k}(e_k) = \lambda$ and $e_k x e_k = E_{N_{k+1}}(x) e_k$, for $x \in N_k$ (so that e_k commutes with N_{k+1}).

It follows that the projections $e_0, e_1, \ldots$ satisfy the axioms (*a*), (*b*), (*c*). So we may denote $R = \{e_0, e_1, \ldots\}''$, $R_\lambda = \{e_1 e_2, \ldots\}''$, $A_{0,n} = \{1, e_0, e_1, \ldots, e_n\}''$, $A_{1,n} = \{1, e_1, \ldots, e_n\}''$.

5.1. Lemma. — *With the preceding notations we have*

$$E_N E_{A_{0,n}} = E_{A_{1,n}}, \quad n \geqq 1 \qquad \textit{and} \qquad E_N E_R = E_{R_\lambda}.$$

Proof. — From the axioms (*a*), (*b*), (*c*) it is easily seen that $E_{A_{1,n}}(e_0) = \lambda$. As it is proved in [13], any element $x \in A_{0,n}$ is a linear combination of words in $e_0, e_1, \ldots, e_n$ with e_0 appearing at most once, so $x = a + \sum a_i e_0 b_i$ with $a, a_i, b_i \in A_{1,n}$. Since $A_{1,n} \subset N$ we get $E_N(x) = a + \sum a_i E_N(e_0) b_i = a + \lambda a_i b_i \in A_{1,n}$. But $N \cap A_{0,n} \supset A_{1,n}$ so that if $y \in A_{1,n} \subset N$ then $\tau(xy) = \tau(E_N(xy)) = \tau(E_N(x) y)$. This shows that if x is orthogonal to $A_{1,n}$ then $E_N(x)$ is also orthogonal to $A_{1,n}$ and since $E_N(x) \in A_{1,n}$, $E_N(x) = 0$. Thus $E_N E_{A_{0,n}} = E_{A_{1,n}}$. Letting $n \to \infty$ it follows $E_N E_R = E_{R_\lambda}$.

Q.E.D.

5.2. Corollary. — *If* $N \subset M$ *has finite index* $[M, N] = \lambda^{-1}$, *then* $H(M \mid N) \geqq H(R \mid R_\lambda)$.

Proof. — Let R, R_λ be identified with subfactors of M as in the preceding lemma. If $x_1, \ldots, x_n$ is a partition of the unity in R then by 5.1 we get $E_{R_\lambda}(x_i) = E_N E_R(x_i) = E_N(x_i)$ so that

$$\sum (\tau\eta\, E_R(x_i) - \tau\eta\, x_i) = \sum (\tau\eta\, E_N(x_i) - \tau\eta\, x_i) \leqq H(M \mid N).$$

Q.E.D.

5.3. Corollary. — *If* $\lambda \geqq 1/4$ *then* $H(R \mid R_\lambda) = -\ln \lambda$. *If* $\lambda < 1/4$ *then*

$$H(R \mid R_\lambda) = 2\eta\, t + 2\eta\,(1-t),$$

where $t\,(1-t) = \lambda$, *and there exist a projection* $f \in R_\lambda' \cap R$, $\tau(f) = t$, *and an isomorphism* $\theta : R_f \to R_{1-f}$ *such that* $R_\lambda = \{x \oplus \theta(x) \mid x \in R_f\}$.

Proof. — Since $\lambda \geqq 1/4$ implies $R'_\lambda \cap R = \mathbb{C}$ (*cf.* [13]), the first part is a particular case of 4.5. If $\lambda < 1/4$ then let $f_0 \in R$ be such that $\tau(f_0) = t$ and $\theta : R_{f_0} \to R_{1-f_0}$ an isomorphism (*cf.* [18]). Denote by $R_0 = \{x \oplus \theta(x) \mid x \in R_0\}$. Then $[R : R_0] = 1/t + 1/1 - t = \lambda^{-1}$ and by 4.4, $H(R \mid R_0) = 2\eta\, t + 2\eta(1-t)$. Now, taking $M = R$, $N = R_0$ in the preceding corollary we get $2\eta\, t + 2\eta(1-t) = H(R \mid R_0) \geqq H(R \mid R_\lambda)$. Corollary 4.8 then yields the rest of the statement.

Q.E.D.

The preceding corollary contains implicitely one of Jones' results in [13]: that if $\lambda < 1/4$ then $R'_\lambda \cap R \neq \mathbb{C}$. A very short proof of this fact (found independently by V. Jones and the authors) is as follows: Let M be a type II_1 factor, $f \in M$ a projection, $\tau(f) = t$ (as usual $t(1-t) = \lambda$). Assume there exists an isomorphism $\theta : M_f \to M_{1-f}$ and denote $N = \{x \oplus \theta(x) \mid x \in M_f\} \subset M$. Let $p \leqq f$ be a projection of trace $\tau(p) = \lambda$ and denote by $q = (1-f) - \theta(p)$, so that

$$\tau(q) = 1 - t - \lambda \, . \, \frac{1-t}{t} = \lambda.$$

Consider also a partial isometry $v \in M$ such that $v^* v = p$, $vv^* = q$. Finally denote $e_0 = (1-t)p + tq + (t(1-t))^{1/2}(v + v^*)$. It is easy to verify that e_0 is a projection and that $E_N(e_0) = \lambda$. By [13] there exists $N_1 \subset N_0 = N$ such that M is the extension of N_0 by N_1 and $e_0 x e_0 = E_{N_1}(x) e$, $x \in N_0$ (in fact N_1 can be found explicitely without use of [13]). Thus N_0 and e_0 can be considered as a first step of the construction in Lemma 5.1. Using the notations of that lemma we have $\tau(fe_0) = \tau(E_R(f) e_0)$. But

$$\tau(fe_0) = \tau(fe_0 f) = \tau((1-t)p) = (1-t)\lambda,$$

while if $R'_\lambda \cap R = \mathbb{C}$ then in particular $E_R(f) \in R'_\lambda \cap R$ is a scalar so that $\tau(E_R(f) e_0) = \tau(f)\tau(e_0) = \lambda t$. Since $t \neq 1-t$, this is a contradiction. As factor M we can take any factor having $t/1-t$ in its fundamental group, for instance the hyperfinite factor.

We shall now derive a useful consequence of the preceding argument and of Corollary 5.3.

5.4. Corollary. — *Let* M *be a type* II_1 *factor,* $f \in M$ *a projection,* $\tau(f) = t$ *(with* $t(1-t) = \lambda < 1/4$*) and suppose there exists an isomorphism* $\theta : M_f \to M_{1-f}$. *Denote by* $N = \{x \oplus \theta(x) \mid x \in M_f\}$. *If* $e_0, e_1, \ldots$ *are associated with the pair* $N \subset M$ *as in Lemma* 5.1 *then* $f \in \{e_0, e_1, \ldots\}''$.

Proof. — We just showed that $\tau(fe_0) = \lambda(1-t)$ (the particular choice of e_0 doesn't matter, since by 1.8 if $e \in M$ is another projection with $E_N(e) = \lambda$ then there exists a unitary $u \in N$ such that $ue_0 u^* = e$, hence

$$\tau(fe) = \tau(fue_0 u^*) = \tau(u^* fue_0) = \tau(fe_0) = \lambda(1-t)).$$

By 5.3 $R'_\lambda \cap R = \mathbb{C} f_0 + \mathbb{C}(1-f_0)$ for some projection f_0, $\tau(f_0) = t$ (where $R = \{1, e_0, e_1, \ldots\}''$, $R_\lambda = \{1, e_1, e_2, \ldots\}''$). Since $E_R(f) \in R'_\lambda \cap R$, there exist $\alpha, \beta \in \mathbb{R}$ such that $E_R(f) = \alpha f_0 + \beta(1-f_0)$. But $R_\lambda \subset R$ is also a pair of factors of the type considered in the statement, so that $\tau(f_0 e_0)=\lambda(1-t)$. Thus

$$\lambda(1-t)=\tau(fe_0)=\tau(\mathrm{E_R}(f)\,e_0)=\alpha\lambda(1-t)+\beta\lambda t.$$

Moreover

$$t=\tau(f)=\tau(\mathrm{E_R}(f))=\alpha\tau(f_0)+\beta\tau(1-f_0)=\alpha t+\beta(1-t).$$

From these two equalities it follows that if $\beta\neq 0$ then $t=1/2$ so that $\lambda=1/4$, a contradiction so $\beta=0$ and $f=f_0$.

Q.E.D.

Remark now that in the hyperfinite factor one can also find a set of projections satisfying (*a*), (*b*), (*c*) for some λ but indexed on $\mathbb{Z}$ rather than on $\mathbb{N}$, This can be seen by applying recursively the basic construction as follows: Start with $\mathrm{M}=\{1, e_0, e_1, \ldots\}''$, $\mathrm{N}=\{1, e_1, e_2, \ldots\}''$ (so that with the preceding notations $\mathrm{M}=\mathrm{R}$, $\mathrm{N}=\mathrm{R}_\lambda$), and consider the (usual) basic construction, i.e. take M_1 to be the extension of M by N, q_1 the projection in M_1 implementing the conditional expectation of M onto N, $\mathrm{E_M}(q_1)=\lambda$, $q_1 x q_1=\mathrm{E_N}(x)\,q_1$, $x\in\mathrm{M}$. Then by iterating the basic construction we obtain II_1 factors $\mathrm{N}\subset\mathrm{M}\subset\mathrm{M}_1\subset\mathrm{M}_2\ldots$ and projections $q_1, q_2, \ldots$ such that $\mathrm{E}_{\mathrm{M}_{k-1}}(q_k)=\lambda$, $q_k x q_k=\mathrm{E}_{\mathrm{M}_{k-2}}(x)\,q_k$, $x\in\mathrm{M}_{k-1}$. If P denotes the closure of $\bigcup \mathrm{M}_k$ in the weak topology given by τ and $e_k=q_{-k}$ for $k=-1, -2, \ldots$ then it follows that P is the hyperfinite II_1 factor (as it is the union of an increasing sequence of hyperfinite factors) and $\{e_k\}_{k\in\mathbb{Z}}$ satisfy the axioms (*a*), (*b*), (*c*) and generate P.

5.5. Notation. — Let P be isomorphic to the hyperfinite factor and $\{e_k\}_{k\in\mathbb{Z}}$ a set of projections in P satisfying the axioms (*a*), (*b*), (*c*) for some

$$\lambda(0,\ 1/4]\cup\left\{\left(4\cos^2\frac{\pi}{n}\right)^{-1}\Big|\, n\geqq 3\right\}$$

and generating P. We denote by θ_λ the automorphism of P given by $\theta_\lambda(e_i)=e_{i+1}$.

Note that θ_λ is ergodic. In fact if x, y are words in $\{e_i\}_{s\geqq i\geqq r}$ for some intergers $r>s$, then $\tau(x\,\theta_\lambda^n(y))=\tau(x)\,\tau(\theta^n(y))=\tau(x)\,\tau(y)$ whenever $n>s-r$. By Kaplansky density theorem it follows that $\tau(x\,\theta^n(y))\to\tau(x)\,\tau(y)$ when $n\to\infty$, for any x, $y\in\mathrm{P}$.

We shall now use 5.4 to show that if $\lambda<1/4$ then θ_λ is just the Bernouli shift of entropy $\eta\,t+\eta\,(1-t)$ constructed by Connes-Störmer and also by Krieger [9]. Indeed, let M_0 be the algebra of 2 by 2 complex matrices with faithful state φ_0 with eigenvalue list $\{t,\ 1-t\}$, where $t(1-t)=\lambda$. For each $k\in\mathbb{Z}$ let $\mathrm{M}_k=\mathrm{M}_0$ and $\varphi_k=\varphi_0$. Denote $(\mathrm{M},\ \varphi)=\bigotimes_{\mathbb{Z}}(\mathrm{M}_k,\ \varphi_k)$ and P the centralizer of $\varphi=\otimes\,\varphi_k$ in M, so that P is the hyperfinite factor and it is invariant to the Bernoulli shift θ on M (*cf.* [9]). Let e_{ij}^n be the matrix unit in M_n and denote

$$e_n=\ldots 1\otimes((1-t)\,e_{11}^n\otimes e_{22}^{n+1}+t e_{22}^n\otimes e_{11}^{n+1}$$
$$+(t(1-t))^{1/2}\,e_{12}^n\otimes e_{21}^{n+1}+(t(1-t)^{1/2}\,e_{21}^n\otimes e_{12}^{n+1}))\otimes 1\ldots$$

Easy calculations show that $\theta(e_n)=e_{n+1}$, $\{e_n\}_{n\in\mathbb{Z}}\subset P$ and $\{e_n\}_{n\in\mathbb{Z}}$ satisfy the axioms (*a*), (*b*), (*c*). Let $R=P\cap\bigotimes_{k\geqq 0}(M_k,\varphi_k)$ and $R_j=P\cap\bigotimes_{k\geqq j+1}(M_k,\varphi_k)$, $j\geqq 0$. Then R and R_j are factors (by the same argument showing that P is a factor), $\theta(R)=R_0$ and $E_{R_0}(e_0)=\lambda$ (in fact even the Fubini projection of e_0 on $\bigotimes_{h\geqq 1}(M_k,\varphi_k)$ is a scalar and E_{R_0} is just its restriction to R). Moreover e_0 commutes with R_1 and if $f_0=\ldots 1\otimes e^0_{11}\otimes 1\ldots$ then $\tau(f_0)=t$, $f_0\in R'_0\cap R$, and $f_0 R f_0=f_0 R_0 f_0$, $(1-f_0)R(1-f_0)=(1-f_0)R_0(1-f_0)$. *By* 1.3 e_0 implements the conditional expectation of R_0 onto R_1. It follows by induction that $e_0, e_1, \ldots$ are as in 5.1 so that applying the preceding corollary for $M=R$, $N=R_0$, $f=f_0\in N'\cap M$ and the sequence $\{e_0, e_1, \ldots\}$, it follows that $f_0\in\{e_0, e_1, \ldots\}''$. Similarly $f_k\in\{e_k, e_{k+1}, \ldots\}''$ so that $\{f_k\}_{k\in\mathbb{Z}}$ are all in the von Neumann algebra generated by $\{e_k\}_{k\in\mathbb{Z}}$. Since

$$f_j e_j(1-f_j)=(t(1-t))^{1/2}(\ldots 1\otimes e^n_{12}\otimes e^{n+1}_{21}\otimes 1\ldots)$$

it follows that $\ldots 1\otimes e^n_{12}\otimes e^{n+1}_{21}\otimes 1\ldots$ are in $\{e_k\,|\,k\in\mathbb{Z}\}''$.

It is now straightforward to see that $\{e_k\}_{k\in\mathbb{Z}}$ generate P. We have thus proved:

5.6. COROLLARY. — *If* $\lambda<1/4$ *then* θ_λ *is the noncommutative Bernoulli shift of weights* $t, 1-t$, *where* $t(1-t)=\lambda$. *In particular* $H(\theta_\lambda)=\eta\,t+\eta\,(1-t)$.

The computation of $H(\theta_\lambda)$ for $\lambda>1/4$ is entirely different and does not involve the results obtained in the preceding sections.

5.7. PROPOSITION. — *If* $\lambda=(4\cos^2\pi/n)^{-1}$, $n\geqq 3$, *then* $H(\theta_\lambda)=-(1/2)\ln\lambda$.

Proof. — By the Kolmogorov-Sinai type theorem of Connes and Störmer it follows that $H(\theta_\lambda)=\lim\sup(H(A_{0,n})/n)$, where $A_{0,n}=\{1, e_0, \ldots, e_n\}''$. Indeed, using the notations in [9] for the joint entropy, we have:

$$H(\Theta_\lambda)=\sup_k(\lim_n H(A_{0,k}, \theta_\lambda(A_{0,k}), \ldots, \theta^n_\lambda(A_{0,k}))/n)$$

$$=\sup_k(\lim_n H(A_{0,k}, A_{1,k+1}, \ldots, A_{n,k+n})/n)$$

$$\leqq\sup_k(\lim_n\sup H(A_{0,k+n})/n)=\lim_n\sup H(A_{0,n})/n.$$

Moreover

$$(n+2)H(\theta_\lambda)=H(\theta^{n+2}_\lambda)\geqq\lim_k H(A_{0,n}, \theta^{n+2}_\lambda(A_{0,n}), \ldots, \theta^{k(n+2)}_\lambda(A_{0,n}))/k$$

and since $\theta^{s(n+2)}_\lambda(A_{0,n})$ mutually commute and are τ-independent, for $s=0, 1, \ldots, k$, it follows that

$$H(A_{0,n}, \theta^{n+2}_\lambda(A_{0,n}), \ldots, \theta^{k(n+2)}_\lambda(A_{0,n})=\sum_s H(\theta^{s(n+2)}_\lambda(A_{0,n}))=(k+1)H(A_{0,n})$$

and so $(n+2)H(\theta_\lambda)\geqq H(A_{0,n})$.

Now each $A_{0,n}$ is of the form $\sum_{j=1}^{k_n} M_{j,n}$, where $M_{j,n}$ are factors of dimension r_{jn} and with minimal projections of trace $\varepsilon_{j,n}$. By [13] there exists n_0 such that if $n \geqq n_0$ then $k_n = k_{n+2}$ and $\varepsilon_{j,n+2} = \lambda \varepsilon_{jn}$. Since $\sum_j r_{jn} \varepsilon_{jn} = 1$ and $H(A_{0,n}) = \sum_j r_{jn} \eta \varepsilon_{jn}$, it follows that

$$H(A_{0,n}) = -\sum_j r_{jn} \varepsilon_{jn} \ln \lambda^k \varepsilon_{j,n-2k} = \sum_j r_{jn} \varepsilon_{jn} \ln \varepsilon_{j,n-2k} - k \ln \lambda.$$

If k is the integral part of $(n-n_0)/2$ then $\varepsilon_{j,n-2k} \in \{\varepsilon_{j,1}, \ldots, \varepsilon_{j,n_0}\}$ so that, for some $c > 0$

$$-\sum_j r_{jn} \varepsilon_{jn} \ln \varepsilon_{j,n-2k} \leqq c \sum_j r_{jn} \varepsilon_{jn} = c.$$

Thus

$$\lim_n \frac{H(A_{0,n})}{n} = (-\ln \lambda) \lim \frac{n-n_0}{2n} = -(1/2) \ln \lambda.$$

Q.E.D.

5.8. *Remarks*. — 1° We believe that for $\lambda > 1/4$ the automorphisms θ_λ are not Bernoulli shifts and even that they normalize no nontrivial abelian *— subalgebras of R.

2° The computation of $H(\theta_{1/4})$ remains an open problem. However let us note that using the representation [13] of the pair $R_{1/4} \subset R$ and of the corresponding generating projections e_i it is easily seen that $H(\theta_{1/4}) \leqq \ln 2$.

3° For $\lambda = 1/2$ there is a representation of the projections $\{e_n\}_{n \in \mathbb{Z}}$ similar to the one we found for $\lambda < 1/4$: For each $n \in \mathbb{Z}$, let M_n be isomorphic to the algebra of 2 by 2 complex matrices and $\{e_{ij}^n\}$ a matrix unit for M_n as before. Let $P = \bigotimes_{n \in \mathbb{Z}} M_n$ be the tensor product with respect to the trace and $e_{2k} = \ldots 1 \otimes e_{11}^k \otimes 1 \ldots$ and

$$e_{2k+1} = 2^{-1}(\ldots 1 \otimes (e_{12}^k + e_{21}^k) \otimes (e_{12}^{k+1} + e_{21}^{k+1}) \otimes 1 \ldots) + 2^{-1} I.$$

e_n thus defined satisfy (*a*), (*b*), (*c*), for $\lambda = 1/2$. The algebra generated by them is easily seen to be the fixed point algebra $R = P^\sigma$, where σ is the period 2 automorphism implemented by $\bigotimes_{k \in \mathbb{Z}} (2 e_k^{11} - 1)$ Note that $A_1 = \{e_{2k}\}''$ is a Cartan subalgebra both in R and P and that $A_2 = \{e_{2k+1}\}''$ is a Cartan subalgebra in R but not in P. Moreover $\theta_{1/2}^2$ is just the restriction to R of the noncommutative Bernoulli shift on P. Thus $H(\theta_{1/2}^2) \leqq \ln 2$ and since $\theta_{1/2|A}^2$ is the commutative Bernoulli shift, $H(\theta_{1/2}^2) \geqq \ln 2$. We have thus obtained another proof of $H(\theta_{1/2}) = 2^{-1} \ln 2$.

6. Computation of H and λ for finite dimensional algebras

In this section M will be a finite dimensional von Neumann algebra with faithful trace τ, $\tau(1) = 1$, and $N \subset M$ a von Neumann subalgebra. Thus $M = \bigoplus_{l \in L} M_l$, $N = \bigoplus_{k \in K} N_k$ where

M_l is the algebra of $m_l \times m_l$ matrices, N_k the algebra of $n_k \times n_k$ matrices and the sets of indices L and K are finite. We denote by $A=(a_{kl})_{k \in K,\, l \in L}$ the embedding matrix of N in M and by t_l respectively s_k the traces of the minimal projections in M_l respectively N_k. Thus if $m=(m_l)_l$, $n=(n_k)_k$, $t=(t_l)_l$, $s=(s_k)_k$ are column vectors then $At=s$, $A^t n=m$ (A^t is the transpose of A).

6.1. THEOREM. — $(\lambda(M, N))^{-1}=\max_l (\sum_k b_{kl} s_k/t_l)$, *where* $b_{kl}=\min\{a_{kl}, n_k\}$.

6.2. THEOREM :

$$H(M(N)=-\sum m_l t_l \ln t_l+\sum m_l t_l \ln m_l+\sum n_k s_k \ln s_k-\sum n_k s_k \ln n_k+\sum_{k, l} n_k a_{kl} t_l \ln c_{kl},$$

where $c_{kl}=\min\{n_k/a_{kl}, 1\}$.

We now discuss some equivalent forms of the formulas 6.1, 6.2 under the additional hypothesis $a_{kl} \in \{0, 1\}$ and, more generally $a_{kl} \leqq n_k$. Then we give two simple examples and finally proceed with the proof of the theorems.

First some notations that will be used in all the rest of the section.

We denote by e^k and f^l the minimal central projections in N and respectively M (e^k is the support of N_k in N and f^l the support of M_l in M). Thus $e^k.f^l$, $k \in K$, $l \in L$, are the minimal central projection in $N' \cap M$ and $\tau(e^k f^l)=n_k a_{kl} t_l$. Note also that $n_k t_l$ is the trace of the minimal projections in $(N' \cap M)_{e^k f^l}$.

If we assume $a_{kl} \leqq n_k$, the term $\sum_k b_{kl} s_k/t_l$ in the formula for the index 6.1 becomes $\sum_k a_{kl} s_k/t_l=\sum_k a_{kl} n_k s_k/n_k t_l=\sum_k a_{kl} \tau(e^k)/n_k t_l$. In particular if $a_{kl} \in \{0, 1\}$ then:

6.3. $(\lambda(M, N))^{-1}=\max_l (\sum \tau(e^k)/\tau(e^k f^l))$ where the sum is taken over k, for $e^k f^l \neq 0$.

If $a_{kl} \leqq n_k$ the formula for the entropy also gets simplified, because then $c_{kl}=1$ and the last term in 6.2 will vanish. Moreover, as

$$m_l t_l=\sum_k n_k a_{kl} t_l \qquad \text{and} \qquad n_k s_k=\sum_l n_k a_{kl} t_l,$$

we get:

$$H(M \mid N)=\sum_{k, l}\left(\tau(e^k f^l) \ln \frac{s_k}{t_l}+\tau(e^k f^l) \ln \frac{m_l}{n_k}\right).$$

Note also that

$$\frac{m_l}{n_k}=\frac{m_l t_l}{n_k t_l}=\frac{\tau(f^l)}{n_k t_l} \qquad \text{and} \qquad \frac{s_k}{t_l}=\frac{n_k s_k}{n_k t_l}=\frac{\tau(e^k)}{n_k t_l}$$

and so, if $a_{kl} \in \{0, 1\}$ then:

6.4.

$$H(M \mid N) = \sum \left(\tau(e^k f^l) \ln \frac{\tau(e^k)}{\tau(e^k f^l)} + \tau(e^k f^l) \ln \frac{\tau(f^l)}{\tau(e^k f^l)} \right)$$

where the sum is taken over k and l, for $e^k f^l \neq 0$.

6.5. *Examples.* — 1° Let M and N be factors of type I_m and respectively I_n. Then the embedding matrix A is just the number m/n, $t_1 = t = 1/m$, $s_1 = s = 1/n$, so that by 6.1 and 6.2 when $m/n \leqq n$ we have $\lambda(M, N)^{-1} = (m/n)^2$, $H(M \mid N) = \ln(m/n)^2$ and when $m/n > n$ we have $\lambda(M, N)^{-1} = m$, $H(M \mid N) = \ln m$. Note that we allways have $[M : N] = (m/n)^2$.

2° Let M be a factor of type I_m and $N = \sum_{k \in K} N_k$, so that $\sum n_k = m$. If d denotes the cardinal of K then 6.3 and 6.4 give

$$\lambda(M, N) = 1/d \qquad \text{and} \qquad H(M \mid N) = \sum \eta(n_k/m)$$

(*see* also 4.3).

For the proof of 6.1 we first need some technical lemmas. The key result is the following:

6.6. LEMMA. — *Let Q be a finite von Neumann algebra,* $S \subset Q$ *a von Neumann subalgebra and* $e \in Q$ *a projection such that* $eSe = \mathbb{C}e$. *If* $E_S(e) \geqq \lambda e$ *for some positive scalar* λ *then* $E_S(x) \geqq \lambda x$ *for all positive elements* x *in the weak closed* $*$*-algebra generated by* SeS.

Proof. — Let $x = (\sum_i a_i e b_i)^* ((\sum_j a_j e b_j)$, $a_i, b_j \in S$, so that by the hypothesis $x = \sum_{i,j} b_i^* \lambda_{ij} e b_j$ for some scalars λ_{ij}, where $\lambda_{ij} e = e a_i^* a_j e$. Since $(a_i^* a_j)_{i,j}$ is a positive matrix over S it follows that $(\lambda_{ij})_{i,j}$ is a positive matrix. Thus there exist $c_{ij} \in \mathbb{C}$such that $\lambda_{ij} = \sum_k \bar{c}_{ik} c_{kj}$. As in the proof of 2.1 if we denote by b the column matrix $(b_j)_j$, $c = (c_{ij})_{i,j}$, $\tilde{e}$ the diagonal matrix

$$\begin{pmatrix} e & & 0 \\ & \ddots & \\ 0 & & e \end{pmatrix} \quad \text{and} \quad \widetilde{E_S(e)} = \begin{pmatrix} E_S(e) & & 0 \\ & \ddots & \\ 0 & & E_S(e) \end{pmatrix}$$

then we get

$$E_S(x) = \sum_{i,j} b_i^* \lambda_{ij} E_S(e) b_j = b^* c^* \widetilde{E_S(e)} \subset b \geqq b^* c^* (\lambda \tilde{e}) cb = \lambda \sum_{i,j} b_i^* \lambda_{ij} e b_j = \lambda x.$$

Q.E.D.

6.7. LEMMA. — *With the notations at the beginning of this section, let* $e \in M_l \subset M$ *be a minimal projection and f the support of* $E_{N' \cap M}(e)$. *Then* $\mathrm{Alg}(NeN) = fMf = fM_l f$.

Proof. — As M, N are finite dimensional, $\mathrm{Alg}(NeN)$ is a weakly closed $*$-subalgebra in M. Its support in M is the projection $f' = \vee \{ueu^* \mid u \text{ unitary element in } N\}$ and in fact if G is a group of unitaries in N generating N then $f' = \vee \{ueu^* \mid u \in G\}$. Moreover if G is finite and $|G|$ is the cartinality of G then

$$E_{N' \cap M}(e) = \frac{1}{|G|} \sum_{u \in G} ueu^*$$

so that the support of $E_{N' \cap M}(e)$ coincides with f', i. e. $f = f'$.

Let $e_0 = e, e_1, e_2, \ldots, e_n$ be a maximal family of mutually orthogonal equivalent projections in $\mathrm{Alg}(NeN)$. Then $q = \sum e_i \leqq f$, and suppose $q \neq f$. Then $f - q \in \mathrm{Alg}(NeN)$ and $(f-q)Ne \neq 0$, because otherwise $(f-q)\mathrm{Alg}(NeN) = 0$, so that $0 = (f-q)f = f - q$, a contradiction. If $v \in (f-q)Ne$, $v \neq 0$, then $v^* v$ is supported by e and since e is a minimal projection, $v^* v$ is a scalar multiple of e. Thus for a suitable scalar c, $e_{n+1} = cvv^*$ is a projection, $e_{n+1} \in \mathrm{Alg}(NeN)$ and e_{n+1} is equivalent to e. This contradicts the maximality of $e_0, e_1, \ldots, e_n$ and proves the lemma.

Q.E.D.

6.8. LEMMA. — *Let Q be a type Iq factor and* $S \subset Q$ *a type* I_s *subfactor of* Q. *If* $e \in Q$ *is a minimal projection and f is the support of* $E_{S' \cap Q}(e)$ *then*

$$\tau(f) \leqq \frac{s \min\{s, q/s\}}{q}.$$

Proof. — If $q/s \leqq s$ then $s \min(s, q/s) = q$ and obviously $\tau(f) \leqq 1$. By the preceding lemma, f is the support of Alg (SeS) so that it is also the supremum of the left supports $l(xe)$ of the elements xe, $x \in S$, i. e. $f = \vee \{l(xe) \mid x \in S\}$. In fact it is enough to take x in a linear basis of S. As S has dimension s^2 and e is one dimensional in Q, it follows that f has dimension not larger than s^2, i. e. $\tau(f) \leqq s^2/q$.

Q.E.D.

Proof of 6.1. — Let $\lambda_l = (\sum_k b_{kl} s_k/t_l)^{-1}$. By Lemma 6.6 it is enough to show that for any minimal projection $e \in M_l$ there exists a minimal projection $e_0 \in M_l$ such that:

(*a*) $e \in \mathrm{Alg}(Ne_0 N)$;

(*b*) $E_N(e_0) \geqq \lambda_l e_0$;

(*c*) λ_l is the best constant for which (*b*) holds.

Suppose $e \in M_l = Mf^l$ and let e^k be such that $e^k f^l \neq 0$. Then $e^k ee^k$ is a scalar multiple of a one-dimensional projection. Applying the preceding lemma for $Q = M_{e^k f^l} = e^k (M_l) e^k$ and $S = N_{e^k f^l} = f^l N_k f^l$ it follows that the support e'_k of $E_{N' \cap M}(e^k ee^k)$ satisfies

$$\tau(e'_k) \leqq \frac{b_{kl}}{a_{kl}} \tau(e^k f^l) = b_{kl} n_k a_{kl} t_l / a_{kl} = n_k b_{kl} t_l$$

(where $b_{kl} = \min\{a_{k,l}, n_k\}$). Note that $n_{kl} t_l$ is the trace of the minimal projections in $S' \cap Q = (N' \cap M) e^k f^l$. Thus there exist b_{kl} minimal projections in $(N' \cap M)_{e^k f^l}$, say $\{g^k_{ii}\}_{1 \leqq i \leqq b_{kl}}$ such that $e'_k \leqq \sum_i g^k_{ii}$, and let $\{g^k_{ij}\}_{i,j}$ be a set of matrix units in $(N' \cap M)_{e^k f^l}$

having g^k_{ii} as diagonal. Let also $\{e^k_{ij}\}_{1 \leq i, j \leq b_{kl}}$ be a set of matrix units in $N_{e^k f^l}$ (this is possible because $b_{kl} \leq n_k$). Denote by

$$f'_k = \frac{1}{b_{kl}} \sum_{i, j} e^k_{ij} g^k_{ij}.$$

It is easy to verify that f_k is a rank one projection in $M_{e^k f^l}$ (and thus in M_l) and that

$$E_{N' \cap M}(f'_k) = \frac{1}{b_{kl}} \sum_i E_{N' \cap M}(e^k_{ii}) g^k_{ii} = \frac{1}{b_{kl} n_k} \sum_i g^k_{ii},$$

so that the support of $E_{N' \cap M}(f'_k)$ majorizes e'_k. Thus, for each $k \in K$, with $e^k f^l \neq 0$, we find a minimal projection f'_k in M_l, $f'_k \leq e^k$, such that the support of $E_{N' \cap M}(f'_k)$ majorizes the support of $E_{N' \cap M}(e^k e e^k)$. Consider now some positive scalars α_k (to be specified later) such that $\sum \alpha_k = 1$. There exists a projection e_0 in M_l such that $e^k e_0 e^k = \alpha_k f'_k$.

This projection is of rank one in M_l and satisfies

$$E_{N' \cap M}(e_0) = \sum e^k E_{N' \cap M}(e_0) e^k = \sum E_{N' \cap M}(e^k e_0 e^k) = \sum \alpha_k E_{N' \cap M}(f'_k),$$

so that the support of $E_{N' \cap M}(e_0)$ majorizes the support of

$$\sum E_{N' \cap M}(e^k e e^k) = \sum e^k E_{N' \cap M}(e) e^k = E_{N' \cap M}(e).$$

By 6.7 it follows that $e \in \operatorname{Alg} N e_0 N$.

We shall now compute $E_N(e_0)$. Let P be the algebra $\bigoplus_{l, k} f^l N_k f^l$ so that $N \subset P \subset M$. Then $E_P(e_0) = \sum E_P(e^k e_0 e^k) = \sum \alpha_k E_P(f'_k)$. Using the preceding notations, we have

$$E_P(f'_k) = \frac{1}{b_{kl}} \sum_i E_P(e^k_{ii} g^k_{ii}) = \frac{1}{b_{kl}} \sum_i e^k_{ii} E_P(g^k_{ii}) = \frac{1}{b_{kl} a_{kl}} \sum_i e^k_{ii}.$$

But $\sum e^k_{ii} \in N_{e^k f^l} \subset P$ so that there exists a unique projection q_k in N_k such that $q_k f^l = \sum_i e^k_{ii}$. Note that q_k is the sum of b_{kl} minimal projection in N_k. Hence

$$\begin{aligned} E_N(f'_k) &= E_N E_P(f'_k) = \frac{1}{b_{kl} a_{kl}} E_N(q_k f^l) \\ &= \frac{1}{b_{kl} a_{kl}} E_N(q_k e^k f^l) = \frac{1}{b_{kl} a_{kl}} q_k E_N(e^k f^l) \\ &= \frac{1}{b_{kl} a_{kl}} \frac{\tau(e^k f^l)}{\tau(e^k)} q_k e^k = \frac{1}{b_{kl} a_{kl}} \frac{n_k a_{kl} t_l}{n_k s_k} q_k = \frac{1}{b_{kl}} \frac{t_l}{s_k} q_k. \end{aligned}$$

It follows that

$$E_N(e_0) = \sum_k \alpha_k \frac{1}{b_{kl}} \frac{t_l}{s_k} q_k$$

and if we take $\alpha_k = \dfrac{b_{kl} s_k}{\sum_k b_{kl} s_k}$ then $E_N(e_0) = \lambda_l \sum_k q_k$.

Since $\sum q_k$ is a projection and since the support of $E_N(e_0)$ majorizes e_0, we get $E_N(e_0) \geqq \lambda_l e_0$ and λ_l is the best constant for which this inequality holds.

Q.E.D.

From Theorem 6.1 and the last part of its proof we easily get:

6.9. Corollary. — $\lambda(M, N) = \inf \{ \| E_N(e) \| \mid e \text{ nonzero projection in } M \}$.

Proof. — The inequality $\leqq$ allways holds (*see* the remark at the end of Section 2). In the proof of 6.1 it was shown that there exists a projection $e_0 \in M$ such that $E_N(e_0)$ is $\lambda(M, N)$ times a projection. Thus $\| E_N(e_0) \| = \lambda(M, N)$, which yields the opposite inequality.

Q.E.D.

We turn now to the computation of the relative entropy of N in M (Theorem 6.2). From now on the inclusion $N \subset M$ will be described in the following way: For each $k \in K$, $l \in L$ we fix a finite set $A_{k,l}$ of cardinal a_{kl} and identify $[1, m_l]$ with $\bigcup_k (A_{kl} \times [1, n_k])$, where the intervals are integer valued and the $A_{k,l}$'s are supposed to be disjoint. According to the above decomposition we shall fix a system of matrix units for M denoted by $(f^l_{(a,i)(b,j)})$, $l \in L$, $a \in A_{k_1, l}$, $b \in A_{k_2, l}$, $1 \leqq i \leqq k_1$, $1 \leqq j \leqq k_2$, and a system of matrix units (e^k_{ij}), $k \in K$, $1 \leqq i, j \leqq n_k$ for N, and express the inclusion $N \subset M$ by the formula

$$e^k_{i,j} = \sum_{l \in L} \sum_{a \in A_{k,l}} f^l_{(a,i)(a,j)}.$$

The inclusion matrix [2] is easily seen to be $A = (a_{k,l})_{k \in K,\, l \in L}$. In terms of these matrix units the minimal central projections in N and M respectively, i. e. e^k respectively f^l, have the form:

$$e^k = \sum_{i=1}^{n_k} e^k_{ii},$$

$$f^l = \sum_{k \in K} \sum_{i=1}^{n_k} \sum_{a \in A_{k,l}} f^l_{(a,i)(a,i)}.$$

Note also that the conditional expectation E_N acts as follows:

$$E_N(f^l_{(a,i)(a,j)}) = \frac{t_l}{s_k} e^k_{ij},$$

k being the index such that $a \in A_{k,l}$

$$E_N(f^l_{(a,i)(l,j)}) = 0 \qquad \text{if} \quad a \neq b.$$

(Recall that t_l, s_k are the values of the trace on the minimal projections in M_l respectively N_k).

We shall denote by f^l_a, $a \in A_{kl}$, the minimal projections in $N' \cap M$ defined by:

$$f^l_a = \sum_{i=1}^{n_k} f^l_{(a,i)(a,i)} \qquad \text{for} \quad a \in A_{k,l}.$$

6.10 LEMMA. — *Let p be a minimal projection in* M *such that $p \leqq e^k f^l$. If $u_a \in \mathbb{R}_+$ are defined by $pf^l_a p = u_a p$ then*

$$E_N(p) = \sum_{a \in A_{kl}} u_a \frac{t_l}{s_k} q_a$$

with q_a minimal projections in N, $q_a \leqq e^k$.

Proof. — The map $N e^k \ni x \mapsto x f^l_a \in M_{f^l_a}$ being an isomorphism and p being minimal in M it follows that there exist minimal projections q_a in $N e^k$ such that $f^l_a p f^l_a = u_a q_a f^l_a$. It follows that

$$E_N(p) = \sum_{a \in A_{kl}} E_N(f^l_a p f^l_a) = \sum_{a \in A_{kl}} u_a q_a E_N(f^l_a) = \sum_{a \in A_{kl}} u_a \frac{t_l}{s_k} q_a$$

Q.E.D.

6.11. LEMMA. — *Suppose $\{y_i\}_{i \in I}$ is a partition of $e^k f^l$* (i. e. $\sum_i y_i = e^k f^l$) *each y_i being a positive multiple of a minimal projection in* M. *Then*

$$\sum_{i \in I} \tau(\eta(E_N(y_i))) - \tau(\eta(y_i)) \leqq -n_k a_{kl} t_l \ln t_l + n_k a_{kl} t_l \ln s_k + n_k a_{kl} t_l \ln b_{k,l}$$

where $b_{kl} = \min\{a_{kl}, n_k\}$.

Proof. — Write $y_i = c_i p_i$, where $c_i \in \mathbb{R}_+$, and each p_i is a minimal projection smaller than $e^k f^l$.

To prove the inequality with $b_{k,l} = n_k$ note first that for every $z \in N e^k$, $z \geqq 0$, $\tau(\eta(z)) \leqq \eta(\tau(z)) + \tau(z) \ln(n_k s_k)$. To see this denote by τ_k the normalized trace on $N e^k$, that is $\tau_k = (1/n_k s_k)\tau$, and apply the known inequality (property (6) in section 3) $\tau_k(\eta(z)) \leqq \eta \tau_k(z)$.

It follows that:

$$\begin{aligned}\sum_{i \in I} \tau(\eta(E_N(y_i))) - \tau(\eta(y_i)) & \\ \leqq \sum_{i \in I} \eta(\tau(y_i)) + \tau(y_i) \ln(n_k s_k) - \tau(\eta(y_i)) & \\ = \sum_{i \in I} -\tau(y_i) \ln(\tau(y_i)) + \tau(y_i) \ln(n_k s_k) - \eta(c_i) \tau p_i. &\end{aligned}$$

Since p_i is minimal in M_l, $\tau(p_i)=t_l$, and since $\sum y_i=e^k f^l$, $\sum \tau(y_i)=n_k a_{kl} t_l$ so that we finally get

$$\sum_{i \in I} -\tau(y_i)\ln(\tau(p_i))-\tau(y_i)\ln c_i+\tau(y_i)\ln(n_k s_k)+\tau(y_i)\ln c_i$$
$$= -n_k a_{kl} t_l \ln t_l + n_k a_{kl} t_l \ln s_k + n_k a_{kl} t_l \ln n_k.$$

To prove the inequality with $b_{k,l}=a_{kl}$ define the nonnegative numbers $u_{ai}\in\mathbb{R}_+$ by $p_i f^l_a p_i = u_{ai} p_i$.

Note that

$$(*) \qquad \sum_{a \in A_{k,l}} u_{ai}=1 \quad \text{for each } i\in I$$

and that $\sum_i y_i = e^k f^l$ implies

$$\sum_{i\in I} \tau(c_i u_{ai} p_i)=\sum_{i \in I}\tau(y_i f^l_a)=\tau(f^l_a)$$

so that

$$(**) \qquad \sum_{i \in I} c_i u_{ai} t_l = n_k t_l.$$

Moreover the preceding lemma shows that

$$E_N(y_i)=\sum_{a \in A_{kl}} c_i u_{ai}\frac{t_l}{s_k} q_{ai}, \qquad \text{where} \quad \tau(q_{ai})=s_k.$$

It follows that

$$\sum_{i \in I}(\tau(\eta(E_N(y_i)))-\tau(\eta(y_i)))$$
$$=\sum_{i \in I}\tau\left(\eta\left(\sum_{a\in A_{kl}} c_i u_{ai}\frac{t_l}{s_k}q_{ai}\right)\right)-\tau(\eta(y_i))$$
$$\leqq \sum_i \sum_{a\in A_{kl}}\left(\tau\left(\eta\left(c_i u_{ai}\frac{t_l}{s_k}\right)q_{ai}\right)\right)-\sum_i \tau(\eta(c_i)p_i)$$
$$=\sum_i\sum_{a\in A_{k,l}}\eta\left(c_i u_{ai}\frac{t_l}{s_k}\right)s_k-\sum_i \eta(c_i)t_l$$
$$=\sum_i\sum_{a\in A_{k,l}} c_i u_{ai} t_l \ln s_k-\sum_i\sum_{a\in A_{k,l}} c_i u_{ai} t_l \ln c_i$$
$$-\sum_i\sum_{a\in A_{kl}} c_i u_{ai} t_l \ln a_i-\sum_i\sum_{a\in A_{kl}} c_i u_{ai} t_l\ln t_l+\sum_i t_l c_i \ln c_i.$$

The inequality used above is the well known $\tau(\eta(x+y))\leqq\tau(\eta(x))+\tau(\eta(y))$ (property (5), Section 3).

Property $(**)$ implies that the first term equals $n_k a_{kl} t_l \ln s_k$ and the forth term equals $-n_k a_{kl} t_l \ln t_l$ while $(*)$ shows that the second and last term annihilate each other. Finally the concavity of the logarithm, (or the technical lemma 3.6) shows that the third term may be majorized by:

$$-\sum_i c_i t_l \Big(\sum_{a \in A_{kl}} u_{ai} \ln u_{ai} \Big) \leqq \sum_i c_i t_l \ln a_{kl} = n_k a_{kl} t_l \ln a_{kl}$$

Q.E.D.

6.12. PROPOSITION :

$$H(M \mid N) \leqq -\sum_l m_l t_l \ln t_l + \sum m_l t_l \ln m_l + \sum_k n_k s_k \ln s_k - \sum_k n_k s_k \ln n_k + \sum_{k,\, l} n_k a_{kl} t_l \ln c_{k,\, l}$$

where $c_{kl} = \min \{ n_k . a_{kl}^{-1}, 1 \}$.

Proof. — It is sufficient to consider partitions of the unity in M consisting of positive multiples of minimal projections in some M_l. Let $\{ x_{il} \}_{i \in I,\, l \in L}$ be such a partition and write $x_{il} = c_{il} p_{i,\, l}$ with $c_{i,\, l} \in \mathbb{R}_+$ and $p_{i,\, l}$ a minimal projection in M_l. Define the nonnegative numbers $u_{k,\, il}$ by:

(1) $p_{il} e^k f^l p_{il} = u_{kil} p_{il}$,

so that:

(2) $e^k f^l p_{il} e^k f^l = u_{kil} q_{kil}$, q_{kil} being a minimal projection smaller than $e^k f^l$.

Applying the trace in (1) and (2) and using $\sum_i x_{il} = f^l$ $\sum_k e^k f^l = f^l$, $\tau(p_{il}) = \tau(q_{kil}) = t_l$ one gets:

(3) $\sum_{k \in K} u_{kil} = 1$;

(4) $\sum_i c_{il} t_l u_{kil} = n_k a_{kl} t_l$;

(5) $\sum_i c_{il} t_l = m_l t_l$.

For each $k \in K$, $l \in L$ fixed, apply the preceding lemma with $y_i = e^k x_{il} e^k$ to get

$$\sum_i \tau(\eta(E_N(e^k x_{il} e^k))) \leqq \sum_i \tau(\eta(e^k x_{il} e^k)) + n_k a_{kl} t_l \ln \frac{s_k b_{kl}}{t_l}$$

$$= \sum_i \tau(\eta(c_{il} u_{kil}) q_{kil}) + n_k a_{kl} t_l \ln \frac{s_k b_{kl}}{t_l}$$

$$= \sum_i \eta(c_{il} u_{kil}) t_l + n_k a_{kl} t_l \ln \frac{s_k b_{kl}}{t_l},$$

where $b_{kl} = \min \{ a_{kl}, n_k \}$.

It follows that

$$\sum_{i,\, l} (\tau(\eta(E_N(x_{il})) - \tau(\eta(x_{il}))$$

$$= \sum_{i,\,l,\,k} \tau(\eta(\mathrm{E_N}(x_{il}e^k))) - \sum_{i,\,e} \tau(\eta(x_{il}))$$

$$= \sum_{i,\,l,\,k} \tau(\eta(\mathrm{E_N}(e^k x_{il} e^k))) - \sum_{i,\,l} \tau(\eta(x_{il}))$$

$$= \sum_{i,\,l,\,k} \eta(c_{il}u_{kil})\,t_l + \sum_{k,\,l} n_k a_{kl} t_l \ln \frac{s_k b_{kl}}{t_l} - \sum_{i,\,l} \tau(\eta(x_{il}))$$

$$= - \sum_{i,\,l,\,k} c_{il} u_{kil} t_l \ln c_{il} - \sum_{i,\,lk} c_{il} u_{kil} t_l \ln u_{kil}$$

$$+ \sum_{k,\,l} n_k a_{kl} t_l\, l \frac{s_k b_{kl}}{t_l} + \sum_{i,\,l} c_{il} t_l \ln c_{il}.$$

Since $\sum_k u_{kil} = 1$ the first and last term vanish.

Applying the technical lemma 3.6 to the second term, noting by (4) that $\sum_i c_{il} u_{kil} t_l = u_k a_{kl} t_l$ and obviously $\sum_k n_k a_{kl} t_l = m_l t_l$, we can majorize further by

$$\sum_i \left(- \sum_k n_k a_{kl} t_l \ln \left(\frac{n_k a_{kl} t_l}{m_l t_l} \right) \right) + \sum_{k,\,l} n_k a_{kl} t_l \ln \frac{s_k b_{kl}}{t_l}$$

$$= \sum_{k,\,l} n_k a_{kl} t_l \ln \left(\frac{m_l}{t_l} \frac{s_k}{n_k} \frac{b_{kl}}{a_{kl}} \right) = \sum_{k,\,l} n_k a_{kl} t_l \ln \left(\frac{m_l}{t_l} \frac{s_k}{n_k} c_{kl} \right).$$

where $c_{kl} = \min\{n_k/a_{k1},\ 1\}$.

This is the desired estimate since

$$\sum_k n_k a_{kl} t_l = m_l t_l \qquad \text{and} \qquad \sum_l n_k a_{kl} t_l = n_k s_k.$$

Q.E.D.

The rest of this section is devoted to the proof of the opposite inequality, by exhibiting a partition of the unity in M with entropy equal to the right hand side in theorem 6.2. Looking at the proof of the inequality just obtained we see that we have to get equality at two stages, on the one hand under each $e^k f^l$ (Lemma 6.11) and on the other hand in each M_l (Proposition 6.12). This is the content of the following two lemmas:

6.13 Lemma. — *Let* $k \in \mathrm{K}$ *and* $l \in \mathrm{L}$ *be fixed. There exist*

$$\{p_{ai}\}_{i \in [1,\, n_k],\ a \in \mathrm{A_{kl}}},$$

orthogonal projections in M, $p_{ai} \leq e^k f^l$, $\sum_{a,\,i} p_{ai} = e^k f^l$ *and such that*

$$\tau(\eta(\mathrm{E_N}(p_{ai})) = t_l \ln \frac{b_{kl} s_k}{t_l}, \qquad \textit{for every} \qquad i \in [1,\, n_k],\ a \in \mathrm{A}_{kl},$$

where $b_{kl}=\min\{a_{kl}, n_k\}$.

Proof. — Let us identify A_{kl} with $\mathbb{Z}/a_{kl}\mathbb{Z}$ and $[1, n_k]$ with $\mathbb{Z}/n_k\mathbb{Z}$.

If $a_{k,l}\leqq n_k$ let us embedd n_k copies of the matrix algebra of dimension a_{kl} into $M_{e^k f^l}$ by

$$q^i_{s,\, s+t}\mapsto f^l_{(s,\, i+s)\,(s+t,\, i+s+t)},$$

where $q^i_{s,\, s+t}$ denote the matrix units in $\overset{n_k}{\underset{i=1}{\oplus}} M_i$, $M_i\simeq\mathcal{M}_{a_{kl}}$. If λ is a primitive root of order a_{kl} of the unity, $\lambda=\exp(2\pi i/a_{kl})$, then

$$q_{ai}=\frac{1}{a_{kl}}\sum_{s,\, t}\lambda^{at}q^i_{s,\, s+t}, \qquad a\in[i, a_{kl}]$$

are the minimal projections of a maximal abelian subalgebra in each M_i so that

$$p_{ai}=\frac{1}{a_{kl}}\sum_{s,\, t-l}^{a_{kl}}\lambda^{at}f^l(s,\, i+s)(s+t,\, i+s+t)$$

are minimal projections in $M_{e^k f^l}$ such that $\sum_{a,\, i}p_{ai}=e^k f^l$.

Moreover

$$E_N(p_{ai})=\frac{1}{a_{kl}}\sum_{s,\, t=1}^{a_{kl}}\lambda^{at}E_N(f^l_{(s,\, i+s)\,(s+t,\, i+s+t)})=\frac{1}{a_{kl}}\sum_{s=1}^{a_{kl}}\frac{t^l}{s^k}e^k_{i+s,\, i+s}.$$

Since $a_{k,l}\leqq n_k$, $e^k_{i+s,\, i+s}e^k_{i+t,\, i+t}=0$ for $s\neq t$, $l\leqq s$, $t\leqq a_{kl}$, so that

$$\tau(\eta(E_N(p_{ai}))=\left(-\frac{t_l}{a_{kl}s_k}\ln\frac{t_l}{a_{kl}s_k}\right)a_{kl}s^k=t_l\ln\frac{a_{kl}s_k}{t_l}.$$

If $a_{kl}\geqq n_k$ then we embedd in the same way a_{kl} copies of matrix algebras of dimension n_k in $M_{e^k f^l}$ to get the desired projections as

$$p_{aj}=\frac{1}{n_k}\sum_{s,\, t=1}^{n_k}\lambda^{jt}f^l_{(a+s,\, s)\,(a+s+t,\, s+t)},$$

with

$$\lambda=\exp\left(\frac{2\pi i}{n_k}\right)$$

As in the case $a_{kl}\leqq n_k$ we get

$$E_N(p_{aj})=\frac{1}{n_k}\sum_{s=1}^{n_k}\frac{t^l}{s^k}e^k_{ss}$$

and

$$\tau(\eta(\mathrm{E_N}(p_{a,\,j})))=t_l \ln \frac{n_k s_k}{t_l}$$

Q.E.D.

6.14. LEMMA. — *For each $l \in \mathrm{L}$ denote*

$$\mathrm{C}_l=\{k \in \mathrm{K} \mid \mathrm{A}_{kl} \neq \varnothing\} \qquad \textit{and} \qquad \mathrm{B}_{kl}=\mathrm{A}_{kl}\times[1, n_k].$$

For every family $\{p_b\}\, b \in \bigcup_{k \in \mathrm{C}_l} \mathrm{B}_{kl}$ of minimal projections in M_l such that $\sum_{b \in \mathrm{B}_{kl}} p_b=e^k f^l$ for every $k \in \mathrm{C}_l$, there exists a partition $\{x_i\}_{i \in \mathrm{I}}$ of f^l (i. e. $\sum x_i=f^l$; $x_i \geqq 0$) with the properties:

(1) *$x_i=c_i q_i$, $c_i \in \mathbb{R}_+$ and q_i a minimal projection in M_l;*

(2) *$e^k x_i e^k=c_i(n_k a_{kl} t_l/m_l t_l)\, q_{k,\,i}$ with $q_{ki} \in \{p_b \mid b \in \mathrm{B}_{kl}\}$ for each $k \in \mathrm{K}$, $i \in \mathrm{I}$.*

Proof. — Fix $j=(j_k)_{k \in \mathrm{C}_l} \in \prod_{k \in \mathrm{C}_l} \mathrm{B}_{kl}$ a multiindex. Since $p_{j_k} \leqq e^k f^l$ it follows that $p_{j_k} p_{j_{k'}}=0$ if $k \neq k'$, k, $k' \in \mathrm{C}_l$and since the p_{j_k}'s are all minimal in M_l we may embedd in M (nonunotally) a factor M_j pf dimension c_l, equal to the cardinality of C_l, in such a way that the p_{j_k}'s are the minimal projections of a maximal abelian subalgebra in M_j. In M_j we can moreover find c_l unitaries denoted u_{jk}, $k \in \mathrm{C}_l$, such that :

(i) $u_{j,\,k}$ belongs to the maximal abelian subalgebra generated by the projections p_{j_k} and

(ii) $(1/c_l) \sum_{k \in \mathrm{C}_l} u_{jk} x u^*_{j,\,k}=\sum_{k \in \mathrm{C}_l} p_{j_k} x p_{j_k}$ (this means that they define the conditional expectation of M_j onto the considered maximal abelian $*$-subalgebra).

Let us also fix a minimal projection $q_j \in \mathrm{M}_j$ with the property that

(iii) $p_{j_k} q_j p_{j_k}=(u_k a_{kl} t_l/m_l t_l)\, p_{j_k}$ for every $k \in \mathrm{C}_l$. This is possible because $\sum_{k \in \mathrm{C}_l} n_k a_{kl} t_l/m_l t_l=1$.

The partition $\{x_i\}_{i \in \mathrm{I}}$ is defined as follows:

The index set I is equal to $\prod_{k \in \mathrm{C}_l} \mathrm{B}_{kl} \times \mathrm{C}_l$ and

$$x_{jk}=\frac{m_l}{c_l \prod_{s \in \mathrm{C}_l} n_s a_{sl}}\, u_{jk} q_j u^*_{jk}.$$

The first property of the lemma in obvious so we check the second. Since u_{jk} belongs to the C* algebra generated by the projections p_{j_k} and $p_{j_k} \leqq e^k$ it follows that

$e^{k_0} u_{jk}=\alpha p_{j_{k_0}}$, where $\alpha \in \mathbb{C}$, $|\alpha|=1$. Hence

$$e^{k_0} u_{jk} q_j u^*_{j,\,k} e^{k_0}=p_{j_{k_0}} q_j p_{j_{k_0}}=\frac{n_{k_0} a_{k_0 l} t_l}{m_l t_l} p_{j_{k_0}}.$$

Finally we check that $\{x_i\}_{i \in I}$ is a partition of f^l.

Using (ii) we get that

$$\sum_{j \in \Pi B_{kl}} \sum_{s \in C_l} x_{js} = \sum_{j \in \Pi B_{kl}} c_l \sum_{s \in C_l} p_{j_s} q_j p_{j_s} \frac{m_l}{c_l \prod_{k \in C_l} n_k a_{kl}}$$

$$= \sum_{j \in \Pi B_{kl}} \sum_{s \in C_l} \frac{n_s a_{sl} t_l}{m_l t_l} p_{j_s} \left(\frac{m_l}{\prod_{k \in C_l} n_k a_{kl}} \right)$$

$$= \sum_{s \in C_l} \sum_{j \in \Pi B_{kl}} \frac{n_s a_{sl} t_l}{m_l t_l} p_{j_s} \left(\frac{m_l}{\prod_{k \in C_l} n_k a_{kl}} \right)$$

$$= \sum_{s \in C_l} \frac{n_s a_{sl} t_l}{m_l t_l} \left(\prod_{k \neq s} n_k a_{kl} \right) f^l e^s \frac{m_l}{\prod_{k \in C_l} n_k a_{kl}}$$

$$= \sum_{s \in C_l} \frac{n_s a_{sl} t_l}{m_l t_l} \frac{m_l}{n_s a_{sl}} f^l e^s = f^l.$$

Q.E.D.

We can now complete the proof of theorem 6.2. The partition of the unity is obtained in the following way: For each $k \in K$ and $l \in L$ let $\{p_b\}_{b \in B_{kl}}$ $(B_{kl} = A_{kl} \times [l, n_k])$ be the projections constructed in Lemma 6.13. Applying to these projections lemma 6.14, for each l, we get a partition of the unity in M $\{x_{il}\}_{i \in I,\, l \in L}$ with the following properties:

(1) $x_{il} = c_{il} p_{il}$, $c_{il} \in \mathbb{R}_+$, p_{il} is a minimal projection in M_l.

(2) $e^k x_{il} e^k = c_{il} (u_k a_{kl} t_l / m_l t_l) q_{i,\, l,\, k}$ with q_{ilk} a minimal projection in M_l.

(3) $\tau(\eta(E_N(q_{i,\, l,\, k}))) = t_l \ln (b_{kl} s_k / t_l)$ where $b_{kl} = \min \{a_{kl}, n_k\}$. Thus we get also

(4) $\sum_i c_{il} t_l = m_l t_l$ and $\tau(q_{ilk}) = t_l$.

It follows that the entropy of the partition x_{il} is equal to

$$\sum_{i,\, l} \tau(\eta(E_N(c_{il} p_{il}))) - \sum_{i,\, l} ((c_{il} p_{il}))$$

$$= \sum_{i,\, l,\, k} \tau(\eta(E_N(c_{il} e^k p_{il} e^k))) + \sum_{i,\, e} c_{il} t_l \ln c_{il}$$

$$= \sum_{i,\, l,\, k} \tau\left(\eta\left(c_{il} \frac{n_k a_{kl} t_l}{m_l t_l} E_N(q_{ilk})\right)\right) + \sum_{i,\, l} c_{il} t_l \ln c_{il}$$

$$= \sum_{i,\, l,\, k} \tau\left(\eta\left(c_{il} \frac{n_k a_{kl} t_l}{m_l t_l}\right) E_N(q_{ilk})\right)$$

$$+ \sum_{i,\, l,\, k} \tau\left(c_{il} \frac{n_k a_{kl} t_l}{m_l t_l}\right) E_N(q_{ilk}) + \sum c_{il} t_l \ln c_{il}$$

$$= -\sum_{i,\,l,\,k} c_{il}\frac{n_k a_{kl} t_l}{m_l} \ln \frac{c_{il} n_k a_{kl}}{m_l} + \sum_{i,\,l,\,k} c_{il}\frac{n_k a_{kl} t_l}{m_l} \ln \frac{b_{kl} s_k}{t_l}$$

$$+\sum_{i,\,l} c_{il} t_l \ln c_{i,\,l} = -\sum_{i,\,l,\,k} c_{il}\frac{n_k a_{kl} t_l}{m_l} \ln c_{il}$$

$$+\sum_{i,\,l,\,k} c_{il}\frac{n_k a_{kl} t_l}{m_l} \ln \left(\frac{s_k}{n_k}\frac{m_l}{t_l}\frac{b_{kl}}{a_{kl}}\right) + \sum_{i,\,l} c_{il} t_l \ln c_{il}.$$

Since $\sum_k n_k a_{kl} = m_l$ the first and last term disappear and using that $\sum_i c_{il} t_l = m_l t_l$, the second term becomes:

$$\sum_{l,\,k} n_k a_{kl} t_l \ln \frac{s_k}{t_l}\frac{m_l}{t_l} c_{kl},$$

which gives the desired result since

$$\sum_k n_k a_{kl} t_l = m_l t_l \qquad \text{and} \qquad \sum_l n_k a_{kl} t_l = n_k s_k.$$

Q.E.D.

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