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A FEW REMARKS ABOUT THE VARIETY OF IRREDUCIBLE PLANE CURVES OF GIVEN DEGREE AND GENUS

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1. It was stated by Severi [12] that the variety of irreducible plane curves of degree d with δ nodes is irreducible. To this day, Severi's statement remains unproved; however proofs of various special cases have been provided by Several authors [10], [1]. The goal of this paper is to widen the range of values of d and δ for which Severi's assertion is known to hold true.

Throughout this paper we shall always work, without further notice, over the complex field.

Let $\Sigma_{d,g}$ be the variety of irreducible plane curves of degree d and genus g. It is known (*cf.* [2], for example) that a general point of any component of $\Sigma_{d,g}$ corresponds to a curve which has only nodal singularities. Thus, to say that the variety of irreducible plane curves of degree d with δ nodes is irreducible is the same as saying that $\Sigma_{d,g}$ is irreducible, with:

$$g=\binom{d-1}{2}-\delta.$$

The main result of this paper is that $\Sigma_{d,g}$ is irreducible whenever $d\geqq(2/3)g+7/3$. To understand what the bound means we consider the Brill-Noether number:

$$\rho=\rho(r, d, g)=g-(r+1)(g-d+r).$$

The integer $\rho(r, d, g)$, when non-negative, is the dimension of the variety of linear series of degree d and dimension r (g^r_d's, for short) on a general curve of genus g; when $\rho<0$, this variety is empty [7]. This being understood, we can state our main result in the following, more striking, form.

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(1.1) THEOREM. — *The variety* $\Sigma_{d,g}$ *is irreducible whenever*

$$\rho(2, d, g) \geqq 1$$

or, which is the same, when:

$$d \geqq (2/3) g + 7/3.$$

Put otherwise, our theorem states that, denoting by:

$$m: \quad \Sigma_{d,g} \to \mathcal{M}_g$$

the natural morphism of $\Sigma_{d,g}$ to the moduli space of genus g curves, $\Sigma_{d,g}$ is irreducible as soon as a general fiber of $\Sigma_{d,g}$ has dimension at least equal to $1 + \dim \mathbb{P}GL(2) = 9$ (provided of course, that $g > 1$).

To our knowledge, the best previously known bound of this type is due to Alibert and Maltsiniotis [1], who showed that $\Sigma_{d,g}$ is irreducible when:

$$d \geqq (4/3) g,$$

that is, when:

$$\rho(2, d, g) \geqq 2g - 6.$$

One of the main reasons for the appearance of the bound $\rho \geqq 1$ in Theorem (1.1) lies in the essential use we shall make of a beautiful result of Fulton and Lazarsfeld [5]. In the statement $G^r_d(C)$ will denote the variety of g^r_d's on the curve C (*cf.* [3] for a precise definition).

(1.2) THEOREM (Fulton-Lazarsfeld). — *Let* C *be a smooth genus* g *curve. Suppose* $\rho(r, d, g) \geqq 1$. *Then* $G^r_d(C)$ *is connected.*

Combining this with Gieseker's result that $G^r_d(C)$ is smooth of dimension ρ if C is general [6], we get:

(1.3) COROLLARY (Fulton-Lazarsfeld). — *Let* C *be a general smooth curve of genus* g. *If* $\rho(r, d, g) \geqq 1$, $G^r_d(C)$ *is irreducible (and smooth).*

The other essential ingredient of our proof will be the irreducibility of $\mathcal{M}_g$ (*cf.* [13]). Thus, in a way, we shall be deducing the irreducibility of $\Sigma_{d,g}$ from the one of $\mathcal{M}_g$. Incidentally, this is how Severi handles the case $d \geqq g + 2$ in [12], Anhang F, n° 10, although his argument cannot be considered complete.

2. The irreducibility of $\Sigma_{d,g}$ is known for any d when g is small (for example, Alibert and Maltsiniotis'result already disposes of the cases $g \leqq 3$). Therefore it will do no harm, and save a lot of time, if we state most of our auxiliary results under the additional assumption that $g > 1$, without bothering to say how they ought to be modified when $g = 0, 1$.

We shall prove Theorem (1.1) by showing, more exactly, that every point of $\mathcal{M}_g$ has arbitrarily small neighbourhoods U such that $m^{-1}(U)$ is irreducible.

The irreducibility of $\Sigma_{d,g}$ then follows from the one of $\mathcal{M}_g$. We shall find it convenient to translate our problem about plane curves into one about g^2_d's on smooth genus g curves, as we now explain.

A g^r_d $\mathscr{D}$ on a smooth curve C corresponds to an $(r+1)$-dimensional vector subspace $V \subset H^0(C, L)$, for a suitable degree d line bundle L. By a *frame for* $\mathscr{D}$ we shall mean a frame in V up to homothethy. There is a $1-1$ correspondence between couples $(\mathscr{D}, \mathscr{F})$, where $\mathscr{D}$ is a base-point-free g^r_d on C and $\mathscr{F}$ is a frame for $\mathscr{D}$, and non-degenerate degree d maps:

$$\varphi: \quad C \to \mathbb{P}^r.$$

We shall use the symbol $|\mathscr{D}|$ to denote the complete linear series corresponding to the linear series $\mathscr{D}$. Also, if $p_1, \ldots, p_h$ are points of C, we shall denote by $\mathscr{D} - \sum p_i$ the linear series consisting of those effective divisors E such that $E + \sum p_i \in \mathscr{D}$. As is customary, we shall denote by $r(\mathscr{D})$ and $i(\mathscr{D})$ the dimension and index of speciality of $\mathscr{D}$; in other words:

$$r(\mathscr{D}) = \dim V - 1; \qquad i(\mathscr{D}) = h^1(C, L).$$

We shall also use the symbol K_C (or K) to designate the canonical sheaf on the curve C.

We next recall (*cf.* [3]) that, given non-negative integers r, d, for every point p of $\mathscr{M}_g$ and any sufficiently small connected neighbourhood U of p (either in the complex or the Zariski topology), there are a smooth connected variety $\mathscr{M}$, a finite ramified covering:

$$h: \quad \mathscr{M} \to U$$

and two varieties, proper over $\mathscr{M}$:

$$\xi: \quad \mathscr{C} \to \mathscr{M}, \qquad \pi: \quad \mathscr{G}^r_d \to \mathscr{M}$$

with the following properties:

(*a*) $\mathscr{C}$ is a universal curve over $\mathscr{M}$, i. e. for every $p \in \mathscr{M}$, $\xi^{-1}(p)$ is a smooth genus g curve whose isomorphism class is $h(p)$.

(*b*) $\mathscr{G}^r_d$ parametrizes couples $(p, \mathscr{D})$, where $p \in \mathscr{M}$ and $\mathscr{D}$ is a g^r_d on $\xi^{-1}(p)$.

Now suppose $r=2$ and let $\mathscr{U}$ be the open subset of $\mathscr{G}^2_d$ consisting of all points which correspond to couples $(p, \mathscr{D})$ where $p \in \mathscr{M}$ and $\mathscr{D}$ is a g^2_d on $\xi^{-1}(p)$ which has no base points and is not composed with an involution. We denote by $\mathscr{V}$ the variety whose points are the couples $(\gamma, \mathscr{F})$ where $\gamma \in \mathscr{U}$ and $\mathscr{F}$ is a frame for the corresponding g^2_d. Clearly $\mathscr{V}$ parametrizes couples (p, φ), where $p \in \mathscr{M}$ and:

$$\varphi: \quad \xi^{-1}(p) \to \mathbb{P}^2$$

is a non-degenerate degree d map which is not composed with an involution; moreover $\mathscr{V}$ maps *onto* $m^{-1}(U) \subset \Sigma_{d, g}$ *via*:

$$(p, \varphi) \to \varphi(\xi^{-1}(p)).$$

In view of our previous remarks, Theorem (1.1) will be proved if we can prove that $\mathscr{U}$, and hence $\mathscr{V}$, is irreducible. We denote by $\mathscr{G}$ the closure of $\mathscr{U}$ in $\mathscr{G}^2_d$.

Clearly, $\mathscr{G}$ is the union of all the components of $\mathscr{G}^2_d$ whose general points correspond to base-point free g^2_d's which are not composed with an involution. This being understood, what has to be proved is:

(2.1) Proposition. — *Under the assumption of Theorem* (1.1), $\mathscr{G}$ *is irreducible.*

In order to be able to deal effectively with the variety $\mathcal{G}$, we first need to recast, in a form suitable for our needs, some basic results of [3].

To begin with, let l be a point of $\mathcal{G}^r_d$; the point l corresponds to the datum of a degree d line bundle L on the curve $\xi^{-1}(\pi(l))=\mathrm{C}$, plus an $(r+1)$-dimensional subspace $\mathrm{V}\subset\mathrm{H}^0(\mathrm{C},\mathrm{L})$. Recall that a differential operator ∇ of order at most one on L (over an open subset U of C) is a $\mathbb{C}$-linear endomorphism of $\mathrm{L}|_{\mathrm{U}}$ which locally looks as follows:

$$\nabla s=a(z)\frac{\partial s}{\partial z}+b(z)\,s,$$

where z is a local coordinate on C and a and b are holomorphic. The differential operators of order at most one on L make up a locally free rank two $\mathcal{O}_{\mathrm{C}}$-module, henceforth denoted Σ_{L}. The subsheaf of Σ_{L} consisting of algebraic operators is isomorphic to $\mathcal{O}_{\mathrm{C}}$, while the quotient $\Sigma_{\mathrm{L}}/\mathcal{O}_{\mathrm{C}}$ is easily seen to be isomorphic to θ_{C}, the tangent sheaf to C. The vector space $\mathrm{H}^1(\mathrm{C},\Sigma_{\mathrm{L}})$ parametrizes first order one-parameter deformations of C along with L. Secondly there is a mapping:

$$\mu^*:\quad \mathrm{H}^1(\mathrm{C},\Sigma_{\mathrm{L}})\to\mathrm{Hom}(\mathrm{V},\mathrm{H}^1(\mathrm{C},\mathrm{L}))$$

induced by the natural sheaf homomorphism:

$$\lambda:\quad \Sigma_{\mathrm{L}}\to\mathrm{Hom}(\mathrm{V},\mathrm{L})\cong\mathrm{V}^*\otimes\mathrm{L}.$$

Fix an element $\sigma\in\mathrm{H}^1(\mathrm{C},\Sigma_{\mathrm{L}})$, that is, fix a deformation:

$$\tilde{\mathrm{C}}\to\mathrm{Spec}\,\mathbb{C}[\varepsilon]$$

of C together with a line bundle $\mathcal{L}$ on $\tilde{\mathrm{C}}$ which restricts to L on the central fibre. Then an element s of V extends to a section of $\mathcal{L}$ if and only if $\sigma.s=0$. Further, given an open covering $\{\mathrm{U}_\alpha\}$ on C and a cocycle $\{\nabla_{\alpha\beta}\}$ representing σ, any way of writing $\{\nabla_{\alpha\beta}s\}$ as a coboundary:

$$\nabla_{\alpha\beta}s=t_\beta-t_\alpha$$

yields a well defined extension of s. This extension, in a sense which is easy to make precise, "is" the collection $\{s+\varepsilon t_\alpha\}$.

The tangent space $\mathrm{T}_l(\mathcal{G}^r_d)$ fits into an exact sequence:

$$0\to\mathrm{Hom}(\mathrm{V},\mathrm{H}^0(\mathrm{L})/\mathrm{V})\to\mathrm{T}_l(\mathcal{G}^r_d)\to\ker\mu^*\to 0 \tag{2.2}$$

To better understand this, at least when $g>1$, we need to study the cokernel of the basic sheaf homomorphism λ, which we denote by $\mathcal{N}$. We shall do this under the additional assumption that $r>0$, in which case λ is injective.

Clearly, if $\{s_i\}_{i=0,\ldots,r}$ is a basis for V, $\{\xi_i\}$ is the dual basis, and ∇ is a local section of Σ_{L}, we have:

$$\lambda(\nabla)=\sum\xi_i\otimes\nabla s_i.$$

Thus a section of $\mathcal{N}$ is determined by a collection $\{\sum_i \xi_i \otimes t_{i\alpha}\}$, where $t_{i\alpha}$ is a section of L over an open subset U_α of C, subject to the conditions:

$$\sum \xi_i \otimes (t_{i\beta} - t_{i\alpha}) = \sum \xi_i \otimes \nabla_{\alpha\beta} s_i$$

or, which is the same:

$$\nabla_{\alpha\beta} s_i = t_{i\beta} - t_{i\alpha}$$

for suitable sections $\nabla_{\alpha\beta}$ of Σ_L over $U_\alpha \cap U_\beta$. Therefore a global section of $\mathcal{N}$ yields both a class $\sigma \in H^1(C, \Sigma_L)$ (the class of the cocycle $\{\nabla_{\alpha\beta}\}$) and a way of writing $\{\nabla_{\alpha\beta} s_i\}$ as a cohoundary, for each i. Put otherwise, a global section of $\mathcal{N}$ corresponds to the datum of a deformation:

$$\tilde{C} \to \operatorname{Spec} \mathbb{C}[\varepsilon]$$

of C, an extension $\mathcal{L}$ of L, an extension of the g_d^r determined by $V \subset H^0(C, L)$, and an extension of the frame $\{s_i\}$ (up to homothethy, since a change of scale in the s_i modifies the $t_{i\alpha}$'s by the same factor). Two such extensions:

$$s_i + \varepsilon\, t_{i\alpha}, \qquad s_i + \varepsilon t'_{i\alpha}, \qquad i = 0, \ldots, r$$

determine the same section of $\mathcal{N}$, if and only if there is a global section ∇ of Σ_L such that:

$$t'_{i\alpha} - t_{i\alpha} = \nabla s_i, \qquad i = 0, \ldots, r.$$

In case $g > 1$, $H^0(C, \Sigma_L) = H^0(C, \mathcal{O}_C)$, hence ∇ is a constant and the two given extensions are proportional (by the factor $1 + \varepsilon\nabla$). In conclusion, *when* $g > 1$, *if l is a point of* $\mathcal{G}_d^r$ *and* $\mathcal{F}$ *a frame for the corresponding* g_d^r, $H^0(C, \mathcal{N})$ *is the tangent space at* $(l, \mathcal{F})$ *to the variety whose points are the couples* (*point of* $\mathcal{G}_d^r$, *frame for the corresponding* g_d^r). The exact cohomology sequency of:

$$0 \to \Sigma_L \to V^* \otimes L \to \mathcal{N} \to 0$$

gives an exact sequence:

$$0 \to \operatorname{Hom}(V, H^0(C, L))/\mathbb{C} \to H^0(C, \mathcal{N}) \to \ker \mu^* \to 0$$

Dividing $H^0(C, \mathcal{N})$ by $\operatorname{Hom}(V, V)/\mathbb{C}$, one gets the tangent space to $\mathcal{G}_d^r$ at l, whence the exact sequence (2.2), considering that:

$$\operatorname{Hom}(V, H^0(C, L)/V) \cong \operatorname{Hom}(V, H^0(C, L))/\operatorname{Hom}(V, V).$$

To get a hold on $\ker \mu^*$ it now remains to analyze $\mathcal{N}$ in terms of other, better known, sheaves. We shall do this under the assumption that $r > 0$, as before. Let F be the base locus of $V \subset H^0(C, L)$. Choose a basis for V and let:

$$\varphi: \ C \to \mathbb{P}^r$$

be the corresponding mapping: the hyperplane bundle on $\mathbb{P}^r$ pulls back, *via* φ, to $L(-F)$. We then have a commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_C & \longrightarrow & \mathcal{O}_C(F) & \longrightarrow & \mathcal{O}_F(F) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Sigma_L & \longrightarrow & V^*\otimes L & \longrightarrow & \mathcal{N} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \theta(C) & \longrightarrow & \theta(F) & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Here θ is the pullback *via* φ of the tangent bundle to $\mathbb{P}^r$, while the middle column is obtained by tensoring with $\mathcal{O}(F)$ the pullback of the Euler sequence of $\mathbb{P}^r$. As for $\mathcal{Q}$, recalling that the normal sheaf to φ, denoted by N_φ, is the quotient θ/θ_C, it fits into an exact sequence:

(2.4) $$0 \to \theta_C(F)\otimes\mathcal{O}_F \to \mathcal{Q} \to N_\varphi(F) \to 0$$

Since $\mathcal{O}_F(F)$ and $\theta_C(F)\otimes\mathcal{O}_F$ are concentrated on F, we deduce from (2.3) and (2.4) that:

$$\text{Coker}\,\mu^* = H^1(C, \mathcal{N}) = H^1(C, \mathcal{Q}) = H^1(C, N_\varphi(F)).$$

As a special case, we recover the identification, given in [3], between coker μ^* and $H^1(C, N_\varphi)$ when V has no base points. Returning to (2.2) we can now compute the dimension of the tangent space to $\mathcal{G}^r_d$ at l:

$$
\begin{aligned}
\dim T_l(\mathcal{G}^r_d) &= (r+1)(h^0(C, L)-r-1)+\dim\ker\mu^* \\
&= (r+1)(h^0(C, L)-r-1)+h^1(C, \Sigma_L)-(r+1)\,h^1(C, L)+\dim(\text{Coker}\,\mu^*) \\
&= (r+1)\,\chi(L)-(r+1)^2+h^1(C, \mathcal{O})+h^1(C, \theta_C)+h^1(C, N_\varphi(F)) \\
&= \dim\mathcal{M}_g+\rho(r, d, g)+h^1(C, N_\varphi(F))
\end{aligned}
$$

Since it is well-known (*see* [3], for example), that every component of $\mathcal{G}^r_d$ has dimension equal at least to $\dim\mathcal{M}_g+\rho$, the above formula implies.

(2.5) LEMMA. — *If* $r>0$, $\mathcal{G}^r_d$ *is smooth of dimension* $\rho+\dim\mathcal{M}_g$ *at* l *if and only if* $h^1(C, N_\varphi(F))=0$.

(2.6) *Remark.* — When L is non-special, the condition of the above lemma is always satisfied. In fact $h^1(C, N_\varphi)$ is just the dimension of the cokernel of the map μ^*, whose target is $\text{Hom}(V, H^1(C, L))=\{0\}$.

In case $r=1$, the sheaf N_φ is concentrated on a zero-dimensional subset of C, hence Lemma (2.5) implies the well-known.

(2.7) Proposition. — *The variety $\mathscr{G}_d^1$ is smooth of dimension:*

$$\rho(1, d, g)+\dim \mathscr{M}_g.$$

In other words, when $g>1$, $\mathscr{G}_d^1$ is smooth of dimension:

$$2d+2g-5.$$

In general the dimension of a component of $\mathscr{G}_d^r$ may well exceed $\dim \mathscr{M}_g+\rho$. When $r=2$, however, the components of $\mathscr{G}_d^r$ which are of interest to us have the "correct" dimension. The precise result, whose proof is to be found in [2] or [3] is:

(2.8) Proposition. — *Suppose $g>1$. Let X be a component of $\mathscr{G}_d^2$ whose general point corresponds to a g_d^2 (necessarily base-point-free) which is not composed with an involution. Then the dimension of X is:*

$$\dim X=3g-3+\rho=3d+g-9.$$

The singular locus of $\mathscr{G}_d^2$ is known to be relatively "small". The following result, which is implicit in [3], makes this precise.

(2.9) Proposition. — *Let $\mathscr{U}$ be the open subset of $\mathscr{G}_d^2$ consisting of all points $(p, \mathscr{D})$, where $p\in\mathscr{M}$ and $\mathscr{D}$ is a g_d^2 (possibly with base points) on $\xi^{-1}(p)$ which is not composed with an involution. Then the dimension of the singular locus of $\mathscr{U}$ does not exceed $g-8$.*

Proof. — The cases $g=0$, 1 are taken care of by Remark (2.6). Therefore we assume $g>1$ throughout. Let $\mathscr{X}$ be a component of the singular locus of $\mathscr{U}$ and let f be the degree of the fixed divisor of a general point of $\mathscr{X}$. Let $\mathscr{X}'$ be the subvariety of $\mathscr{G}_{d-f}^2$ consisting of all points $(p, \mathscr{D}-F)$ where $(p, \mathscr{D})\in\mathscr{X}$ has fixed divisor F and $\deg(F)=f$. Write:

$$\dim \mathscr{X}'=\dim \mathscr{X}-f'$$

Let $l=(p, \mathscr{D}')$ be a general point of $\mathscr{X}'$ and set $C=\xi^{-1}(p)$. Choose a frame $\mathscr{F}$ for $\mathscr{D}'$ and let:

$$\varphi:\quad C\to\mathbb{P}^2$$

be the corresponding morphism. Denote by $\tilde{\mathscr{X}}$ the variety whose points are the couples $(l', \mathscr{F}')$ where l' is a point of $\mathscr{X}'$ and $\mathscr{F}'$ is a frame for the corresponding g_{d-f}^2. If we assume that:

$$\dim \mathscr{X}>g-8$$

the tangent space to $\tilde{\mathscr{X}}$ at $(l, \mathscr{F})$ is a vector subspace of $H^0(C, N_\varphi)$ of dimension at least:

$$g-7-f'+\dim \mathbb{P}\,GL(2)=g-f'+1.$$

To reach a contradiction we now make use of what we might call the "ramification trick". Let Z be the ramification divisor of φ, and set:

$$\mathscr{K}_\varphi = \theta_C(Z)/\theta_C, \qquad N'_\varphi = \varphi^* \theta_{P^2}/\theta_C(Z)$$

so that we have an exact sequence:

$$0 \to \mathscr{K}_\varphi \to N_\varphi \to N'_\varphi \to 0$$

Clearly N'_φ is a line bundle, while $\mathscr{K}_\varphi$ is concentrated on Z, hence $h^1(C, N_\varphi) = h^1(C, N'_\varphi)$. Furthermore, since $(l, \mathscr{F})$ is a general point of $\tilde{\mathfrak{X}}$, if follows from [4], Lemma (1.4), that:

$$T_{(l, \mathscr{F})}(\tilde{\mathfrak{X}}) \cap H^0(C, \mathscr{K}_\varphi) = \{0\},$$

an hence that·

$$r(N'_\varphi) \geqq g - f'.$$

Now either N'_φ is non-special, or else, by Clifford's theorem:

$$i(N'_\varphi) = r(N'_\varphi) + g - \deg(N'_\varphi) \leqq g - r(N'_\varphi) \leqq f'.$$

Since $\dim \mathfrak{X} = \dim \mathfrak{X}' + f'$, there is an f'-dimensional algebraic system of degree f divisors F such that $(p, \mathscr{D}' + F) \in \mathfrak{X}$. In particular, given f' general points $p_1, \ldots, p_f$, we may choose an F that contains all of them and therefore:

$$h^1(C, N_\varphi(F)) = h^1(C, N'_\varphi(F)) = 0.$$

Thus by Lemma (2.5) $(p, \mathscr{D}' + F)$ is a smooth point of $\mathscr{G}^2_d$, a contradiction.

Q.E.D.

We now turn to the variety of g^r_d's on a fixed smooth curve C, which we denote by $G^r_d(C)$ (*cf.* [3] for more details). Associating to each g^r_d, $\mathscr{D}$ the corresponding complete linear series $|\mathscr{D}|$, maps $G^r_d(C)$ into $\mathrm{Pic}^d(C)$; the image is $W^r_d(C)$, the variety of complete linear series of degree d and dimension at least r on C. The description of the tangent spaces to $\mathscr{G}^r_d$ which we have just given has an exact analogue for the variety $G^r_d(C)$. Fix a point l on $G^r_d(C)$, corresponding to $V \subset H^0(C, L)$. The role of λ and μ^* is now played by the sheaf homomorphism:

$$\lambda_0 : \quad \mathscr{O}_C \to \mathrm{Hom}(V, L) \cong V^* \otimes L$$

and by the corresponding homomorphism in cohomology:

$$\mu_0^* : \quad H^1(C, \mathscr{O}_C) \to \mathrm{Hom}(V, H^1(C, L)).$$

Thus $T_l(G^r_d(C))$ fits into an exact sequence:

$$0 \to \mathrm{Hom}(V, H^0(C, L)/V) \to T_l(G^r_d(C)) \to \ker \mu_0^* \to 0$$

and we have:

$$\dim T_l(G_d^r(C)) = \rho(r, d, g) + \dim \ker \mu_0$$

where μ_0, the transpose of μ_0^*, is the cup-product mapping:

$$\mu_0 : \quad V \otimes H^0(C, K_C L^{-1}) \to H^0(C, K_C)$$

Since $G_d^r(C)$, when non-empty, is known to have dimension at least ρ, we have:

(2.10) LEMMA. — *The variety $G_d^r(C)$ is smooth of dimension ρ at l if and only if* $\ker \mu_0 = \{0\}$.

(2.11) *Remark.* — The condition of the above lemma is obviously satisfied when l corresponds to a non-special g_d^r, i. e. when $i(L) = 0$.

The kernel of μ_0 is easily computable when $r = 1$. In this case:

$$\ker \mu_0 \cong H^0(C, K_C L^{-2}(\Delta)),$$

where Δ is the base locus of the g_d^1 corresponding to l (*cf.* [3], for example). This simple fact can be put to work to gather information about $G_d^2(C)$.

The following result has been obtained also by Accola, Griffiths, Harris. Their proof will appear in [8].

(2.12) PROPOSITION. — *Let* C *be a smooth curve of genus* g. *Let* G *be an irreducible subvariety of* $G_d^2(C)$. *Let* $\mathscr{D}$ *be a general point of* G. *Suppose* $|\mathscr{D}|$ *is base-point-free and not composed with an involution. Let* D *be a divisor belonging to* $\mathscr{D}$. *Then:*

(i) *if* $|\mathscr{D}|$ *is non-special:*

$$\dim G \leqq \rho(2, d, g)$$

(ii) *if* $|\mathscr{D}|$ *is special:*

$$\dim G \leqq 2d - g - 5 + i(2D) = \rho(2, d, g) + g - d + 1 + i(2D).$$

Statement i) follows at once from Remark (2.11). The proof of ii) makes use of the following standard:

(2.13) LEMMA. — *Let* $\mathscr{D}$ *be a linear series of degree* d *and dimension* r *on a smooth curve* C. *Then:*

(*a*) *if* p *is a general point of* C and $1 \leqq t \leqq r$, $r(\mathscr{D} - tp) = r - t$, *i. e.* $i(\mathscr{D} - tp) = i(\mathscr{D})$.

(*b*) *If, moreover,* $\mathscr{D}$ *is base-point-free and not composed with an involution, and* $p_1, \ldots, p_{r-1}$ *are general points of* C, $\mathscr{D} - \sum p_i$ *is a base-point-free* g_{d-r+1}^1.

We now prove part (ii) of Proposition (2.12). Set $r = r(\mathscr{D})$. Let G′ be the open subset of G consisting of all linear series $\mathscr{D}'$ such that $r(|\mathscr{D}'|) = r$ and such that $|\mathscr{D}'|$ is base-point-free and not composed with an involution. Associating to each $\mathscr{D}' \in G'$ the complete linear series $|\mathscr{D}'|$, maps G′ into $G_d^r(C)$; let S be the image of G′. The fibre of $G' \to S$ over $|D'| \in S$ is contained in the grassmannian of projective 2-planes in the r-dimensional projective space $|D'|$. Thus:

$$\dim G \leqq \dim S + 3r - 6 \qquad (2.14)$$

By Lemma (2.13), part (*b*), there are open subsets U of the $(r-1)$-fold symmetric product C_{r-1} and V of S such that if $|D'| \in S$ and $\sum p_i \in U$, the series $|D' - \sum p_i|$ is base-point-free of dimension one. Hence there is a well defined map:

$$\psi : \quad U \times V \to G^1_{d-r+1}(C)$$
$$\psi(\sum p_i, |D'|) = |D' - \sum p_i|.$$

We fix $\sum p_i$ and D′. In case:

$$\psi(\sum p_i, |D'|) = \psi(\sum q_i, |D''|)$$

the linear series $|D' - \sum p_i + \sum q_i|$ has dimension r. Since $|D' - \sum p_i|$ is one-dimensional, we must have:

$$i(D' - \sum p_i + \sum q_i) = i(D' - \sum p_i) = i(D') \neq 0.$$

Thus if F is the base locus of $|K(-D + \sum p_i)|$, we must have:

$$\sum q_i \leqq F$$

and there are only a finite number of possibilities for $\sum q_i$ (and hence for $|D''|$). In conclusion ψ is finite-to-one. Hence if $\sum p_i \in U$ is general and $\mathscr{E} = |D - \sum p_i|$, we have:

$$(2.15) \quad \dim S \leqq \dim T_{\mathscr{E}}(G^1_{d-r+1}(C)) - r + 1$$
$$= \rho(1, d-r+1, g) + i(2D - 2\sum p_i) - r + 1$$
$$= 2d - g - 3r + 1 + i(2D - 2\sum p_i).$$

Now, if p is a general point of C:

$$i(2D) \leqq i(2D - 2\sum p_i) \leqq i(2D - (2r-2)p).$$

Since $r(2D) \geqq 2r+1$, part(a) of Lemma 1 applies and gives:

$$i(2D - (2r-2)p) = i(2D) = i(2D - 2\sum p_i).$$

Combining this with (2.14) and (2.15) we find:

$$\dim G \leqq 2d - g - 5 + i(2D),$$

as desired.

The bound in part (ii) of Proposition (2.12) has various consequences, among which we single out the following.

(2.16) COROLLARY. — *Let C be a smooth genus g curve. Let X be a component of* $G^2_d(C)$ *whose general point* $\mathscr{D}$ *is such that* $|\mathscr{D}|$ *is base-point-free and not composed with an involution. Then, when* $d \geqq g+1$:

$$\dim X = \rho(2, d, g)$$

Moreover if $d > g+1$, $|\mathscr{D}|$ *is not special.*

In fact, if $|\mathscr{D}|$ is special, part (ii) of Proposition (2.12) gives us the bound:

$$\dim X \leqq 2d-g-5=\rho(2, d, g)-(d-g-1)$$

which is compatible with the known bound:

$$\dim X \geqq \rho$$

only when $d=g+1$. Otherwise $|\mathscr{D}|$ is not special, and part (i) of Proposition (2.12) applies.

3. We are now ready to proceed with the proof of (2.1), and hence of Theorem (1.1). As we already observed, we may, and will, assume that $g>1$. We also fix an integer d such that:

(3.1) $$d \geqq (2/3)g+(7/3).$$

i.e., such that:

$$\rho(2, d, g) \geqq 1.$$

We shall keep the notation of the preceding section. In particular $\mathscr{M}$ will be a suitable ramified covering of an open subset of $\mathscr{M}_g$:

$$\xi: \mathscr{C} \to \mathscr{M}; \qquad \pi: \mathscr{G}_d^2 \to \mathscr{M}$$

will be a universal curve on $\mathscr{M}$ and the variety parametrizing g_d^2's on the fibres of ξ, respectively, while:

$$\mathscr{G} \subset \mathscr{G}_d^2$$

will stand for the union of all the components of $\mathscr{G}_d^2$ whose general point corresponds to a (necessarily base-point-free) g_d^2 which is not composed with an involution. We recall that $\mathscr{G}$ is of pure dimension $3d+g-9$ [Proposition (2.8)]. If G is a subvariety of $\mathscr{G}_d^2$, we shall denote by π_G the restriction of π to G. The first step in the proof is:

(3.2) Lemma. — *Let* X *be a component of* $\mathscr{G}$. *Suppose* $d \leqq g+1$. *Let a general point on* X *correspond to a* g_d^2, $\mathscr{D}$ *on the curve* C. *Then* $\mathscr{D}$ *is complete.*

Proof. — Suppose $r(\mathscr{D})=r>2$: in particular $\mathscr{D}$ is special. Let $\mathscr{C}_{r-2}$ be the $(r-2)$-fold symmetric product *over* $\mathscr{M}$ of $\mathscr{C}$. Then, by Lemma (2.13):

$$\psi(\textstyle\sum p_i, \mathscr{D}')=|\mathscr{D}'|-\sum p_i$$

yields a well-defined morphism:

$$\psi: A \to \mathscr{G}_{d-r+2}^2$$

where A is an open subset of $\mathscr{C}_{r-2} \times_{\mathscr{M}} X$; moreover, ψ maps A into a component Y of $\mathscr{G}_{d-r+2}^2$ whose general member is not composed with an involution. Fix $\sum p_i$ and $\mathscr{D}'$, and suppose:

$$\psi(\textstyle\sum p_i, \mathscr{D}')=\psi(\sum q_i, \mathscr{D}')$$

If $D' \in \mathscr{D}'$, $D'' \in \mathscr{D}''$, this implies that:

$$D'' \in |D'-\textstyle\sum p_i+\sum q_i|$$
$$r(D'')=r.$$

Since $r(D'-\sum p_i)=2$, the q_i's must be base points for $|K(-D'+\sum p_i)|$. But $i(D'-\sum p_i)=i(D')\neq 0$, hence there are finitely many possibilities for $\sum q_i$. Moreover, for any choice of $\sum q_i$, $\mathscr{D}''$ belongs to the grassmannian of projective 2-planes in the projective r-space $|D'-\sum p_i+\sum q_i|$. Thus the dimension of the fibres of ψ does not exceed $3(r-2)$ and we conclude:

$$3d+g-9-3(r-2)=\dim Y \geqq \dim A-3(r-2)=3d+g-9+(r-2)-3(r-2)$$

a contradiction.

Q.E.D.

Next, we notice that, since $\rho(2, d, g)\geqq 0$, by the existence theorem for special divisors (*cf.* [9], for example) $\mathscr{G}^2_d$ maps *onto* $\mathscr{M}$. Actually, even more is true. Since it is known (*cf.* [3], Proposizione (5.8) and the following remark) that a general g^2_d on a general curve is not composed with an involution, $\mathscr{G}$ also surjects onto $\mathscr{M}$. Furthermore, since $\rho(2, d, g)\geqq 1$, by Corollary (1.3) we conclude:

(3.3) LEMMA. — *There is one and only one component of $\mathscr{G}$ which surjects onto $\mathscr{M}$.*

This Lemma, coupled with the results of the previous section, is already sufficient to conclude when $d\geqq g+1$.

(3.4) LEMMA. — *The conclusion of Proposition* (2.1) *is valid under the additional assumption that $d\geqq g+1$.*

Proof. — Every component of $\mathscr{G}$ has dimension $3g-3+\rho$. On the other hand, by Corollary (2.16), every fibre of $\pi_{\mathscr{G}}:\mathscr{G}\to\mathscr{M}$ has pure dimension ρ. Thus every component of $\mathscr{G}$ surjects onto $\mathscr{M}$, and, by Lemma (3.3), $\mathscr{G}$ is irreducible.

Q.E.D.

From now on we shall assume that:

$$d\leqq g.$$

To complete the proof, we shall argue by contradiction, assuming that $\mathscr{G}$ is reducible. We shall denote by X the unique component of $\mathscr{G}$ that maps onto $\mathscr{M}$. Let Y be any other component of $\mathscr{G}$, and set:

$$\pi(Y)=Z\subset\mathscr{M}.$$

Also, let:

$$(p, |D|)$$

be a general point of Y, and set

$$C=\xi^{-1}(p)$$

We now begin an analysis of the irreducible variety Z.

(3.5) LEMMA. — $\dim Z\geqq d+2g-4-i(2D)$.

Proof. — It suffices to apply part (ii) of Proposition (2.12) to a general fibre of π_Y. We find:

$$\dim(Z) \geqq \dim(Y) - (2d - g - 5 + i(2D))$$
$$= 3d + g - 9 - 2d + g + 5 - i(2D) = d + 2g - 4 - i(2D)$$

Q.E.D.

We next recall the well-known "base-point-free pencil trick" (*cf.* [11], p. 162).

(3.6) LEMMA. — *Let* Γ *be a smooth curve. Let* Δ *be an effective divisor on* Γ, *and let* V *be a vector subspace of* $H^0(\Gamma, \mathcal{O}(\Delta))$. *Suppose* V *has no base points. Let* L *be a line bundle on* Γ *and let:*

$$\varphi: \quad V \otimes H^0(\Gamma, L) \to H^0(\Gamma, L(\Delta))$$

be the cup-product mapping. Then:

$$h^0(\Gamma, L(\Delta)) \geqq 2h^0(\Gamma, L) - h^0(\Gamma, L(-\Delta)).$$

The next three lemmas are aimed at sharpening the lower bound on the dimension of Z given by (3.5), while at the same time describing the involutions, both rational and irrational, that a general member of Z might have. For this reason, in the statement of the next lemma and in the sequel of this section, $\mathscr{X}_{n,\gamma}$ will stand for the closure of the subvariety of $\mathscr{M}$ consisting of all points corresponding to curves which are n-fold ramified coverings of smooth genus γ curves. By Hurwitz' formula, the degree of the ramification divisor of such a covering is:

$$w = 2g + 2n(1-\gamma) - 2$$

and Riemann's moduli count yields:

$$\dim \mathscr{X}_{n,\gamma} \leqq 3\gamma - 3 + w = 2g + (2n-3)(1-\gamma) - 2.$$

(3.7) LEMMA. — *Assume* $Z \subset \mathscr{X}_{n,\gamma}$. *Then:*

$$\gamma = 0; \qquad n \geqq (3/8)d + (7/8).$$

Proof. — By Lemma (3.5) we have:

$$2g + (2n-3)(1-\gamma) - 2 \geqq d + 2g - 4 - i(2D).$$

In order to estimate $i(2D)$ we shall apply Lemma (3.6) with $\Delta = D$, $L = K(-2D)$. The result is:

$$i(D) = h^0(C, K(-2D)(D)) \geqq 2h^0(C, K(-2D)) - h^0(C, K(-3D)).$$

Now, the degree of 3 D, by our assumption, is larger than $2g$, hence $i(3D) = 0$ and the above inequality yields:

$$i(D) \geqq 2i(2D).$$

On the other hand, by Lemma (3.2), $r(\mathrm{D})=2$, hence:

$$i(2\,\mathrm{D}) \leqq \frac{g-d}{2}+1.$$

Substituting into the first inequality we find:

$$(4n-6)(1-\gamma) \geqq 3d-g-6;$$

By our assumptions, the right-hand side is always positive, hence $\gamma=0$, and the above inequality reduces to

$$4n \geqq 3d-g.$$

Since, by our assumptions:

$$g \leqq (3/2)\,d-(7/2)$$

the conclusion of the lemma follows.

Q.E.D.

(3.8) LEMMA. — $i(2\,\mathrm{D}) \leqq 2$.

Proof. — We argue by contradiction, and assume $i(2\,\mathrm{D}) \geqq 3$. We set $i(2\,\mathrm{D})=s$, $e=\deg \mathrm{K}(-2\,\mathrm{D})$. Notice that, by our basic assumption (3.1):

$$e<d. \tag{3.9}$$

Next, we denote by Y′ the variety of couples $(\eta, \mathscr{E})$, where $\eta=(q, |\mathrm{D}'|) \in \mathrm{Y}$, $i(2\,\mathrm{D}')=s$, and $\mathscr{E}$ is a g^2_e contained in $|\mathrm{K}(-2\,\mathrm{D}')|$. The variety Y′ has two projections:

$$\psi: \quad \mathrm{Y}' \to \mathscr{G}^2_d$$
$$\varphi: \quad \mathrm{Y}' \to \mathscr{G}^2_e$$

clearly,

$$\dim \mathrm{Y}' = \dim \mathrm{Y} + 3(s-3)$$

On the other hand, since $\mathscr{E}$ determines $|2\,\mathrm{D}'|$, it determines $|\mathrm{D}'|$, up to points of order two in the Jacobian of $\xi^{-1}(q)$. Hence φ is finite-to-one. Thus if we set $\mathrm{Y}''=\varphi(\mathrm{Y}')$, we have:

$$\dim \mathrm{Y}'' = \dim \mathrm{Y} + 3(s-3) = 3d+g-9+3(s-3).$$

Two cases are possible. If $|\mathrm{K}(-2\,\mathrm{D})|$ is not composed with an involution, by Proposition (2.8):

$$3e+g-9 \geqq \dim \mathrm{Y}'' \geqq 3d+g-9+3(s-3)$$

contradicting (3.9). Otherwise, let $\mathscr{E}$ be a general g^2_e inside $|\mathrm{K}(-2\,\mathrm{D})|$, and denote by F its fixed divisor. Set:

$$f=\deg \mathrm{F}, \qquad e=f+e'.$$

Thus $\mathscr{E}-\mathrm{F}$ is base-point-free and the corresponding morphism represents C as a ramified n-sheeted covering of a plane curve of degree m. By Lemma (3.7) this curve has to be rational, and we have:

$$d/m > e'/m = n \geqq (3/8)\,d.$$

In particular $m=2$. Thus to a general point $(q, \mathscr{E})$ of Y″ and each frame $\mathscr{F}$ in $\mathscr{E}$ we may associate a smooth conic Γ plus a ramified degree n covering:

$$h:\quad \mathrm{C} \to \Gamma$$

Moreover, Γ and h determine both $\mathscr{F}$ and $\mathscr{E}-\mathrm{F}$, where F stands for the fixed divisor of $\mathscr{E}$. Since the dimension of the space of degree n morphisms from genus g curves to $\mathbb{P}^1$ is:

$$2n+2g-2,$$

this implies that:

$$\dim \mathrm{Y}'' + 8 \leqq 2\mathrm{n} + 2\mathrm{g} - 2 + 5 + \mathrm{f}.$$

In other terms, since:

$$2n+f=e=2g-2-2d,$$

we find that:

$$3d+g+3s-10 \leqq 4g-2d+1.$$

Combining this with our basic assumption:

$$d \geqq (2/3)\,g+(7/3)$$

we conclude that:

$$10g \leqq 9g-9s-2,$$

a contradiction.

Q.E.D.

(3.10) LEMMA. — *Assume* $\mathrm{Z} \subset \mathscr{X}_{n,\gamma}$. *Then:*

$$\gamma=0, \qquad n \geqq \frac{d-1}{2}.$$

Proof. — We already known that $\gamma=0$. Also:

$$2g+(2n-3)-2 \geqq \dim \mathscr{X}_{n,0} \geqq \dim \mathrm{Z} \geqq d+2g-4-i(2\mathrm{D}) \geqq d+2g-6,$$

hence:

$$2n \geqq d-1.$$

Q.E.D.

Let us now go back to $\mathscr{G}$ and $\mathscr{G}^2_d$. We set:

$$\mathscr{G}^2_d = X \cup (\bigcup_i Y_i) \cup (\bigcup_j T_j),$$

where X and the Y_i's are the components of $\mathscr{G}^2_d$ whose general point is not composed, and the T_j's are the components whose general point is composed. In particular

$$\mathscr{G} = X \cup (\bigcup_i Y_i).$$

By Fulton and Lazarsfeld's connectedness Theorem (1.2), we must have:

$$\bigcup_i \pi(Y_i \cap X) \cup \bigcup_j \pi(T_j \cap X) = \pi(\bigcup_i Y_i \cup \bigcup_j T_j) = (\bigcup_i \pi(Y_i)) \cup (\bigcup_j \pi(T_j)).$$

Let i be such that $\pi(Y_i)$ has maximal dimension. Since $\pi(Y_i)$ is irreducible, two cases are, *a priori*, possible:

(α) There is an h such that:

$$\pi(Y_i) \subset \pi(Y_h \cap X).$$

Then, by the maximality of $\pi(Y_i)$, we get:

$$\pi(Y_h) = \pi(Y_h \cap X).$$

(β) There is j such that:

$$\pi(Y_i) \subset \pi(T_j \cap X).$$

We want to show that case (β) cannot occur. We set:

$$Y = Y_i; \qquad T = T_j; \qquad Z = \pi(Y).$$

Let $(C, \mathscr{D})$ be a general point of T. By the very definition of T, the series $\mathscr{D}$ is composed with an involution:

$$\varphi: \quad C \to \Gamma.$$

We denote by n the degree of φ and by γ the genus of Γ. We know that:

$$\pi(T \cap X) \supset Z.$$

We may then apply Lemma (3.10), concluding that $\gamma = 0$ and that n equals either $d/2$ or $(d-1)/2$, depending on the parity of d. In the first case $\mathscr{D}$ is base-point-free, while in the second one it has a single base point. Thus:

$$\dim T = \dim \mathscr{G}^1_n$$

if d is even, or:

$$\dim T = \dim \mathscr{G}^1_n + 1$$

if d is odd. In both cases:

$$\dim T \leqq 2n+2g-4 \leqq d+2g-4 < 3d+g-9,$$

by our basic assumption:

$$g \leqq (3/2)d - 7/2.$$

Since, as we already recalled, every component of $\mathcal{G}_d^2$ has dimension at least equal to $3d+g-9$, we have reached a contradiction.

We are thus reduced to case (α). We first introduce a few pieces of notation, which will be used thoughout this section. Looking back at the statement of case (α), we set $Y = Y_h$. The basic property of Y is:

$$\pi(Y \cap X) = \pi(Y).$$

We also set:

$$Z = \pi(Y)$$

and denote by W a fixed component of $X \cap Y$ such that:

$$\pi(W) = Z.$$

We shall also denote by:

$$(p, |D|) \in Y,$$

$$(p, \mathcal{D}) \in W$$

general members of Y and W, respectively, and set, as usual:

$$C = \xi^{-1}(p).$$

(3.11) LEMMA. — *$\mathcal{D}$ is composed with an involution.*

Proof. — By Proposition (2.9), it suffices to show that the dimension of W is strictly larger than $g-8$. We can do much better. By Clifford's theorem (which applies to series of degree not exceeding $2g$, such as $|2D|$):

$$i(2D) = r(2D) + g - 2d \leqq g - d.$$

By Lemma (3.5), we have:

$$\dim W \geqq \dim Z \geqq 2d + g - 4 \gg g - 8$$

Q.E.D.

Now (3.10) applies and the morphism of C into $\mathbb{P}^2$ induced by $\mathcal{D}$ factors through an n-fold covering of a conic; moreover, if f stands for the degree of the fixed divisor of $\mathcal{D}$, two cases are, *a priori*, possible.

Case A: $d = 2n, f = 0$.

Case B: $d = 2n+1, f = 1$.

The inequality used in the proof of Lemma (3.10) yields:

$$d+2g-5 \geqq \dim Z \geqq d+2g-6$$

in case A, and:

$$\dim Z = d+2g-6$$

in case B.

Before going on with the proof of Proposition (2.1) we state and prove two general deformation-theoretic lemmas.

(3.12) LEMMA. — *Let* Q *be a smooth quadric surface. Denote by* L_1 *and* L_2 *lines in the two rulings of* Q, *and fix integers* $n_1 \geqq 2$, $n_2 \geqq 2$. *Then the variety of morphisms:*

$$\varphi : \quad \Gamma \to Q$$

such that Γ *is a smooth genus g curve,* φ *is birational onto its image and:*

$$\deg \varphi^* (L_i) = n_i, \qquad i=1, 2,$$

has dimension:

$$g+2n_1+2n_2-1.$$

(3.13) LEMMA. — *Let* Q *be the blow-up at the vertex of a quadric cone in* $\mathbb{P}^3$. *Denote by* L *and* E *a line of the ruling of* Q *and the exceptional divisor on* Q, *respectively. Fix an integer* $n \geqq 2$ *and a non-negative integer f. Then the variety of morphisms:*

$$\varphi : \quad \Gamma \to Q$$

such that Γ *is a smooth genus g curve,* φ *is birational onto its image and:*

$$\deg \varphi^* (L) = n,$$

$$\deg \varphi^* (E) = f.$$

has dimension:

$$g+4n+2f-1.$$

We prove both lemmas at the same time. Let $\mathscr{V}$ be the variety of morphisms in question. The proof is based on two principles. The first is that the tangent space to $\mathscr{V}$ at φ is $H^0(\Gamma, N_\varphi)$ where $N_\varphi = \varphi^*(\theta_Q)/\theta_\Gamma$, and moreover, by general deformation theory nonsense:

$$\dim \mathscr{V} \geqq h^0(\Gamma, N_\varphi) - h^1(\Gamma, N_\varphi).$$

A straightforward computation shows that the right-hand side is precisely the postulated dimension of $\mathscr{V}$. The second point is the "ramification trick" we already used in the proof of (2.9). Namely we set:

$$N_\varphi = \varphi^*(\theta_Q)/\theta_\Gamma(Z), \qquad \mathscr{K}_\varphi = \theta_\Gamma(Z)/\theta_\Gamma,$$

where Z is the ramification divisor of φ, so that there is an exact sequence:

$$0 \to \mathscr{K}_\varphi \to N_\varphi \to N'_\varphi \to 0.$$

Clearly N'_φ is a line bundle, $\mathscr{K}_\varphi$ is concentrated on Z and hence $h^1(\Gamma, N_\varphi)=h^1(\Gamma, N'_\varphi)$. Also let $T_\varphi(\mathscr{V}_{red})\subset H^0(\Gamma, N_\varphi)$ be the tangent space to $\mathscr{V}_{red}$ at the point corresponding to φ. The point is that if φ is general:

$$T_\varphi(\mathscr{V}_{red})\cap H^0(\Gamma, \mathscr{K}_\varphi)=\{0\},$$

[*cf.* [4], Lemma (1.4)]. Thus:

$$h^0(\Gamma, N'_\varphi)\geqq \dim \mathscr{V}\geqq g+1$$

in either case. Therefore $h^1(\Gamma, N_\varphi)=0$, which proves our contention.

We now rejoin the main course of the proof of Proposition (2.1) with a final series of lemmas. From now on n and f will be as defined before Lemma (3.12).

(3.14) LEMMA. — *On* C *there is only one* g^1_n.

Proof. — By Lemma (3.10) every g^1_n on C is base point free and is not composed. If there were two of them, they would provide a morphism of C into a smooth quadric Q which is **birational onto its image. By Lemma 3.12 this would show that:**

$$\dim Z\leqq g+4n-1-\dim \operatorname{Aut}(Q)=g+4n-7$$

which implies:

$$\dim Z\leqq g+2d-7$$

both in case A and case B. Because of our assumption that $d\leqq g$, this is not compatible with the known inequality:

$$\dim Z\geqq d+2g-6.$$

Q.E.D.

(3.15) LEMMA. — $\mathscr{D}$ *is complete.*

Proof. — We argue by contradiction. Suppose $\mathscr{D}$ is not complete. Choose a general g^3_d in $|\mathscr{D}|$ and let:

$$\varphi:\quad C\to \mathbb{P}^3$$

be the corresponding morphism; φ cannot be composed with an involution, for otherwise, since the moving part of $|\mathscr{D}|$ is the double of a g^1_n, its image would be a conic. On the other hand $\varphi(C)$ is contained in a quadric cone Q, cuts every line of its ruling n times and passes through the vertex of Q at most once. In any case Lemma (3.13) gives:

$$\dim Z\leqq g+4n+2f-1-\dim \operatorname{Aut}(Q)=g+4n+2f-8=g+2d-8.$$

This again contradicts the bound:

$$\dim Z\geqq d+2g-6,$$

since $d\leqq g$.

Q.E.D.

(3.16) LEMMA. — $\dim(W) \geqq 4d-11$.

Proof. — We denote by $\mathscr{G}'$ the open subset of $\mathscr{G}$ consisting of complete g_d^2's. By the preceding lemma $\mathscr{G}' \cap W \neq \emptyset$. We also consider the incidence correspondence:

$$\mathscr{G}'' \subset \mathscr{G}' \times_{\mathscr{M}} \mathscr{G}_d^1$$

consisting of couples:

$$(g_d^2,\ g_d^1 \text{ contained in the } g_d^2).$$

There are two projections:

$$\varphi:\ \mathscr{G}'' \to \mathscr{G}' \subset \mathscr{G}_d^2$$
$$\psi:\ \mathscr{G}'' \to \mathscr{G}_d^1.$$

The fibers of φ are projective 2-spaces, while ψ is injective by the very definition of $\mathscr{G}'$. Thus $\varphi^{-1}(W)$ is a component of $\varphi^{-1}(X) \cap \varphi^{-1}(Y)$ and $\psi\varphi^{-1}(W)$ is a component of $\psi\varphi^{-1}(X) \cap \psi\varphi^{-1}(Y)$.

Since $\mathscr{G}_d^1$ is smooth:

$$\dim W = \dim \psi\varphi^{-1}(W) - 2 \geqq \dim \psi\varphi^{-1}(X) + \dim \psi\varphi^{-1}(Y) - \dim \mathscr{G}_d^1 - 2$$
$$= 2(3d+g-7) - (2d+2g-5) - 2 = 4d-11$$

Q.E.D.

With this lemma we have finally reached a contradiction. In fact Lemma (3.15) tells us, in particular, that the projection from W to Z is generically $1-1$ or, at worst, could have one-dimensional fibres when $2n=d-1$. At any rate, contrasting the inequality:

$$\dim W \geqq 4d-11$$

with:

$$\dim Z \leqq d+2g-5 \quad \text{if} \quad 2n=d$$
$$\dim Z = d+2g-6 \quad \text{if} \quad 2n=d-1$$

we find, in any case:

$$d+2g-5 \geqq 4d-11,$$

that is,

$$3d \leqq 2g+6$$

or

$$\rho \leqq 0.$$

4. In the course of the proof of Theorem (1.1), we made a systematic use of the variety of g_d^2's instead of working directly with the variety of plane curves of degree d and genus g. Here we would like to explain, by means of examples, why one should except this to be technically more convenient. First of all, let us look at the irreducible plane curves of degree d and genus g. Let Γ be one such curve. It is easy to show that Γ is a smooth point of $\Sigma_{d,g}$ when all of its singular points are union of smooth branches. In fact let $f=0$ be an equation for Γ, and denote by:

$$\varphi:\ C \to \Gamma \subset \mathbb{P}^2$$

the normalization map. The important point is that N_φ has no torsion. To compute the tangent space to $\Sigma_{d,g}$ at Γ we have to look for degree d forms h and infinitesimal deformations φ_ε of φ such that:

$$(4.1) \qquad (f+\varepsilon h)(\varphi_\varepsilon) \equiv 0 \pmod{\varepsilon^2}.$$

Write:

$$\varphi_\varepsilon = \varphi + \varepsilon\psi,$$

where $\psi \pmod{\dot\varphi}$ is a section of N_φ. Then (4.1) reads:

$$f(\varphi)=0$$

$$(4.2) \qquad h(\varphi)+\Sigma \frac{\partial f}{\partial x_i}\psi_i = 0.$$

If $h \equiv 0 \pmod f$ we find that ψ is proportional to $\dot\varphi$ at smooth points of Γ. Thus ψ, as a section of N_φ, vanishes almost everywhere and hence everywhere. Therefore the natural map:

$$H^0(C, N_\varphi) \to T_\Gamma(\Sigma_{d,g})$$

sending:

$$\psi \pmod{\dot\varphi} \mapsto h \pmod f$$

is injective. On the other hand $H^0(C, N_\varphi)$ is the tangent space to the universal deformation space B of φ, which is smooth since $H^1(C, N_\varphi)$ vanishes, by a trivial degree computation. Moreover B maps in $1-1$ fashion onto a neighborhood of Γ in $\Sigma_{d,g}$, showing that $\Sigma_{d,g}$ is smooth at Γ.

In the previous case nothing is to be gained by working on $\mathcal{G}^2_d$ instead of $\Sigma_{d,g}$. Things are different if Γ has singular branches. The same degree computation as above shows that $H^1(C, N_\varphi)=0$ if there are no more then $3d$ such singular branches (counted with appropriate multiplicity), showing that B is smooth, and hence $\mathcal{G}^2_d$ is smooth at the point corresponding to φ. By contrast such a Γ is *never* a smooth point of $\Sigma_{d,g}$. To prove this, since B maps in $1-1$ fashion onto a neighborhood of Γ in $\Sigma_{d,g}$, it suffices to show that, in third case:

$$\alpha: \quad H^0(C, N_\varphi) \to T_\Gamma(\Sigma_{d,g})$$

is not injective. Here again the torsion subsheaf $\mathcal{K}_\varphi$ of N_φ, already used in the proofs of (2.9), (3.12), (3.13), comes to our help, for (4.2) shows that the kernel of the differential α is just $H^0(C, \mathcal{K}_\varphi)$.

A second instance in which one can observe a real difference between $\mathcal{G}^2_d$ and $\Sigma_{d,g}$ is the one of linear series with base points. These do not correspond to points of $\Sigma_{d,g}$; rather, their counterpart is represented by those reducible degree d plane curves in the closure of $\Sigma_{d,g}$ which are made up of an irreducible genus g curve of degree $d-f$ plus f lines. Any direct proof of the irreducibility of $\Sigma_{d,g}$ is likely to involve at least the consideration of these degenerate curves.

Now if $\mathcal{D}$ is a g^2_d on a curve C which is not composed with an involution, F its base locus, $f = \deg F$ and:

$$\varphi: \quad C \to \mathbb{P}^2$$

is the corresponding map, there are very weak conditions which insure the smoothness of $\mathscr{G}_d^2$ at $\mathscr{D}$. For example it suffices to know that the degree of the ramification divisor of φ does not exceed $3(d-f)$. By contrast, even if we limit ourself to the simplest case, namely $f=1$, the couterpart of this situation from the plane curve point of view is provided by an irreducible genus g plane curve Γ of degree $d-1$ plus a line l. This is a singular point of $\overline{\Sigma}_{d,g}$; in fact there are $d-1$ distinct ways of "approximating" $\Gamma+l$ by means of irreducible curves in $\Sigma_{d,g}$, namely we can smooth any one of the $d-1$ intersection points of Γ and l. This shows, indeed, that in the neighborhood of a general curve of the type $\Gamma+l$, $\overline{\Sigma}_{d,g}$ is made up of $d-1$ smooth branches. The situation becomes much more entagled for curves $\Gamma+l_1+\ldots+l_f$, especially when the lines $l_1,\ldots,l_f$ are not in general position. Nothing of this is visible in $\mathscr{G}_d^2$.

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