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SOME REGULARITY THEOREMS IN RIEMANNIAN GEOMETRY

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This paper consists of some variations on a theme. We begin with an artificial example. Consider the metrics given in local coordinates in $\mathbb{R}^2$ by:

$$g_1=(1+3x|x|)^2\,dx^2+[1+(x+|x|^3)^2]\,dy^2,$$

and:

$$g_2=(1+3x|x|)^2(dx^2+dy^2).$$

For each of them one may ask if there is a coordinate system (containing the origin) in which the metric is smooth. For the first metric, the change of variables $t=x+|x|^3$ reveals that:

$$g_1=dt^2+(1+t^2)\,dy^2,$$

which is certainly smooth. On the other hand, the metric g_2 cannot be made smoother in any other coordinates, as we shall see in Example 2.2. This question of smoothness of the metric and other tensors when the coordinate system is changed is the underlying theme of the paper.

We systematically investigate this local question in Section 2. The key idea is that there are natural coordinate systems in which a metric is as smooth as it can be. The first candidate for such a coordinate system that comes to mind is geodesic normal coordinates. As we shall see, this intuitive notion is *false* in general: changing to geodesic normal coordinates may involve loss of two derivatives. For optimal regularity properties one should use *harmonic coordinates*, in which each coordinate function is harmonic. These were first used by Einstein [E] in a special situation, and subsequently by Lanczos [L], who observed that they simplify the formula for the Ricci tensor. We learned of this from [FM], where harmonic coordinates were used to study the Cauchy problem for the homogeneous Einstein equation of general relativity. One should also note that in two dimensions, isothermal coordinates are harmonic, that the coordinates of a minimal immersion in $\mathbb{R}^n$ are harmonic, and the work of Greene and Wu [GW], who used global harmonic coordinates to embed open manifolds.

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In Section 3, we show that if a connection comes from a metric and if the connection is smooth, then so is the metric (this is also proved for non-Riemannian metrics). Regularity of Ricci curvature is treated in Section 4. There, we show that if $\mathrm{Ricc}(g)$ is smooth in harmonic coordinates, then so is the metric g. This is false in arbitrary coordinates as one can see from the example $g=\varphi^*(g_0)$, where g_0 is the standard flat metric and the map φ is not smooth (*see* Remark 4.6). However, if $\mathrm{Ricc}(g)$ is smooth *and invertible*, then we can prove that g must be smooth. We also discuss the cases where $\mathrm{Ricc}(g)$ is in the Hölder class $C^{k,\alpha}$ or C^{ω} (=real analytic).

These ideas are applied to Einstein metrics in Section 5. The results are: (i) Einstein metrics are real analytic in harmonic coordinates, (ii) a unique continuation theorem: if two Einstein metrics agree on an open set, then up to a diffeomorphism they are globally identical, and (iii) a local isometric embedding statement. Kähler and Kähler-Einstein metrics are briefly treated in Section 6.

Our approach here was strongly influenced by Malgrange's proof [M] of the Newlander-Nirenberg theorem. All of the regularity statements ultimately boil down to showing that some differential operator is elliptic. For simplicity, we have not stated the corresponding regularity results for Sobolev spaces; no new ideas are needed for this extension.

NOTATION. — We say that a function f defined on an open subset of $\mathbb{R}^n$ is of class $C^{k,\alpha}$ (written $f\in C^{k,\alpha}$) if all of its derivatives up to order k are continuous and if its k-th derivatives satisfy a Hölder condition with exponent α, $0<\alpha\leqq 1$. We write $f\in C^{\omega}$ if f is a real analytic function. We use standard tensor notation and write $\sqrt{g}$ for $(\det(g_{ij}))^{1/2}$. A metric $g=\sum g_{ij}\,dx^i\,dx^j$ is said to be of class $C^{k,\alpha}$ (or C^{ω}) in the coordinate chart $(x^1, \ldots, x^n)$ if the coefficients $g_{ij}\in C^{k,\alpha}$ (or C^{ω}). The analogous definition is used for any tensor to be of class $C^{k,\alpha}$.

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1. Harmonic coordinates

We recall some basic facts, some of which do not seem to appear explicitly in the literature. A coordinate chart $(x^1, \ldots, x^n)$ on a Riemannian manifold $(\mathcal{M}, g)$ is called *harmonic* if $\Delta x^j=0$ for $j=1, \ldots, n$. Our first lemma relates this to the Christoffel symbols.

LEMMA 1.1. — *In a local coordinate chart* $(x^1, \ldots, x^n)$, *let* $\Gamma^k=g^{ij}\Gamma^k_{ij}$. *A coordinate function* x^k *is harmonic if and only if* $\Gamma^k=0$. *In fact*, $\Delta x^k=-\Gamma^k$.

Proof. — By a straightforward computation:

$$\Gamma^k = -\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g}\, g^{ik}).$$

Thus, for any function u, its Laplacian is given by:

$$\Delta u = g^{ij}\frac{\partial^2 u}{\partial x^i \partial x^j} - \Gamma^j \frac{\partial u}{\partial x^j}.$$

In particular, $\Delta x^k = -\Gamma^k$.

Q.E.D.

Next, we prove the existence of harmonic coordinates.

LEMMA 1.2. — *Let the metric on a Riemannian manifold $(\mathcal{M}, g)$ be of class $C^{k,\alpha}$ (for $k \geqq 1$) (resp. C^ω) in a local coordinate chart about some point p. Then there is a neighborhood of p in which harmonic coordinates exist, these new coordinates being $C^{k+1,\alpha}$ (resp. C^ω) functions of the original coordinates. Moreover, all harmonic coordinate charts defined near p have this regularity.*

Proof. — Since:

$$\Delta u = g^{ij}\frac{\partial^2 u}{\partial x^i \partial x^j} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g}\, g^{ij})\frac{\partial u}{\partial x^j}, \tag{1.3}$$

and $g \in C^{k,\alpha}$, the coefficients of this differential operator are of class $C^{k-1,\alpha}$. By a standard result ([BJS], p. 228, § 5.4), in some neighborhood of p there is a solution $u \in C^{k+1,\alpha}$ of $\Delta u = 0$ with $u(p)$ and $\partial u/\partial x^i|_p$ prescribed. Let $y^j = u^j(x)$, $j = 1, \ldots, n$ be the solution with $u^j(p) = 0$ and $\partial u^j/\partial x^i|_p = \delta^j_i$. These functions y^j are the desired harmonic coordinates. The regularity is a consequence of the standard elliptic regularity theorems, *see* [BJS], p. 136, § 5.8 and [Mo], Thm. 5.8.6.

Q.E.D.

On a two-dimensional manifold, one can use a special type of harmonic coordinates: isothermal coordinates. To obtain them, one begins as above with one harmonic function u with grad $u(p) \neq 0$. Then let the second coordinate v be the harmonic function *conjugate* to u, so that (by definition):

$$dv = \star\, du,$$

where $\star$ is the Hodge star operator mapping one-forms to one-forms. Having already found u, then $\star\, du$ is closed (since u is harmonic), so v exists. The function v is harmonic because $\Delta v = \star\, d \star dv = -\star\, ddu = 0$. In higher dimensions, there is no known adequate generalization of the harmonic conjugate, so harmonic coordinates are all one has. Note however that in any dimension we can find harmonic coordinates such that $g_{ij}(p) = \delta_{ij}$ by composing with an auxiliary linear transformation.

The following corollary gives the regularity of various tensors in the change to harmonic coordinates.

COROLLARY 1.4. — *Let the metric $g \in C^{k, \alpha}$, $k \geqq 1$ in the coordinate chart $(x^1, \ldots, x^n)$ and let $(y^1, \ldots, y^n)$ be a harmonic coordinate chart. If a tensor $\mathcal{T}$ is of class $C^{l, \beta}$ with $l \geqq k$ and $\beta \geqq \alpha$ in the original coordinates, then it is of class at least $C^{k, \alpha}$ in harmonic coordinates. If both g and $\mathcal{T}$ are in C^ω in the original coordinates then $\mathcal{T}$ will be in C^ω in harmonic coordinates.*

Proof. — This is clear from the last part of Lemma 1.2 and the fact that the expression for $\mathcal{T}$ in the new coordinates involves at most the first derivatives of those coordinates.

Q.E.D.

2. Regularity of metrics in various coordinates

We prove that a metric has optimal regularity in any harmonic coordinate chart, i.e., that it is no smoother in any other coordinates. By an example, we see that a metric may have less than optimal regularity in geodesic normal coordinates.

THEOREM 2.1. — *If a metric $g \in C^{k, \alpha}$, $1 \leqq k \leqq \infty$ (or C^ω) in some coordinate chart, then it is also of class $C^{k, \alpha}$ (or C^ω) in harmonic coordinates, while it is of at least class $C^{k-2, \alpha}$ (or C^ω) in geodesic normal coordinates.*

Proof. — (*a*) *harmonic coordinates.* This is a special case of Corollary 1.4 where the tensor $\mathcal{T}$ is the metric tensor itself;

(*b*) *geodesic normal coordinates.* We show that the (isometric) map f from geodesic coordinates is of class $C^{k-1, \alpha}$. The components of f clearly satisfy the geodesic ordinary differential equations:

$$\frac{d^2 f^s}{dr^2} + \Gamma^s_{ij} \frac{df^i}{dr} \frac{df^j}{dr} = 0,$$

where r is arc length along the geodesics from the origin. The Christoffel symbols involve the first derivatives of the metric, so if the metric is in $C^{k, \alpha}$ then Γ^k_{ij} is in $C^{k-1, \alpha}$. Hence, f is a $C^{k+1, \alpha}$ function of r; however the "angular" variables arise in the equation only as parameters, so there is no gain of differentiability of f in these variables. Thus, all one can assert is that f is of class $C^{k-1, \alpha}$ in these angular variables.

Q.E.D.

Example 2.2. — If $0 < p(x, y) \in C^{k, \alpha}$ in an open set, then the metric $g = p(x, y)(dx^2 + dy^2)$ is also of class $C^{k, \alpha}$ in that set. Moreover, its differentiability cannot be increased by changing coordinates. To see this, simply note that isothermal coordinates are harmonic. This proves the assertion made in the Introduction concerning the metric g_2.

Example 2.3. — If $g \in C^{k, \alpha}$ in some coordinates, then all we were able to assert in Theorem 2.1 is that $g \in C^{k-2, \alpha}$ in geodesic normal coordinates. This example shows that one can do no better, in general. Near the origin in $\mathbb{R}^2$, consider the metric:

$$g = (1 + |y|^{k+\alpha})(dx^2 + dy^2),$$

where $0<\alpha<1$. Clearly $g \in C^{k,\alpha}$ in these (harmonic) coordinates. We claim that g is not of class C^{k-1} in geodesic normal coordinates. To prove this, we recall that Hartman [H] has shown that the map f from these coordinates to normal coordinates is in $C^{k-1,\alpha}$ but *not* in C^k (Hartman's proof, which only addresses the case $k=2$, can be directly applied for $k>2$). But a result of Calabi-Hartman [CH] says that any isometry between metrics of class C^{k-1} must itself be of class C^k. Therefore, g could not be of class C^{k-1} in geodesic normal coordinates.

3. Regularity of metrics with smooth connections

The canonical torsion-free connection Γ of a metric g involves the first derivatives of the metric. Thus, if $\Gamma \in C^k$ then the most one can expect is that $g \in C^{k+1}$. This is essentially what we will prove.

LEMMA 3.1. — *Let g be a C^1 metric, and let T be the operator that maps metrics to their connections, so $T(g)=\Gamma$. Then, at g, T is an overdetermined elliptic partial differential operator.*

Proof. — In local coordinates, the equation $T(g)=\Gamma$ can be rewritten as the linear equation:

$$A(g) \equiv \frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} - 2\, g_{ks}\, \Gamma^s_{ij} = 0.$$

The principal symbol of A, $\sigma_A(\xi)$, for any vector ξ is:

$$\sigma_A(\xi)\, h = (\sqrt{-1})(h_{jk}\,\xi_i + h_{ik}\,\xi_j - h_{ij}\,\xi_k), \tag{3.2}$$

where h_{ij} is any symmetric matrix. Overdetermined elliptic means that the map $\sigma_A(\xi)$ is injective for any real $\xi \neq 0$. Thus, given that $\sigma_A(\xi)\, h=0$, we must show that $h=0$. We sum (3.2) over the terms where $i=j$ to obtain:

$$\sum_i 2\, h_{ik}\, \xi_i = \sum_i h_{ii}\, \xi_k. \tag{3.3}$$

Also, multiply (3.2) by ξ_i and sum over i. Then use (3.3) to conclude that:

$$0 = h_{jk}\,|\xi|^2 + \sum_i (h_{ik}\,\xi_i\,\xi_j - h_{ij}\,\xi_i\,\xi_k) = h_{jk}\,|\xi|^2,$$

where $|\xi|^2 = \sum \xi_j^2$. Therefore $h=0$, i.e., $\sigma_A(\xi)$ is injective if $\xi \neq 0$.

Q.E.D.

THEOREM 3.4. — *Let Γ be the connection of a C^2 metric g. If in some local coordinates $\Gamma (=\Gamma^l_{ij})$ is of class $C^{k,\alpha}$ for some $k \geqq 1$ (resp. C^ω), then in these coordinates the metric g is of class $C^{k+1,\alpha}$ (resp. C^ω).*

Proof. — Let A^* be the formal L_2 adjoint (using the Euclidean inner product, say) of the operator A above. Then g also satisfies the second-order linear system:

$$A^* A\, g = 0.$$

But $\sigma_{A^*}(\xi)=[\sigma_A(\xi)]^*$, so that:

$$\sigma_{A^*A}(\xi)=[\sigma_A(\xi)]^*\,\sigma_A(\xi),$$

which is bijective for all $\xi\neq 0$ since $\sigma_A(\xi)$ is injective. Therefore $A^*A\,g=0$ is a (determined) elliptic system. Since the coefficients of A^*A involve first derivatives of Γ, they are of class $C^{k-1,\alpha}$ (or C^ω). Therefore, by the usual elliptic regularity ([Mo], Thms. 6.8.1 and 6.7.6) we obtain the asserted regularity of g.

Q.E.D.

In this proof, we used a simple general device to obtain regularity for overdetermined elliptic systems from the corresponding results for systems that are elliptic in the usual sense. Because of this device, we needed to assume that $g\in C^2$ and $\Gamma\in C^{k,\alpha}$, $k\geqq 1$, although it would have been more natural if this were relaxed to just assuming that $g\in C^1$ and $k\geqq 0$. This extension is also true because the equation $A^*A\,g=0$ is linear. We can thus use the regularity for weak solutions in [Mo], Thm. 6.4.3. The details are left to the interested reader (Hint: in the notation of [Mo], let $A^*A\,g=0$ be (6.4.5) with $f=0$, $t_k=2$, $s_j=0$, $m_j=1$, $h_0=-1$, and $h=0$).

Remark 3.5. — Nowhere in this section did we use the fact that g was a Riemannian metric. Thus, Theorem 3.4 is true for nonsingular metrics of any signature, for instance, Lorentz metrics.

4. Regularity of metrics with smooth Ricci tensors

Our first task is to find the formula for $\operatorname{Ricc}(g)$ in harmonic coordinates.

LEMMA 4.1 (Lanczos). — *Let $\Gamma^k=g^{ij}\Gamma^k_{ij}$. Then:*

$$\operatorname{Ricc}(g)_{ij}=-\frac{1}{2}g^{rs}\frac{\partial^2 g_{ij}}{\partial x^r\,\partial x^s}+\frac{1}{2}\left(g_{ri}\frac{\partial\Gamma^r}{\partial x^j}+g_{rj}\frac{\partial\Gamma^r}{\partial x^i}\right)+\dots \tag{4.2}$$

where the dots indicate lower-order terms involving at most one derivative of the metric. In particular, in harmonic coordinates:

$$\operatorname{Ricc}(g)_{ij}=-\frac{1}{2}g^{rs}\frac{\partial^2 g_{ij}}{\partial x^r\,\partial x^s}+\dots \tag{4.3}$$

Proof. — From the standard formula for $\operatorname{Ricc}(g)$ in terms of the Christoffel symbols one has:

$$\operatorname{Ricc}(g)_{ij}=-\frac{1}{2}g^{rs}\frac{\partial^2 g_{ij}}{\partial x^r\,\partial x^s}+\frac{1}{2}g^{rs}\left[\frac{\partial^2 g_{ri}}{\partial x^s\,\partial x^j}+\frac{\partial^2 g_{rj}}{\partial x^s\,\partial x^i}-\frac{\partial^2 g_{rs}}{\partial x^i\,\partial x^j}\right]+\dots \tag{4.4}$$

Also, by a direct calculation:

$$g^{rs}\frac{\partial^2 g_{ri}}{\partial x^s\,\partial x^j}=g_{ri}\frac{\partial\Gamma^r}{\partial x^j}+\frac{1}{2}g^{rs}\frac{\partial^2 g_{rs}}{\partial x^i\,\partial x^j}+\dots$$

Substitution of this into (4.4) yields (4.2). The formula (4.3) is then an immediate consequence of Lemma 1.1.

Q.E.D.

The next theorem considers the question: "If Ricc(g) is smooth, then is g smooth?" This problem is more subtle than those considered up to now in this paper. Following the statement of the theorem we make some remarks containing examples which help clarify the picture. Normal coordinates are briefly discussed after the proof of the theorem.

THEOREM 4.5. — *Let $g \in C^2$ be a Riemannian metric with Ricci tensor $\mathscr{R}$ in some neighborhood of a point p:*

(*a*) *if $\mathscr{R}(p)$ is invertible and if in some local coordinate chart $\mathscr{R} \in C^{k, \alpha}$ (or C^ω) in a neighborhood of p, then in these coordinates $g \in C^{k, \alpha}$ (or C^ω);*

(*b*) *if in harmonic coordinates $\mathscr{R} \in C^{k, \alpha}$, $k \geqq 0$ (or C^ω) near p, then in these coordinates $g \in C^{k+2, \alpha}$ (or C^ω);*

(*c*) *if in some coordinates $g \in C^{k, \alpha} (k \geqq 2)$ and $\mathscr{R} \in C^{l, \alpha} (l \geqq k)$, then $g \in C^{k+2, \alpha}$ in harmonic coordinates;*

(*d*) *if, in addition to the hypotheses in part (c), $\mathscr{R}$ is invertible, then $g \in C^{l+2, \alpha}$ in harmonic coordinates.*

Remark 4.6. — The assumption in part (*a*) that $\mathscr{R}(p)$ is invertible is clearly needed, as the following example shows: let g be the standard flat metric near p and let φ be a diffeomorphism (keeping p fixed) of class C^3. Then Ricc$(\varphi^*(g)) \equiv 0 \in C^\omega$ near p, but the metric $\varphi^*(g)$ is only twice differentiable. By taking products like $\mathscr{M}^n = T^2 \times \mathscr{N}^{n-2}$ (here T^2 is the flat torus) with φ the identity on $\mathscr{N}$, one obtains more complicated examples. Note that this example also shows that the metric of a smooth Ricci tensor need not be smooth in arbitrary coordinates.

Remark 4.7. — In part (*a*) one is tempted to try to prove that, in fact, $g \in C^{k+2, \alpha}$. This must fail. To see this, let g be an Einstein metric, say the canonical metric on S^n, and let $\varphi : S^n \to S^n$ be a $C^{k+1, \alpha}$ diffeomorphism. Then $g_1 = \varphi^*(g) \in C^{k, \alpha}$, so Ricc$(\varphi^*(g)) \in C^{k, \alpha}$ because Ricc$(\varphi^*(g)) = c\, \varphi^*(g)$ for some constant $c \neq 0$. This gives an example where in some local coordinates Ricc$(g_1) \in C^{k, \alpha}$ with $g_1 \in C^{k, \alpha}$, but such that the regularity of g_1 cannot be improved in these coordinates.

Remark 4.8. — If Riem(g) is the full (sectional) curvature tensor, then its smoothness is reflected in that of Ricc(g). Thus, Theorem 4.5 also shows how regularity of Riem(g) implies regularity of the metric. Note, too, that the obvious modification of Remarks 4.6 and 4.7 apply to Riem(g). Thus, in some local coordinates one can have Riem(g) smooth – say identically zero – but g is not smooth. For the analog of Remark 4.7 one uses constant sectional curvature metrics instead of Einstein metrics.

Proof of Theorem 4.5. — (*a*) Since g is a solution of Ricc$(g) = \mathscr{R}$, then the Bianchi identity must hold:

$$0 = \text{Bian}(g, \mathscr{R}) = \mathscr{R}^i{}_{j;i} - \frac{1}{2} \mathscr{R}^i{}_{i;j}.$$

Consequently (as was first observed in [D1] and [D2]), since $\mathscr{R}(p)$ is invertible then g and $\mathscr{R}$ must also satisfy:

$$\text{Ricc}(g)+\text{div}^*[\mathscr{R}^{-1}\,\text{Bian}(g,\mathscr{R})]-\mathscr{R}=0. \tag{4.9}$$

Here, we define $\text{div}^*(v)=(v_{i;j}+v_{j;i})/2$ for a covector field v. Of course, for a metric g with $\text{Ricc}(g)=\mathscr{R}$, equation (4.9) is obvious because $\text{Bian}(g,\mathscr{R})$ is identically zero. If one writes (4.9) in local coordinates, then one finds that it is of the form:

$$-\frac{1}{2}g^{rs}\frac{\partial^2 g_{ij}}{\partial x^r\,\partial x^s}+\text{H}(g,\mathscr{R})=0, \tag{4.9)'$$

where H is a real analytic function of its variables and involves at most *first* order partial derivatives of g and second order partial derivatives of $\mathscr{R}$. The virtue of this more complicated (4.9) becomes evident when written out as (4.9)′, namely, it is an elliptic differential operator and is uncoupled in the second order derivatives of g. To be even more specific, from (4.9)′ one observes that the principal part of the linearization of (4.9) (varying g, keeping $\mathscr{R}$ fixed) is simply half the Laplacian. The conclusion now follows by elliptic regularity.

(*b*) By Lemma 4.1, Equation (4.3), we see that $\text{Ricc}(g)$ is elliptic in harmonic coordinates, and the result follows.

(*c*) By Corollary 1.4, applied where $\mathscr{T}$ there is the Ricci tensor, we find that $\mathscr{R}\in C^{k,\alpha}$ in harmonic coordinates. One now uses part (*b*) to complete the proof.

(*d*) Since $\text{Ricc}(g)$ is invertible, then by part (*a*), $g\in C^{l,\alpha}$ in the original coordinates. By part (*c*), we get that $g\in C^{l+2,\alpha}$ in harmonic coordinates.

Q.E.D.

Remark 4.10. — The analog of Theorem 4.5 (*b*) for geodesic normal coordinates is false, as one can see from the following example. The coordinates (r,θ) for the metric:

$$g=dr^2+(1+r^2\,\Psi(r,\theta))\,r^2\,d\theta^2, \tag{4.11}$$

on $\mathbb{R}^2$ are geodesic polar coordinates. In geodesic normal coordinates $x=r\cos\theta$ and $y=r\sin\theta$, this metric becomes:

$$g=(1+y^2\,\Psi)\,dx^2-2xy\,\Psi\,dx\,dy+(1+x^2\,\Psi)\,dy^2. \tag{4.11)'$$

If we set $\Phi^2=(1+r^2\,\Psi)\,r^2$, then it is classical that the Gauss curvature $\mathscr{K}(r,\theta)$ of the metric g satisfies Jacobi's equation:

$$\Phi''+\mathscr{K}\,\Phi=0, \tag{4.12}$$

where the primes denote differentiations with respect to r. In particular, if:

$$\Psi=|y|^{\lambda}+o(r^{\lambda}), \tag{4.13}$$

near the origin, then:

$$\mathscr{K}=-\frac{1}{2}(\lambda^2+5\lambda+6)\,|y|^{\lambda}+o(r^{\lambda}). \tag{4.14}$$

If $\lambda = k + \alpha$ with k an integer and $0 < \alpha < 1$, then the function Ψ in (4.13) is clearly of class $C^{k,\alpha}$ near the origin, but is not smoother than $C^{k,\alpha}$. From (4.11)′, we see that the coefficients of the metric g are thus of at least class $C^{k,\alpha}$, and the coefficient of dy^2 is of class exactly $C^{k,\alpha}$. Since the function $\mathscr{K}$ in (4.14) is also of class $C^{k,\alpha}$, we have $\mathrm{Ricc}(g) = 2\mathscr{K}\, g$ is in class $C^{k,\alpha}$ as well. Thus, we have an example of a metric for which both g and $\mathrm{Ricc}(g)$ have the *same* degree of differentiability in geodesic normal coordinates.

It would be interesting to further clarify the relation between regularity of $\mathrm{Ricc}(g)$ and of the metric g in normal coordinates. In view of Theorem 4.5 (a), the unresolved situation is when $\mathrm{Ricc}(g)$ is not invertible. For example, if $\mathrm{Ricc}(g) \in C^\infty$ in normal coordinates, is $g \in C^\infty$? (In two dimensions the answer is "yes" by (4.11) and (4.12) along with the proof of Lemma 4.4 in [KW].) If $\mathrm{Ricc}(g) \in C^\omega$ is $g \in C^\omega$?

Note that these latter questions are not addressed by Theorems 2.1 and 4.5, since we do not know that $\mathrm{Ricc}(g) \in C^\infty$ in some coordinates implies that $g \in C^\infty$ in harmonic coordinates. Our guess is that this is not necessarily the case, although (again) by 4.5 (a), one need only consider the case where $\mathrm{Ricc}(g)$ is not invertible at the point in question. One should first resolve the case where $\dim \mathscr{M} = 2$.

As is clear from Remarks 4.6 and 4.7, in any discussion of regularity one must control the group of diffeomorphisms, which acts as a "gauge group". In 4.5 (b) this was evident since we explicitly restricted our attention to harmonic coordinates. In 4.5 (a), control was obtained by using the Bianchi identity to obtain (4.9). The relation between the Bianchi identity and the group of diffeomorphisms was made more specific in [K]. In fact, both the present paper and the results in [D1] and [D2] were the main motivation to seek the proof of the Bianchi identites contained in [K]. Note also that invertibility of $\mathrm{Ricc}(g)$ is a key issue in [D1] and [D2] where local existence and non-existence of metrics with prescribed Ricci curvature is discussed.

Since the equation $\mathrm{Ricc}(g) = \mathscr{R}$ is elliptic in harmonic coordinates, one might try to use them to solve for the metric g given a Ricci candidate $\mathscr{R}$. Now, one can always solve $\mathrm{Ricc}^h(g) = \mathscr{R}$ locally, where Ricc^h is the expression (not tensorial) for the Ricci tensor in harmonic coordinates, as indicated by equation (4.3). The catch is that the given coordinates may not be harmonic for the metric obtained. Instead one must solve the overdetermined elliptic system:

$$\mathrm{Ricc}^h(g) = \mathscr{R}, \qquad \Gamma^1(g) = 0, \qquad \Gamma^2(g) = 0, \qquad \ldots, \qquad \Gamma^n(g) = 0.$$

Presumably this can be treated as in [D2], but because of the counterexample in [D1] to local existence if $\mathscr{R}$ is not invertible, somewhere one will have to invoke the invertibility of $\mathscr{R}$.

5. Einstein metrics

Throughout this section we assume $\dim \mathscr{M} \geqq 3$. Then a (Riemannian) Einstein metric satisfies:

$$\mathrm{Ricc}(g) = c\,.\,g, \tag{5.1}$$

for some constant c. Let φ be a diffeomorphism of $\mathcal{M}$. If g is Einstein, then so is $\varphi^*(g)$; so that by varying the differentiability of φ one sees that in various coordinates, Einstein metrics can have various degrees of smoothness. By using "natural" coordinates, one can do very well indeed.

THEOREM 5.2. — *Let $(\mathcal{M}, g)$ be a connected Einstein manifold of class* C^2 *with* $\dim \mathcal{M} \geqq 3$. *Then g is real analytic in harmonic and geodesic normal coordinates.*

Proof. — By Lemma 4.1, Equation (5.1) is quasi-linear elliptic in harmonic coordinates. Since (5.1) is an analytic function of all its dependent and independent variables, then all of its solutions are real analytic in harmonic coordinates ([Mo], Thm. 6.7.6). Analyticity in geodesic normal coordinates now follows from Theorem 2.1.

Q.E.D.

An immediate corollary concerns local isometric embedding.

COROLLARY 5.3. — *An Einstein manifold $(\mathcal{M}, g)$ of class* C^2 *with* $\dim \mathcal{M} = n \geqq 3$ *is locally isometrically embeddable in* $\mathbb{R}^{n(n+1)/2}$.

Proof. — By Theorem 5.2, we may pick local coordinates in which g is real analytic. The assertion now follows from the Cartan-Janet theorem ([J], [C], also [S], p. 230), which states that any real analytic metric can be locally isometrically embedded in $\mathbb{R}^{n(n+1)/2}$.

Q.E.D.

It would be interesting to determine the optimal dimension for local isometric embeddings of Einstein manifolds.

Our final theorem of this section concerns the unique continuation of Einstein metrics.

THEOREM 5.4. — *Let $\mathcal{M}$ be a simply connected manifold and let g_1 and g_2 be Einstein metrics on $\mathcal{M}$. If $g_1 = g_2$ (as tensor fields) on some open set, then up to a diffeomorphism, $g_1 = g_2$ on all of $\mathcal{M}$. In other words there is a diffeomorphism $f : \mathcal{M} \to \mathcal{M}$ such that $g_1 = f^*(g_2)$.*

This is a consequence of a result of Myers ([My], Thm. 3; *see* also [KN], Cor. 6.4, p. 256), which applies to analytic metrics.

6. Kähler manifolds and their Ricci curvature

It is straightforward to apply the results of the previous sections to a Kähler manifold with metric $g_{\alpha\bar{\beta}}$. Since in this case the Laplacian is:

$$\Delta u = \sum_{\alpha,\,\beta} g^{\alpha\bar{\beta}} \frac{\partial^2 u}{\partial z^\alpha \partial \bar{z}^\beta},$$

one immediately observes that *the coordinate functions in a local chart* $(z^1, \ldots, z^n, \bar{z}^1, \ldots, \bar{z}^n)$ *are always harmonic*. In other words, the corresponding real coordinates x^α, y^α given by $z^\alpha = x^\alpha + i y^\alpha$ are harmonic. Therefore we can apply Theorems 4.5 and 5.2 above to conclude the following:

THEOREM 6.1. — *Let g be a Kähler metric of class C^2 with Ricci tensor* $\mathrm{Ricc}(g)$;

(*a*) *if* $\mathrm{Ricc}(g)$ *is in* $C^{k,\alpha}$, *then g is in* $C^{k+2,\alpha}$;

(*b*) *if* $\mathrm{Ricc}(g)$ *is analytic in* $(z^1,\ldots,z^n,\bar{z}^1,\ldots,\bar{z}^n)$, *then so is* g;

(*c*) *A* C^2 *Kähler-Einstein metric is analytic in the coordinate variables* $(z^1,\ldots,z^n,\bar{z}^1,\ldots,\bar{z}^n)$.

Note that if the Kähler metric is analytic in $(z^1,\ldots,z^n)$—so it does not involve $\bar{z}^1,\ldots,\bar{z}^n$—then by the usual formulas it is flat. Analyticity of a metric in the variables $(z^1,\ldots,z^n,\bar{z}^1,\ldots,\bar{z}^n)$ is equivalent to the real and imaginary parts of $g_{\alpha\bar{\beta}}$ being real analytic in $(x^1,\ldots,x^n,y^1,\ldots,y^n)$.

One can give a direct proof of Theorem 6.1, without resorting to harmonic coordinates. The ingredients are:

1. Locally:

$$g_{\alpha\bar{\beta}}=\frac{\partial^2 f}{\partial z^\alpha\,\partial\bar{z}^\beta},$$

for some real-valued function f. This is an overdetermined elliptic system for f.

2. Therefore:

$$\mathrm{Ricc}(g)=-\frac{\partial^2 \log\det g}{\partial z^\alpha\,\partial\bar{z}^\beta},$$

can be written as a fourth-order nonlinear equation for the function f. Since this equation is also overdetermined elliptic, one can use the regularity results for elliptic equations to prove Theorem 6.1.

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