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QUASI-ELLIPTIC SURFACES IN CHARACTERISTIC THREE

BY WILLIAM E. LANG

Introduction

This paper deals with quasi-elliptic surfaces over fields of characteristic three. Quasi-elliptic surfaces reflect almost all of the known pathologies of algebraic geometry in characteristic p, such as the failure of Bertini's theorem, the Hodge theorem, and the Kodaira vanishing theorem. On the other hand, they can be described very explicitly, and therefore are useful in testing conjectures.

The first three sections are devoted to the classification of quasi-elliptic surfaces in characteristic three. In the first section, we classify Jacobian quasi-elliptic surfaces, and show that they are essentially determined by an exact rational differential form on the base curve. We classify degenerate fibres in Jacobian quasi-elliptic pencils and prove a formula for the arithmetic genus in terms of the degenerate fibres on X by counting zeroes of the differential form, thus generalizing a result of Miyanishi [15]. Section 1 C gives the relationship between our theory and Raynaud's counterexamples to the Kodaira vanishing theorem.

Sections 2 and 3 deal with the classification of non-Jacobian surfaces. Any non-Jacobian surface is a twisted form of a Jacobian surface, and one may use either etale cohomology or naive computation to compute the group of twisted forms of a fixed Jacobian surface. We have mingled these two approaches here. From the cohomological point of view, the most important result is exact sequence (2) in Section 3, which gives a two-step resolution by vector groups of the group scheme associated to a quasi-elliptic surface, and thus enables us to reduce the computation of etale cohomology to computation of Zariski sheaf cohomology. (Over a field, this was noticed by Russell [27].) Following the program of Ogg-Saferevic-Artin, we compute the group of locally trivial twisted forms in Theorem 3.1, the local group at each point in Theorem 2.1, and the obstruction to finding twisted forms with prescribed multiple fibres at the end of Section 3 A.

In section 2 B, we state an important conjecture relating the arithmetic theory of multiple fibres to the geometric theory of Bombieri-Mumford. (Such theorems would be important in the elliptic case also.) A special case is applied to a question of Zariski on unirational surfaces.

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As an application of our general theory, we redo the classification of quasi-hyperelliptic surfaces given in Bombieri-Mumford III in Section 3 B.

In Section 4, we compute the cohomology of the tangent and cotangent bundles of quasi-hyperelliptic surfaces, and obtain some partial results for Raynaud surfaces. We show that the torsion numbers of crystalline cohomology of hyperelliptic surfaces and quasi-hyperelliptic surfaces in characteristic $\neq 2$ are the same as those of the integral cohomology of analogous surfaces over the complex numbers. Combining these results with those of Illusie [10] for Enriques surfaces and Rudakov-Saferevic [26] for K 3 surfaces, we see that the torsion of crystalline cohomology is well-behaved for all surfaces of Kodaira dimension 0 (with the possible exception of quasi-hyperelliptic surfaces in characteristic two). We also see that certain Raynaud surfaces are pathological with respect to crystalline cohomology.

We conclude this introduction by mentioning two open problems suggested by our work.

(1) what is the relation between the moduli of hyperelliptic and quasi hyperelliptic surfaces in characteristic three and the reduction mod 3 of $X_0(3)$? And how are the pathologies in the cohomology of the tangent bundle accounted for deformation-theoretically ? (Some fragmentary results were obtained in [13], but we are far from a complete solution);

(2) is there a scheme structure on the group of locally trivial principal homogeneous spaces of an elliptic or quasi-elliptic surface which would account for "missing" non-Jacobian hyperelliptic or quasi-hyperelliptic surfaces ?

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1. Weierstrass Models

A. Construction of the Weierstrass model. — Let X be a smooth complete surface over an algebraically closed field k. X is said to be elliptic if there is a morphism $f: X \to C$, where C is a smooth curve, such that the general fibre is a smooth curve of arithmetic genus 1. X is said to be quasi-elliptic if there is a map from X to a smooth curve such that the general fibre is geometrically reduced and irreducible of arithmetic genus 1 and is *not* smooth. Note that X may be both elliptic and quasi-elliptic. The map f is called an elliptic (or quasi-elliptic) fibration.

PROPOSITION 1.1. — *The general fibre of a quasi-elliptic fibration has only one ordinary cusp.*

Proof. — Let $X_g/k(C)$ be the generic fibre. Since X_g has arithmetic genus 1, it has either exactly one node or exactly one ordinary cusp. However, X_g must be a regular scheme (*see* Zariski [37], paper 48 and Mumford's introduction) and hence analytically irreducible. Hence the singularity cannot be a node.

PROPOSITION 1.2. — *Quasi-elliptic surfaces exist only over fields of characteristic two and three.*

Proof. — *See* Bombieri-Mumford III.

We want to generalize the Weierstrass theory of elliptic curves to elliptic and quasi-elliptic surfaces in characteristic 3. Our treatment is quite similar to Mumford-Suominen [19]. (*See* also Deligne [5].) We give the construction in some detail in order to fix notation.

Let C be a scheme (almost always a smooth complete curve over k, or a deformation of such a curve). Let $f: X \to C$ be a flat and proper morphism such that the geometric fibres are reduced and irreducible, all fibres have arithmetic genus 1, and such that there is a section $s: C \to X$ such that $fs = 1_C$, and such that S, the image of s, is contained in the smooth points of f.

LEMMA 1.1. — (*a*) *The canonical map* $0_C \to f_* 0_X$ *is an isomorphism.*
(*b*) $f_* 0_X(nS)$ *is locally free of rank* n *for* $n > 0$.
(*c*) $R^1 f_* 0_X(nS) = 0$ *for* $n > 0$ *and is locally free of rank* 1 *for* $n = 0$.
(*d*) $R^i f_* 0_X(nS) = 0$ *for* $i > 1$ *and all integers* n.
(*e*) *The natural inclusion* $0_C \to f_* 0_X(S)$ *is an isomorphism.*
(*f*) $R^1 f_* 0_X \simeq 0_X(S)/0_X$ (*both sides considered as sheaves on* C).

Proof. — *See* Mumford-Suominen [19].

The invertible sheaf $R^1 f_* 0_X$ will play an extremely important role throughout and will always be denoted by $\mathscr{L}$. Note that Lemma 1.1 (*f*) implies that $\mathscr{L}$ is isomorphic to the normal bundle of any section of f, with values in the smooth part.

LEMMA 1.2. — *We have an exact sequence*

$$0 \to f_* 0_X((n-1)S) \to f_* 0_X(nS) \to \mathscr{L}^n \to 0$$

for $n > 1$.

Proof. — Use Lemma 1.1 and the observation that

$$f_*(0_X(nS)/0_X((n-1)S) = s^* 0_X(nS) \simeq \mathscr{L}^n.$$

Now we construct the Weierstrass model. Pick an affine open cover $\{U_\alpha\}$ of C, such that $\mathscr{L} \mid U_\alpha$ is free, with basis t_α. Then t_α^n is a basis for $\mathscr{L}^n \mid U_\alpha$ for all n. Let $V = f_* 0_X(2S)$, and choose $x \in \Gamma(U_\alpha, V)$ such that x_α projects to t_α^2 in the exact sequence

$0 \to 0_C \to V \to \mathscr{L}^2 \to 0$. Pick $y_\alpha \in \Gamma(U_\alpha, f_* 0_X(3S))$ such that y_α projects to t_α^3 in the sequence $0 \to V \to f_* 0_X(3S) \to \mathscr{L}^3 \to 0$. Then 1, x_α, y_α form a basis for $f_* 0_X(3S) | U_\alpha$. Consider the following sections of $f_* 0_X(6S) | U_\alpha$: 1, x_α, y_α, x_α^2, x_α, y_α, x_α^3. By considering leading terms, we see that these sections are a basis of $f_* 0_X(6S) | U_\alpha$. Since y_α^2 is also a section of this sheaf, we get a relation

$$y_\alpha^2 + a_1 x_\alpha y_\alpha + a_3 y_\alpha = a_0 x_\alpha^3 + a_2 x_\alpha^2 + a_4 x_\alpha + a_6,$$

with $a_i \in \Gamma(U_\alpha, 0_C)$. The a_i are uniquely determined by the choice of x_α and y_α; furthermore, by checking leading parts, we find that $a_0 = 1$. If 2 is invertible, we may replace y_α by $y_\alpha + 1/2(a_1 x_\alpha + a_3)$ and obtain a new equation (we abuse notation by retaining the same symbols) with $a_1 = a_3 = 0$, The homogenized equation

$$(1) \qquad y_\alpha^2 z_\alpha = x_\alpha^3 + a_2 x_\alpha^2 z + a_4 x_\alpha z_\alpha^2 + a_6 z_\alpha^3$$

defines a subscheme of $P^2 \times U_\alpha$, and it is easy to see that $0_X(3S)$ is relatively very ample and that $f^{-1}(U_\alpha)$ is isomorphic to the subscheme defined by (1). (1) is called the local Weierstrass equation.

The next step is to patch these schemes together, that is, to see how the Weierstrass equation changes when we move from U_α to U_β. This computation has been done by Tate in the "Formulaire" (Tate [35], [36]; Deligne [5]). Retracing our steps, we see that the changes of coordinates are forced to be

$$(2) \qquad \begin{cases} t_\alpha = u_{\alpha\beta} t_\beta, \\ x_\alpha = u_{\alpha\beta}^2 x_\beta + r_{\alpha\beta}, \\ y_\alpha = u_{\alpha\beta}^3 y_\beta + s_{\alpha\beta} u_{\alpha\beta}^2 x_\beta + t_{\alpha\beta}, \end{cases}$$

where $u_{\alpha\beta} \in \Gamma(U_\alpha \cap U_\beta, 0_C^*)$ and $r_{\alpha\beta}$, $s_{\alpha\beta}$, $t_{\alpha\beta} \in \Gamma(U_\alpha \cap U_\beta, 0_C)$. Let the Weierstrass equations on U_α and U_β be respectively

$$y_\alpha^2 = x_\alpha^3 + a_2 x_\alpha^2 + a_4 x_\alpha + a_6,$$
$$y_\beta^2 = x_\beta^3 + a'_2 x_\beta^2 + a'_4 x_\beta + a'_6.$$

Since $a_1 = a'_1 = a_3 = a'_3 = 0$, the Formulaire shows that $s_{\alpha\beta} = t_{\alpha\beta} = 0$. Putting $a_0 = a'_0 = 1$, we find that the a_i are related by the matrix equation (we suppress the subscripts α, β):

$$(3) \qquad \begin{bmatrix} a_6 \\ a_4 \\ a_2 \\ a_0 \end{bmatrix} = \begin{bmatrix} u^6 & -ru^4 & r^2 u^2 & -r^3 \\ 0 & u^4 & -2ru^2 & 3r^2 \\ 0 & 0 & u^2 & -3r \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} a'_6 \\ a'_4 \\ a'_2 \\ a'_0 \end{bmatrix}.$$

Thus, the a_i give a section of a certain rank 4 vector bundle. But the equations (2) imply that the transition matrices for $\mathscr{L}$ are $[u_{\alpha\beta}^{-1}]$, for V are

$$\begin{bmatrix} 1 & -u_{\alpha\beta}^{-2} r_{\alpha\beta} \\ 0 & u_{\alpha\beta}^{-2} \end{bmatrix}$$

and for $f_* 0_X(3S)$ are

$$\begin{bmatrix} 1 & -u_{\alpha\beta}^{-2} r_{\alpha\beta} & 0 \\ 0 & u_{\alpha\beta}^{-2} & 0 \\ 0 & 0 & u_{\alpha\beta}^{-3} \end{bmatrix}.$$

We see that $f_* 0_X(3S) \simeq V \oplus \mathscr{L}^3$.

[We digress to explain our conventions for the transition matrices of a vector bundle W. We let φ_α be an isomorphism from $W|U_\alpha$ to $0^n|U_\alpha$. We put $\varphi_{\alpha\beta} = \varphi_\alpha . \varphi_\beta^{-1}$. A global section of W can be described as a system of sections w_α of 0^n satisfying $w_\alpha = \varphi_{\alpha\beta} w_\beta$. So to work out the transition matrices of $\mathscr{L}$, we note that a typical section of $\mathscr{L}|U_\alpha$ is at_α, $a \in \Gamma(U_\alpha, 0_C)$, and we may define $\varphi_\alpha(at_\alpha) = a$. Hence

$$\varphi_{\alpha\beta}(a) = \varphi_\alpha . \varphi_\beta^{-1}(a) = \varphi_\alpha(at_\alpha) = \varphi_\alpha(au_{\alpha\beta}^{-1} t_\beta) = u_{\alpha\beta}^{-1} a.]$$

Now we compute that the transition matrices for $\mathrm{Symm}^3(V)$ are

$$\begin{bmatrix} 1 & -u^{-2} r & u^{-4} r^2 & -u^{-6} r^3 \\ 0 & u^{-2} & -2ru^{-4} & 3u^{-6} r^2 \\ 0 & 0 & u^{-4} & -3u^{-6} r \\ 0 & 0 & 0 & u^{-6} \end{bmatrix}.$$

Comparing with (3), we see that the a_i *give a section of* $\mathrm{Symm}^3(V) \otimes \mathscr{L}^{-6}$.

Thus, given C, X, f, and s, we get:

(1) a line bundle $\mathscr{L}$ on C;

(2) a rank 2 vector bundle V on C with an exact sequence

$$0 \to 0_C \to V \to \mathscr{L}^2 \to 0;$$

(3) a section a of $\mathrm{Symm}^3(V) \otimes \mathscr{L}^{-6}$ projecting to 1 under the obvious map $\mathrm{Symm}^3(V) \otimes \mathscr{L}^{-6} \to 0_C$, unique up to isomorphisms of (1) and (2). Conversely, given this data, we may recover X, f, and s by using the Weierstrass equation (1) to define X locally as a subscheme of $P^2 \times U_\alpha$. The compatibility (3) implies that these schemes patch together to give a subscheme of $P(V \oplus \mathscr{L}^3)$. Note that if 3 is invertible as well, we may eliminate a_2. This forces $r=0$ and we get a splitting $V \simeq 0_C \oplus \mathscr{L}^2$. Then $\mathrm{Symm}^3(V) \oplus \mathscr{L}^{-6}$ splits up as $\mathscr{L}^{-6} \oplus \mathscr{L}^{-4} \oplus \mathscr{L}^{-2} \oplus 0_C$, and since $a_2 = 0$, a lies in the subbundle $\mathscr{L}^{-6} \oplus \mathscr{L}^{-4} \oplus 0_C$. A section of this bundle projecting to 1 is specified by giving $a_6 \in H^0(\mathscr{L}^{-6})$ and $a_4 \in H^0(\mathscr{L}^{-4})$. Thus, when 2 and 3 are invertible, our theory reduces to that outlined in Mumford-Suominen.

Now we specialize to the case where C is a smooth curve over k, an algebraically closed field of characteristic three.

PROPOSITION 1.3. — *The surface* X *is quasi-elliptic if and only if* $a_2 = a_4 = 0$ *and* a_6 *is not a cube.*

Proof. — We compute the singular locus of the generic fibre, which has the affine equation $y^2 = x^3 + a_2 x^2 + a_4 x + a_6$ ($a_i \in k(\mathrm{C})$), and is smooth at ∞. The singular locus is defined by the equations $\partial/\partial y = 2y = 0$ and $\partial/\partial x = 2a_2 x + a_4 = 0$. But since the generic fibre is a regular scheme, the singularity must be inseparable over $k(\mathrm{C})$. This forces $a_2 = a_4 = 0$. If $a_2 = a_4 = 0$, and a_6 is not a cube, then da_6 is not identically 0, and at points P, where a_6 is regular and da_6 does not vanish, the surface looks like the surface defined by the equation $y^2 = x^3 + t$, where t is a local coordinate at P. A surface with this equation is immediately seen to be smooth. If a_6 is a cube, we may make the substitution $x = x' - a_6^{1/3}$. Then the surface has the equation $y^2 = x'^3$, which is clearly not smooth.

We see that if X is quasi-elliptic, the section a lies in the subbundle of $\mathrm{Symm}^3(\mathrm{V}) \otimes \mathscr{L}^{-6}$ defined by $a_2 = a_4 = 0$, and the equation (3) reduces to

$$\begin{bmatrix} a_6 \\ 1 \end{bmatrix} = \begin{bmatrix} u^6 & -r^3 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} a'_6 \\ 1 \end{bmatrix}. \tag{4}$$

This subbundle is clearly $\mathrm{V}^{(3)} \otimes \mathscr{L}^{-6}$, where $\mathrm{V}^{(3)}$ is obtained from V by raising all entries in the transition matrices to the third power.

Expanding out our matrix equation, we find that

$$a_6 = u_{\alpha\beta}^6 a'_6 - r_{\alpha\beta}^3. \tag{5}$$

Differentiating (5), we see that $da_6 = u^6 da'_6$, which means that da_6 defines a section of $\mathrm{K_C} \otimes \mathscr{L}^{-6}$. ($\mathrm{K_C}$ is the canonical bundle of C.)

We may recover the bundle V from da_6 as follows. Consider the exact sequence

$$0 \to 0_{\mathrm{C}} \xrightarrow{\mathrm{F}} \mathrm{F}_* 0_{\mathrm{C}} \xrightarrow{d} \mathrm{B}^1 \to 0, \tag{6}$$

where F is Frobenius, d is differentiation, and B^1 is the image of $d\colon \mathrm{F}_* 0_{\mathrm{C}} \to \mathrm{F}_* \Omega^1_{\mathrm{C}}$. Tensor (6) with $\mathscr{L}^{-2}$ and note that $\mathrm{B}^1 \otimes \mathscr{L}^{-2} \subseteqq \mathrm{F}_* \Omega^1_{\mathrm{C}} \otimes \mathscr{L}^{-2} \simeq \Omega^1_{\mathrm{C}} \otimes \mathscr{L}^{-6}$. (The last isomorphism of sheaves of abelian groups, not an isomorphism of 0_{C} modules. *See* Tango [33] for further discussion.) We get an exact sequence

$$0 \to \mathscr{L}^{-2} \to \mathscr{L}^{-6} \to \mathrm{B}^1 \otimes \mathscr{L}^{-2} \to 0. \tag{7}$$

Now da_6 is by definition a section of $\mathrm{B}^1 \otimes \mathscr{L}^{-2}$, and the extension class of V, which is an element of $\mathrm{H}^1(\mathscr{L}^{-2})$ is easily seen to be the image of da_6 under the connecting homomorphism in cohomology of exact sequence (7).

We summarize our results in Theorem 1.1.

THEOREM 1.1. — *Let* C *be a scheme such that* 2 *is invertible on* C. *Given:*

(A) *a diagram* $\mathrm{X} \underset{s}{\overset{f}{\rightleftarrows}} \mathrm{C}$, *where* f *is a proper flat morphism with all geometric fibres reduced and irreducible curves of genus* 1, *and* $fs = 1_{\mathrm{C}}$, *we obtain:*

(B) (1) *a line bundle* $\mathscr{L}$ *on* C;

(2) *an exact sequence of locally free sheaves*

$$0 \to 0_C \to V \to \mathscr{L}^2 \to 0;$$

(3) *a section a of* $\mathrm{Symm}^3(V) \otimes \mathscr{L}^{-6}$ *uniquely determined up to automorphisms of* $\mathscr{L}$ *and* V *preserving the filtration* (2), *projecting to* 1 *under* $\mathrm{Symm}^3(V) \otimes \mathscr{L}^{-6} \to 0_C$.

Conversely, given (B), *we may recover* (A) *by using a to define* X *as a subscheme of* $P(V \oplus \mathscr{L}^3)$.

If C *is a smooth proper curve over an algebraically closed field of characteristic* 3, *and* $f: X \to C$ *is quasi-elliptic, then the second and third parts of* (B) *may be obtained from:*

(4) *a non-zero section* da_6 *of* $B^1 \otimes \mathscr{L}^{-2}$, *or equivalently, a non-zero element of* $H^0(K_C \otimes \mathscr{L}^{-6})$ *killed by the Cartier operator. The local Weierstrass equation is obtained by taking local integrals* a_6 *of* da_6, *and the extension class of* V *is obtained by pushing* da_6 *into* $H^1(\mathscr{L}^{-2})$.

B. Degenerate fibres. — From now on, C will be a smooth curve over an algebraically closed field of characteristic three. An elliptic or quasi-elliptic fibration with a section is called a Jacobian fibration. Given a relatively minimal model $X \underset{s}{\overset{f}{\rightleftarrows}} C$ of a Jacobian fibration, we may blow those components of fibres not meeting $S = s(C)$ and obtain the situation of Theorem 1.1. Furthermore, f has no multiple fibres, and Kodaira's classification of curves of canonical type shows that the singularities obtained are rational. (Note that Kodaira's proof [12] works without change in characteristic p, except for the assertion that certain types of fibres are not multiple, which is false in characteristic p, as we shall see later.) In order for a Weierstrass fibration described by Theorem 1.1 to be obtainable from the relatively minimal model by blowing down components of fibres, a non-degeneracy condition must be imposed on a. We will study this only in the case of quasi-elliptic surfaces in characteristic three.

Locally near a point P, our Weierstrass fibration looks like $y^2 = x^3 + a_6$, where a_6 is regular at P. Pick a local parameter t at P and expand a_6 in a power series. Since we work over an algebraically closed field, the constant term is a cube, which may be eliminated by changing x. Let $a_6 = c_n t^n + c_{n+1} t^{n+1} + \ldots$, $c_n \neq 0$, $n \geqq 1$. Since a_6 *is not a cube*, $c_r \neq 0$ for at least one $r \not\equiv 0 \pmod 3$. Let r_0 be the smallest such r. By changing x again, we may kill all terms above r_0, and get an expansion $a_6 = c_{r_0} t^{r_0} + \ldots$. Furthermore, if $r_0 \geqq 7$, we may replace y by $t^{3m} y$, and x by $t^{2m} x$ to get a new equation with $r_0 = 1, 2, 4$, or 5. Using Hensel's lemma, we may change the parameter t such that the equation becomes $y^2 = x^3 + t^r$, $r = 1, 2, 4$, or 5. The singularity in the fibre $t = 0$ is rational. We see that in order to get the best model, we should impose the non-degeneracy condition that da_6 have zeroes of no worse than order 5 at each point. (Actually, we should require that it have zeroes of no worse than order 4, but since exact order 5 is impossible for an exact differential, our condition is equivalent to this.)

The resolution of the singularity and the determination of the Kodaira-Neron type of each fibre is straightforward, and is worked out in detail in Neron [21]. We get the following four types of fibres. The notation comes from Saferevic [28], p. 172, and a picture of each type of fibre may be found there. The third column indicates the order of zero of da_6 for each type of fibre.

$$\begin{array}{lll} r=1 & B_2 & 0, \\ r=2 & B_4 & 1, \\ r=4 & B_8 & 3, \\ r=5 & B_{10} & 4. \end{array}$$

(Note that these are precisely the types such that the connected component of the group of nonsingular points is G_a and the group modulo the connected component is killed by 3.)

THEOREM 1.2. — *Let X be a relatively minimal Jacobian quasi-elliptic surface over a smooth curve C of genus g. Let c_j be the number of fibres of type B_j for $j=4, 8, 10$. Then*

$$12\chi(0_X)=2(2-2g)+2c_4+6c_8+8c_{10}. \tag{8}$$

Proof. — Recall that the canonical bundle formula for a relatively minimal quasi-elliptic fibration without multiple fibres implies that $\deg \mathscr{L}=-\chi(0_X)$. Since da_6 is a section of $K_C \otimes \mathscr{L}^{-6}$, the number of zeroes of da_6 is $2g-2+6\chi(0_X)$. The number of zeroes is also equal to $c_4+3c_8+4c_{10}$. Equate these expressions, multiply both sides by 2, and rearrange to get Theorem 1.2.

COROLLARY. — $\chi(0_X) \geqq -((g-1)/3)$ *with equality if and only if all fibres are irreducible.*

Now we know that $\chi(0_X)=K_X^2+e(X)/12$, where $e(X)$ is the etale Euler number of X. For a relatively minimal quasi-elliptic surface $K_X^2=0$; observe also that the Euler number of a generic fibre is 2 and the Euler number of a fibre of type B_j is j. Hence we may rewrite (8) in the form

$$e(X)=e(F_g)e(C)+\sum(e(F_b)-e(F_g)), \tag{9}$$

where F_g is the generic fibre, the sum runs over the closed points of C, and F_b is the fibre over b. Dolgacev has pointed out in this form, our formula can be obtained (and generalized to arbitrary quasi-elliptic surfaces) using the result of his paper [6]. For if $f: X \to C$ is a proper flat morphism of a smooth surface onto a smooth curve such that all fibres are geometrically connected, then

$$e(X)=e(F_g)e(C)+\sum(e(F b)-e(F_g)+\delta_b(f; l)),$$

where $\delta_b(f; l)$ is Serre's measure of wild ramification applied to the generic stalk of the sheaf $R^1 f_* \mu_l$. But, in the case of a quasi-elliptic surface, the generic stalk of $R^1 f_* \mu_l$ is 0, therefore $\delta_b(f; l)=0$ for all b.

C. EXAMPLES. — We have seen that to construct a Jacobian quasi-elliptic surface over a curve C such that $R^1 f_* 0_X \simeq \mathscr{L}$, it is enough to give a section of $B^1 \otimes \mathscr{L}^{-2}$ with zeroes of no worse than order 5 at each point. Let $\mathscr{L}=0_X(D)$, then such sections are the same as exact rational differentials df such that (*a*) $\operatorname{div}(df) \geqq 6D$ (*see* Tango [33]) and (*b*) $\operatorname{div}(df)$ -6 D contains no point more than 5 times. It is obvious that given df, there exists a unique D satisfying (*a*) and (*b*).

We are interested in examples such that deg $\mathscr{L}>0$, since then $\chi(0_X)<0$ and we get a counterexample in characteristic p to the well-known characteristic 0 theorem of Castelnuovo-de Franchis "$\chi(0_X)<0$ implies X ruled" (*see* Bombieri [4]). (A quasi-elliptic surface over an irrational base is obviously *not* ruled, since all rational curves must be in the fibres.) We are even more interested in examples such that deg $\mathscr{L}>0$ and all fibres are irreducible, because the following theorem of Mumford-Raynaud implies that such surfaces provide smooth counterexamples to the Kodaira vanishing theorem. Such surfaces will be called Raynaud surfaces in the sequel.

THEOREM 1.3. — *Let $f: X \to C$ be a proper morphism of a smooth surface* X *onto a smooth curve* C *over a field k of arbitrary characteristic. Let* S *be a section of f and assume that the fibres of f have positive arithmetic genus. Let* X_0 *be the normal surface obtained by blowing down all components of fibres of f not meeting* S. *Then if* $S^2>0$ *and* $\mathscr{L}=0_X(S)/0_X$ *(an invertible sheaf on* C*), then* $H=0_X(S)\otimes f^*\mathscr{L}$ *is ample on* X_0 *and* $H^1(X_0, -H)\neq 0$.

Proof. — An application of the Leray spectral sequence. *See* Mumford [20].

To get a surface in characteristic 3 satisfying the hypotheses of Theorem 1.3, let C be the hyperelliptic curve with affine equation $y^2=x^{3n}+x^7+1$, n odd, $n\geqq 3$. Then $-y\,dy=x^6\,dx$, or $dx/y=d(-y/x^6)$. Therefore, dx/y is locally exact, and $\operatorname{div}(dx/y)=3(n-1)\infty$ (*see* Saferevic [29], p. 175). Let $\mathscr{L}=0_C(1/2(n-1)\infty)$. Then dx/y gives a nowhere vanishing, locally exact section of $K_C\otimes\mathscr{L}^{-6}$, which gives us a quasi-elliptic surface with all fibres irreducible, and with a section S such that $S^2=1/2(n-1)$. By Theorem 1.3, this is a smooth counterexample to Kodaira vanishing. The genus of C is $1/2(3n-1)$ and we see that we can find curves of arbitrarily high genus which are base curves for Raynaud quasi-elliptic surfaces.

2. Local Theory

A. NORMAL FORMS. — In this section, we will study the local group of twisted forms of quasi-elliptic surfaces over fields of characteristic three (i.e., multiple fibres), using the birational methods of C. S. Queen. In principle, this leads to a complete classification of formal neighborhoods of multiple fibres. However, what we have described here are normal forms for singular surfaces, formally birationally isomorphic to formal neighborhoods of each type of multiple fibre. For a complete theory, it will be necessary to resolve these singularities. I hope to return to this in another paper. In the next section, we apply a simple special case to a question of Zariski, originally answered by P. Blass.

PROPOSITION 2.1. — *Let* $F=nE$ *be a multiple fibre on a quasi-elliptic surface over a field of characteristic p, where E is an indecomposable curve of canonical type and* $n>1$. *Let* D *be the curve of cusps. Then* $n=p$ *and* D *intersects* E *transversally.*

Proof. — Let F_g be the generic fibre of the quasi-elliptic fibration. It is proved in Bombieri-Mumford III that $(F_g.D)=p$. Therefore $(F.D)=p$. The result follows easily.

Proposition 2.1 implies that any multiple fibre will be contracted by the rational map associated to $0_X(D)$. This is the reason that our normal forms are singular.

PROPOSITION 2.2 (Queen). — *Let $f: X \to C$ be a quasi-elliptic fibration in characteristic three, let $X_g/k(C)$ be the generic fibre. Then $0_X(D)$ defines a birational morphism between X_g and a plane curve over $k(C)$ with affine equation of the form*

$$z+a+x^3+cz^3,$$

where $a, c \in k(C)$, and $c \notin k(C)^3$.

Proof. — *See* Queen [24 I].

We call the equation given by Proposition 2.2 the generic (Queen) equation of X.

PROPOSITION 2.3 (Queen). — *If X has generic equation $z+a+x^3+cz^3$, then the Jacobian of X has generic equation $z+x^3+cz^3$.*

Proof. — *See* Queen [24 II].

Note that the generic Weierstrass equation of the Jacobian can be obtained by homogenizing with respect to a new variable y and then dehomogenizing with respect to z to get y^2+x^3+c.

We now want to study the Queen equation over the completion of $k(C)$ at a point P. By picking a local parameter t at P, we may identify this field with $k((t))$.

THEOREM 2.1. — *Let $X/k((t))$ be a formal neighborhood of a multiple fibre of a quasi-elliptic fibration. Then X is birationally isomorphic over $k((t))$ to a surface with affine equation of one of the following types:*

(1) $$t^{2m}z+t^2(f_{m-1}(t))^3+x^3+tz^3;$$

(2) $$t^{2m}z+t(f_{m-1}(t))^3+x^3+t^2z^3;$$

(3) $$t^{2m}z+t^2(f_{m-2}(t))^3+x^3+t^4z^3;$$

(4) $$t^{2m}z+t(f_{m-2}(t))^3+x^3+t^5z^3,$$

where $f_n(t)$ is a polynomial of degree $\leqq n$ with non-zero constant term. After choosing a formal birational isomorphism of Jac(X) *with the surface defined by $y^2+x^3+t^r$ ($r=1, 2, 4, 5$), m and f are uniquely determined by* X.

Proof. — We start with the Queen equation $z+a+x^3+cz^3$, $a, c \in k((t))$. After making the substitutions $z=t^{3n}z$, $x=t^n x$, for suitable n, we may assume $0 \leqq \operatorname{ord} c \leqq 5$. By making a substitution of the form $x=x+fz$, and then making a change of parameter, we may assume that $c=t$, t^2, t^4, or t^5. (In other words, we put the Jacobian of X into one of the normal forms of Section 1 B.) Note that $h(z)=z+cz^3$ is a homomorphism of the additive group of $k((t))$ into itself, and that the substitutions $z=z+d$, $x=x+e$ show that a can be replaced by anything in the coset $a+(h(K)+K^3)$, where $K=k((t))$. Therefore, we need only find a set of coset representatives for $K^+/(h(K^+)+K^{+3})$. We do the case $c=t$ in detail, leaving the other three cases (which are quite similar) to the reader.

We expand a into a Laurent series. Note that if d is holomorphic, $h(d)=d+$(higher order terms); therefore $k[[t]]$ is contained in the image of h. Now if g is a constant,

$$h(gt^{-1})=gt^{-1}+g^3t^{-2},$$
$$h(gt^{-2})=gt^{-2}+g^3t^{-5},$$
$$h(gt^{-3})=gt^{-3}+g^3t^{-8}, \dots$$

Since k is perfect, we see that every Laurent series is congruent modulo $h(\mathrm{K})+\mathrm{K}^3$ to a Laurent series such that all terms have negative exponent $\equiv 2 \pmod 3$, and no Laurent series such that all terms have negative exponent $\equiv 2 \pmod 3$ is congruent to 0 mod $h(\mathrm{K})+\mathrm{K}^3$. Therefore a normal form for a multiple fibre on a quasi-elliptic surface with Jacobian $z+x^3+t$ is

$$z+(a_0t^{-1}+a_1t^{-4}+\dots+a_nt^{-3n-1})+x^3+tz^3, \qquad a_n\neq 0.$$

We want a normal form with holomorphic coefficients, so we make the substitution $z=t^{-m}z$, $x=t^{-m}x$, where $m=n+1$. The equation is replaced by the equation

$$t^{2m}z+t^2(a_n+a_{n-1}t^3+\dots+a_0t^{3n})+x^3+tz^3.$$

Since k is perfect, we have the desired normal form.

[Compare this result with the theorem of Russell-Queen that $\mathrm{H}^1_{et}(\mathrm{K}, \mathrm{G})\cong \mathrm{K}^+/(h(\mathrm{K}^+)+\mathrm{K}^{+3})$, where K is an arbitrary field of characteristic three and G is the K-group scheme defined by the equation $z+x^3+cz^3$. Our only original contribution in this section is the explicit description of this group in the case $\mathrm{K}=k((t))$, k an algebraically closed field.]

B. Applications. — Recall that if $f: \mathrm{X}\to\mathrm{C}$ is an elliptic or quasi-elliptic fibration, $\mathrm{R}^1 f_* 0_{\mathrm{X}} \simeq \mathscr{L}\oplus \mathrm{T}$, where $\mathscr{L}$ is an invertible sheaf and T is a torsion sheaf. If $\mathrm{P}\in \mathrm{Supp}(\mathrm{T})$, then $f^{-1}(\mathrm{P})$ is a multiple fibre (Bombieri-Mumford II). Such multiple fibres are called wild, and all others are called tame.

Conjecture 2.1. — Let $f: \mathrm{X}\to\mathrm{C}$ be a quasi-elliptic surface with a multiple fibre over a point P. Then the length at P is $[2m/3]$ in cases (1) and (2) and $[2(m-1)/3]$ in cases (3) and (4), where [ ] denotes the greatest integer function, and m is the integer associated with the multiple fibre by Theorem 2.1.

I have verified this in a number of cases, including all irreducible multiple fibres. I hope that the method (resolving singularities and examining the effect on the dualizing sheaf) will work in all cases. For the purposes of this paper, we need only the following special case.

Proposition 2.4. — *Multiple fibres with normal form $t^2z+at^2+x^3+tz^3$ (where a is a non-zero constant) are tame.*

Proof. — Our fibre is formally isomorphic to a fibre on a quasi-elliptic surface X over P^1 birationally isomorphic to the quartic hypersurface in P^3 defined by the equation $t^2zw+at^2w^2+x^3w+tz^3$. One checks immediately that this quartic has only isolated singularities. It is well known that a smooth model of a quartic hypersurface in P^3 with only isolated singularities has Kodaira dimension $\leqq 0$. Using the basic table in Bombieri-Mumford II and the fact that $H^1(0_X)$ is a birational invariant of a smooth surface, we see that $H^1(0_X)=0$. Proposition 2.4 now follows from Proposition 2.5, which was first proved by Dolgacev by a more complicated method.

PROPOSITION 2.5. — *Let $f: X \to C$ be a relatively minimal elliptic or quasi-elliptic surface such that $H^1(0_X)=0$. Then all multiple fibres on X are tame.*

Proof. — The Leray spectral sequence gives us an exact sequence

$$(1) \qquad 0 \to H^1(C, 0_C) \to H^1(X, 0_X) \to H^0(C, R^1 f_* 0_X) \to 0.$$

If $R^1 f_* 0_X \simeq \mathscr{L} \oplus T$, where T is a non-zero torsion sheaf, then $H^0(C, R^1 f_* 0_X) \neq 0$, so $H^1(0_X) \neq 0$.

We digress to give two more results in the same vein.

PROPOSITION 2.6. — *Let $f: X \to C$ be a relatively minimal elliptic or quasi-elliptic fibration. Assume that f is the Albanese map of X and $\chi(0_X) \geqq 1$. Then $\mathrm{Pic}(X)$ is reduced if and only if f has no wild fibres.*

Proof. — $\mathrm{Pic}^0(X)$ has dimension $g=h^1(C, 0_C)$. Exact sequence (1) implies that $\mathrm{Pic}(X)$ is reduced if and only if $H^0(C, R^1 f_* 0_X)=0$. Therefore, if $\mathrm{Pic}(X)$ is reduced, there are no wild fibres. If there are no wild fibres, then $R^1 f_* 0_X \simeq \mathscr{L}$, where $\mathscr{L}$ is an invertible sheaf of degree $-\chi(0_X)<0$. Therefore $H^0(C, R^1 f_* 0_X)=0$ and $\mathrm{Pic}(X)$ is reduced.

The next proposition is implicit in Bombieri-Mumford III.

PROPOSITION 2.7. — *There exist no quasi-elliptic fibrations on Enriques surfaces over fields of characteristic three.*

Proof. — If X is an Enriques surface over a field of characteristic three, then $\chi(0_X)=1$ and $H^1(0_X)=H^2(0_X)=0$, so there are no wild multiple fibres, and the base of any quasi-elliptic pencil must be P^1. Now we use the formula of Bombieri-Mumford (for X a relatively minimal elliptic or quasi-elliptic surface over P^1):

$$(2) \qquad \dim |n K_X| = nr + \sum_{\lambda} [n a_\lambda / m_\lambda],$$

where λ runs over the multiple fibres and $r=-2+\chi(0_X)+\text{length } T$. In our case, $T=0$, $r=1$, and for all λ, $a_\lambda/m_\lambda=2/3$. We see that if X is a quasi-elliptic surface with only tame multiple fibres over P^1 such that $\chi(0_X)=1$, then X is rational if the number of multiple fibres is $\leqq 1$, and X is properly quasi-elliptic if there are 2 or more multiple fibres. Therefore there are no quasi-elliptic Enriques surfaces.

Surfaces birationally isomorphic to irreducible hypersurfaces in A^3 with equation $z^p - f(x, y)=0$ ($p=\mathrm{char}\, k$) are called Zariski surfaces (because they were first studied in Zariski's paper on Castelnuovo's criterion). All Zariski surfaces are clearly unirational. Zariski gave examples of Zariski surfaces with $p_g>0$, and later posed the following question: If X is a Zariski surface with $p_g=0$, is X rational? P. Blass [2] answered this question negatively in characteristic 2. His counterexample is quite complicated. Dolgacev suggested that it might be possible to use quasi-elliptic surfaces in characteristic three with two multiple fibres to give a simpler counterexample. (Note that the Queen equation implies that every quasi-elliptic surface over P^1 is a Zariski surface.) We carry out this program below.

Consider the quasi-elliptic surface over P^1 with generic Queen equation

$$z+t^{-1}(t-1)^{-1}+x^3+tz^3, \tag{3}$$

which is birationally isomorphic to the surface with equation

$$t^2(t-1)^2 z+t^2(t-1)^2+x^3+tz^3. \tag{4}$$

Let X be the relatively minimal smooth model. The Jacobian of X is the rational surface Y with generic equation $z+x^3+tz^3$. Y has a fibre of type B_{10} at ∞, and all other fibres are of type B_2. X is clearly formally isomorphic to Y in a formal neighborhood of each fibre except possibly 0, 1, and ∞. At 0 and 1, X has a tame irreducible multiple fibre, by Proposition 2.4. To see what happens at ∞, we make the substitution $t=u^{-1}$. Then u is a coordinate at ∞ and X now has the equation

$$z+u^2(\text{unit power series in } u)+x^3+u^{-1}z^3.$$

Make the substitution $z=u^3 z$, $x=ux$ to get the equation

$$z+u^1(\text{unit})+x^3+u^5 z^3.$$

After making the substitution $z-au^{-1}$, $a\in k$, we get an equation of the form

$$z+(\text{holomorphic power series in } u)+x^3+u^5 z^3.$$

By the method of proof of Theorem 2.1, we see that X is formally isomorphic to its Jacobian near $f^{-1}(\infty)$, so the fibre over ∞ is reduced of type B_{10}.

X has exactly two tame irreducible multiple fibres. Using Dolgacev's formula for $\chi(0_X)$ (Section 1 B) we find that $\chi(0_X)=1$. As in the proof of Proposition 2.7, we use formula (2) to find that X has geometric genus 0, but X is not rational.

Finally, we note that the simplest special cases of Conjecture 2.1 give unirational surfaces with $H^1(0_X)\neq 0$. I believe that these are the first known surfaces with these properties. (*See* Shioda [31] and the references given there for known examples of unirational surfaces.)

3. Twisted Forms

A. Twisted forms and the Picard scheme. — Let X be a Jacobian quasi-elliptic surface over a smooth curve C over an algebraically closed field of characteristic three. The set of nonsingular points in each fibre of the quasi-elliptic fibration f form a group scheme G over C, and G has a subgroupscheme G^0 consisting of the identity component of each fibre. We have an exact sequence

$$0 \to G^0 \to G \to T \to 0,$$

where T is a finite group scheme supported on a finite subset of C. Taking (etale or flat) cohomology, we get an exact sequence

$$H^1(C, G^0) \to H^1(C, G) \to H^1(C, T).$$

But $H^1(C, T)=0$ so the map $H^1(C, G^0) \to H^1(C, G)$ is surjective. Classes in $H^1(C, G^0)$ correspond to quasi-elliptic surfaces with no multiple fibres together with a choice of irreducible component of multiplicity 1 in each fibre. After this choice, we may blow down all other components and obtain a new surface X with only rational singularities. We note here for future reference that $H^2(C, G^0) \simeq H^2(C, G)$.

LEMMA 3.1. — *Let* $f: X \to C$ *be a quasi-elliptic fibration with no multiple fibres. Choose an irreducible component of multiplicity* 1 *in each fibre, and blow down all other components to get* $\tilde{f}: \tilde{X} \to \tilde{C}$. *Let* D *be the curve of cusps on* X. *Then over a sufficiently fine affine open cover* $\{U_\alpha\}$ *of* C, $f^{-1}(U_\alpha)-D$ *is isomorphic to the subscheme of* $A^2 \times U_\alpha$ *defined by the equation*

$$z_\alpha - b_\alpha - x_\alpha^3 - a_\alpha z_\alpha^3,$$

where a_α, $b_\alpha \in \Gamma(U_\alpha, 0_C)$. *The obvious compactification in* $P^2 \times U_\alpha$ *has only rational singularities, and is therefore isomorphic to* $\tilde{f}^{-1}(U_\alpha)$. *The coefficients may be chosen so that* $y_\alpha^2 z_\alpha - x_\alpha^3 - a_\alpha z_\alpha^3$ *is the Weierstrass normal form of the Jacobian of* X.

Proof. — We saw in Section 2 that the linear system D defines a birational isomorphism between X and the subscheme of $A^2_{k(C)}$ defined by the equation $z-b-x^3-az^3$ where the coefficients are in $k(C)$. Pick a point P on C. Using the method of proof of Theorem 2.1, we may change the equation so that it is of the form $z-b'-x^3-a_\alpha z^3$, where a_α has order between 0 and 5 at P and $z-x^3-a_\alpha z^3$ is a Weierstrass equation of the Jacobian of X in a neighborhood U_α of P. Since there are no multiple fibres, there exist d and e in the completion of $k(C)$ at P such that the substitution $x=x+d$, $z=z+e$ kills the constant term b'. We approximate d and e by elements of $k(C)$ so that the Laurent tail of b' is killed. We now have the desired equation in a neighborhood of P. There are only rational singularities at ∞ because b' can be killed after completing the local ring of C at P.

We want to patch these schemes together so that the "twisting" comes from translations in the Jacobian. After dehomogenizing the Weierstrass equation $y_\alpha^2 z_\alpha - x_\alpha^3 - a_\alpha z_\alpha^3$ with respect to y_α, equation (2) in Chapter 1 tells us that we should patch together the surfaces defined by $z_\alpha - b_\alpha - x_\alpha^3 - a_\alpha z_\alpha^3$ by the rules

$$(1) \qquad \left\{ \begin{aligned} z_\alpha &= u_{\alpha\beta}^{-3} z_\beta + s_{\alpha\beta}, \\ x_\alpha &= u_{\alpha\beta}^{-1} x_\beta + u_{\alpha\beta}^{-3} r_{\alpha\beta} z_\beta + t_\beta, \\ a_\alpha &= u_{\alpha\beta}^{6} a_\beta - r_{\alpha\beta}^{3}. \end{aligned} \right.$$

Therefore the transition matrices for $\tilde{f}_* 0_{\tilde{X}}(D)$ are

$$\begin{bmatrix} 1 & -u_{\alpha\beta}^3 s_{\alpha\beta} & u_{\alpha\beta} r_{\alpha\beta} s_{\alpha\beta} - t_{\alpha\beta} u_{\alpha\beta} \\ 0 & u_{\alpha\beta}^3 & -u_{\alpha\beta} r_{\alpha\beta} \\ 0 & 0 & u_{\alpha\beta} \end{bmatrix}$$

Comparing with Section 1, we find that there is an exact sequence

$$0 \to 0_C \to \tilde{f}_* 0_{\tilde{X}}(D) \to V \otimes \mathscr{L}^{-3} \to 0,$$

where V and $\mathscr{L}$ are the sheaves associated to Jac(X) in Section 1. (Note that if X is Jacobian, the sequence splits by Lemma 4.2.) Therefore X gives us a class in $H^1(V^* \otimes \mathscr{L}^3)$.

We could continue as in Section 1, and find that our equation gives us a section of a subbundle of $\mathrm{Symm}^3(W) \otimes \mathscr{L}^3$ $(W = f_* 0_X(D))$, analyze the condition that a class in $H^1(V^* \otimes \mathscr{L}^3)$ come from a quasi-elliptic surface, etc. Since this "geometric" theory of locally trivial twisted forms is used only once in the sequel, we leave this to the reader, together with the check that it is compatible with the cohomological approach discussed below.

We note that the equation of a Jacobian quasi-elliptic surface $z_\alpha = x_\alpha^3 - a_\alpha z_\alpha^3$ defines an open subset of our surface not only as a subscheme of $A^2 \times U_\alpha$ but also as a subgroupscheme of the vector group $S(V \otimes \mathscr{L}^{-3})$. [If F is a locally free sheaf on C, we let S(F) = Spec Symm (F), which is a group scheme over C.] In fact, it is clear that we have an exact sequence of group schemes

$$0 \to G^0 \to S(V \otimes \mathscr{L}^{-3}) \to S(\mathscr{L}^{-3}) \to 0, \tag{2}$$

where G^0 is the group scheme obtained from our Jacobian surface by taking the identity component of each fibre. The map $S(V \otimes \mathscr{L}^{-3}) \to S(\mathscr{L}^{-3})$ is defined locally by $(x_\alpha, z_\alpha) \to z_\alpha - x_\alpha^3 - a_\alpha z_\alpha^3$. If F is a locally free sheaf, then the sheaf of sections of S(F) is isomorphic to F* (Hartshorne [7], p. 129) and $H^i_{fl}(S(F)) \simeq H^i_{Zar}(F^*)$. Therefore, writing down the flat (or etale, since everything is smooth) cohomology sequence of (2), we get the following theorems.

THEOREM 3.1. — *Let* X *be a Jacobian quasi-elliptic surface, and let* G^0 *be the group scheme obtained by taking the connected component of the identity in the nonsingular points of each fibre. Then there is an exact sequence*

$$0 \to D \to H^1_{f1}(C, G^0) \to B \to 0,$$

where $D = \mathrm{coker}(H^0(V^* \otimes \mathscr{L}^3) \to H^0(\mathscr{L}^3))$, $B = \ker(H^1(V^* \otimes \mathscr{L}^3) \to H^1(\mathscr{L}^3))$. *The maps are induced by* $p - \bigcup a . F$, *where* $p \colon V^* \otimes \mathscr{L}^3 \to \mathscr{L}^3$ *is projection (coming from the dual of the exact sequence* $0 \to 0_C \to V \to \mathscr{L}^2 \to 0$) $F \colon V^* \otimes \mathscr{L}^3 \to V^{*(3)} \otimes \mathscr{L}^9$ *is Frobenius and* $\bigcup a$ *is cup-product with the section of* $V^{(3)} \otimes \mathscr{L}^{-6}$ *used to define* X (*Section* 1).

THEOREM 3.2. — *Let* X *be as in Theorem* 3.1, *and assume in addition that all fibres are irreducible. Then we have an exact sequence*

$$0 \to \mathrm{Pic}^0(C) \to \mathrm{Pic}^\tau(X) \to K \to 0,$$

where K *is the scheme-theoretic kernel of* $p - \bigcup a \circ F$ *on* $H^0(V^* \otimes \mathscr{L}^3)$.

Proof. — We work with S-valued points. Note that $\mathrm{Pic}^\tau(X/C)=G=G^0$, and that $H^0(C\times S, G^0\times S)=K(S)$, using exact sequence (2). By definition of $\mathrm{Pic}^\tau(X/C)$, we have an exact sequence for any S:

$$0\to \mathrm{Pic}^0(C)(S)\to \mathrm{Pic}^\tau(X)(S)\xrightarrow{\alpha}\mathrm{Pic}^\tau(X/C)(C\times S)$$

and it is clear that since G is an open subscheme of X, α is surjective.

Now we return to an arbitrary Jacobian quasi-elliptic surface X. The group $H^2(C, G)\simeq H^2(C, G^0)$ is interesting also, although its interpretation is not as simple. First note that exact sequence (2) shows that $H^2(C, G)$ is isomorphic to coker $(g: H^1(V^*\otimes\mathscr{L}^3)\to H^1(\mathscr{L}^3))$, where $g=p-\bigcup a.F$. Recall that if W is a vector bundle over a curve C, and if $K=k(C)$, then the exact sequence

$$0\to W\to W\otimes K\to \oplus W\otimes K/0_P\to 0$$

shows that $H^1(W)$ may be interpreted as the quotient of the group of "principal parts" modulo those coming from meromorphic sections of W.

Now let $z_\alpha-b_\alpha-x_\alpha^3-a_\alpha z_\alpha^3$ be a generic Queen equation for a quasi-elliptic surface, and assume that $z_\alpha-x_\alpha^3-a_\alpha z_\alpha^3$ is in Weierstrass normal form for an open subset U_α of C. Then the "reduction to normal form" of Section 2 A shows that the type of multiple fibre at a point P of U depends only on the principal part of b modulo the image of the group of two-variable principal parts under the map $(x, z)\to z-x^3-a_\alpha z^3$. If we go over to another open subset U, then using the transformation rules (1), we find that

$$b_\beta=u_{\alpha\beta}^3(b_\alpha+s_{\alpha\beta}+t_{\alpha\beta}^3-a_\alpha s_{\alpha\beta}^3),$$

or

$$u_{\alpha\beta}^{-3}b_\beta=b_\alpha+s_{\alpha\beta}+t_{\alpha\beta}^3-a_\alpha s_{\alpha\beta}^3).$$

But the reduced Laurent expansion of the right-hand side is the same as that of b_α. Therefore, the collection of reduced Laurent expansions coming from a quasi-elliptic surface must come from a meromorphic section of $\mathscr{L}^3$. This shows that we should think of $H^2(C, G)$ as the obstruction group for the following problem: given a Jacobian quasi-elliptic fibration, construct a twisted form with prescribed multiple fibres.

Of course, this is well known from the point of view of etale cohomology. Let $x=\mathrm{Spec}\,k(C)$ be the generic point of C, let $i: x\to C$ be the inclusion, and let $F=i^*G$. Then we have the following exact sequence (*see* Artin [1], p. 106, for details):

$$0\to H^1(C, i_*F)\to H^1(x, F)\to \bigoplus_p H^1(x_p, F_p)\xrightarrow{\delta} H^2(C, i_*F)\to 0,$$

and we have just computed δ explicitly. (Note that by the "Neronian" property of the relatively minimal model, $i_*i^*G\simeq G$.)

B. CLASSIFICATION OF QUASI-HYPERELLIPTIC SURFACES. — As an application, we check the classification of quasi-elliptic surfaces given in Bombieri-Mumford III. Quasi-hyperelliptic surfaces are singled out among general quasi-elliptic surfaces by the following properties:

(1) the base curve is elliptic;

(2) all fibres are reduced and irreducible.

First, we classify Jacobian quasi-hyperelliptic surfaces. We need a line bundle $\mathscr{L}$ and a nowhere vanishing section of $K_C \otimes \mathscr{L}^{-6}$ killed by the Cartier operator. Therefore, $\mathscr{L}^6 \approx 0_C$. Since proportional differentials give isomorphic surfaces, there is at most one surface X for each choice of C and $\mathscr{L}$. Let $n = \operatorname{ord} \mathscr{L} = \operatorname{ord} K_X$.

Case 1: $n=1$. – Our surface exists if and only if $H^0(B^1) \neq 0$. Therefore, using the sequence

$$0 \to H^0(B^1) \to H^1(0_X) \overset{F}{-} H^1(0_X),$$

we see that the surface exists if and only if C is supersingular. This corresponds to case (*d*) in Bombieri-Mumford III.

Case 2: $n=2$. – Similar arguments to those given above show that C is supersingular. This case corresponds to case (*e*) of Bombieri-Mumford III. The different possibilities for $\mathscr{L}$ correspond to different isogenies $0 \to Z/2 \to E_1 \to C \to 0$.

Case 3: $n=3$. – Since $\mathscr{L} \not\simeq 0_C$, and $\mathscr{L}^3 \simeq 0_C$, C must be ordinary, and the map $H^0(\mathscr{L}^{-6}) \to H^0(B^1 \otimes \mathscr{L}^{-2})$ is an isomorphism. This case corresponds to case (*a*) in Bombieri-Mumford III, and the different choices for $\mathscr{L}$ correspond to different isogenies $0 \to \mu_3 \to E_1 \to C \to 0$.

Case 4: $n=6$. – This is related to case 3 in the same way that case 2 is related to case 1, and is case (*b*) in Bombieri-Mumford.

Now we compute the non-Jacobian surfaces associated with each Jacobian surface.

Case 1: $n=2$ or 6. – Then $\mathscr{L}^3$ is not trivial. Since there is an exact sequence $0 \to \mathscr{L} \to V^* \otimes \mathscr{L}^3 \to \mathscr{L}^3 \to 0$, we see that $H^0(\mathscr{L}^3) = H^1(V^* \otimes \mathscr{L}^3) = 0$. Hence there are no twisted forms.

Case 2: $n=3$. – Here the sequence $0 \to \mathscr{L} \to V^* \otimes \mathscr{L}^3 \to \mathscr{L}^3 \to 0$ induces isomorphisms in cohomology $H^0(V^* \otimes \mathscr{L}^3) \simeq H^0(\mathscr{L}^3)$, $H^1(V^* \otimes \mathscr{L}^3) \simeq H^1(\mathscr{L}^3)$. We also know that the sequence $0 \to 0_C \to V \to \mathscr{L}^2 \to 0$ splits and that the base is an ordinary elliptic curve, so $\bigcup a \circ F$ is also an isomorphism, but a *p*-linear one, on both H^0 and H^1. The difference $p - \bigcup a \circ F$ between our linear and *p*-linear isomorphisms has trivial cokernel on H^0 (since we are over an algebraically closed field and everything is 1-dimensional) and on H^1 has kernel isomorphic to $Z/3$. By applying inversion to G, we may change the homogeneous space structure of a twisted form without changing the underlying surface. Therefore, there is essentially only one non-trivial twisted form, corresponding to case (*c*) in Bombieri-Mumford III.

Case 3: $n=1$. – Here $\mathscr{L}$ is trivial and we have an exact sequence $0 \to 0_C \to V^* \to 0_C \to 0$ where the extension class is non-trivial. Therefore $p: H^0(V^*) \to H^0(0_C)$ is 0 while $\bigcup a \circ F$ is surjective on H^0. Therefore $D=0$. Now $p: H^1(V^*) \to H^1(0_C)$ is bijective, and it is known that $F: H^1(V^*) \to H^1(V^{*(3)})$ is zero (Oda [23]). Therefore the group of twisted forms is zero.

Case (f) in the table of Bombieri-Mumford does not exist, since the group scheme of order 9 listed there is not a subgroupscheme of an elliptic curve.

4. Cohomology of the Tangent and Cotangent Bundles

A. COHOMOLOGY OF THE TANGENT BUNDLE. — Throughout this section, we will work with Jacobian quasi-elliptic surfaces over an algebraically closed field of characteristic three such that all fibres are irreducible. For such surfaces the Weierstrass model studied in Chapter 1 is equal to the relatively minimal smooth model. We know that $\chi(0_X) = -((g-1)/3)$, where g is the genus of the base curve. Such surfaces are of two types: those with $g > 1$, which are Raynaud surfaces, and those with $g = 1$, which are quasi-hyperelliptic surfaces.

We keep the notation of Section 1. We let D be the curve of cusps of the quasi-elliptic fibrations, S the section, K_C the canonical bundle of C, and $\omega_{X/C}$ the relative dualizing sheaf of $f: X \to C$. Note that $\omega_{X/C}$ is isomorphic to $f^* \mathscr{L}^{-1}$.

We start with two basic lemmas.

LEMMA 4.1. — *Let X be a relatively minimal quasi-elliptic surface with all fibres reduced and irreducible (not necessarily Jacobian). Then there is an exact sequence*

$$(1) \qquad 0 \to f^* K_C \otimes 0_X(D) \to \Omega_X^1 \to \omega_{X/C} \otimes 0_X(-D) \to 0.$$

Proof. — This is clear on $X - D$, for there f is smooth, and we have the usual sequence $0 \to f^* \Omega_C^1 \to \Omega_C^1 \to \Omega_{X/C}^1 \to 0$. Let P be a point of D. Then in a formal neighborhood of P, $f: X \to C$ is formally isomorphic to the surface defined by the equation $y^2 = x^3 + t$, where t is a coordinate on C, x and y are coordinates on X, and y is a local equation for D (Bombieri-Mumford III). Differentiating, we get $-y\,dy = f^* dt$ or $dy = -dt/y$. This shows that we have an injection $f^* K_C \otimes 0_X(D) \to \Omega_X^1$ with locally free quotient M. To see the isomorphism $M \simeq \omega_{X/C} \otimes 0_X(-D)$, we note that (1) gives $K_X \simeq f^* K_C \otimes 0_X(D) \otimes M$. On the other hand, we know (Bombieri-Mumford II) that $K_X \simeq f^* K_C \otimes \omega_{X/C}$. Our isomorphism follows by rearrangement.

LEMMA 4.2. — *Let* X *be as in Lemma* 4.1, *and assume also that* X *is Jacobian. Then* $0_X(D) \simeq 0_X(3S) \otimes f^* \mathscr{L}^{-3}$.

Proof. — In the Weierstrass normal form for our surface $y_\alpha^2 = x_\alpha^3 + t_\alpha$, y_α has a pole of order 3 along $S \cap f^{-1}(U_\alpha)$, vanishes to first order along $D \cap f^{-1}(U_\alpha)$ and has no other zeroes or poles. We saw in Chapter 1 that the sheaf generated by y_α on $f^{-1}(U_\alpha)$ for each α is isomorphic to $f^* \mathscr{L}^3$, hence $0_X(3S - D) \simeq f^* \mathscr{L}^3$. The lemma follows by rearrangement.

Now we use Serre duality to compute the cohomology of the tangent bundle. We know that $H^i(\theta_X)^* = H^{2-i}(\Omega_X^1 \otimes K_X)$. We tensor exact sequence (1) with $K_X \simeq f^* K_C \otimes \mathscr{L}^{-1}$ to get

$$0 \to 0_X(D) \otimes f^*(K_C^2 \otimes \mathscr{L}^{-1}) \to \Omega_X^1 \otimes K_X \to 0_X(-D) \otimes f^*(K_C \otimes \mathscr{L}^{-2}).$$

Using Lemma 4.2, this becomes

$$0 \to 0_X(3S) \otimes f^*(K_C^2 \otimes \mathscr{L}^{-4}) \to \Omega_X^1 \otimes K_X \to 0_X(-3S) \otimes f^*(K_C \otimes \mathscr{L}) \to 0.$$

We get a nine-term sequence in cohomology

$$\begin{aligned}
(2) \quad 0 &\to H^0(0_X(3S) \overset{a}{\otimes} f^*(K_C^2 \otimes \mathscr{L}^{-4}) \to H^0(\Omega_X^1 \otimes K_X) \to H^0(0_X(-3S) \overset{b}{\otimes} f^*(K_C \otimes \mathscr{L})) \\
&\to H^1(0_X(3S) \otimes f^*(K_C^2 \overset{c}{\otimes} \mathscr{L}^{-4}) \to H^1(\Omega_X^1 \otimes K_X) \to H^1(0_X(-3S) \overset{d}{\otimes} f^*(K_C \otimes \mathscr{L})) \\
&\to H^2(0_X(3S) \otimes f^*(K_C^2 \overset{e}{\otimes} \mathscr{L}^{-4}) \to H^2(\Omega_X^1 \otimes K_X) \to H^2(0_X(-3S) \overset{f}{\otimes} f^*(K_C \otimes \mathscr{L})) \to 0.
\end{aligned}$$

We know that $f_* 0_X(3S) \simeq V \oplus \mathscr{L}^3$. $R^1 f_* 0_X(3S)=0$, and $f_* 0_X(-3S)=0$. Therefore groups (*b*) and (*e*) are 0. By Serre duality and the Leray spectral sequence

$$h^0(\theta_X)=h^2(0_X(-3S)\otimes f^*(K_C\otimes\mathscr{L}))=h^0(0_X(3S)\otimes\mathscr{L}^{-2})=h^0(V\otimes\mathscr{L}^{-2})+h^0(\mathscr{L}).$$

Case I: Raynaud surfaces. — We know that deg $\mathscr{L}=(g-1)/3>0$. Hence the map $H^0(B^1\otimes\mathscr{L}^{-2})\to H^1(\mathscr{L}^{-2})$ is injective. Therefore the extension class of the exact sequence

$$0\to 0_C\to V\to\mathscr{L}^2\to 0$$

is non-trivial. Therefore $h^0(V\otimes\mathscr{L}^{-2})=0$, so that $h^0(\theta_X)=h^0(\mathscr{L})$.

The computation of all other groups in exact sequence (2) is easily done using the Leray spectral sequence and the Riemann-Roch theorem for curves. The result is:

THEOREM 4.1. — *The cohomology of the tangent bundle of a Jacobian Raynaud quasi-elliptic surface is given by the following Table:*

$h^0(\theta_X)$	$h^1(\theta_X)$	$h^2(\theta_X)$
$h^0(\mathscr{L})$	$8\left(\frac{g-1}{3}\right)+h^1(\mathscr{L})$	$20\left(\frac{g-1}{3}\right)$

We gave examples in Section 1 of Raynaud surfaces with $h^0(\mathscr{L})>0$, therefore there exist Raynaud surfaces with non-zero vector fields.

We will call a curve which is a base curve for a Raynaud surface a Tango curve. The connection between cohomology of the tangent bundle and deformation theory suggests the following question: do Tango curves form a smooth subvariety of the moduli space of curves of genus g of dimension $8((g-1)/3)$?

Case II: Quasi-hyperelliptic surfaces. — First, assume X is a Jacobian quasi-hyperelliptic surface. Then the exact sequence $0\to 0_C\to V\to\mathscr{L}^2\to 0$ splits if ord $K_X=3$ or 6 and does not if ord $K_X=1$ or 2. Using this, the computation of the cohomology of the tangent bundle of X is straightforward.

THEOREM 4.2. — *The cohomology of the tangent bundle of a Jacobian quasi-hyperelliptic surface over an algebraically closed field of characteristic three is described by the Table below:*

ord K_X	$h^0(\theta_X)$	$h^1(\theta_X)$	$h^2(\theta_X)$
6	1	1	0
2	1	2	1
3	1	1	0
1	2	4	2

In the non-Jacobian case, we use the fact that there is an exact sequence

(3) $$0\to 0_C\to f_* 0_X(D)\to V\otimes\mathscr{L}^{-3}\to 0,$$

whose extension class is described by Theorem 3.1. The only case in which we have a twisted form is ord $K_X = 3$, so $V \simeq 0_C \oplus \mathscr{L}^2$. The extension class of exact sequence (3) is a non-zero element of $H^1(\mathscr{L}^3) = H^1(0_C)$. Therefore, we find that if X is a non-Jacobian quasi-hyperelliptic surface in characteristic 3, then $h^0(\theta_X) = 1$, $h^1(\theta_X) = 1$, $h^2(\theta_X) = 0$.

B. De Rham cohomology of quasi-hyperelliptic surfaces. — First, we need the Hodge numbers $h^{p,q} = h^q(X, \Omega^p)$, where X is a quasi-hyperelliptic surface over a field of characteristic three. These are computed easily using the methods of Section 4 A. The notations 3 J and 3 N stand for Jacobian or non-Jacobian surfaces respectively such that ord $K_X = 3$.

Theorem 4.3. — *The Hodge numbers of a quasi-hyperelliptic surface in characteristic three are given by the Table below:*

ord K_X	$h^{0,0}$	$h^{0,1}$	$h^{0,2}$	$h^{1,0}$	$h^{1,1}$	$h^{1,2}$
6	1	1	0	1	2	1
2	1	1	0	1	2	1
3 N.	1	1	0	1	2	1
3 J	1	1	0	2	4	2
1	1	2	1	2	4	2

Theorem 4.4. — *The de Rham cohomology of a quasi-hyperelliptic surface in characteristic three is given by the Table below:*

ord K_X	h^0_{DR}	h^1_{DR}	h^2_{DR}	h^3_{DR}	h^4_{DR}
6.	1	2	2	2	1
2.	1	2	2	2	1
3 N	1	2	2	2	1
3 J.	1	3	4	3	1
1.	1	3	4	3	1

Proof. — The conjugate spectral sequence $E_2^{p,q} = H^p(X, \Omega^q) \Rightarrow H^{p+q}_{DR}(X)$ degenerates in all cases but the case where K_X is trivial. In this case, we know from Theorem 3.2 that there is an exact sequence $0 \to \mathrm{Pic}^0(C) \to \mathrm{Pic}^\tau(X) \to \alpha_3 \to 0$. To find a basis for the global 1-forms on X, we use the Weierstrass equation $y_\alpha^2 z_\alpha = x_\alpha^3 + t_\alpha z_\alpha^3$, where $z_\alpha = z_\beta$ by construction, $y_\alpha = y_\beta$ since $\mathscr{L}$ is trivial, and the t_α are chosen such that $dt_\alpha = dt_\beta$ is a holomorphic global 1-form on the base curve C. Therefore a basis for the 1-forms on X is $\{dt;\ d(y/z) = -(z/y)\,dt\}$. Both these forms are exact, hence closed. The proof that $h^1_{DR}(X) = 3$ may be finished by using Oda's theorem [22], which says that on a surface such that all 1-forms are closed, $H^1_{DR}(X) = DM({}_p\mathrm{Pic}^\tau(X))$. (Our result also follows from Theorem 4.8.)

C. 1-Forms on Raynaud surfaces. — Because of the non-splitting of the sequence $0 \to 0_C \to V \to \mathscr{L}^2 \to 0$, it is difficult to compute the de Rham cohomology of Raynaud surfaces. In this section, we will show only that certain Raynaud surfaces are pathological with respect to p-adic cohomology.

We see from exact sequence (1) in Section 4 A that if X is a Raynaud surface (Jacobian quasi-elliptic surface with all fibres irreducible over a curve of genus $g > 1$), then

$H^0(\Omega^1)=H^0(\mathcal{O}_X(D)\otimes f^*K_C)$. A generator of $\mathcal{O}_X(D)\otimes f^*K_C$ near $D\cap f^{-1}(U_\alpha)$ is $dt_\alpha/y_\alpha=-dy_\alpha$.

Now

$$H^0(\mathcal{O}_X(D)\otimes f^*K_C)\simeq H^0(f^*\mathcal{O}_X(D)\otimes K_C)$$
$$\simeq H^0(f^*0_X(3S)\otimes \mathcal{L}^{-3}\otimes K_C)\oplus H^0(\mathcal{L}^3\otimes\mathcal{L}^{-3}\otimes K_C).$$

The second isomorphism comes from Lemma 4.2. The second group is made up of 1-forms pulled up from C. There are g such 1-forms. The first group fits into an exact sequence

$$0\to H^0(0_C\otimes\mathcal{L}^{-3}\otimes K_C)\to H^0(V\otimes\mathcal{L}^{-3}\otimes K_C)$$
$$\xrightarrow{\gamma} H^0(\mathcal{L}^2\otimes\mathcal{L}^3\otimes K_C)\xrightarrow{\delta} H^1(0_C\otimes\mathcal{L}^{-3}\otimes K_C).$$

The connecting homomorphism δ is cup-product with the extension class of V, which spans the 1-dimensional subspace of $H^1(\mathcal{L}^{-2})$ killed by Frobenius. Recalling the construction of V, we see that the kernel of γ is the space of 1-forms which are locally of the form $f(t_\alpha)\,dy_\alpha$, while the 1-forms which are not in the kernel of γ are of the form $f(t_\alpha)\,dy_\alpha+g(t_\alpha)x_\alpha\,dy_\alpha$, where $g(t_\alpha)\neq 0$. Since x_α and y_α are coordinates near the origin, and since $dt_\alpha=-y_\alpha\,dy_\alpha$, the closed 1-forms are precisely those pulled up from the base plus those in the kernel of γ.

Using the fact that $K_C\simeq\mathcal{L}^6$, we get the following theorem.

THEOREM 4.5. — *Let* X *be a Raynaud surface over a curve* C *of genus* $g>1$. *Then* $h^0(Z^1)=h^0(K_C)+h^0(\mathcal{L}^3)$, *where* Z^1 *is the sheaf of closed* 1-*forms, and* $h^0(\Omega_X^1)=h^0(Z^1)+\dim\ker H^0(\mathcal{L}^5)\xrightarrow{\cup a} h^1(\mathcal{L}^3)$, *where a spans the* 1-*dimensional subspace of* $h^1(\mathcal{L}^{-2})$ *killed by Frobenius.*

THEOREM 4.6. — *Let* X *be a Raynaud surface. Then if* $H^1_{DR}(X)^{alg}=H^1_{DR}(X)$, *then all closed* 1-*forms on* X *are indefinitely closed.*

THEOREM 4.7. — *There exist Raynaud surfaces* X *with closed* 1-*forms which are not indefinitely closed, therefore such that* $H^1_{DR}(X)^{alg}\subsetneq H^1_{DR}(X)$.

By $H^1_{DR}(X)^{alg}$, I mean the image of the Oda injection $DM({}_p\mathrm{Pic}(X))\to H^1_{DR}(X)$. Recall Oda's description of $H^1_{DR}(X)^{alg}$ by the following diagram,

$$\begin{array}{ccccccc}
0\to H^1(0_X) & \to & H^1_{DR}(X) & \longrightarrow & H^0(\Omega^1)_{d_2=0} & \longrightarrow & 0\\
\| & & \uparrow & & \uparrow & & \\
0\to H^1(0_X) & \to & H^1_{DR}(X)^{alg} & \to & H^0(\Omega^1)_{d_2=0}\cap H^0(\Omega^1)_{dC_\infty=0} & \to & 0
\end{array}$$

where $d_2\colon H^0(\Omega^1)\to H^2(0_X)$ is the differential in the conjugate spectral sequence of de Rham cohomology, and $H^0(\Omega^1)_{dC_\infty=0}$ denotes the space of indefinitely closed 1-forms. If α is a closed 1-form, then $dC\alpha$ is defined, where C is the Cartier operator. If this is zero, then $dC^2\alpha$ is defined, etc. 1-forms that are always closed after repeated applications of C are called indefinitely closed.

Illusie has informed me that, using the de Rham-Witt complex, Theorem 4.6 can be proved for general varieties over algebraically closed fields. However, we will give instead an elementary proof for Raynaud surfaces.

Let $f: X \to C$ be a Raynaud surface. Then $h^0(Z^1) \simeq f^* H^0(C, \Omega_C^1) \oplus H^0(\mathscr{L}^3)$. Forms in the second factor are locally of the form $f(t_\alpha)\, dy_\alpha$, where t_α is a coordinate on the base. There are $h^0(\mathscr{L})$ exact 1-forms, in the second factor, which are locally of the form $f(t_\alpha)^3\, dy_\alpha$.

We know that $h^0(\Omega^1)=h^0(Z^1)+r$, $h^0(Z^1)=g+h^0(\mathscr{L}^3)$, $h^1(0_X)= g+h^0(\mathscr{L})$. Therefore, the Hodge-de Rham spectral sequence shows that $h^1_{\text{Hodge}}-h^1_{\text{DR}} \leqq r+h^0(\mathscr{L})$.

Now assume $H^1_{DR}(X)^{\text{alg}}=H^1_{DR}(X)$ and use the conjugate spectral sequence. Our hypothesis implies that $H^0(\Omega^1)_{d_2=0} \subseteqq H^0(Z^1)$. On the other hand, I claim that d_2 restricted to the $h^0(\mathscr{L})$ dimensional space of exact 1-forms described above is injective. To see this, note that Theorem 3.2 gives us a split exact sequence

$$0 \to \operatorname{Pic}^0(C) \to \operatorname{Pic}^0(X) \to \alpha_3(H^0(\mathscr{L})) \to 0.$$

Therefore, $H^1(0_X)=f^* H^1(C, 0_C) \oplus H^0(\mathscr{L})$, and on the second factor $F=0$ and β_1 is injective, where β_1 is the first Bockstein operation. Our claim follows from:

THEOREM 4.8. — *Let* X *be a non-singular variety over a perfect field k of characteristic* $p>0$. *Then the following diagram is commutative:*

$$\begin{array}{ccc} H^0(X, B^1) & \to & H^0(X, \Omega^1) \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle d_2} \\ H^1(X, 0_X) & \underset{-\beta_1}{\to} & H^2(X, 0_X) \end{array}$$

The horizontal arrow is the obvious inclusion, β_1 *is the first Bockstein operation of Witt vector cohomology, the left vertical arrow comes from the exact sequence* $0 \to 0_X \xrightarrow{F} F_* 0_X \xrightarrow{d} B^1 \to 0$, *and the right vertical arrow is the differential in the conjugate spectral sequence of de Rham cohomology.*

Proof. — This is Theorem 1 of [14],

Therefore $h^1_{\text{Hodge}}-h^1_{\text{DR}} \geqq r+h^0(\mathscr{L})$. Putting our inequalities together, we find that $h^1_{\text{Hodge}}-h^1_{\text{DR}}=r+h^0(\mathscr{L})$, and we find a splitting

$$H^0(Z^1)=\ker d_2 \oplus H^0(\mathscr{L}).$$

Everything in the first factor is indefinitely closed by assumption, and everything in the second is exact, therefore clearly indefinitely closed. Theorem 4.6 is proved.

For Theorem 4.7, we compute $dC(f(t_\alpha)\, dy_\alpha)$ by expanding f in a power series and suppressing α in the notation. I claim that

$$dC\left(\sum_{n=0}^{\infty} a_n t^n\, dy\right)=\sum_{n=0}^{\infty} a_{3n+2}^{1/3}\, t^n\, d\, x\, dy.$$

To see this, note that $C(a_n t^{3n+3} dy) = a_n^{1/3} t^n C(t^a dy)$, $a=0, 1, 2$. Now $C(dy)=0$, $C(t\,dy) = C((y^2 - x^3)\,dy) = dy$, and $C(t^2\,dy) = C((y^4 + y^2 x^3 + x^6)\,dy) = x\,dy$. Now since $dt\,dy = 0$, our assertion follows immediately.

Since x and y are coordinates near the cusps, $dx\,dy \neq 0$. Therefore, to find a closed form which is not indefinitely closed, we need only find $f \in H^0(\mathscr{L}^3)$ such that the power series expansion of f has terms of the form $a_n t^n$, $a_n \neq 0$, $n \equiv 2 \pmod 3$.

Let C be a hyperelliptic Tango curve with affine equation of the form $z^2 = f_{3n}(w)$, f monic of degree $3n$, n odd, $n \geqq 5$, and such that $\mathscr{L} \simeq 0_C(1/2(n-1)\infty)$. For instance, we can take $f = w^{15} + w^7 + 1$. Let P be a point on C such that $z(P) \neq 0$. Then $a = w - w(P)$ is a local coordinate at P, and we may choose t in a neighborhood of P so that t is also a local coordinate at P. (Recall that t is a local integral of a nowhere vanishing 1-form.) Therefore the power series expansion of a is $bt + $(higher order), $b \neq 0$, and so the expansion of a^2 starts with $b^2 t^2$. Since a^2 has a pole of order 4 at ∞, and no other poles, it defines a section of $\mathscr{L}^3$, and therefore $a^2\,dy$ extends to a global closed form on X which is not indefinitely closed.

D. Hyperelliptic surfaces. — In this section, we compute the cohomology of the tangent and cotangent bundles of hyperelliptic surfaces. Many of these results are known (Suwa [32]; Iitaka [8]; Jensen [11]), but complete proofs do not appear in the literature. We include them so that the reader may compare the cohomology of quasi-hyperelliptic surfaces to that of hyperelliptic surfaces.

First, assume the characteristic is not 2. If X is a hyperelliptic surface, the Albanese map $f: X \to C$ is smooth, hence there is an exact sequence

$$0 \to f^*(\Omega_C^1) \to \Omega_X^1 \to \Omega_{X/C}^1 \to 0. \tag{4}$$

In fact, we see from the representation $X = E_1 \times E_2 / G$, where G is a finite group acting "diagonally" on $E_1 \times E_2$ that exact sequence (4) splits. By duality, we know that $\Omega_{X/C}^1 \simeq f^* \mathscr{L}^{-1}$, where $\mathscr{L} = R^1 f_* 0_X$. We also know that $K_X \simeq f^*(K_C \otimes \mathscr{L}^{-1})$. The order of K_X is the order of $\mathscr{L}$ and this order is 1, 2, 3, 4, or 6. Using these facts, the computation of the cohomology of the tangent and cotangent bundles is straightforward. We summarize below.

Theorem 4.9. — *The cohomology of the tangent and cotangent bundles of a hyperelliptic surface* X *over a field of characteristic* $\neq 2$ *is described by the following Table.*

ord K_X	$h^0(\theta_X)$	$h^1(\theta_X)$	$h^2(\theta_X)$	$h^{0,1}$	$h^{0,2}$	$h^{1,0}$	$h^{1,1}$	$h^{1,2}$
6	1	1	0	1	0	1	2	1
4	1	1	0	1	0	1	2	1
3	1	1	0	1	0	1	2	1
2	1	2	1	1	0	1	2	1
1	2	4	2	2	1	2	4	2

(Note that the cases ord $K_X = 3$ or 6 do not occur in characteristic three.)

THEOREM 4.10. — *Let X be as in Theorem* 4.9. *Then the de Rham cohomology of* X *is described by the following Table:*

ord K_X	$h^0_{DR}(X)$	$h^1_{DR}(X)$	$h^2_{DR}(X)$	$h^3_{DR}(X)$	$h^4_{DR}(X)$
6.	1	2	2	2	1
4.	1	2	2	2	1
3.	1	2	2	2	1
2.	1	2	2	2	1
1.	1	3	4	3	1

Proof. — The conjugate spectral sequence for de Rham cohomology degenerates in all cases but ord $K_X = 1$ (which only occurs in characteristic two or three). In this case, we use Oda's theorem and a result on the Picard scheme of a Jacobian hyperelliptic surface.

To state this result, we need some notation. We know from Bombieri-Mumford II that if X is a Jacobian hyperelliptic surface, then $X \simeq E_1 \times E_2/G$, where E_1 and E_2 are elliptic curves and G is a finite cyclic group. G acts on E_1 by translations, and $C = \text{Alb}(X) \simeq E_1/G$. On E_2, G acts as a group of group-scheme automorphisms. Let $F = E_2^G$, a finite subgroupscheme of E_2.

PROPOSITION 4.1. — *If* X *is a Jacobian hyperelliptic surface, then there is a split exact sequence*

$$0 \to \text{Pic}^0(C) \to \text{Pic}^\tau(X) \to F \to 0.$$

To prove this, we need the following result of S. T. Jensen [11].

PROPOSITION 4.2 (Jensen). — *Let* G *be a finite cyclic group acting freely on a projective variety over an algebraically closed field k. Let* $X = Y/G$. *Then there is an exact sequence*

$$0 \to G^D \to \text{Pic}(X) \to \text{Pic}(Y)^G \to 0,$$

where G^D *is the Cartier dual of* G.

Proof of Proposition 4.1. — Apply Proposition 4.2, where $Y = E_1 \times E_2$. There is an exact sequence $0 \to \text{Pic}^0(Y)^G \to \text{Pic}(Y) \to \text{NS}(Y) \to 0$, which gives an exact sequence $0 \to \text{Pic}^0(Y)^G \to \text{Pic}(Y)^G \to \text{NS}(Y)^G$. Since NS(Y) is torsion-free and discrete, the composition $\text{Pic}^\tau(X) \to \text{Pic}(Y)^G \to \text{NS}(Y)^G$ is zero. Now

$$\text{Pic}^0(Y) \simeq \text{Pic}^0(E_1) \times \text{Pic}^0(E_2).$$

The action of G on $\text{Pic}^0(E_1)$ is trivial, and since G fixes a point of E_2, we have a G-isomorphism $\text{Pic}^0(E_2) \simeq E_2$. This gives us a split exact sequence

$$0 \to \text{Pic}^0(E_1) \to \text{Pic}^0(Y)^G \to F \to 0.$$

It is clear that the following diagram commutes.

$$\begin{array}{ccccccc} 0 \to G^D \to & \text{Pic}(X) & \to & \text{Pic}^0(Y)^G & \to & 0 \\ \| & \uparrow & & \uparrow & & \\ 0 \to G^D \to & \text{Pic}^0(C) & \to & \text{Pic}^0(E_1) & \to & 0 \end{array}$$

This gives us the desired exact sequence

$$0 \to \mathrm{Pic}^0(C) \to \mathrm{Pic}(X) \to F \to 0.$$

Conclusion of the Proof of Theorem 4.9. — If X is a hyperelliptic surface with K_X trivial over a field of characteristic three, then X is Jacobian (Bombieri-Mumford II) and one computes easily that $F \simeq \alpha_3$. X has an etale cover by an abelian surface, so all 1-forms on X are closed. Therefore, Oda's theorem shows that $h^1_{DR}(X) = 3$.

Note that Theorem 4.9 and its proof go over without change to hyperelliptic surfaces in characteristic two, except for surfaces of type $a\,3$ (using the notation of Bombieri-Mumford II). Note also that Oda's theorem, together with Proposition 4.1, can be used to compute the de Rham cohomology of Jacobian hyperelliptic surfaces over fields of characteristic two.

To compute the Hodge and de Rham cohomology in case $a\,3$, $(X = E_1 \times E_2/\mu_2 . Z/2$, μ_2 acts by translations on both factors, $Z/2$ acts by translations on the first factor and by inversion on the second), we apply Proposition 4.2 to the abelian surface $A = E_1 \times E_2/\mu_2$. $\mathrm{Pic}^0(A)$ is generated by two abelian subvarieties $\mathrm{Pic}^0(E_1/\mu_2)$ and $\mathrm{Pic}^0(E_2/\mu_2)$ whose intersection is $Z/2$. The action of $Z/2$ on $\mathrm{Pic}^0(E_1/\mu_2)$ is trivial, and $Z/2$ acts by inversion on $\mathrm{Pic}^0(E_2/\mu_2)$. Therefore, we get an exact sequence $0 \to \mathrm{Pic}^0(E_1/\mu_2) \to \mathrm{Pic}^0(A)^G \to \mu_2 \to 0$. Using Proposition 4.2, we get an exact sequence $0 \to \mathrm{Pic}^0(C) \to \mathrm{Pic}(X) \to \mu_2 \to 0$. Therefore $h^1_{DR}(X) = 3$, and since C is ordinary, $F: H^1(0_X) \to H^1(0_X)$ is bijective. Since X has an etale cover by A, all 1-forms on X are closed. Therefore the Hodge-de Rham spectral sequence degenerates, and we see that $h^0(\Omega^1_X) = 1$.

There is no further difficulty in working out the cohomology of the tangent and cotangent bundles in all cases, and we leave this to the reader. We list the most important information in Theorem 4.11.

THEOREM 4.11. — *The cohomology of the tangent and cotangent bundles and the de Rham cohomology of a hyperelliptic surface* X *over a field of characteristic two is given by the following Table:*

	$h^0(\theta_X)$	$h^1(\theta_X)$	$h^0(\Omega^1)$	$h^1(\Omega^1)$	h^1_{DR}
a^1	2	4	2	4	4
a^3	1	2	1	2	3
b	1	1	1	2	2
c^1	2	4	2	4	3
d	1	1	1	2	2

E. CRYSTALLINE COHOMOLOGY. — In this section, we use the theory of the de Rham-Witt complex developed by Bloch-Deligne-Illusie to compute the crystalline cohomology of hyperelliptic surfaces, and of quasi-hyperelliptic surfaces in characteristic $\neq 2$.

If X is a hyperelliptic or quasi-hyperelliptic surface, then since the crystalline Betti numbers are the same as the l-adic Betti numbers, $b_1 = b_2 = 2$. The Poincaré duality theorem shows that the underlying W-module structure of the crystalline cohomology of any smooth projective surface is known once we know the Betti numbers and $H^2(X/W)_{\text{tors}}$. For $H^0(X/W) = H^4(X/W) = W$, $H^1(X/W)$ is torsion-free, and $H^3(X/W)_{\text{tors}} \simeq \text{Ext}^1_W(H^2(X/W), W)$. Recall that if $H^2(X/W)_{\text{tors}} \simeq \bigoplus (W/p^j)^{t_j}$, $j \geqq 1$, then

$$h^1_{\text{DR}}(X) = b_1 + \sum t_j. \quad \text{(Illusie [9] 3.4).}$$

The next proposition was inspired by Illusie's analysis of Enriques surfaces in [10], II 7.3.2.

PROPOSITION 4.3. — *Let X be an algebraic surface, and suppose the first Bockstein operation of Witt vector cohomology* $\beta_1 : H^1(0_X) \to H^2(0_X)$ *is surjective. Then* $H^2(W\,0_X) \simeq k^{p_g}$.

Proof. — Let W_n denote the sheaf of Witt vectors of length n. Since $H^2(W\,0_X) = \lim H^2(W_n)$, it is enough to show that the map $H^2(W_n) \to H^2(0_X)$ is an isomorphism for all n. Since the Bockstein operation is the connecting homomorphism in cohomology of the exact sequence $0 \to 0_X \to W_2 \to 0_X \to 0$, we have it for $n = 2$. Now we use the exact sequence $0 \to W_{n-1} \to W_n \to 0_X \to 0$. A class in $H^1(0_X)$ lifts to W_n if and only if it lifts to W_2 (since all higher Bockstein operations are necessarily zero), so we get an injection $k^{p_g} \to H^2(W_{n-1})$, which is an isomorphism by induction. Therefore $H^2(W_n)$ maps isomorphically onto $H^2(0_X)$.

Using the connection between the Bockstein operation and the Picard scheme ([17], lecture 27), we see that the hypotheses of Proposition 4.3 are are satisfied for all the surfaces in question except for one special case of type $a\,1$ in characteristic two. This is $E_1 \times E_2/Z/2$, where E_2 is supersingular. Then there is a split exact sequence $0 \to \text{Pic}^0(E_1/Z/2) \to \text{Pic}^\tau(X) \to M_2 \to 0$ (M_2 = kernel of multiplication by 2 on E_2) and the first Bockstein operation is 0. However, the second Bockstein operation is surjective, and using an argument similar to the proof of Proposition 4.3, we see that

$$H^2(W\,0_X) \simeq H^2(W_2) \simeq k \oplus k.$$

In all cases, $H^2(W\,0_X)$ is torsion and finitely generated, so the slope spectral sequence degenerates, by a result of Nygaard ([10], II 3.14). This gives an exact sequence

$$0 \to P^1 H^2(X/W) \to H^2(X/W) \to H^2(W\,0_X) \to 0.$$

But $P^1 H^2(X/W)_{\text{tors}} \simeq NS(X)_{p-\text{tors}}$ (Illusie [10], II 6.8.1), where $NS(X)_{p-\text{tors}}$ denotes the p-primary part of $\text{Pic}^\tau(X)/\text{Pic}^0(X)$. (This is an abuse of notation, since our convention is that NS(X) is torsion-free.)

Now $\text{Pic}^\tau(X)$ is computable using Proposition 4.2 and Theorem 3.2. Putting together this information with our knowledge of the de Rham cohomology of X and the second paragraph of Section 4 E, the reader may check the following result, case by case.

THEOREM 4.11. — *If* X *is a hyperelliptic surface, or a quasi-hyperelliptic surface in characteristic* $\neq 2$, *then* $\mathrm{H}^2(\mathrm{X}/\mathrm{W})_{\mathrm{tors}}$ *is killed by p, and its rank as a vector space over k is* $h^1_{\mathrm{DR}}(\mathrm{X})-2$.

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