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HITTING PROBABILITIES OF KILLED BROWNIAN MOTION; A STUDY ON GEOMETRIC REGULARITY

BY CHRISTER BORELL

1. Introduction

Consider a Brownian motion X in n-space with first hitting times $\tau_A = \tau_A(X) = \inf\{t>0;\ X(t)\in A\}$ and let $\mathscr{U}(\mathbb{R}^n)$ denote the class of all non-empty, open, and convex subsets of $\mathbb{R}^n$. Then, if $x_0,\ x_1 \in \mathbb{R}^n$ and $A_0,\ A_1,\ B_0,\ B_1 \in \mathscr{U}(\mathbb{R}^n)$:

$$(1.1)\quad \mathbb{P}_{x_\lambda}(\tau_{B^c_\lambda} \geqq \tau_{A_\lambda} < +\infty) \geqq \mathbb{P}_{x_0}(\tau_{B^c_0} \geqq \tau_{A_0} < +\infty) \wedge \mathbb{P}_{x_1}(\tau_{B^c_1} \geqq \tau_{A_1} < +\infty), \qquad 0<\lambda<1,$$

where $\xi_\lambda = (1-\lambda)\xi_0 + \lambda\xi_1$, $\xi = x$, A, B, and $B^c = \mathbb{R}^n \setminus B$, respectively (Borell [4]).

In this paper, the basic diffusion process is a Brownian motion Y in $\mathbb{R}^n \cup \{\varphi\}$, which starts in $\mathbb{R}^n$ and behaves as an ordinary Brownian motion up till a certain random point of time when it jumps to φ and remains there. More explicitly, conditioned on X, the event $Y(t)\in\mathbb{R}^n$, has the probability $\exp\left(-\int_0^t V(X(s))\,ds\right)$, where $V:\mathbb{R}^n \to [0, +\infty]$ is such that $V^{-1/2}_{|\operatorname{dom} V}$ is concave and $\operatorname{dom} V = \{V < +\infty\} \in \mathscr{U}(\mathbb{R}^n)$. Under these assumptions (1.1) still holds with $\tau_\cdot = \tau_\cdot(Y)$ (Theorem 3.1). In fact, the same result remains true if $\mathbb{R}^n$ is replaced by an arbitrary Banach space.

About half the paper deals with various interpretations of Theorem 3.1. Thus, we discuss convexity properties of:

(i) V-harmonic measures (Section 6, Example 7.1);

(ii) V-Newtonian potentials (Theorem 7.1);

(iii) V-equilibrium measures (Example 7.2) and

(iv) logarithmic and Newtonian capacities (examples 7.3-7.5).

In the above list, perhaps the single most interesting point is the following: let $B\in\mathscr{U}(\mathbb{R}^n)$, $n\geqq 2$, be bounded and suppose $V: B\to[0, +\infty[$ is $-1/2$-concave, that is, $V^{-1/2}: B \to]0, +\infty]$ is concave. Moreover, let g denote the Green function of

$-1/2\,\Delta+V$ in B with the Dirichlet boundary condition zero. Then g is quasi-concave if $n=2$, and if $n\geqq 3$ the function $g^{-1/(n-2)}$ is convex. Theorem 7.1 expresses these facts as a Brunn-Minkowski inequality of appropriate potentials of g. For comparison, we here only mention that the 3-dimensional potential $g\mu$, μ being the uniform distribution of a line segment, turns out to have convex equipotential surfaces. The same thing is known to be true for a point mass if $V=0$ (Gabriel [15], [16]). Needless to say, the beautiful works of Gabriel have played a decisive role for this and some other closely related papers of the author ([4], [5]).

Finally, in this section, let us make some remarks on the potential V above.

Again, consider a Y-process in $\mathbb{R}^n$ now with a convex potential V. Moreover, suppose $\operatorname{dom} V\in\mathscr{U}(\mathbb{R}^n)$ is bounded. Then, by Brascamp and Lieb ([9], [10]), the transition densities p_t, $t>0$, of Y are log-concave for each fixed $t>0$. From this we expect nice geometrical properties of the corresponding Green function:

$$g=\int_0^\infty p_t\,dt.$$

In fact, our fruitless attempts to understand this puzzling problem have finally led us to $-1/2$-concave potentials. The reader should note that a $-1/2$-concave function is convex. (The log-concavity of the p_t for convex V turns out to be an algebraic consequence of (1.1) but that is another uniformity!) Below we will also see that $-1/2$-concave potentials enter quite naturally in the hyperbolic potential theory of plane convex domains (Example 3.1).

2. Definitions

Throughout, E denotes a separable Banach space and $\mathscr{C}_E([0,+\infty[)$ is the standard Fréchet space of all continuous maps of $[0,+\infty[$ into E. A centered Gaussian random vector X in $C_E([0,+\infty[)$ is called a Brownian motion in E or an E-valued Brownian motion if X possesses stochastically independent increments and if, for every $t>0$, the law of $X_t=[X(\,.\,)](t)$ equals the law of $t^{1/2}X_1$ (*see* Gross [18] (potential theory) and Chow [12] (noise theory)).

Example 2.1. – Suppose S is a compact metric space and let $G=(G(s), s\in S)$ be a real-valued, centered Gaussian stochastic process with continuous paths. Then there exists an unique real-valued centered Gaussian process X with time set $S\times[0,+\infty[$ and covariance $[\mathbb{E}(G(s)G(s'))](t\wedge t')$. Moreover, a version of $X=(X(s,t),$ $(s,t)\in S\times[0,+\infty[)$ has continuous paths with probability one and, accordingly, induces a Brownian motion in $\mathscr{C}(S)$ (for details, *see* Carmona [11]). □

The above example brings out the most general form of a Banach space-valued Brownian motion.

An E-valued Brownian motion is said to be non-degenerated if $\operatorname{supp}\mathscr{L}(X_1)=E$. If F is a separable Banach space and $A: E\to F$ is a bounded linear map, then each E-valued Brownian motion X defines an F-valued Brownian motion by the rule $[AX]_t=AX_t$.

In what follows, X is supposed to be a fixed *non-degenerated* Brownian motion in E and, as usual, we let $\mathbb{P}_x = \mathscr{L}(x+X)$ and $\mathbb{E}_x = \int (\ \ ) \, d\mathbb{P}_x$.

Below $\mathscr{U}(E)$ denotes the class of all non-empty, open, and convex subsets of E. Moreover, $\bar{\mathscr{U}}(E) = \{\bar{A};\ A \in \mathscr{U}(E)\}$, $\mathscr{U}_\infty(E) = \{A \in \mathscr{U}(E);\ A \text{ bounded}\}$, and $\bar{\mathscr{U}}_\infty(E)\} = \{\bar{A};\ A \in \mathscr{U}_\infty(E)\}$, respectively. If A_0, $A_1 \subseteq E$, and $0 < \lambda < 1$, we write $A_\lambda = (1-\lambda) A_0 + \lambda A_1$. The same convention will be used for vectors in E. Given $A_i \in \mathscr{U}(E)$, concave functions $f_i : A_i \to [0, +\infty]$, $i = 0, 1$, and $\lambda \in [0, 1]$, the so-called λ-supremum convolution :

$$f_0 \,|\underline{\lambda}|\, f_1 : \quad A_\lambda \to [0, +\infty],$$

of f_0 and f_1 is defined by:

$$(f_0 \,|\underline{\lambda}|\, f_1)(x_\lambda) = \sup\{(1-\lambda)\, f_0(x_0) + \lambda f_1(x_1);\ x_0 \in A_0,\ x_1 \in A_1\}.$$

Here $0.(+\infty) = 0$. Of course, $f_0 \,|\underline{\lambda}|\, f_1$ is concave and by simple means one verifies:

$$f_0 \,|\underline{\theta_\lambda}|\, f_1 = (f_0 \,|\underline{\theta_0}|\, f_1) \,|\underline{\lambda}|\, (f_0 \,|\underline{\theta_1}|\, f_1), \qquad \theta_0, \theta_1 \in [0, 1]. \tag{2.1}$$

Next suppose $\alpha \in \mathbb{R} \setminus \{0\}$. Using the conventions $0^\alpha = +\infty$ and $(+\infty)^\alpha = 0$, if $\alpha < 0$, a function $f : A \to [0, +\infty]$ $(A \subseteq E)$ is said to be α-convex (α-concave) if f^α is convex (concave). For this reason, a quasi-concave (log-concave) function is sometimes called $-\infty$-convex (0-convex or 0-concave). The same terminology is used for set functions on vector spaces. For future reference, recall that a Gaussian Radon measure on a locally convex Hausdorff vector space is log-concave (Borell [6]).

3. The main result

Consider the Feynman-Kac semi-group:

$$S_t f = \mathbb{E}_\cdot \left(f(X(t)) \exp\left(-\int_0^t V(X(s))\, ds \right) \right), \qquad t > 0,$$

where the potential $V : E \to [0, +\infty]$ is Borel measurable. If, in addition, V is convex, the log-concavity of Gaussian measures may be used to show that each S_t preserves log-concavity. Indeed, this property has many nice consequences (Brascamp, Lieb [9], [10], Lions [20]). The reader should note that if $B = \operatorname{dom} V \in \mathscr{U}(E)$, then:

$$S_t f = \mathbb{E}_\cdot \left(f(X(t)) \exp\left(-\int_0^t V(X(s))\, ds \right);\ \tau_{B^c} \geqq t \right), \qquad t > 0.$$

THEOREM 3.1. — *For* $i=0, 1$, *suppose* A_i, $B_i \in \mathcal{U}(E)$, $x_i \in B_i$, *and let* $V_i : B_i \to [0, +\infty[$ *be* $-1/2$-*concave. Set* $V_\lambda = (V_0^{-1/2} \underline{|\lambda|} V_1^{-1/2})^{-2}$ *and:*

$$M(\lambda) = \mathbb{E}_{x_\lambda}\left(\exp\left(- \int_0^{\tau_{A_\lambda}} V_\lambda(X(s))\, ds \right); \tau_{B_\lambda^c} \geqq \tau_{A_\lambda} < +\infty \right), \quad 0<\lambda<1,$$

respectively. Then M *is quasi-concave.*

Interestingly enough, there are several relations between Theorem 3.1 and the Brunn-Minkowski theory of convex bodies but the interplay is not yet fully understood. In particular, one may ask if the log-concavity of Gaussian measures (on all measurable sets!) and Theorem 3.1 have a common source.

For some other geometrical estimates on Feynman-Kac semi-groups, *see* Borell [7] and Ehrhard [14].

Before giving the proof of Theorem 3.1, which is rather lengthy, we should like to discuss an example where $-1/2$-concave potentials arise in a natural way.

First, however, recall that if X is the usual Brownian motion in $\mathbb{R}^n$, then the expectation:

$$u(x) = \mathbb{E}_x\left(\exp\left(- \int_0^{\tau_A} V(X(s))\, ds; \tau_{B^c} \geqq \tau_A < +\infty \right) \right), \qquad x \in \bar{B},$$

solves the V-equilibrium potential equation:

$$\begin{cases} \frac{1}{2}\Delta u - V u = 0 & \text{in } B, \\ u = 1 & \text{on } \bar{A}, \\ u = 0 & \text{on } \partial B, \end{cases}$$

where, for example, A, $B \in \mathcal{U}(\mathbb{R}^n)$, $\bar{A} \subseteq B$, and $V : B \to [0, +\infty[$ is continuous (*see* e. g. Dynkin [13], Chap. 13). (Here and elsewhere $\Delta = \partial^2/\partial x_1^2 + \ldots + \partial^2/\partial x_n^2$.)

Example 3.1. — Consider a $B \in \mathcal{U}(\mathbb{C})$, $B \neq \mathbb{C}$, equiped with the hyperbolic metric:

$$ds = \left| \frac{f'(z)}{\operatorname{Im} f(z)} \right| |dz|,$$

f being an arbitrary one-to-one conformal map onto the upper half plane in $\mathbb{C}$. Note that:

$$\frac{1}{2}\left| \frac{f'(z)}{\operatorname{Im} f(z)} \right| = \lim_{\zeta \to z} d(z, \zeta)/|z-\zeta|,$$

where $d(z, \zeta) = |(f(z)-f(\zeta))/(f(z)-\overline{f(\zeta)})|$, z, $\zeta \in B$, is a strictly increasing function of the hyperbolic distance in B (*see* e. g. Ahlfors [1], [2]).

The following discussion is based on the fact that the Green function $g(z, \zeta)$ of $-1/2\Delta$ in B with the Dirichlet boundary condition zero is quasi-concave in (z, ζ) (this may be known; for safety's sake the result is proved in Theorem 7.1). Equivalently, if $B(z; r)$ denotes the open d-ball with center $z \in B$ and radius $r > 0$, then:

$$(1-\lambda) B(z_0; r) + \lambda B(z_1; r) \subseteq B(z_\lambda; r), \qquad 0 < \lambda < 1.$$

Accordingly, for reals $t \neq 0$ close to zero:

$$\frac{1}{|t|}\left|\frac{f(z_\lambda + th_\lambda) - f(z_\lambda)}{f(z_\lambda + th_\lambda) - \overline{f(z_\lambda)}}\right| \leqq \frac{1}{|t|}\left\{\left|\frac{f(z_0 + th_0) - f(z_0)}{f(z_0 + th_0) - \overline{f(z_0)}}\right| \vee \left|\frac{f(z_1 + th_1) - f(z_1)}{f(z_1 + th_1) - \overline{f(z_1)}}\right|\right\},$$

and in the limit as $t \to 0$:

$$\left|\frac{f'(z_\lambda) h_\lambda}{\operatorname{Im} f(z_\lambda)}\right| \leqq \left|\frac{f'(z_0) h_0}{\operatorname{Im} f(z_0)}\right| \vee \left|\frac{f'(z_1) h_1}{\operatorname{Im} f(z_1)}\right|.$$

By choosing:

$$h_\nu = \left|\frac{\operatorname{Im} f(z_\nu)}{f'(z_\nu)}\right|, \quad \nu = 0, 1,$$

the resulting inequality states that the function $|(\operatorname{Im} f(z))/f'(z)|$ is concave.

Now recall that the Laplace-Beltrami operator Δ_B in the hyperbolic B equals:

$$\Delta_B = \left|\frac{\operatorname{Im} f(z)}{f'(z)}\right|^2 \Delta.$$

Consequently, if $A \in \mathcal{U}_\infty(\mathbb{C})$ and $\bar{A} \subseteq B$, Theorem 3.1 applies to the 1-equilibrium potential equation:

$$\begin{cases} \Delta_B u - u = 0 & \text{in } B \setminus \bar{A}, \\ u_{|\bar{A}} = 1, \end{cases}$$

and we conclude that u is quasi-concave. Moreover, if u_z denotes the 1-equilibrium potential of $B(z; r)$, then the map $(z, \zeta) \rightsquigarrow u_z(\zeta)$ is quasi-concave too. □

4. Reduction of Theorem 3.1 to finite dimension

To begin with, we list a series of Lemmas, which are all well-known and easy to prove.

LEMMA 4.1. — *Suppose* F_n, $n \in \mathbb{N}$, *are closed and* $F_n \downarrow F$. *Then* $\mathbb{P}.(\tau_{F_n} \downarrow \tau_F) = 1$ *on* $F^r \cup F^c$.

Here $F^r = \{\mathbb{P}.(\tau_F = 0) = 1\}$ is the set of all regular points for F. Recall that $\mathbb{P}.(\tau_F = 0)$ vanishes on $(F^r)^c$ by Blumenthal's zero-one law (*see* e. g. Port and Stone [22]).

LEMMA 4.2. — If $A \in \mathcal{U}(E)$, then $A^r = \bar{A}^r = \bar{A}$. *In addition*, $\tau_A = \tau_{\bar{A}}$ a. s. $\mathbb{P}.$.

The reader should note that the last part of Lemma 4.2 depends on the strong Markov property of X. The next Lemma is a consequence of continuity of paths only.

LEMMA 4.3. — *Let* F_n, $n \in \mathbb{N}$, *be closed and* $F_n \downarrow F$. *If* $B_n \in \mathscr{B}(E)$, $n \in \mathbb{N}$, *and* $B_n \downarrow B$, *then:*

$$\{\tau_{B_n^c} \geqq \tau_{F_n} < +\infty\} \downarrow \{\tau_{B^c} \geqq \tau_F < +\infty\}, \qquad \text{a. s. } \mathbb{P}.((\quad) \cap \{\tau_{B^c} < +\infty\}),$$

on $F^r \cup F^c$.

Here $\mathscr{B}(E)$ denotes the Borel field in E.

LEMMA 4.4. — *Suppose* $0 < \lambda < 1$:

(*a*) *If* A_0, $A_1 \in \mathscr{U}_\infty(E)$ *and* A_λ *is contained in an open affine half-space* H, *then there exist open affine half-spaces* H_0, H_1, *satisfying* $H \supseteqq H_\lambda$, $H_0 \supseteqq A_0$, *and* $H_1 \supseteqq A_1$.

(*b*) *Let* $B_i \in \mathscr{U}_\infty(E)$ *and suppose* $f_i : B_i \to [0, +\infty[$, $i=0, 1$, *are concave. If* ζ *is a continuous affine function on* E *and* $\zeta_{|B_\lambda} \geqq f_0 \underline{|\lambda|} f_1$, *then there exist continuous affine functions* ζ_0, ζ_1 *on* E *satisfying* $\zeta \geqq \zeta_0 \underline{|\lambda|} \zeta_1$, $\zeta_{0\,|B_0} \geqq f_0$, *and* $\zeta_{1\,|B_1} \geqq f_1$.

LEMMA 4.5:

(*a*) $\bar{A}_0 + \bar{A}_1 \subseteqq \overline{A_0 + A_1}$, A_0, $A_1 \subseteqq E$;

(*b*) *If* A_n, $A \in \mathscr{U}(E)$, $n \in \mathbb{N}$, *and* $A_n \downarrow A$, *then* $\bar{A}_n \downarrow \bar{A}$.

Proof of Theorem 3.1, $\dim E < +\infty \Rightarrow$ *Theorem* 3.1. In view of (2.1) it is enough to establish the following inequality:

$$M(\lambda) \geqq M(0) \wedge M(1),$$

where $0 < \lambda < 1$ is fixed. Furthermore, we may assume B_0, $B_1 \in \mathscr{U}_\infty(E)$.

Let $j = 0, 1$, or λ and set $f_j = V_j^{-1/2}$. By monotone convergence, there is no loss of generality if we only treat the case when the f_j are finite-valued. Suppose:

$$f_j = \inf_{n \in \mathbb{N}} \zeta_{jn\,|B_j}$$

and $\zeta_{\lambda\,n} \geqq \zeta_{0\,n} \underline{|\lambda|} \zeta_{1\,n}$, where the ζ_{jn} are finite infimums of continuous affine functions on E. This construction is possible due to Lemma 4.4. By the same Lemma here exist open polyedrons C_{jn}, $n \in \mathbb{N}$, C = A, B, satisfying:

$$C_{jn} \downarrow C_j \quad \text{as } n \to +\infty$$

and $C_{\lambda\,n} \supseteqq (1-\lambda) C_{0\,n} + \lambda C_{1\,n}$.

We now introduce:

$$f_{jn} = \inf_{0 \leqq k \leqq n} \zeta_{jk\,|B_{jn} \cap \{\zeta_{j0} > 0, \ldots, \zeta_{jn} > 0\}}$$

and:

$$M_n(j) = \mathbb{E}_{x_j}\left(\exp\left(-\int_0^{\tau_{A_{jn}}} f_{jn}^{-2}(X(s))\,ds\right); \tau_{B_{jn}^c} \geqq \tau_{A_{jn}} < +\infty\right).$$

Granted the validity of Theorem 3.1 in the finite-dimensional case, we have:

$$M_n(\lambda) \geqq M_n(0) \wedge M_n(1)$$

and (4.1) follows from Lemmas 4.1-4.4 and monotone convergence. □

5. Proof of Theorem 3.1, *dim* $E < +\infty$

In the following lemma, the V_j, $j=0, 1$, or λ, are as in Theorem 3.1.

LEMMA 5.1. — *If* $J(r)=r$, $r>0$, *then:*

$$J^3 \otimes V_\lambda \leqq (J^3 \otimes V_0) \underline{|\lambda|} (J^3 \otimes V_1), \; 0<\lambda<1.$$

Proof. — By the Hölder inequality the function $(J \otimes 1)^3/(1 \otimes J)^2$ is convex and the result follows at once. □

LEMMA 5.2. — *Suppose* A, $B \in \mathscr{U}_\infty(\mathbb{R}^n)$ *and* $0 \in \bar{A} \subsetneqq B$. *Let* $f: B \to]0, \infty[$ *be* $\mathscr{C}^\infty$ *and concave and set* $V=f^{-2}$. *Then the solution of the Dirichlet problem:*

$$\begin{cases} \Delta u - V u = 0 & in\ B \setminus \bar{A}, \\ u=1 & on\ \partial A \\ u=0 \quad on\ \partial B, & u \in \mathscr{C}(\bar{B}), \end{cases}$$

has a non-vanishing gradient in $B \setminus \bar{A}$.

Proof. — The solution u is $\mathscr{C}^\infty$ (*see* e. g. Gilbarg and Trudinger [17], Theorem 6.17).

We first prove that the function $v(x)=x; \nabla u(x)$, $x \in B \setminus \bar{A}$, is non-positive.

To see this, let $\alpha>1$ satisfy $\alpha \bar{A} \subsetneqq B$ and note that:

$$\Delta [u(x/\alpha)] - \alpha^{-2} V(x/\alpha) u(x/\alpha) = 0 \quad \text{in } B \setminus \alpha \bar{A}.$$

Moreover, as:

$$f(x/\alpha) \geqq \alpha^{-1} f(x) + (1-\alpha^{-1}) f(0) \quad \text{in } B,$$

we have $\alpha f(x/\alpha) \geqq f(x)$, $x \in B$, and hence:

$$\Delta [u(x/\alpha)] - V(x) u(x/\alpha) \leqq 0 \quad \text{in } B \setminus \alpha \bar{A}.$$

Thus:

$$\Delta [u(x) - u(x/\alpha)] - V(x) [u(x) - u(x/\alpha)] \geqq 0 \quad \text{in } B \setminus \bar{A}$$

and as $(u-u(\,.\,/\alpha))_{|\partial(B \setminus \alpha \bar{A})} \leqq 0$, the maximum principle ([17], cor. 3.2) gives

$$(u-u(\,.\,/\alpha))_{|B \setminus \alpha \bar{A}} \leqq 0.$$

But then $v \leqq 0$.

In the next step we show that v is strictly negative.

A computation yields:

$$\Delta v = x; \nabla(\Delta u) + 2\Delta u = x; \nabla(\mathrm{V}u) + 2\mathrm{V}u = (x; \nabla\mathrm{V})u + \mathrm{V}(x; \nabla u) + 2\mathrm{V}u,$$

that is:

$$\Delta v - \mathrm{V}v = (2\mathrm{V} + x; \nabla\mathrm{V})u.$$

But:

$$2\mathrm{V} + x.\nabla\mathrm{V} = \frac{2}{f^3}(f - x; \nabla f) \geqq \frac{2}{f^3} f(0) > 0$$

and so $\Delta v - \mathrm{V}v > 0$. Since $v \leqq 0$, the strong maximum principle ([17], Th. 35) gives $v < 0$ and accordingly $v \neq 0$ in $\mathrm{B} \setminus \bar{\mathrm{A}}$. □

The main points in the proof which follows are due to Gabriel ([15], [16]). The Brunn-Minkowski aspect was added for the first time in [4]. The Gabriel differential method also applies to certain time-dependent [5] and non-linear (Lewis [19]) problems.

Proof of Theorem 3.1, $\dim \mathrm{E} < +\infty$. — There is no loss of generality in assuming:

(i) X *is the usual Brownian motion in* $\mathbb{R}^n$, $n \geqq 1$;

(ii) $0 \in \bar{\mathrm{A}}_0 \cap \bar{\mathrm{A}}_1$, $\mathrm{B}_0, \mathrm{B}_1 \in \mathscr{U}_\infty(\mathbb{R}^n)$;

(iii) *the functions* $f_i = \mathrm{V}_i^{-1/2}$ *have concave* $\mathscr{C}^\infty$ *extensions* $\tilde{f}_i : \mathrm{B}_i + \mathrm{B}(0; \delta) \to]0, +\infty[$, $i = 0, 1$ ($\delta > 0$ fixed) and from (iii) and Lemma 4.3;

(iv) $$\bar{\mathrm{A}}_i \subseteq \mathrm{B}_i, \; i = 0, 1.$$

Next let $0 < \lambda < 1$ be fixed. Moreover, suppose:

$\tilde{\mathrm{V}}_\lambda : \mathrm{B}_\lambda \to]0, +\infty[$ is $-1/2$-concave and $\mathscr{C}^\infty$ and $\tilde{\mathrm{V}}_\lambda \leqq \mathrm{V}_\lambda$.

Set:

$$u_i(x) = \mathbb{E}_x\left(\exp\left(-\int_0^{\tau_{\mathrm{A}_i}} \tilde{\mathrm{V}}_i(\mathrm{X}(s))\,ds\right); \tau_{\mathrm{B}_i^c} \geqq \tau_{\mathrm{A}_i}\right), \qquad x \in \bar{\mathrm{B}}_i, \qquad i = 0, 1,$$

and:

$$u_\lambda(x) = \mathbb{E}_x\left(\exp\left(-\int_0^{\tau_{\mathrm{A}_\lambda}} \tilde{\mathrm{V}}_\lambda(\mathrm{X}(s))\,ds\right); \tau_{\mathrm{B}_\lambda^c} \geqq \tau_{\mathrm{A}_\lambda}\right), \qquad x \in \bar{\mathrm{B}}_\lambda.$$

It now only remains to prove that:

$$u_\lambda(x_\lambda) \geqq u_0(x_0) \wedge u_1(x_1), \qquad x_0 \in \bar{\mathrm{B}}_0, \qquad x_1 \in \bar{\mathrm{B}}_1.$$

Let $u_\lambda^*(x_\lambda) = \sup\{u_0(x_0) \wedge u_1(x_1); x_0 \in \bar{\mathrm{B}}_0, x_1 \in \bar{\mathrm{B}}_1\}$. If $\neg(u_\lambda^* \leqq u_\lambda)$, then:

$$\sup(u_\lambda^* - u_\lambda) = u_\lambda^*(\hat{x}_\lambda) - u_\lambda(\hat{x}_\lambda) > 0,$$

for a suitable $\hat{x}_\lambda \in \bar{\mathrm{B}}_\lambda$. Suppose $u_\lambda^*(\hat{x}_\lambda) = u_0(\hat{x}_0) \wedge u_1(\hat{x}_1)$, where $\hat{x}_\lambda = (1-\lambda)\hat{x}_0 + \lambda\hat{x}_1$. Certainly, $(\hat{x}_0, \hat{x}_1) \in (\mathrm{B}_0 \times \mathrm{B}_1) \setminus (\bar{\mathrm{A}}_0 \times \bar{\mathrm{A}}_1)$. Also it is easy to see that the relation $\hat{x}_0 \notin \mathrm{A}_0$,

$\hat{x}_1 \in A_1$ is contradictory. Indeed, arbitrarily close to $\hat{x}_0$ there are points where u_0 exceeds $u_0(\hat{x}_0)$, by the maximum principle. Thus, by symmetry, $(\hat{x}_0, \hat{x}_1) \in (B_0 \setminus \bar{A}_0) \times (B_1 \setminus \bar{A}_1)$.

In the following, let $i=0$ or 1 and $j=0,1$, or λ.

Suppose $h \in \mathbb{R}^n$ and $h; \nabla u_i(\hat{x}_i) > 0$ (i fixed). Then, if $s>0$ is small, $u_i(\hat{x}_i + sh) > u_i(\hat{x}_i)$ and, hence, $u_\lambda^*(\hat{x}_\lambda + s\lambda_i h) > u_\lambda^*(\hat{x}_\lambda)$, where $\lambda_i = (2i-1)\lambda + 1 - i$, so that $u_\lambda(\hat{x}_\lambda + s\lambda_i h) \geqq u_\lambda(\hat{x}_\lambda)$. Accordingly, $h; \nabla u_\lambda(\hat{x}_\lambda) \geqq 0$ and it follows that the non-zero vectors $\nabla u_i(\hat{x}_i)$ and $\nabla u_\lambda(\hat{x}_\lambda)$ are parallel. Let $a_j = |\nabla u_j(\hat{x}_j)|$ and $\nu = \nabla u_j(\hat{x}_j)/a_j$.

From now on we assume that $u_\lambda^*(\hat{x}_\lambda) = u_0(\hat{x}_0)$. The case $u_\lambda^*(\hat{x}_\lambda) = u_1(\hat{x}_1)$ may be treated in a similar way.

Let $h \in \mathbb{R}^n$ be such that $\kappa = h; \nu \neq 0$. For each s close to 0 there exists a unique $r = r(s)$, with $|r|$ minimal, satisfying the equation :

$$u_0(\hat{x}_0 + sh/a_0) - u_0(\hat{x}_0) = u_1(\hat{x}_1 + rh/a_1) - u_1(\hat{x}_1).$$

Writing:

$$\hat{x}_\lambda(s) = (1-\lambda)(\hat{x}_0 + sh/a_0) + \lambda(\hat{x}_1 + r(s)h/a_1) = \hat{x}_\lambda + [(1-\lambda)s/a_0 + \lambda r(s)/a_1]h,$$

we have:

$$u_0(\hat{x}_0 + sh/a_0) - u_\lambda(\hat{x}_\lambda(s)) \leqq u_\lambda^*(\hat{x}_\lambda(s)) - u_\lambda(\hat{x}_\lambda(s)) \leqq u_0(\hat{x}_0) - u_\lambda(\hat{x}_\lambda)$$

and, in particular:

$$D_s^k(u_0(\hat{x}_0 + \mathrm{sh}/a_0) - u_\lambda(\hat{x}_\lambda(s)))|_{s=0} = \begin{cases} 0, & k=1 \\ \leqq 0, & k=2. \end{cases}$$

Next suppose:

$$u_j(\hat{x}_j + sh/a_j) = u_j(\hat{x}_j) + \kappa s + b_j s^2 + o(s^2) \quad \text{as } s \to 0.$$

Then:

$$r(s) = s + \kappa^{-1}(b_0 - b_1)s^2 + o(s^2) \quad \text{as } s \to 0$$

and introducing $p = (1-\lambda)/a_0 + \lambda/a_1$ we have:

$$\begin{cases} a_\lambda p = 1, \\ \left(1 - \lambda \dfrac{a_\lambda}{a_1}\right) b_0 + \lambda \dfrac{a_\lambda}{a_1} b_1 - b_\lambda \leqq 0. \end{cases}$$

Thus:

$$\sum_{1 \leqq \alpha,\, \beta \leqq n} \left[\frac{1-\lambda}{a_0^3} D_{\alpha\beta} u_0(\hat{x}_0) + \frac{\lambda}{a_1^3} D_{\alpha\beta} u_1(\hat{x}_1) - \frac{1}{a_\lambda^3} D_{\alpha\beta} u_\lambda(\hat{x}_\lambda) \right] h_\alpha h_\beta \leqq 0$$

and, accordingly:

$$\frac{1-\lambda}{a_0^3} V_0(\hat{x}_0) u_0(\hat{x}_0) + \frac{\lambda}{a_1^3} V_1(\hat{x}_1) u_1(\hat{x}_1) \leqq p^3 \tilde{V}_\lambda(\hat{x}_\lambda) u_\lambda(\hat{x}_\lambda).$$

Finally, noting that $u_\lambda(\hat{x}_\lambda) < u_0(\hat{x}_0) \wedge u_1(\hat{x}_1)$ we get:

$$\frac{1-\lambda}{a_0^3} \mathrm{V}_0(\hat{x}_0) + \frac{\lambda}{a_1^3} \mathrm{V}_1(\hat{x}_1) < p^3 \mathrm{V}_\lambda(\hat{x}_\lambda),$$

which contradicts Lemma 5.1. Hence $u_\lambda^* \leqq u_\lambda$. □

6. Quasi-concavity of V-harmonic measures restricted to supporting hyperplanes

We first recall some known properties of quasi-concave measures on Banach spaces. All the results may be found in the author's papers [6] and [8].

A non-negative finite Borel measure μ on E is quasi-concave if:

$$\mu(\mathrm{A}_\lambda) \geqq \mu(\mathrm{A}_0) \wedge \mu(\mathrm{A}_1), \tag{6.1}$$

for all $0<\lambda<1$ and all $\mathrm{A}_0, \mathrm{A}_1 \in \mathscr{B}(\mathrm{E})$ = the Borel field in E. It turns out that a non-negative finite Borel measure μ on E is quasi-concave if (6.1) holds for all $0<\lambda<1$ and all $\mathrm{A}_0, \mathrm{A}_1 \in \mathscr{U}(\mathrm{E})$.

Next suppose $0<\lambda<1$ is fixed and suppose $\mu_0, \mu_1, \mu_\lambda$ are quasi-concave measures on E. If:

$$\mu_\lambda(\mathrm{A}_\lambda) \geqq \mu_0(\mathrm{A}_0) \wedge \mu_1(\mathrm{A}_1), \tag{6.2}$$

for all $\mathrm{A}_0, \mathrm{A}_1 \in \mathscr{U}(\mathrm{E})$, then (6.2) is true for all $\mathrm{A}_0, \mathrm{A}_1 \in \mathscr{B}(\mathrm{E})$. Moreover, if $\mathrm{E} = \mathbb{R}^n$ and $d\mu_j = f_j\,dx$, $j = 0, 1, \lambda$, where the $f_j : \mathrm{E} \to [0, +\infty]$ are semi-continuous from below, then (6.2) holds for all Borel sets $\mathrm{A}_0, \mathrm{A}_1$ in $\mathbb{R}^n$ if and only if:

$$f_\lambda^{-1/n}(x_\lambda) \leqq (1-\lambda) f_0^{-1/n}(x_0) + \lambda f_1^{-1/n}(x_1), \qquad x_0, x_1 \in \mathbb{R}^n.$$

The above makes it possible to pass from convex bodies to Borel sets in a very special but still interesting case of Theorem 3.1.

THEOREM 6.1. — *Let* $\mathrm{B} \in \mathscr{U}(\mathrm{E})$ *and suppose* F *is a supporting hyperplane* $(0 \in \mathrm{F})$ *of* $\bar{\mathrm{B}}$. *If* $\mathrm{V} : \mathrm{B} \to [0, +\infty[$ *is* $-1/2$*-concave, then the* V*-harmonic measure:*

$$\kappa_x(\mathrm{A}) = \mathbb{E}_x\left(\exp\left(- \int_0^{\tau_{\mathrm{B}^c}} \mathrm{V}(\mathrm{X}(s))\,ds \right); \mathrm{X}(\tau_{\mathrm{B}^c}) \in \mathrm{A} \right), \ \mathrm{A} \in \mathscr{B}(\mathrm{B}^c),$$

at $x \in \mathrm{B}$ *satisfies:*

$$\kappa_{x_\lambda}(\mathrm{A}_\lambda) \geqq \kappa_{x_0}(\mathrm{A}_0) \wedge \kappa_{x_1}(\mathrm{A}_1), \qquad 0<\lambda<1, \qquad \mathrm{A}_0, \mathrm{A}_1 \in \mathscr{B}(\mathrm{F}).$$

In particular, $\kappa_{x \mid \mathscr{B}(\mathrm{F})}$ *is quasi-concave.*

Proof. — First note that for any closed $\mathrm{A} \subseteq \mathrm{B}^c$:

$$\kappa_x(\mathrm{A}) = \mathbb{E}_x\left(\exp\left(- \int_0^{\tau_{\mathrm{A}}} \mathrm{V}(\mathrm{X}(s))\,ds \right); \tau_{\mathrm{B}^c} \geqq \tau_{\mathrm{A}} < +\infty \right),$$

because $x \in B$ is non-regular for B^c. Hence the inequality we shall prove is true for all A_0, $A_1 \in \overline{\mathcal{U}}(F)$ and Theorem 6.1 follows from what we said above. □

Example 6.1. — Let G be a Borel measurable additive subgroup of F, where we abide by the various assumptions in Theorem 6.1. Then $\kappa_x(G)$ or $\kappa_x(F \setminus G) = 0$ from the zero-one law of quasi-concave measures [6]. A direct proof of this fact is rather simple but we do not know any proof independent of the zero-one law of quasi-concave measures. □

Example 6.2. — Let $E = \mathbb{R}^n$ but otherwise assume the same conditions as in Theorem 6.1.

If $\bar{B} \cap F = C$ is $(n-1)$-dimensional, then an appropriate version of the restricted Poisson kernel $(d\kappa_x/d\sigma_{\partial B})(y)$, $(x, y) \in B \times C$, is $-1/(n-1)$-convex. □

Example 6.3. — If C_0, $C_1 \in \overline{\mathcal{U}}_\infty(\mathbb{R}^n)$, then the original Brunn-Minkowski inequality states that:

$$|C_0 + C_1|^{1/n} \geqq |C_0|^{1/n} + |C_1|^{1/n}. \tag{6.3}$$

To deduce this estimate from (1.1) we let $B_0 = B_1 = \{x_{n+1} > 0\} \subseteqq \mathbb{R}^{n+1}$, $x = x_0 = x_1 = (\alpha, \ldots, \alpha, 1)$, and get:

$$|\alpha|^{n+1} \int_{C_\lambda} \frac{dy}{\|x-y\|^{n+1}} \geqq |\alpha|^{n+1} \left(\int_{C_0} \frac{dy}{\|x-y\|^{n+1}} \wedge \int_{C_0} \frac{dy}{\|x-y\|^{n+1}} \right), \quad 0 < \lambda < 1.$$

As $|\alpha| \to +\infty$, we obtain $|C_\lambda| \geqq |C_0| \wedge |C_1|$ or, due to homogeneity, (6.3). In fact, already Minkowki's ideas entail (6.3) for arbitrary Borel sets but the Gabriel differential method seems to collapse beyond star-shaped bodies. □

7. Quasi-concavity of V-Newtonian potentials of very thin bodies

Consider, for $\dim E \geqq 3$, the Newtonian potential of $A \in \mathcal{B}(E)$:

$$\nu_\cdot(A) = \mathbb{E}_\cdot \left(\int_0^\infty 1_A(X(t))\, dt \right),$$

that is, the expected amount of time the Brownian motion spends in A. If $x \in E$ is fixed, the measure ν_x is not quasi-concave, although, by ([8], Th. 5.1):

$$\nu_{x_0 + x_1}(A_0 + A_1) \geqq \nu_{x_0}(A_0) \wedge \nu_{x_1}(A_1),$$

or, stated otherwise:

$$\nu_{x_{1/2}}(A_{1/2}) \geqq \frac{1}{4} [\nu_{x_0}(A_0) \wedge \nu_{x_1}(A_1)],$$

for all x_0, $x_1 \in E$ and all A_0, $A_1 \in \mathscr{B}(E)$. The convexity behaviour of $\nu_{\cdot}(A)$, with $A \in \mathscr{U}(E)$ fixed, is unknown to us.

The main questions we focus on in this section have no direct meaning without restriction on dim E. We therefore assume throughout that $E = \mathbb{R}^n$, $n \geqq 2$.

Now suppose $B \in \mathscr{U}(\mathbb{R}^n)$ and that $V : B \to [0, +\infty[$ is $-1/2$-concave. Moreover, we suppose $B \neq \mathbb{R}^2$ if $n=2$ and $V=0$ so that B becomes a Greenian domain for the operator $-1/2\,\Delta + V$ with the Dirichlet boundary condition zero. Let:

$$\nu_x(A) = \mathbb{E}_x\left(\int^{\tau_{B^c}} 1_A(X(t)) \exp\left(- \int_0^t V(X(s))\, ds \right) dt \right) = \int_A g(x, y)\, dy, \qquad x \in B,$$

be the V-Newtonian potential of $A \in \mathscr{B}(B)$, g being the corresponding Green function. The reader should note that $g : B \times B \to [0, +\infty]$ is continuous (*see* e. g. [13], Chap. 13). In particular, given a k-dimensional affine manifold F in $\mathbb{R}^n$ possessing Lebesgue measure m^F $(m^{\{a\}} = \delta_a)$, the V-Newtonian potential of any $A \in \mathscr{B}(F \cap B)$, *viz*:

$$\nu_x^F(A) = \int_A g(x, y)\, dm^F(y), \qquad x \in B,$$

becomes well-defined.

THEOREM 7.1. — *If* $\dim F = n-2$, *then*:

$$\nu_{x_\lambda}^{c_\lambda + F}(A_\lambda) \geqq \nu_{x_0}^{c_0+F}(A_0) \wedge \nu_{x_1}^{c_1+F}(A_1), \qquad 0 < \lambda < 1,$$

where $A_i \in \mathscr{B}((c_i + F) \cap B)$, $c_i \in \mathbb{R}^n$, *and* $x_i \in B$, $i = 0, 1$, *are arbitrary.*

Before presenting the proof of Theorem 7.1, we recall some basic facts from potential theory.

Suppose $A \in \mathscr{U}_\infty(\mathbb{R}^n)$ and $\bar{A} \subseteq B$. Then there exists a unique non-negative measure μ_A in $\bar{A}$, called the V-equilibrium measure of A, such that:

$$\int g(x, y)\, d\mu_A(y) = \mathbb{E}_x\left(\exp\left(- \int_0^{\tau_A} V(X(s))\, ds \right);\ \tau_{B^c} \geqq \tau_A < +\infty \right), \qquad x \in B.$$

The total mass $\mu_A(\bar{A}) = \mathscr{C}(A)$ is termed the V-capacity of A and, moreover, writing $g\mu = (\mu(g(x, .)))_{x \in B}$ if μ is a non-negative measure in B:

$$\mathscr{C}(A) = \sup\{\mu(B);\ \operatorname{supp} \mu \subseteq A,\ g\mu \leqq 1\}$$

(*see* e. g. Blumental, Getoor [3], Chap. 6.4).

Proof of Theorem 7.1. — We shall prove that g is $-1/(n-2)$-convex. By eventually diminishing V and using the Dini theorem, there is no loss of generality in assuming $\sup V = q < +\infty$.

In the following, we sometimes write $g^{B,V}$, $\mathscr{C}^{B,V}$, $\mu_A^{B,V}$ instead of g, $\mathscr{C}$, and μ_A, respectively, and asume, as we may, that X is the standard Brownian motion.

Case $n \geqq 3$. — Letting $\mathscr{C}^{\mathbb{R}^n}(\mathrm{B}(0:r)) = c_n r^{n-2}$, we claim that:

$$(7.1) \qquad \lim_{r \to 0^+} \frac{\mathscr{C}^{\mathrm{B}, \mathrm{V}}(\mathrm{B}(y;r))}{c_n r^{n-2}} = 1, \qquad y \in \mathrm{B}.$$

To *see* this, let $y \in \mathrm{B}$ be fixed and write $\mathrm{B}_r = \mathrm{B}(y;r)$ for brevity. Then, if $\mathrm{B}_r \subseteq \bar{\mathrm{B}}_\mathrm{R} \subseteq \mathrm{B}$, certainly:

$$c_n r^{n-2} \leqq \mathscr{C}^{\mathrm{B}, \mathrm{V}}(\mathrm{B}_r) \leqq \mathscr{C}^{\mathrm{B}_\mathrm{R}, q}(\mathrm{B}_r).$$

We next integrate:

$$g^{\mathrm{B}_\mathrm{R}, 0} = g^{\mathrm{B}_\mathrm{R}, q} + q g^{\mathrm{B}_\mathrm{R}, 0} g^{\mathrm{B}_\mathrm{R}, q},$$

with respect to $\mu_{\mathrm{B}_r}^{\mathrm{B}_\mathrm{R}, 0} \otimes \mu_{\mathrm{B}_r}^{\mathrm{B}_\mathrm{R}, q}$, arriving at:

$$\mathscr{C}^{\mathrm{B}_\mathrm{R}, q}(\mathrm{B}_r) \leqq \mathscr{C}^{\mathrm{B}_\mathrm{R}, 0}(\mathrm{B}_r) + q d_n \mathrm{R}^n,$$

where $d_n = \mathrm{Vol}\, \mathrm{B}(0;1)$. Moreover, by integrating:

$$g^{\mathrm{B}_\mathrm{R}, 0}(x, \xi) = g^{\mathbb{R}^n, 0}(x, \xi) - \mathbb{E}_x g^{\mathbb{R}^n, 0}(\mathrm{X}(\tau_{\mathrm{B}_\mathrm{R}^c}), \xi)$$

with respect to $\mu_{\mathrm{B}_r}^{\mathrm{B}_\mathrm{R}, 0}(dx) \otimes \mu_{\mathrm{B}_r}^{\mathbb{R}^n, 0}(d\xi)$, we get:

$$\mathscr{C}^{\mathrm{B}_\mathrm{R}, 0}(\mathrm{B}_r) = c_n r^{n-2}(1 - (r/\mathrm{R})^{n-2})^{-1}.$$

Finally, by choosing $\mathrm{R} = r^{1-1/n}$ in the above estimates (7.1) follows at once.

Writing $g = g^{\mathrm{B}, \mathrm{V}}$ as above we have for all r_0, $r_1 > 0$, $0 < \lambda < 1$, and $\varepsilon > 0$:

$$\varepsilon^{2-n}(g \mu_{\mathrm{B}(y_\lambda; \varepsilon r_\lambda)})(x_\lambda) \geqq \varepsilon^{2-n}[(g \mu_{\mathrm{B}(y_0; \varepsilon r_0)})(x_0) \wedge (g \mu_{\mathrm{B}(y_1; \varepsilon r_1)})(x_1)],$$

by Theorem 3.1, and in the limit as $\varepsilon \to 0^+$:

$$g(x_\lambda, y_\lambda) r_\lambda^{n-2} \geqq g(x_0, y_0) r_0^{n-2} \wedge g(x_1, y_1) r_1^{n-2}.$$

Thus, choosing $r_i = (g(x_i, y_i))^{-1/(n-2)}$, if $x_i \neq y_i$, $i = 0,1$, the resulting inequality becomes:

$$g^{-1/(n-2)}(x_\lambda, y_\lambda) \leqq (1-\lambda) g^{-1/(n-2)}(x_0, y_0) + \lambda g^{-1/(n-2)}(x_1, y_1),$$

and it follows at once that g is $-1/(n-2)$-convex.

Case $n = 2$. — If Theorem 7.1 is true in $\mathbb{R}^{n_0+1}$, $n_0 \geqq 2$, then we may use the theory of α-convex measures to prove Theorem 7.1 in $\mathbb{R}^{n_0}$. Indeed, set $\tilde{\mathrm{V}}(x, \xi) = \mathrm{V}(x)$, $(x, \xi) \in \mathrm{B} \times \mathbb{R}$ and note that:

$$g(x, y) = \int_{-\infty}^{\infty} g^{\mathrm{B} \times \mathbb{R}, \tilde{\mathrm{V}}}(x, 0, y, \eta)\, d\eta.$$

If $g^{\mathrm{B} \times \mathbb{R}, \tilde{\mathrm{V}}}$ is $-1/(n_0 - 1)$-convex it follows from ([8], Th. 3.1) that g is $-1/(n_0 - 2)$-convex. □

In the following two examples we suppose in addition to the above assumptions that ∂B is $\mathscr{C}^\infty$ and that V has a $\mathscr{C}^\infty$ extension to a neighbourhood of $\bar{B}$.

Example 7.1. — For each $y \in \partial B$, let $n_i(y) = n_i^B(y)$ denote the inner unit normal of $\bar{B}$ at y and set:

$$p(x, y) = \lim_{\varepsilon \to 0^+} g(x, y + \varepsilon n_i(y))/2\varepsilon.$$

If $n_i(y_0) = n_i(y_1)$, then $n_i(y_\lambda) = n_i(y_0)$, $0 < \lambda < 1$, and the $-1/(n-2)$-convexity of g gives:

$$p^{-1/(n-1)}(x_\lambda, y_\lambda) \leqq (1-\lambda) p^{-1/(n-1)}(x_0, y_0) + \lambda p^{-1/(n-1)}(x_1, y_1),$$

employing the same type of argument as in the proof of Theorem 7.1. Noting that $p(x, y)\, d\sigma_{\partial B}(y)$ is the V-harmonic measure at x (use ([17], Th. 6.14) and the Green formula) we have thus complemented Example 6.2. □

Example 7.2. — Let $A \in \mathscr{U}_\infty(\mathbb{R}^n)$, $\bar{A} \subseteq B$, and assume $\partial A \in \mathscr{C}^\infty$. Moreover, suppose F is a supporting hyperplane of $\bar{A}$ such that $\bar{A} \cap F = C$ is $(n-1)$-dimensional. Then:

$$d\mu_{A \mid \beta(C)} = f\, d\sigma_C,$$

where f is -1-concave.

To see this, we apply the Green formula once more to get:

$$-\frac{1}{2} \frac{\partial u_A}{\partial n_e} d\sigma_{\partial A} = d\mu_A - 1_A V\, dm,$$

where m is Lebesgue measure, $u_A = g\mu_A$, and $n_e = -n_i^A$. However, as u_A is quasi-concave $-\partial u_A/\partial n_e$ is -1-concave on C. □

In the planar case, we shall complement Theorem 7.1 in the following way.

THEOREM 7.2. — *Let for $A \in \bar{\mathscr{U}}_\infty(\mathbb{C})$, $g_A \leqq 0$ be the Green function of Δ in $\mathbb{C} \setminus A$ with pole at ∞ and with the Dirichlet boundary condition zero. Then:*

$$g_{A_\lambda}(z_\lambda) \geqq g_{A_0}(z_0) \wedge g_{A_1}(z_1), \qquad 0 < \lambda < 1.$$

Proof. — Assuming $0 \in A$, $g = g_A$ possesses the following characteristic properties:

(i) g is harmonic in $\mathbb{C} \setminus A$;

(ii) g is continuous in $\mathbb{C}$ and $g|_A = 0$,

(iii) $g(z) = \ln \frac{1}{|z|} - \ln \frac{1}{\mathscr{C}_2(A)} + \mathscr{O}\left(\frac{1}{|z|}\right)$ as $|z| \to +\infty$.

The constant $\mathscr{C}_2(A)$ is the logarithmic capacity of A [1]. If $B(0; R) \supseteq A$ and $u_A^{B(0;R)}$ denotes the equilibrium potential of A relative to $B(0; R)$ we thus have:

$$(u_A^{B(0;R)}(z) - 1) \ln \frac{R}{\mathscr{C}_2(A)} - g_A(z) = \mathscr{O}\left(\frac{1}{R}\right) \quad \text{as } R \to +\infty$$

and, consequently:

$$g_A(z) = \lim_{R \to +\infty} (u_A^{B(0;R)}(z) - 1) \ln \frac{R}{\mathscr{C}_2(A)}.$$

From this representation formula Theorem 7.2 follows at once using Theorem 3.1. □

Example 7.3. — $\mathscr{C}_2$ is concave on $\overline{\mathscr{U}}_\infty(\mathbb{C})$:

$$\mathscr{C}_2(A_0 + A_1) \geqq \mathscr{C}_2(A_0) + \mathscr{C}_2(A_1), \quad A_0, A_1 \in \overline{\mathscr{U}}_\infty(\mathbb{C}). \tag{7.2}$$

Indeed, as:

$$\ln \mathscr{C}_2(A) = \lim_{|z| \to +\infty} (g_A(z) + \ln |z|).$$

Theorem 7.2 gives:

$$\mathscr{C}_2(A_\lambda) \geqq \mathscr{C}_2(A_0) \wedge \mathscr{C}_2(A_1),$$

and (7.2) follows by homogeneity. □

The next example is mainly a preparation for Example 7.5.

Example 7.4. — By an excercise in Pólya and Szegö [21], Aufg. [124] :

$$\mathscr{C}_2(A) \leqq \frac{1}{2\pi} \text{length } \partial A, \qquad A \in \overline{\mathscr{U}}_\infty(\mathbb{C}). \tag{7.3}$$

A possible solution reads as follows.

Let H_A be the support function of A:

$$H_A(\xi) = \sup_{x \in A} \langle x, \xi \rangle, \qquad \xi \in \mathbb{C},$$

and remember that:

$$\int_0^{2\pi} H_A(e^{i\theta}\xi)\, d\theta/2\pi = \frac{\|\xi\|}{2\pi} \text{length } \partial A, \qquad \xi \in \mathbb{C}. \tag{7.4}$$

We next approximate the average in the left-hand side by:

$$\sum_{k=1}^{p} H_A(e^{i\theta_k}\xi)\lambda_k \qquad (0 < \lambda_k < 1, \lambda_1 + \ldots + \lambda_p = 1),$$

that is, by the support function of $\sum_{k=1}^{p} \lambda_k e^{-i\theta_k} A$. However,

$$\mathscr{C}_2\left(\sum_{k=1}^{p} \lambda_k e^{-i\theta_k} A\right) \geqq \mathscr{C}_2(A)$$

from Example 7.3 and as the right-hand side of (7.4) is the support function of a ball of radius $1/2\pi$ length ∂A, we have (7.3). □

Example 7.5. — Consider an $A \in \overline{\mathscr{U}}_\infty(\mathbb{R}^3)$ with principal radius R_1 and R_2 and mean curvature:

$$\mathscr{M}(A) = \frac{1}{2}\int_{\partial A}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) d\sigma(\xi).$$

Then by a Theorem of Szegö [23], Satz III:

$$\mathscr{C}_3(A) \leqq \frac{1}{4\pi}\mathscr{M}(A), \tag{7.5}$$

where $\mathscr{C}_3$ is the Newtonian capacity normalized so that $\mathscr{C}_3(B(0;1))=1$. A very important ingredient in Szegö's proof is the following inequality for mixed volumes due to Minkowski:

$$\mathscr{M}^2(A) \geqq 4\pi \quad \text{area } \partial A.$$

Noting that $\mathscr{C}_3$ is concave on $\overline{\mathscr{U}}_\infty(\mathbb{R}^3)$ [4] due to (1.1) we, alternatively, obtain (7.5) as in the previous example. The n-dimensional counterpart of (7.5) is now obvious: if $\mathscr{C}_n$ denotes the Newtonian capacity in $\mathbb{R}^n$ $(n \geqq 3$, $\mathscr{C}_n(B(0;1))=1)$ and if Z_n is a uniformly distributed random vector on S^{n-1}, then:

$$\mathbb{E}\, H_A(Z_n) \geqq \mathscr{C}_n^{1/(n-2)}(A), \quad A \in \overline{\mathscr{U}}_\infty(\mathbb{R}^n).$$

Certainly, the Szegö line of reasoning leads to the same estimate. □

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