ANNALES SCIENTIFIQUES DE L'É.N.S.

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Whitney (a)-faults which are hard to detect

WHITNEY (a)-FAULTS WHICH ARE HARD TO DETECT

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Let X, Y be adjacent strata of a stratification of a C^1 manifold by C^1 submanifolds, and let $y \in Y$. If X fails to be Whitney (a)-regular over Y at y, we say that $(X, Y)_y$ is an (a)-*fault* (*see* § 1 of [2]). Theorem A of [2] says that every (a)-fault can be detected by transverse C^1 foliations: i. e. we can find a C^1 foliation transverse to Y at y, but not transverse to X in any neighbourhood of y. In example 2 below we present an (a)-fault which cannot be detected by transverse C^2 foliations (in the notation of paragraph 2 of [2], $(\mathscr{F}^2)$ holds). This implies that Theorem A of [2] is sharp.

The construction of example 2 uses certain semialgebraic building blocks called *barrows*. Example 1 is another (a)-fault defined by means of barrows, which serves to show that the existence of (a)-faults not detectable by transverse submanifolds does not require that the sequence of points giving the (a)-fault lie on a curve tangent to the base stratum, as in [1].

In both examples X is topologically trivial along Y-in fact $X \cup Y$ is a topological manifold-with-boundary with interior X and boundary Y.

Construction 1. Hills, or Round Barrows

The examples of [1] used a simple construction of a C^1 semialgebraic hill which is useful as a building block for both examples and proofs of theorems. Consider the curve $\{x = (y^2 - 1)^2\}$ in $\mathbb{R}^2$: it has tangent parallel to the y-axis for $y = \pm 1$.

Rotating in $\mathbb{R}^3$ about the x-axis, and cutting around the circle $\{y^2 + z^2 = 1, x = 0\}$ and then inserting in the plane $\{x = 0\}$ with the disc $\{y^2 + z^2 \leq 1, x = 0\}$ removed, gives a C^1 semialgebraic manifold. The vital property of the curve $\{x = (y^2 - 1)^2\}$ which will be used again and again is that in the region $\{y^2 \leq 1\}$ the tangent to the curve is furthest from $\{x = 0\}$ when $y = \pm 1/\sqrt{3}$, and at the points $(4/9, \pm 1/\sqrt{3})$ the normal is $(1 : \pm 8/3\sqrt{3})$.
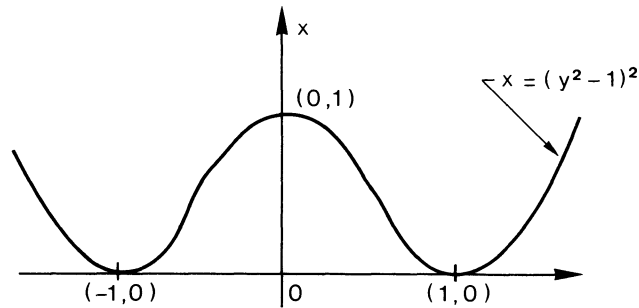
Hill of dimension one.

Construction 2.   Long Barrows

Consider the surface in $\mathbb{R}^3$ with coordinates x, y, z:

$$m^7 r^3 x = (m^2 - z^2)^2 (m^2 r^2 - y^2)^2,$$

where m, $r \in [0, \infty)$.   The normal to the surface at (x, y, z) is

$$(m^7 r^3 \colon 4(m^2 - z^2)^2 (m^2 r^2 - y^2) y \colon 4(m^2 r^2 - y^2)^2 (m^2 - z^2) z).$$

On $\{ z^2 = m^2, \, x = 0 \}$ and $\{ y^2 = m^2 r^2, \, x = 0 \}$ the normal is $(1 : 0 : 0)$,
and thus we can cut along these lines to obtain the surface

$$B(m, r) = \{ \, m^7 r^3 x = (m^2 - z^2)^2 (m^2 r^2 - y^2)^2, \; z^2 \leqq m^2, \; y^2 \leqq m^2 r^2 \}$$

and we can insert $B(m, r)$ into the plane $\{ x = 0 \}$ with a rectangle

$$\{ x = 0, \, z^2 \leqq m^2, \, y^2 \leqq m^2 r^2 \}$$

removed, to give a C^1 submanifold which is also a semialgebraic set.

At (mrx, mry, mz) for $z^2 \leqq 1$, $y^2 \leqq 1$, the normal is now

$$(1 \, \colon \, 4 y (1 - z^2)^2 (1 - y^2) \, \colon \, 4 rz (1 - z^2)(1 - y^2)^2).$$

Thus as m varies $B(m, r)$ varies in size, but the tangent structure (that is the set of points in $P^2(\mathbb{R})$ defined by the normals or tangents to the surface) remains the same.   But as r varies the normals change, and as r tends to 0 the normals tend to lie in the arc of lines

$$\left\{ \left(1 : \frac{8\lambda}{3\sqrt{3}} : 0 \right) : \lambda \in [-1, 1] \right\}.$$

We call this surface $B(m, r)$ a *(long) barrow of magnitude m, ratio r, with axis $O z$, and centre O, and base $y O z$.*   The axis, centre, and base will always be specified.   Calculation shows that for $r < \sqrt{3}/4$, the normal to the surface is furthest from $(1 : 0 : 0)$ when $y = \pm mr/\sqrt{3}$ and

$z=0$, and at these points $(4\,mr/9,\ \pm mr/\sqrt{3},\ 0)$, the normal is $(1\ :\ \pm 8/3\sqrt{3}\ :\ 0)$ (compare Construction 1).

LINGUISTIC NOTE. — The term *barrow* is used because of the resemblance of the surface to the ancient burial mounds called by that name in England, when r is small.

Recall that a C^1 submanifold X is (*t*)-regular over a C^1 submanifold Y at $y \in$ Y if every C^1 submanifold transverse to Y at y is transverse to X near y. The first horn of [1] is an (*a*)-fault which is (*t*)-regular: the sequences for which (*a*) fails lie in a region bounded by a "horn" tangent to Y, so that transversals to Y at y fail to detect them. As an illustration of the use of barrows we now give an example showing that this is not the only way in which transverse submanifolds can fail to detect an (*a*)-fault.

EXAMPLE 1. — *An* (*a*)-*fault not detectable by transverse* C^1 *submanifolds.*

In $\mathbb{R}^3$ with coordinates $(x,\ y,\ z)$ let Y be the y-axis, and let X be

$$\left(\bigcup_{n=1}^{\infty} \{f_n=0,\ g_n \leqq 0\} \right) \cup \left(\bigcap_{n=1}^{\infty} \{x=0,\ g_n \geqq 0,\ z>0\} \right)$$

where $\{f_n=0\}$ is the equation defining the barrow $B(m_n, r_n)$ with centre $(0,\ 1/n,\ 1/n)$ and axis $\{x=0,\ z+y=2/n\}$, with base in the plane $\{x=0\}$, and $\{g_n \leqq 0\}$ defines the interior of the rectangular base of the barrow. Then X is a C^1 manifold, and is semialgebraic on the complement of the origin in $\mathbb{R}^3$. We choose $\{(m_n,\ r_n)\}_{n=1}^{\infty}$ such that,

(1) r_n tends to 0 as n tends to ∞;

(2) the barrows are pairwise disjoint (in particular m_n tends to 0);

(3) m_n tends to 0 fast enough so that the n-th barrow $B(m_n,\ r_n)$ is contained in the 2-sphere with centre $(0,\ 1/n,\ 1/n)$ and radius $1/2n^2$ (so $m_n=1/4n^2$ will do).

By (1) the set of limiting normals is exactly

$$\{(1\ :\ (4\sqrt{2}/3\sqrt{3})\lambda\ :\ (4\sqrt{2}/3\sqrt{3})\lambda)\ :\ 0 \leqq |\lambda| \leqq 1\}$$

(*cf.* Construction 2). Thus (*a*) fails, since for (*a*) to hold *all* limiting normals must be of the form $(c_1:\ 0:\ c_2)$.

By (3) the set of barrows is contained in the horn which is tangent to $\{z=y,\ x=0\}$ and which intersects the plane $\{z+y=2\,t\}$ in a circle of radius t^2. Hence a C^1 submanifold S transverse to Y at 0 intersects infinitely many barrows only if $\{z=y,\ x=0\} \subset T_0 S$. But then S will be transverse to all barrows in some neighbourhood of 0. For, suppose S were nontransverse to infinitely many barrows; then $N_0 S$ would be one of the limiting

$$(1\ :\ (4\sqrt{2}/3\sqrt{3})\lambda\ :\ (4\sqrt{2}/3\sqrt{3})\lambda).$$

But $\{z=y,\ x=0\} \subset T_0 S$, and S is transverse to $\{x=0,\ z=0\}$ at 0, thus $N_0 S$ is of the form $(\mu\ :\ v\ :\ -v)$ with $v \neq 0$, which is not a limiting normal to X.

Thus we have shown that (*t*) holds and that (*a*) fails along sequences which are not tangent to Y.

By smoothing near the base of each barrow we obtain a C^∞ example.

$x = 0.$
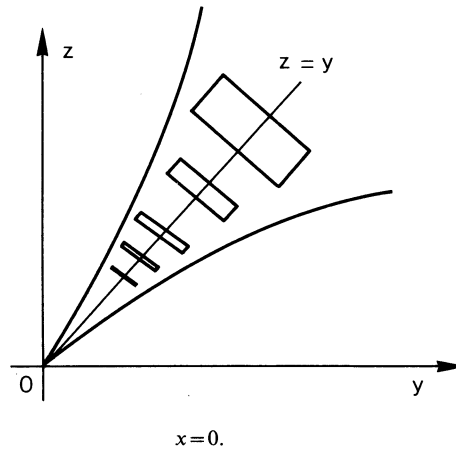
EXAMPLE 2. — An (a)-fault not detectable by transverse C^2 foliations.

In $\mathbb{R}^3$ let (x, y, z) be coordinates, and let Y be the y-axis, and let X be

$$\left(\bigcap_{n=1}^{\infty} \{ x=0, g_n \geqq 0, z>0 \} \right) \cup \left(\bigcup_{n=1}^{\infty} \{ f_n=0, g_n \leqq 0 \} \right),$$

where g_n is a function of y and z and $\{ g_n \leqq 0 \}$ intersects $\{ x=0 \}$ in a rectangle of length m_n, width $m_n r_n$, and $\{ f_n=0 \}$ defines the barrow B_n of magnitude m_n, ratio r_n, axis

$$\{ x=0, \ y+\tan(\theta_n)z=(1/2n)+(\tan \theta_n)/2n \},$$

and centre $p_n=(0, 1/2n, 1/2n)$ with base in the plane $\{ x=0 \}$.

First choose a monotonic decreasing sequence $\{ m_n \}$ such that for any choice of θ_n, and any $r_n \leqq 1$, the barrows are pairwise disjoint (and do not intersect Y). Now let δ_n be the radius of the largest 2-sphere $S^2_\delta(0)$ such that $S^2_\delta(0) \cap B_n \neq \emptyset$ when $r_n=1$ and θ_n takes all values in $[-\pi/2, \pi/2]$. Then set

$$r_n=(3\sqrt{3}/8)\delta_n^{2/3} \qquad \text{and} \qquad \theta_n=\sin^{-1}((3\sqrt{3}/8)(\delta_n^{1/3}+\delta_n^{2/3})),$$

so defining B_n completely, and hence specifying X.

[Note that $(3\sqrt{3}/4)\delta_n^{1/3}<1$, i.e. $\delta_n<64/81\sqrt{3}$, and so this choice of θ_n is possible for all $n \geqq 1$, by the choice of the centre $p_1=(0, 1/2, 1/2)$ of B_1.]

Since $\{ \delta_n \}$ is a monotonic decreasing sequence, tending to 0, both $\{ r_n \}$ and $\{ \theta_n \}$ are monotonic decreasing to 0. Thus (cf. Construction 2) the set of limiting normals to X at 0 is

$$\{ (1 : \lambda : 0) : -8/3\sqrt{3} \leqq \lambda \leqq 8/3\sqrt{3} \}.$$

Hence (a) fails at 0 for the pair (X, Y).

Suppose ($\mathscr{F}^2$) does not hold at 0 for (X, Y).   Then there is a C^2 foliation $\mathscr{F}$ which is transverse to Y at 0 and which is not transverse to X in any neighbourhood of 0. Necessarily $\mathscr{F}$ is of codimension 1 and $T_0\mathscr{F}$ (the tangent at 0 to the leaf of $\mathscr{F}$ passing through 0) must be of the form $(1 : \alpha : 0)$ where $0 < |\alpha| \leq 8/3\sqrt{3}$.

We shall show that there is a constant $C > 0$ and an n_0 such that for all $n \geq n_0$ and for all $p \in B_n$,

($\star$) $|N_p X - (1 : \alpha : 0)| > C\delta_n^{1/3}$

($N_p X$ is the normal space to X at p.)   The proof of ($\star$) will be given later.

$$x = 0.$$
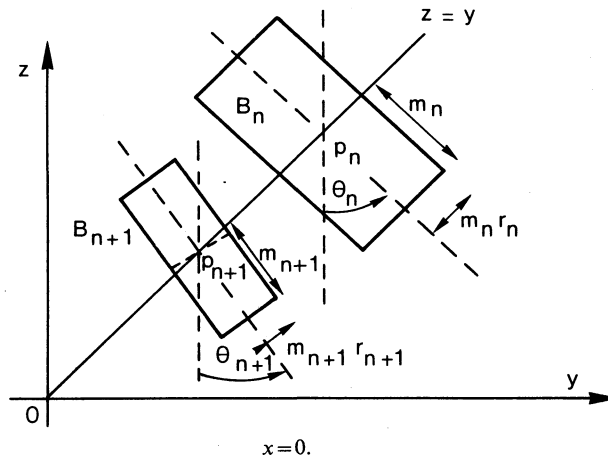
Let $\varphi : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ denote the C^2 diffeomorphism defining $\mathscr{F}$ so that the leaves of $\mathscr{F}$ are the images of $\{\mathbb{R}^2 \times u \}_{u \in \mathbb{R}}$.   Then $d\varphi(0)(\mathbb{R}^2 \times 0)$ is the plane with normal $(1 : \alpha : 0)$.

Since φ is C^2, the map $p \mapsto d\varphi(\varphi^{-1}(p))$ from $(\mathbb{R}^3, 0)$ to $(GL_3(\mathbb{R}), d\varphi(0))$ is C^1 and thus there exist $\varepsilon > 0$ and $M > 0$ such that

$$|d\varphi(\varphi^{-1}(p)) - d\varphi(0)| < M|p|,$$

for all $p \in B_\varepsilon(0)$.   It follows at once that

$$|(d\varphi(\varphi^{-1}(p)) - d\varphi(0))|_{\mathbb{R}^2 \times 0}| < M|p|,$$

for all $p \in B_\varepsilon(0)$, or in other words that

$$|T_p\mathscr{F} - T_0\mathscr{F}| < M|p|,$$

for all $p \in B_\varepsilon(0)$.

Now, by hypothesis, $\mathscr{F}$ is nontransverse to X at some point of B_n, for infinitely many n, i.e. for infinitely many n, there exists $p \in B_n$ such that $T_p\mathscr{F} = T_pX$.   Let $n_1 \geq n_0$ be such that for all $n \geq n_1$, if $p \in B_n$, then $|p| < \varepsilon$.   Then for infinitely many $n \geq n_1$, there exists $p \in B_n$ such that $M|p| > |N_p X - (1 : \alpha : 0)|$.   But assuming ($\star$) and using the choice of δ_n, we

know that for all $n \geq n_0$, and for all $p \in B_n$,

$$|N_p X - (1 : \alpha : 0)| > C |p|^{1/3}.$$

These last two inequalities are absurd, since there is some n_2 such that for all $n \geq n_2$, and for all $p \in B_n$, $|p| < (C/M)^{3/2}$, i.e. $M|p| < C|p|^{1/3}$. Thus we obtain a contradiction, showing that $(\mathscr{F}^2)$ holds, and that transverse C^2 foliations cannot detect this (a)-fault.
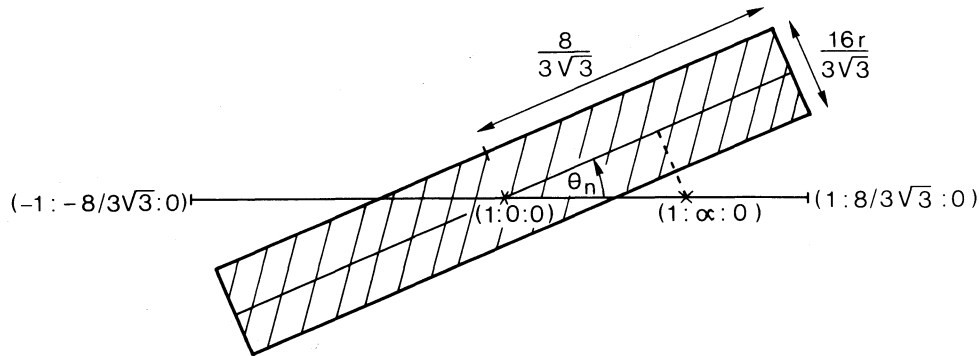

Chart for $P^2(\mathbb{R})$ at $(1 : 0 : 0)$.

Proof of $(\star)$. — A short calculation shows that for all n the set of normals to B_n (rotated back through θ_n), is contained in

$$\left\{ (1 : \lambda : \mu) : \lambda \in [-8/3\sqrt{3}, 8/3\sqrt{3}], \mu \in [-8 r_n/3\sqrt{3}, 8 r_n/3\sqrt{3}] \right\}.$$

It will suffice to establish $(\star)$ in the Euclidean norm $| \quad |_e$ in the usual chart for $P^2(\mathbb{R})$ centred at $(1 : 0 : 0)$ given by the homogeneous coordinates $(v : \lambda : \mu) \mapsto (\lambda/v, \mu/v)$, since this norm is equivalent to the standard one.

$$\left(|(1 : \lambda' : \mu') - (1 : \lambda'' : \mu'')|_e = ((\lambda' - \lambda'')^2 + (\mu' - \mu'')^2)^{1/2} \right)$$

It is evident from the figure above and the choice of r_n and θ_n that there exists n' such that for all $n \geq n'$, $(1 : \alpha : 0)$ is outside the shaded region which contains the normals to B_n. We calculate the minimal distance of $(1 : \alpha : 0)$ from a normal of B_n. This is clearly $(\alpha \sin \theta_n - 8 r_n/3\sqrt{3})$. Thus for all $n \geq n'$ and all $p \in B_n$,

$$|N_p X - (1 : \alpha : 0)|_e \geq \alpha \sin \theta_n - 8 r_n/3\sqrt{3}$$

$$= \alpha (3\sqrt{3}/8)(\delta_n^{1/3} + \delta_n^{2/3}) - \delta_n^{2/3} = \delta_n^{1/3}((3\sqrt{3}\alpha/8) - \delta_n^{1/3}(1 - (3\sqrt{3}\alpha/8))).$$

Since δ_n tends to 0 as n tends to ∞, there exists $n_0 \geq n'$ such that for all $n \geq n_0$, and all $p \in B_n$,

$$|N_p X - (1 : \alpha : 0)|_e > (3\sqrt{3}\alpha/16)\delta_n^{1/3}.$$

Thus we obtain $(\star)$.

Note. — We have in fact proved slightly more by the above example. Namely that a transverse foliation, with C^1 leaves, which is C^1 with a Lipschitz derivative in the direction transverse to the leaves, cannot detect this (a)-fault. If $(\mathscr{F}^{1,p})$ denotes the condition similar to $(\mathscr{F}^1)$ but restricting to foliations defined by a C^1 diffeomorphism C^1 along the leaves and C^p transverse to the leaves, then clearly $(\mathscr{F}^{1,p})$ implies $(\mathscr{F}^{1,q})$ if $p < q$ [and $(\mathscr{F}^{1,p})$ implies (t) for all $p \leqq \infty$]. Also it is (now) easy to construct examples showing $(\mathscr{F}^{1,q})$ does not imply $(\mathscr{F}^{1,p})$ when $p < q$. Simply set

$$\theta_n = \sin^{-1}(3\sqrt{3}(\delta_n^{p-(2/3)} + \delta_n^{2p-(4/3)})/8),$$

$$r_n = (3\sqrt{3}\,\delta_n^{2p-(1/3)})/8,$$

and repeat the argument above.

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