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A THEOREM OF GIESEKER-PETRI TYPE FOR PRYM VARIETIES

GERALD E. WELTERS [1]

Much information—if not all—about the geometry of special subvarieties of Jacobi varieties comes from Brill-Noether theory. In particular, this is the case for the theta divisor of the polarization and its various strata of multiple points, on a generic jacobian.

This latter aspect is suited for extension to another class of principally polarized abelian varieties, the Prym varieties.

A priori one might hope that a more or less straight application of the results of Brill-Noether theory to the class of curves of odd genus with a fixed point free involution could be of use in this sense. However, many statements of this theory are only known to hold generically in the moduli space of curves, and they may fail at the locus we are interested in. This happens actually for one of the main results, the Gieseker-Petri theorem [*cf.* (1.12)].

In this paper we prove a natural analogue of this theorem for Prym varieties [Theorem (1.11)], following the proof given by Eisenbud-Harris [3] of Gieseker's original result [4]. As in the standard theory, this leads to a proof of the smoothness and to a computation of the dimension ρ_k of the loci $\mathrm{Sing}_k\,\Xi$ of k-ple points of the theta divisor on a general Prym variery (Section 3), provided that these loci are nonempty. The latter condition (existence assumption) is fulfilled, by Brill-Noether theory, if the dimension is big enough w.r.t. k. (By analogy with that theory, it is natural to expect that $\rho_k \geqq 0$ actually implies the existence assumption, but this question will not be considered in this paper.)

1. Preliminaries

(1.1) We recall some known facts [7] and make some definitions. Let C be a smooth projective irreducible curve defined over an algebraically closed field k of characteristic different from 2. Let π: $\tilde{C} \to C$ be an irreducible étale (2:1) covering of C. We put

([1]) Partially supported by the Institut d'Estudis Catalans.

$g=g(C)$, and hence $g(\tilde{C})=2g-1$. The map π induces in particular a map

$$\mathrm{Nm}: \quad \mathrm{Pic}^{2g-2}(\tilde{C}) \to \mathrm{Pic}^{2g-2}(C).$$

Let $\omega_C \in \mathrm{Pic}^{2g-2}(C)$ be the canonical class. The (scheme-theoretic) inverse image

$$\mathrm{Nm}^{-1}(\omega_C) \subset \mathrm{Pic}^{2g-2}(\tilde{C})$$

breaks up in two connected components

$$P^{+}=\{L \in \mathrm{Pic}^{2g-2}(\tilde{C}) \mid \mathrm{Nm}(L) \simeq \omega_C,\ h^0(L) \equiv 0 \pmod 2\},$$
$$P^{-}=\{L \in \mathrm{Pic}^{2g-2}(\tilde{C}) \mid \mathrm{Nm}(L) \simeq \omega_C,\ h^0(L) \equiv 1 \pmod 2\}.$$

Both components are translates of an abelian subvariety $P=P(\tilde{C}, C)$ of $J\tilde{C}=\mathrm{Pic}^0(\tilde{C})$, which is the connected component of the origin in the kernel of

$$\mathrm{Nm}: \quad \mathrm{Pic}^0(\tilde{C}) \to \mathrm{Pic}^0(C).$$

One has $\dim P=g-1$, and the canonical polarization of $J\tilde{C}$ restricts to twice a principal polarization on P. Together with this polarization, P is called the Prym variety defined by the couple $(\tilde{C}, C)$.

We define the following closed subsets in $\mathrm{Nm}^{-1}(\omega_C)$, for $r \in \mathbb{Z}$, $r \geqq -1$:

(1.2) $$V^r=\{L \in \mathrm{Pic}^{2g-2}(\tilde{C}) \mid \mathrm{Nm}(L) \simeq \omega_C,\ h^0(L) \geqq r+1,\ h^0(L) \equiv r+1 \pmod 2\}.$$

Thus one has inclusions:

$$P^{+}=V^{-1} \supset V^1 \supset V^3 \supset \ldots,$$
$$P^{-}=V^0 \supset V^2 \supset V^4 \supset \ldots.$$

For technical reasons, which will become apparent in a moment, we shall consider a specific scheme structure on each V^r. This is defined by taking V^r as the scheme-theoretical intersection

$$V^r=W^r_{2g-2}(\tilde{C}) \cap P^{+} \quad \text{if} \quad r \text{ is odd},$$
$$V^r=W^r_{2g-2}(\tilde{C}) \cap P^{-} \quad \text{if} \quad r \text{ is even}.$$

Here the W^r_d are endowed with their natural scheme structure, as defined in [1].

(1.3) *Remark*. — Set-theoretically, one has equalities

$$W^r_{2g-2}(\tilde{C}) \cap P^{+}=W^{r-1}_{2g-2}(\tilde{C}) \cap P^{+} \quad \text{if} \quad r \text{ is odd},$$
$$W^r_{2g-2}(\tilde{C}) \cap P^{-}=W^{r-1}_{2g-2}(\tilde{C}) \cap P^{-} \quad \text{if} \quad r \text{ is even}.$$

However, the scheme-theoretical intersections are generally different. For example, by a translation identifying P^+ with the Prym variety P, the intersection $W^0_{2g-2}(\tilde{C}) \cap P^{+}$ goes over into 2Ξ, where Ξ is the theta divisor of the polarized Prym variety (*cf.* [7]). On the other side, $V^1=W^1_{2g-2}(\tilde{C}) \cap P^{+}$ is generically reduced, by the results below.

A first result about the loci V^r follows immediately from [8] and [5], p. 613:

(1.4) PROPOSITION. — *Let* $L \in \mathrm{Nm}^{-1}(\omega_C)$ *and put* $h^0 L = r+1$. *The dimension of* V^r *at* L *satisfies*

$$\dim_L V^r \geqq g-1-\binom{r+1}{2}.$$

(1.5) Next, we perform some infinitesimal computations. Write $\varepsilon \in {}_2JC$ the class of order two defining the covering $\pi: \tilde{C} \to C$, thus $\varepsilon = c_1 R^0_\pi \mathcal{O}_{\tilde{C}}$. One has a decomposition

$$H^0 \omega_{\tilde{C}} = H^0 \omega_C \oplus H^0 \omega_C(\varepsilon)$$

into invariant and anti-invariant forms under the action of the covering involution ι of $\tilde{C}$.

Let $L \in \mathrm{Nm}^{-1}(\omega_C)$ and write $h^0 L = r+1$. The tangent map of Nm at L is given canonically by $2({}^t\pi^*)$, where

$${}^t\pi^*: \quad (H^0 \omega_{\tilde{C}})^\vee \to (H^0 \omega_C)^\vee$$

is the transpose of the pullback of differential forms. By taking kernels we obtain

$$T_{\mathrm{Nm}^{-1}(\omega_C)}(L) = (H^0 \omega_C(\varepsilon))^\vee \subsetneq (H^0 \omega_{\tilde{C}})^\vee,$$

the inclusion being taken as the transpose of the projection map

$$(1.6) \qquad H^0 \omega_{\tilde{C}} \to H^0 \omega_C(\varepsilon), \qquad \lambda \mapsto 1/2(\lambda - \iota\lambda).$$

On the other side one has ([1], p. 189):

$$T_{W^r_{2g-2}(\tilde{C})}(L) = (\mathrm{Im}\,\alpha)^\perp \subset (H^0 \omega_{\tilde{C}})^\vee,$$

where

$$\alpha: \quad H^0 L \otimes H^0(\omega_{\tilde{C}} \otimes L^\vee) \to H^0 \omega_{\tilde{C}},$$

is the Petri morphism, $\alpha(s \otimes t) = s.t$.

Combining these facts one has that

$$T_{V^r}(L) = (\mathrm{Im}\,\alpha)^\perp \cap (H^0 \omega_C(\varepsilon))^\vee$$

equals the orthogonal subspace of the image of the composition of α with the projection map (1.6).

Now $\mathrm{Nm}L \simeq \omega_C$ implies $L \otimes \iota L \simeq \omega_{\tilde{C}}$, hence $\omega_{\tilde{C}} \otimes L^\vee \simeq \iota L$.

Consider the composition

$$(1.7) \qquad H^0 L \otimes H^0 L \xrightarrow{1\otimes\iota} H^0 L \otimes H^0(\omega_{\tilde{C}} \otimes L^\vee) \xrightarrow{\alpha} H^0 \omega_{\tilde{C}} \xrightarrow{(1.6)} H^0 \omega_C(\varepsilon),$$

sending $s \otimes t \in H^0 L \otimes H^0 L$ to $1/2(s.(\iota t) - (\iota s).t)$. This is skew-symmetric, hence it defines a morphism

$$(1.8) \qquad \beta: \quad \Lambda^2 H^0 L \to H^0 \omega_C(\varepsilon), \qquad s \wedge t \mapsto 1/2(s.(\iota t) - (\iota s).t),$$

which we shall call, for easy reference, the "Prym-Petri" map for L.

The preceding conclusion may be rephrased now as follows:

(1.9) PROPOSITION. — *Let* $L \in Nm^{-1}(\omega_C)$, *and put* $h^0 L = r+1$. *Then the Zariski tangent space* $T_{V^r}(L)$ *of* V^r *at* L *equals* $(\operatorname{Im}\beta)^{\perp}$.

(1.10) COROLLARY. — *Under the same assumptions one has:*

$$\dim_L(V^r) \leqq g-1-\binom{r+1}{2}+\dim(\ker\beta).$$

Thus, if β were injective for L, then V^r would be smooth of dimension $g-1-\binom{r+1}{2}$ at L. The main result of this paper is:

(1.11) THEOREM. — *There exists a non-empty Zariski open subset in* M_g *such that, for all curves* C *in this set and for all irreducible étale double coverings* $\pi: \tilde{C} \to C$, *the Prym-Petri map* β *is injective for all* $L \in Nm^{-1}(\omega_C)$.

(1.12) *Remark.* — Identifying $\Lambda^2 H^0 L$ with the subspace of skew-symmetric tensors in $\otimes^2 H^0 L$,

$$\Lambda^2 H^0 L \subsetneq H^0 L \otimes H^0 L, \qquad s \wedge t \mapsto 1/2(s\otimes t - t\otimes s),$$

the Prym-Petri map β can be identified with the restriction of the Petri map α:

(1.13)
$$\begin{array}{ccc} \Lambda^2 H^0 L \subsetneq H^0 L \otimes H^0 L \xrightarrow[\simeq]{1\otimes\iota} H^0 L \otimes H^0(\omega_{\tilde{C}} \otimes L^{\vee}) \xrightarrow{\alpha} & H^0 \omega_{\tilde{C}} \\ \searrow^{\beta} & \cup \\ & H^0 \omega_C(\varepsilon). \end{array}$$

Thus (1.11) would be a consequence of the Gieseker-Petri theorem, if the Petri condition did hold for a general curve of genus $2g-1$ with a fixed point free involution. But this does not happen: it suffices to take, for example, given $C \in M_g$ and $\varepsilon \in {}_2JC \setminus \{0\}$, two odd theta characteristics M_1 and M_2 such that $M_2 = M_1(\varepsilon)$. The Petri condition fails for $L = \pi^* M_1$, because this is a theta characteristic on $\tilde{C}$ with at least two independent sections.

2. Proof of Theorem (1.11)

We shall adapt the ideas and the proof of D. Eisenbud and J. Harris in their reinterpretation [3] of D. Gieseker's work [4]. Also we shall refer freely to the former paper, for definitions and for some arguments, when using (or following) them.

For $\pi: \tilde{C} \to C$ as in (1.1), we say that $(\tilde{C}, C)$ satisfies the Prym-Petri condition if, for all $L \in Nm^{-1}(\omega_C)$, the Prym-Petri map β is injective.

We start with a discussion of the Zariski openness of the Prym-Petri condition. This is standard (*cf.* [4], Lemma 8.1 and Proposition 8.4, or [3], p. 271).

(2.1) PROPOSITION. — *Let* $\mathscr{C} \to T$ *and* $\tilde{\mathscr{C}} \to T$ *be smooth projective families of irreducible curves over an algebraic k-scheme* T, *and let* $\pi: \tilde{\mathscr{C}} \to \mathscr{C}$ *be an étale* (2:1) *morphism. The set of points* $t \in T$ *such that the Prym-Petri condition fails for* $(\tilde{\mathscr{C}} \otimes \overline{k(t)}, \mathscr{C} \otimes \overline{k(t)})$ *is closed.*

Proof. — We may assume that T is the spectrum of a discrete valuation ring, and that the Prym-Petri condition is violated at the generic point $\eta \in T$. It has to be shown that the same happens at the special point $0 \in T$. Up to performing a base change first, there is a line bundle $\mathscr{L}_\eta$ on $\tilde{\mathscr{C}}_\eta$ such that $\mathrm{Nm}(\mathscr{L}_\eta) \simeq \omega_{\mathscr{C}_\eta/k(\eta)}$ and such that the Prym-Petri map [*cf.* (1.13)] $\Lambda^2 H^0 \mathscr{L}_\eta \to H^0 \omega_{\tilde{\mathscr{C}}_\eta/k(\eta)}$ is non-injective.

The line bundle $\mathscr{L}_\eta$ extends to a line bundle $\mathscr{L}$ on $\tilde{\mathscr{C}}$, and $\mathrm{Nm}(\mathscr{L}) \simeq \omega_{\mathscr{C}/T}$. Furthermore, since T is the spectrum of a discrete valuation ring, the direct image sheaf $R^0 \mathscr{L}$ is a free $\mathscr{O}_T$-module and, secondly,

$$(2.2) \qquad (R^0 \mathscr{L})(0) \subsetneq H^0(\mathscr{L}_0).$$

Now, since the (relative) Prym-Petri map $\Lambda^2 R^0 \mathscr{L} \to R^0 \omega_{\tilde{\mathscr{C}}/T}$ is a generically non-injective morphism of free $\mathscr{O}_T$-modules, it is non-injective at the special fibers, too. Therefore, by (2.2), the same holds for the Prym-Petri map $\Lambda^2 H^0 \mathscr{L}_0 \to H^0 \omega_{\tilde{\mathscr{C}}_0/k(0)}$,

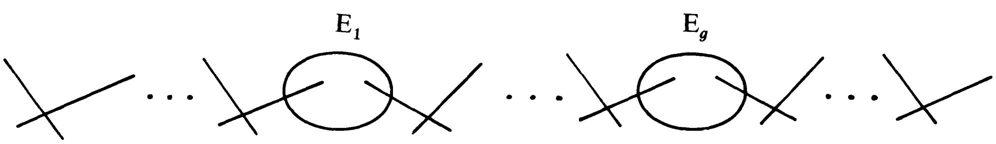
Q.E.D.

In view of (2.1) and the irreducibility of the moduli space of pairs consisting of a smooth curve of genus g and an irreducible étale (2:1) covering [2], Theorem (1.11) will be proved if we exhibit a single pair like this, defined over an algebraically closed field $K \supset k$, and satisfying the Prym-Petri condition.

As in [3], we shall take as such the geometric generic fiber of a suitable family of double coverings defined over the spectrum of a discrete valuation ring:

Let $\mathscr{C} \to T$ be a flat projective k-morphism with $\mathscr{C}$ a smooth surface and T the spectrum of a discrete valuation ring over k having k as residue field, and such that: (a) the generic fiber $\mathscr{C}_\eta$ is smooth and geometrically irreducible, and: (b) the special fiber $\mathscr{C}_0$ is a reduced curve with ordinary double points as only singularities, as described below:

(2.3)



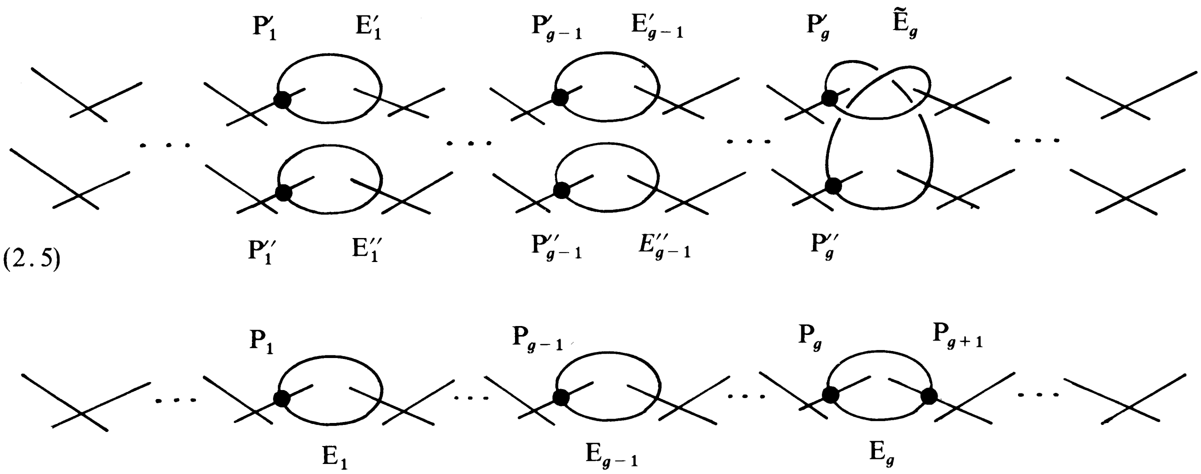
That is to say, $\mathscr{C}_0$ consists of a string of smooth components; the straight lines and the dots stand for rational components; $E_1, \ldots, E_g$ are elliptic curves. Thus $\mathscr{C}_0$ is a curve of arithmetic genus equal to g. We ask furthermore that, for all $i=1, \ldots, g$, the two points at which E_i meets the remaining components of $\mathscr{C}_0$ be $\mathbb{Z}$-independent in $\mathrm{Pic}(E_i)$. In other words, calling these points P and Q for a moment, $P-Q \in \mathrm{Pic}^0(E_i)$ is not a torsion class.

The existence of $\mathscr{C} \to T$ as above follows e.g. from [2] (starting with such a family with $\sum E_i$ as special fiber, the rest is achieved by means of blowing-ups and, if wanted, by means of base extension as below).

(2.4) A feature of such families, to be used here, is that, if $T' \to T$ is a dominant morphism of spectra of discrete valuation rings as above, then the family $\mathscr{C}' \to T'$ obtained from $\mathscr{C} \to T$ by base extension and minimally resolving the singularities satisfies the same requirements as $\mathscr{C} \to T$ (*cf.* [4], p. 271).

Extending the base if necessary (*cf.* above), there exists a line bundle ε on $\mathscr{C}$ such that $\varepsilon^2 \simeq \mathcal{O}_{\mathscr{C}}$ and such that ε restricts on $\mathscr{C}_0$ to a line bundle which is trivial on every component of $\mathscr{C}_0$ except on E_g, where it is non-trivial. Define $\tilde{\mathscr{C}} = \mathrm{Spec}(\mathcal{O}_{\mathscr{C}} \oplus \varepsilon)$, the ring structure coming from the isomorphism $\varepsilon^2 \simeq \mathcal{O}_{\mathscr{C}}$. In this way, $\pi: \tilde{\mathscr{C}} \to \mathscr{C}$ is an étale double covering; the geometric generic fiber $\tilde{\mathscr{C}}_\eta \otimes \overline{k(\eta)}$ is smooth and irreducible, and the special fiber $\tilde{\mathscr{C}}_0$ may be depicted as follows over $\mathscr{C}_0$:

(2.5)



(2.6) Proposition. — *The couple* $(\tilde{\mathscr{C}}_\eta \otimes \overline{k(\eta)}, \mathscr{C}_\eta \otimes \overline{k(\eta)})$ *satisfies the Prym-Petri condition.*

This will prove Theorem (1.11). The proof of (2.6) will occupy the rest of this section. Suppose that the couple violates the Prym-Petri condition. We shall derive a contradiction.

As in [3], p. 272, after possibly base extending and minimally resolving singularities (*cf.* (2.4)), there exists a line bundle $\mathscr{L}$ on $\tilde{\mathscr{C}}$ such that $\mathrm{Nm}\,\mathscr{L}_\eta \simeq \omega_{\mathscr{C}_\eta/k(\eta)}$ and such that the Prym-Petri map $\Lambda^2 H^0 \mathscr{L}_\eta \to H^0 \omega_{\tilde{\mathscr{C}}_\eta/k(\eta)}$ is non-injective.

(2.7) The line bundles $\mathrm{Nm}\,\mathscr{L}$ and $\omega_{\mathscr{C}/T}$ differ by a twist with the line bundle attached to some linear combination of the components of $\mathscr{C}_0$, and one may suppose that the component E_g does not appear in this linear combination (the bundle associated with the sum of all components is trivial). Since $\mathrm{Nm}(\mathcal{O}_{\tilde{\mathscr{C}}}(E'_i)) = \mathrm{Nm}(\mathcal{O}_{\tilde{\mathscr{C}}}(E''_i)) \simeq \mathcal{O}_{\mathscr{C}}(E_i)$ and similarly for the rational components of $\tilde{\mathscr{C}}_0$, we may replace $\mathscr{L}$ by a suitable twist with a linear combination of the components of $\tilde{\mathscr{C}}_0$ and suppose from now on that $\mathrm{Nm}(\mathscr{L}) \simeq \omega_{\mathscr{C}/T}$ holds. This implies $\mathscr{L} \otimes \iota\,\mathscr{L} \simeq \omega_{\tilde{\mathscr{C}}/T}$, where ι stands now for the covering involution of $\tilde{\mathscr{C}}$.

We apply the theory of limit linear series of [3]. Since the curve $\tilde{\mathscr{C}}_0$ (resp. $\mathscr{C}_0$) has no loops in its graph, Section 1 of [3] applies in this case. Given a line bundle $\mathscr{L}$ on $\tilde{\mathscr{C}}$ (resp. on $\mathscr{C}$), we may twist it by a suitable linear combination of the components of $\tilde{\mathscr{C}}_0$ (resp. $\mathscr{C}_0$) and obtain a line bundle whose limit is concentrated at any prescribed component Y of $\tilde{\mathscr{C}}_0$ (resp. $\mathscr{C}_0$), this meaning that the bundle restricts to a bundle of degree 0 on any component of the special fiber other than Y. The so obtained line bundle $\mathscr{L}_Y$ is determined up to isomorphism by this condition. The $\mathcal{O}_T$-module $R^0 \mathscr{L}_Y$ is free and

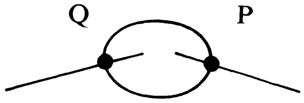
$$(R^0 \mathscr{L}_Y)(0) \subsetneq H^0(\mathscr{L}_Y \otimes \mathcal{O}_{\tilde{\mathscr{C}}_0}) \subset H^0(\mathscr{L}_Y \otimes \mathcal{O}_Y)$$

(the second term being replaced by $H^0(\mathscr{L}_Y \otimes \mathcal{O}_{\mathscr{C}_0})$ in the resp. case). For convenience, we shall use a different notation than [3]: instead of V_Y, we shall write $L_Y \subset H^0(\mathscr{L}_Y \otimes \mathcal{O}_Y)$ for the image of $(R^0 \mathscr{L}_Y)(0)$. This is the vector space defining the limit linear series of $\mathscr{L}$ on Y. If $\sigma \in H^0 R^0 \mathscr{L}_Y$, we shall write $\bar{\sigma} \in L_Y$ for its image.

In our case, we have furthermore: if $\mathscr{L}$ is a line bundle on $\tilde{\mathscr{C}}$ and $\mathscr{L}_Y$ is its associated bundle (as above) w. r. t. Y, then: $\iota(\mathscr{L}_Y) \simeq (\iota \mathscr{L})_{\iota Y}$, and $\mathrm{Nm}(\mathscr{L}_Y) \simeq \mathrm{Nm}(\mathscr{L})_{\pi(Y)}$.

We may use Section 1 of [3] as it stands, except for Proposition 1.5 and its Corollary 1.6. Things change in our favour, however, in the sense than one can remove the characteristic zero hypothesis of [3] (*cf.* p. 275 of *Loc. Cit.*) in the modified statement:

(2.8) PROPOSITION. — *Let* E *be an elliptic component of* $\mathscr{C}_0$ *or* $\tilde{\mathscr{C}}_0$, *other than* $\tilde{E}_g$. *Call* P, Q *the intersection points of* E *with the adjacent components, and call* Y *the component meeting* E *at* P:



(*a*) *There is in* L_E, *up to scalars, at most one section vanishing only at* P *and* Q; (*b*) *For all but at most one value of i, one has the strict inequality between the following vanishing orders:* $a_i(L_E, Q) < a_i(L_Y, P)$.

Proof. — Part (*a*) follows from the assumption that P and Q are $\mathbb{Z}$-independent in Pic(E), and part: (*b*) follows from this and Proposition 1.3 of [3].

(2.9) *Remark.* — The inverse images in $\tilde{E}_g$ of each of the two points of intersection of E_g with the adjacent components do not satisfy this requirement; the difference between the two points is a non-zero element of the kernel of $\mathrm{Nm} : \mathrm{Pic}^0(\tilde{E}_g) \to \mathrm{Pic}^0(E_g)$, i. e., a class of order two. It may be noticeable that this is the point where things would break down, when trying to prove the Petri condition for curves with an involution, along the lines of [3] [*cf.* (1.12)].

To keep following [3], we need information about the vanishing sequences for the limit series of $\omega_{\tilde{\mathscr{C}}/T}$. The vanishing sequences for the limit series of $\omega_{\mathscr{C}/T}$ are the same as in the setting of [3], and Lemma 2.2 of *Loc. Cit.* applies verbatim (with aid of (2.8) above):

(2.10) LEMMA. — *Call* $a_0^h < a_1^h < \ldots < a_{g-1}^h$, $h = 1, \ldots, g$ *the vanishing sequence at* P_h *of the limit series of* $\omega_{\mathscr{C}/T}$ *on* E_h. *Similarly, for* $h = g+1$, *let this be the vanishing sequence at* P_{g+1} *of the limit series of* $\omega_{\mathscr{C}/T}$ *on* Y, *where* Y *is the component of* $\mathscr{C}_0$ *meeting* E_g *at* P_{g+1} [see *Fig.* (2.5)]. *Then:*

$$a_i^h = i + h - 2 \qquad \text{if} \quad i \leqq h-2,$$
$$a_i^h = i + h - 1 \qquad \text{if} \quad i \geqq h-1.$$

In particular, the vanishing sequence for $h=g$ is:

$$(2.11)\qquad (g-2|g-1|\ldots|2g-5|2g-4|2g-2).$$

Since $\omega_{\tilde{\mathscr{C}}/T}\simeq\pi^*(\omega_{\mathscr{C}/T})$, one has: $(\omega_{\tilde{\mathscr{C}}/T})_{\tilde{E}_g}\simeq\pi^*((\omega_{\mathscr{C}/T})_{E_g})$. Therefore the values in (2.11) are a part of the vanishing sequence at P'_g (resp. at P''_g) for the limit series of $\omega_{\tilde{\mathscr{C}}/T}$ on $\tilde{E}_g$.

(2.12) PROPOSITION. — *For $h=1,\ldots,g-1$, the first g terms of the vanishing sequence at P'_h (resp. at P''_h) of the limit linear series of $\omega_{\tilde{\mathscr{C}}/T}$ on E'_h (resp. on E''_h) are given by the formulae of Lemma* (2.10). *Similarly for $h=g$ and the vanishing sequence at P'_g (resp. P''_g) of the limit series of $\omega_{\tilde{\mathscr{C}}/T}$ on $\tilde{E}_g$.*

Proof. — Consider, for each $h=1,\ldots,g$, the terms of the corresponding vanishing sequence which are ranged at the same level as the values of (2.11) in the vanishing sequence for $h=g$. Call Σ_h the sum of these terms. By [3], Proposition 1.3 and by Proposition (2.8) above, one has $\Sigma_{h+1}\geqq\Sigma_h+(g-1)$. Thus $\Sigma_g\geqq\Sigma_1+(g-1)^2$. By direct computation, one finds $\Sigma_g=1/2\,(3g-2)\,(g-1)$, hence $\Sigma_1\leqq 1/2\,g\,(g-1)$. This determines the terms of the vanishing sequence for $h=1$ which are involved: these are: $0|1|\ldots|g-2|g-1$. This implies that the levels involved are the first g ones and, secondly, that, in passing from h to $h+1$, all terms but exactly one are increased by 1. As in the proof of Lemma 2.2 of [3], this yields the desired conclusion.

Q.E.D.

(2.13) COROLLARY. — *For $h=1,\ldots,g-1$, the limit linear series of $\omega_{\tilde{\mathscr{C}}/T}$ on E'_h (resp. E''_h) has precisely one section, up to a scalar factor, vanishing only at P'_h (resp. P''_h) and at the second point of intersection of E'_h (resp. E''_h) with the remaining components of $\tilde{\mathscr{C}}_0$. This section vanishes at P'_h (resp. P''_h) to order exactly $2h-2$.*

This corollary follows from (2.12) by using [3], Proposition 1.3, and replaces for us Proposition 2.1 of *Loc. Cit.*

Now we go back to our particular $\mathscr{L}$, which has been fixed in (2.7), and write $\mathscr{M}=\iota\mathscr{L}$. We recall that $\mathscr{L}\otimes\mathscr{M}\simeq\omega_{\tilde{\mathscr{C}}/T}$. By assumption, the Prym-Petri map

$$(2.14)\qquad \Lambda^2 H^0\mathscr{L}_\eta \hookrightarrow H^0\mathscr{L}_\eta\otimes H^0\mathscr{L}_\eta \xrightarrow[\simeq]{1\otimes\iota} H^0\mathscr{L}_\eta\otimes H^0\mathscr{M}_\eta \xrightarrow{\alpha} H^0\omega_{\tilde{\mathscr{C}}_\eta/k(\eta)},$$

has a non-zero kernel. The composition of mappings in (2.14) is the fibre at the generic point η of the following one:

$$(2.15)\quad \Lambda^2 R^0\mathscr{L}_{\tilde{E}_g} \hookrightarrow R^0\mathscr{L}_{\tilde{E}_g}\otimes R^0\mathscr{L}_{\tilde{E}_g} \xrightarrow[\simeq]{1\otimes\iota} R^0\mathscr{L}_{\tilde{E}_g}\otimes R^0\mathscr{M}_{\tilde{E}_g} \xrightarrow{\alpha} R^0(\omega_{\tilde{\mathscr{C}}/T})_{\tilde{E}_g}.$$

Therefore there exists an element

$$\rho\in H^0R^0\mathscr{L}_{\tilde{E}_g}\otimes H^0R^0\mathscr{M}_{\tilde{E}_g}\setminus t(H^0R^0\mathscr{L}_{\tilde{E}_g}\otimes H^0R^0\mathscr{M}_{\tilde{E}_g})$$

such that

(2.16) $$\alpha(\rho)=0,$$

(2.17) $$(1\otimes\iota)(\rho)\in\otimes^2 H^0 R^0 \mathscr{L}_{\tilde{E}_g} \text{ is skew-symmetric.}$$

Write $\bar{\rho}\in L_{\tilde{E}_g}\otimes M_{\tilde{E}_g}$ for the image of ρ in this vector space. By assumption we have:

(2.18) $$\bar{\rho}\neq 0,$$

(2.19) $$(1\otimes\iota)(\bar{\rho})\in\otimes^2 L_{\tilde{E}_g} \text{ is skew-symmetric.}$$

We are now in position of applying Section 3 of [3], with (2.13) [resp. (2.8)] above replacing Proposition 2.1 (resp. Corollary 1.6) of *Loc. Cit.*, and we conclude from (2.16) that

(2.20) $$\mathrm{ord}_{P'_g}(\bar{\rho})\geqq 2g-2 \quad\text{and}\quad \mathrm{ord}_{P''_g}(\bar{\rho})\geqq 2g-2.$$

In other words (*cf.* [3], p. 277), one has $\bar{\rho}=\sum s_i\otimes t_i=\sum u_j\otimes v_j$, with

$$\mathrm{ord}_{P'_g}(s_i)+\mathrm{ord}_{P'_g}(t_i)\geqq 2g-2 \quad\text{for all } i,$$

and

$$\mathrm{ord}_{P''_g}(u_j)+\mathrm{ord}_{P''_g}(v_j)\geqq 2g-2 \quad\text{for all } j.$$

The proof of Proposition (2.6) and, hence, of Theorem (1.11) will be ended now by showing that (2.19) and (2.20) together imply that $\bar{\rho}=0$, thereby contradicting (2.18). This will be achieved through the following two lemmas.

(2.21) LEMMA. — *The line bundles* $\mathscr{L}_{\tilde{E}_g}\otimes\mathscr{O}_{\tilde{E}_g}$ *and* $\mathscr{M}_{\tilde{E}_g}\otimes\mathscr{O}_{\tilde{E}_g}$ *on* $\tilde{E}_g$ *are both isomorphic either to* $\mathscr{O}_{\tilde{E}_g}((2g-2)P'_g)$ *or to* $\mathscr{O}_{\tilde{E}_g}((2g-3)P'_g+P''_g)$.

Proof. — We have $\mathrm{Nm}(\mathscr{L}_{\tilde{E}_g})\simeq(\omega_{\mathscr{C}/T})_{E_g}$ and $\mathrm{Nm}(\mathscr{M}_{\tilde{E}_g})\simeq(\omega_{\mathscr{C}/T})_{E_g}$. It is easy to check that $(\omega_{\mathscr{C}/T})_{E_g}\otimes\mathscr{O}_{E_g}=\mathscr{O}_{E_g}((2g-2)P_g)$ (*cf.* e.g. [4], p. 257). Therefore

$$\mathrm{Nm}(\mathscr{L}_{\tilde{E}_g}\otimes\mathscr{O}_{\tilde{E}_g})\simeq\mathscr{O}_{E_g}((2g-2)P_g), \quad\text{and}\quad \mathrm{Nm}(\mathscr{M}_{\tilde{E}_g}\otimes\mathscr{O}_{\tilde{E}_g})=\mathscr{O}_{E_g}((2g-2)P_g).$$

Now, $\mathrm{Nm}:\mathrm{Pic}^{2g-2}(\tilde{E}_g)\to\mathrm{Pic}^{2g-2}(E_g)$ is (2:1). Item,

$$\pi_*((2g-3)P'_g+P''_g)=(2g-2)P_g \quad\text{and}\quad \pi_*(2g-2)P'_g=(2g-2)P_g,$$

and these two divisor classes on $\tilde{E}_g$ are different. Also, since $2P'_g\equiv 2P''_g$ [*cf.* (2.9)], both classes are invariant under the involution of $\tilde{E}_g$ w.r.t. E_g. Finally, since $\iota(\mathscr{L}_{\tilde{E}_g}\otimes\mathscr{O}_{\tilde{E}_g})\simeq\mathscr{M}_{\tilde{E}_g}\otimes\mathscr{O}_{\tilde{E}_g}$, the result follows,

Q.E.D.

(2.22) LEMMA. — *Let* $V=H^0\mathscr{O}_{\tilde{E}_g}((2g-2)P'_g)$ (*resp.* $V=H^0\mathscr{O}_{\tilde{E}_g}((2g-3)P'_g+P''_g)$). *Suppose that* $\bar{\rho}\in V\otimes V$ *satisfies*

(i) $\mathrm{ord}_{P'_g}(\bar{\rho})\geqq 2g-2$ *and* $\mathrm{ord}_{P''_g}(\bar{\rho})\geqq 2g-2$;

(ii) $(1\otimes\iota)(\bar{\rho})\in V\otimes V$ *is skew-symmetric.*

Then $\bar{\rho}=0$.

Proof. — Remark that, if $\{e_i\}$ is a basis of V whose vectors have distinct vanishing orders at P'_g (resp. P''_g), then, writing $\bar{\rho}=\sum c_{ij} e_i \otimes e_j$, one has: $c_{ij}=0$ for all i, j such that $\mathrm{ord}_{P'_g}(e_i)+\mathrm{ord}_{P'_g}(e_j)<2g-2$ (resp. $\mathrm{ord}_{P''_g}(e_i)+\mathrm{ord}_{P''_g}(e_j)<2g-2$) (*cf.* [3], proof of Lemma 3.2).

We shall find a basis $\{e_i\}$ of V whose vanishing orders at P'_g are all distinct and, simultaneously, whose vanishing orders at P''_g are all distinct. We separate the two cases for V.

(*a*) $V=H^0\,\mathcal{O}_{\tilde{E}_g}((2g-2)P'_g)$. For $k=0, \ldots, g-1$, let e_{2k} be an equation for the divisor $(2g-2-2k)P'_g+2kP''_g$. Furthermore, put $2P'_g \equiv R+S$ with R, $S \notin \{P'_g, P''_g\}$. Then $(2g-2)P'_g \equiv (2g-5)P'_g+P''_g+\iota R+S$. For $k=0, \ldots, g-3$, we write e_{2k+1} for an equation of $(2g-5-2k)P'_g+(2k+1)P''_g+\iota R+S$. The $2g-2$ vectors so obtained in V exhaust the possible vanishing orders at P'_g and at P''_g (both vanishing sequences are equal to $0|1|\ldots|2g-5|2g-4|2g-2$), hence, in particular, $\{e_i\}$ is a basis of V.

A straightforward computation now shows that $\mathrm{ord}_{P'_g}(e_i \otimes e_j) \geqq 2g-2$ and $\mathrm{ord}_{P''_g}(e_i \otimes e_j) \geqq 2g-2$ happens exactly if $i=2k$, $j=21$, and $k+1=g-1$. On the other side, by construction, we may assume that $\iota(e_{2k})=e_{2(g-k-1)}$ for all $k=0, 1, \ldots, g-1$.

Thus by the remark above, in case (*a*) we deduce from (i) that

$$\bar{\rho}=\sum_{k=0}^{g-1} c_k e_{2k} \otimes e_{2(g-k-1)}=\sum_{k=0}^{g-1} c_k e_{2k} \otimes \iota e_{2k}.$$

So $(1 \otimes \iota)(\bar{\rho})=\sum c_k e_{2k} \otimes e_{2k}$ and, by (ii), $c_k=0$ for all k, i. e. $\bar{\rho}=0$.

(*b*) $V=H^0\,\mathcal{O}_{\tilde{E}_g}((2g-3)P'_g+P''_g)$. Here we write, for $k=0, \ldots, g-2$, e_{2k+1} for an equation of the divisor $(2g-3-2k)P'_g+(2k+1)P''_g$. Also, we put $P'_g+P''_g \equiv R+S$ with R, $S \notin \{P'_g, P''_g\}$. Then $(2g-3)P'_g+P''_g \equiv (2g-4)P'_g+R+S$. For $k=0, \ldots, g-2$, we write e_{2k} for an equation of the divisor $(2g-4-2k)P'_g+2kP''_g+R+S$. As before, we obtain a basis $\{e_i\}$ of V, exhausting the vanishing sequences at P'_g and P''_g (both equal to $0|1|\ldots|2g-4|2g-3$). A direct computation gives: $\mathrm{ord}_{P'_g}(e_i \otimes e_j) \geqq 2g-2$ and $\mathrm{ord}_{P''_g}(e_i \otimes e_j) \geqq 2g-2$ happens exactly if $i=2k+1$, $j=21+1$, and $k+1=g-2$. We may assume again that $\iota(e_{2k+1})=e_{2(g-2-k)+1}$, for $k=0, \ldots, g-2$. So, in case (*b*) we obtain from (i) that

$$\bar{\rho}=\sum_{k=0}^{g-2} c_k e_{2k+1} \otimes e_{2(g-2-k)+1}=\sum_{k=0}^{g-2} c_k e_{2k+1} \otimes \iota e_{2k+1}.$$

Hence, again, $(1 \otimes \iota)(\bar{\rho})=\sum c_k e_{2k+1} \otimes e_{2k+1}$ implies, by (ii), that $\bar{\rho}=0$,

Q.E.D.

3. Application

Write, for $r \geqq -1$, $Q^r=V^r \setminus V^{r+2} \subset \mathrm{Nm}^{-1}(\omega_C)$; this is a locally closed subset of $\mathrm{Pic}^{2g-2}(\tilde{C})$, and is described as

(3.1) $$Q^r=\{L \in \mathrm{Pic}^{2g-2}(\tilde{C}) \mid \mathrm{Nm}(L) \simeq \omega_C,\ h^0(L)=r+1\}.$$

As remarked below Corollary (1.10), Theorem (1.11) implies that, for a general couple $(\tilde{C}, C)$ as in (1.1) and any $r \in \mathbb{Z}$, $r \geqq -1$, Q^r is either empty, or smooth, of dimension $g-1-\binom{r+1}{2}$ everywhere. (Of course, this is clear for the lowest values of r.)

(3.2) LEMMA. — *Let $(\tilde{C}, C)$ be as in (1.1). With the notations of (1.2) and (3.1), if $V^s \neq \emptyset$ then, for all $r \leqq s$, $Q^r \neq \emptyset$.*

Proof. — Use the following fact, *cf.* [9], p. 955 (it is stated there with an obvious misprint): if $\mathrm{Nm}(L) \simeq \omega_C$ and if P is not a base point of $|L|$, then $h^0 L(\iota P - P) = h^0 L - 1$; if P is a base point of $|L|$, then $h^0 L(\iota P - P) = h^0 L + 1$.

The assumption implies (in fact: is equivalent to) that there exists L with $\mathrm{Nm}(L) \simeq \omega_C$ and $h^0 L \geqq s+1$. We may drop the dimension one by one, getting points of Q^r, for all $r \leqq s$,

Q.E.D.

The Brill-Noether number for genus $\tilde{g}$, degree $\tilde{g}-1$ and dimension r equals $\rho = \tilde{g} - (r+1)^2$. This implies in particular that, if $g-1 \geqq (s+1)^2$, then $V^s \neq \emptyset$. Summarizing, we obtain:

(3.3) COROLLARY. — *Let $(\tilde{C}, C)$ be as in (1.1), with $C \in M_g$ general. If $g \geqq (r+1)^2 + 1$, then Q^r is smooth, of dimension $g-1-\binom{r+1}{2}$ everywhere.*

(3.4) *Remarks.* — (i) The fact that the inequality $g-1 \geqq (s+1)^2$ implies $V^s \neq \emptyset$ follows for instance from the fact that $W^s_{\tilde{g}-1}$ for a general curve of genus $\tilde{g}$ is homologous to a rational multiple of a self-intersection of the ample divisor $W^0_{\tilde{g}-1}$. By continuity, one obtains effective cycles inside $W^s_{\tilde{g}-1}(\tilde{C})$, which, by ampleness arguments, have to meet P^+ (resp. P^-). Alternatively, as the referee has pointed out, the results of W. Fulton and R. Lazarsfeld (On the connectedness of degeneracy loci and special divisors, *Acta Math.*, 146, 1981, p. 271-283) do much better: it follows immediately from their Theorem 1.1 and Lemma 2.2 [*cf.* also [1], Chapter VII, Propositions (3.1) and (2.2)] that $V^s \neq \emptyset$ if $g-1 \geqq (s+1)^2$, and that V^s is *connected* if $g-1 > (s+1)^2$.

(ii) As it has been said in the Introduction, it is plausible that a suitable analogue of the Existence Theorem of Brill-Noether theory would allow one to replace the inequality of the preceding corollary by the inequality $g \geqq \binom{r+1}{2} + 1$.

Finally, we turn to the loci $\mathrm{Sing}_k \Xi$. For any couple $(\tilde{C}, C)$ as in (1.1), we write [*cf.* (1.3)] Ξ for the theta divisor of the associated polarized Prym variety. Put, for $k \geqq 2$:

$$\mathrm{Sing}_k \Xi = \{ x \in \Xi \mid (\text{multiplicity of } \Xi \text{ at } x) = k \}.$$

Recall also that a suitable translation identifies the divisor Ξ of P with a divisor $\tilde{\Xi}$ of P^+ such that, scheme-theoretically: $2\tilde{\Xi} = \tilde{\Theta} \cap P^+$, where $\tilde{\Theta} = W^0_{2g-2}(\tilde{C})$. We may identify Ξ with $\tilde{\Xi}$.

(3.5) PROPOSITION. — *Let* $(\tilde{C}, C)$ *be as in* (1.1), *with* $C \in M_g$ *a general curve* [*e.g. as in Theorem* (1.11)]. *Then, for* $k \geqq 2$, *one has*

$$\mathrm{Sing}_k \tilde{\Xi} = Q^{2k-1} = \{ L \in \mathrm{Pic}^{2g-2}(\tilde{C}) \mid \mathrm{Nm}(L) = \omega_C,\ h^0 L = 2k \}.$$

Proof (cf. [7]). — At any point $x \in \tilde{\Xi}$, if the tangent cone of $\tilde{\Theta}$ at x does not contain the tangent space of P^+ at x, then it intersects this space along twice the tangent cone of $\tilde{\Xi}$ at x. Hence, in this case, the multiplicity of $\tilde{\Theta}$ at x is twice the multiplicity of $\tilde{\Xi}$ at x.

If $x \in \tilde{\Xi}$ corresponds to the bundle L, with $h^0 L = 2k$, the Riemann Singularity Theorem [6] implies that the tangent cone of $\tilde{\Theta}$ at x is given by the equation $\Delta = 0$, where $\Delta \in S^{2k} H^0 \omega_{\tilde{C}}$ is the determinant of the bilinear map (associated with the Petri map).

$$(3.6) \qquad H^0 L \times H^0 (\omega_{\tilde{C}} \otimes L^{\vee}) \to H^0 \omega_{\tilde{C}}.$$

Thus (*cf.* Section 1) the intersection of this cone with $T_{P^+}(x)$ is given by $\bar{\Delta} = 0$, with $\bar{\Delta} \in S^{2k} H^0 \omega_C(\varepsilon)$ the determinant of the composition of (3.6) with the projection map (1.6): $H^0 \omega_{\tilde{C}} \to H^0 \omega_C(\varepsilon)$. Or, what amounts to the same (*cf.* Section 1), it is the determinant of the skew-symmetric bilinear map described by (Ψ is the obvious map):

$$\begin{array}{ccc} H^0 L \times H^0 L & \longrightarrow & H^0 \omega_C(\varepsilon) \\ & {\scriptstyle \Psi} \searrow \quad \nearrow {\scriptstyle \beta} & \\ & \Lambda^2 H^0 L. & \end{array}$$

By assumption, β is injective. Since $H^0 L$ is even-dimensional, this implies that $\bar{\Delta} \neq 0$ (the determinant of Ψ is $\neq 0$, since there exist non-degenerate skew-symmetric bilinear forms on $H^0 L$). So, the hypothesis at the beginning of this proof is fulfilled, and the result follows,

Q.E.D.

Combining (3.3) and (3.5)—*cf.* (3.4), to put things into perspective—we obtain:

(3.7) COROLLARY. — *Let* $(\tilde{C}, C)$ *be as in* (1.1), *with* $C \in M_g$ *a general curve* [*e.g. as in* (1.11)]. *If* $g \geqq 4k^2 + 1$, *then the locus* $\mathrm{Sing}_k \Xi$ *is smooth, of dimension* $g - 1 - \binom{2k}{2}$ *everywhere.*

REFERENCES

[1] E. ARBARELLO, M. CORNALBA, P. A. GRIFFITHS and J. HARRIS, *Geometry of Algebraic Curves*, Vol. I, Springer-Verlag, 1985.

[2] P. DELIGNE and D. MUMFORD, *The Irreducibility of the Space of Curves of Given Genus*, (*Publ. Math. I.H.E.S.*, Vol. 36, 1969, pp. 75-110).

[3] D. EISENBUD and J. HARRIS, *A Simpler Proof of the Gieseker-Petri Theorem on Special Divisors* (*Inv. Math.*, Vol. 74, 1983, pp. 269-280).

[4] D. GIESEKER, *Stable Curves and Special Divisors: Petri's Conjecture* (*Inv. Math.*, Vol. 66, 1982, pp. 251-275).

[5] J. HARRIS, *Theta-characteristics on Algebraic Curves* (*Trans. A.M.S.*, Vol. 271, 1982, pp. 611-638).

[6] G. KEMPF, *On the Geometry of a Theorem of Riemann* (*Annals of Math.*, Vol. 98, 1973, pp. 178-185).

[7] D. MUMFORD, *Prym varieties I, in: Contributions to Analysis*, Academic Press, 1974, pp. 325-350.

[8] D. MUMFORD, *Theta Characteristics of an Algebraic Curve* (*Ann. scient. Éc. Norm. Sup.*, Vol. 4, 1971, pp. 181-192).

[9] A. N. TJURIN, *The Geometry of the Poincaré Theta Divisor of a Prym Variery* (*Math. U.S.S.R. Izv.*, Vol. 9, 1975, pp. 951-986).

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