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LENGTH SPECTRUM FOR COMPACT LOCALLY SYMMETRIC SPACES OF STRICTLY NEGATIVE CURVATURE (1)

BY DAVID L. DEGEORGE

Introduction

Let M be a compact, connected, locally symmetric space with negative sectional curvatures. The purpose of this paper is to study the function which assigns to each real number x the number of free homotopy classes of loops in M which have a representative of length less than or equal to x. We will denote this integer by $E(x)$. We will also study a related function $\pi(x)$ which is the enumerating function for the lengths of closed geodesics (parameterized by arclength) in M. These functions are important for the study of the spectrum of M (*see* Berger [1], p. 129).

We will obtain asymptotic estimates for $E(x)$ and $\pi(x)$ together with error terms. Both the estimates and error terms will depend on standard differential-geometric data associated with M.

As is well known M is isometric to a double coset space $\Gamma\backslash G/K$; where G is a rank-one semi-simple Lie group with trivial center, Γ a discrete torsion-free co-compact subgroup, and K a maximal compact subgroup. Our approach to the problem will be through the harmonic analysis of G.

The problem appears to have been first studied by Huber [7] for the case $G = SL(2, \mathbf{R})$. A paper of McKean [11] considers this same case using a somewhat different technique of proof. Bérard-Bergery [2] examines the problem when M has constant curvature. Margulis [10] has announced a result for all negatively curved manifolds. His proof uses ergodic theory. Our result is better (the error is less) and more specific in the case at hand. All the previous authors except Huber do not have error terms. Their proofs, except for Margulis, use the Selberg trace formula and a Tauberian theorem.

(1) The results of this paper are part of the author's thesis submitted to Yale University.

Our approach also will use the trace formula. However by following (and generalizing) a method used by Langlands [9] for $G = SL(2, \mathbf{R})$, we will be able to obtain the error term for the asymptotic estimate.

A brief outline is as follows: In Section I we state the main theorem and its corollaries. In Section II we obtain the necessary geometric information and in Section III we assemble those theorems and facts from the theory of group representations we will need. In Section IV we make the trace computation and prove most of the main theorem. Section V is devoted to a calculation of curvature which completes the proof.

I would like to thank my advisors R. Szczarba and J. Arthur for suggesting the problem, their help and encouragement, and making available Langland's manuscript.

I. Statement of the Main Theorem

From this point on we shall consider M as $\Gamma\backslash G/K$ with Γ acting as isometries on the universal Riemannian covering space G/K.

Since the free homotopy classes of loops in $\Gamma\backslash G/K$ are in one to one correspondence with the conjugacy classes of Γ, we may define the non-negative real number $L(\{\gamma\})$ for each conjugacy class $\{\gamma\}$ by $L(\{\gamma\}) = L(\alpha_\gamma)$, where α_γ is the minimal representative in the free homotopy class determined by γ, and $L(.)$ is the Riemannian length. Such a minimal representative exists and in fact:

$$L(\{\gamma\}) = \inf_{x\in G/K} d(x, \gamma x).$$

Here $d(.,.)$ is the Riemannian distance function (*see* Dieudonné [4], p. 396).

Thus L is a function defined on the conjugacy classes of Γ and $E(x)$ is given by

$$E(x) = \operatorname{card}\{\{\gamma\} \mid L(\{\gamma\}) \leqq x\}.$$

Here card denotes cardinality. We will see that the conjugacy classes of Γ are given by $\bigcup_{\gamma\in P}\bigcup_{k=1}^{\infty}\{\gamma^k\}$, where P is a maximal set of non-conjugate primitive elements. An element γ is primitive if $\tau^k = \gamma$ for $\tau\in\Gamma$ implies that $k = \pm 1$. (P is also the set of periodic geodesics.) It is also true that if γ is a primitive element then $L(\{\gamma^k\}) = k\,L(\{\gamma\})$. [From now on we will abuse notation and write $L(\{\gamma\}) = L(\gamma)$.]

We consider the following functions: $(N = 1, 2, \ldots)$

$$\Phi(x) = \sum_{\substack{\gamma\in P\\ k\in N\\ kL(\gamma)\leqq x}} L(\gamma),$$

$$\theta(x) = \sum_{\substack{\gamma\in P\\ L(\gamma)\leqq x}} L(\gamma),$$

$$\pi(x) = \operatorname{card}\{\gamma : \gamma\in P\, L(\gamma)\leqq x\}.$$

The definitions of these functions are motivated by the definitions of similar functions in the elementary theory of numbers. The function $\pi(x)$ is the enumerating function for periodic geodesics.

The Riemannian metric on $\Gamma\backslash G/K$ is a positive multiple of the one induced by the Killing form of G. We shall prove our theorem for $\Gamma\backslash G/K$ when this multiple is 1, and then in the last section connect our result to other multiples. The main theorem will be stated for the general case.

Consider the following numbers:

n, dimension of $\Gamma\backslash G/K$;

ξ, sup of the sectional curvatures of $\Gamma\backslash G/K$;

δ, inf of the sectional curvatures of $\Gamma\backslash G/K$;

S, scalar curvature of $\Gamma\backslash G/K$.

We define α as follows

$$\alpha=(n-1)(-\delta)^{1/2}, \qquad \text{if } \xi=\delta$$

and

$$\alpha=\frac{4n(n-1)(-\xi)+S}{6n(-\xi)^{1/2}}=\frac{2n(n-1)(-\delta)+S}{3n(-\delta)^{1/2}}, \qquad \text{if } \xi\neq\delta.$$

We are now able to state the main theorem.

THEOREM 1. — *There is a constant* η *(depending on* Γ*) such that* $[1-(1/2n)]\,\alpha\leqq\eta<\alpha$, *and*

(i) $$\Phi(x)=\frac{e^{\alpha x}}{\alpha}+O(e^{\eta x});$$

(ii) $$\theta(x)=\frac{e^{\alpha x}}{\alpha}+O(e^{\eta x});$$

(iii) $$\pi(x)=\int_1^x\frac{e^{\alpha u}}{u}\,du+O(e^{\eta x});$$

(iv) $$E(x)=\int_1^x\frac{e^{\alpha u}}{u}\,du+O(e^{\eta x}).$$

[*Here we are using the convention that* $O(f)$ *is a function such that* $|(Of)/f|$ *is bounded.*]

The remark that η depends on Γ is significant because α depends only on G/K. Integrating (iv) by parts one easily deduces Corollary 2. (Compare Huber [7], McKean [11], Bérard-Bergery [2], Margulis [10].)

COROLLARY 2:

$$E(x)\sim\frac{e^{\alpha x}}{\alpha x}; \qquad \pi(x)\sim\frac{e^{\alpha x}}{\alpha x}.$$

(*Here* $f\sim g$ *means* $\lim_{x\to+\infty}[f(x)/g(x)]=1$.)

If we define $G_E(n)$ ($G_\pi(n)$) as the nth $L(\gamma)$, $\{\gamma\} \in \Gamma$ ($\gamma \in P$) in increasing order and counted with multiplicities, then using standard techniques one can prove:

COROLLARY 3:

$$G_E(n) \sim \log n^{1/\alpha}, \qquad G_\pi(n) \sim \log n^{1/\alpha}.$$

Remark. — The formula for α above involves only *metric* data and does not use the classification theorem. Simpler possible expressions are:

1. $\alpha = 2\|\rho\|$, when $\Gamma \backslash G/K$ has the metric from B the Killing form of G. Here ρ is as defined in Section III and $\|.\|$ is the norm induced by B.

2. $\alpha = (n+q-1)(-\xi)^{1/2}$, where $q = 0, 1, 3, 7$ according to whether G/K is a real, complex, quaternionic, or Cayley hyperbolic space.

The first step in proving Theorem 1 is to show that the estimate for Φ implies the remaining estimates.

PROPOSITION 4. — *In Theorem* 1 (i) *implies* (ii), (ii) *implies* (iii), *and* (iii) *implies* (iv).

Proof of (i) *implies* (ii). — Clearly $\Phi(x) \geqq \theta(x)$, and $\theta(x)$ is an increasing function. Let $M = \inf_{\gamma \in P} L(\gamma)$, then $M > 0$ (*see* Prop. 17). Fix x and let $h = [x/M]$. Then

$$\Phi(x) = \sum_{k=1}^{h} \sum_{\substack{\gamma \in P \\ kL(\gamma) \leqq x}} L(\gamma) = \sum_{k=1}^{h} \theta\left(\frac{x}{k}\right).$$

Thus

$$0 \leqq \Phi(x) - \theta(x) \leqq \sum_{k=2}^{h} \theta\left(\frac{x}{k}\right) \leqq \frac{x}{M}\theta\left(\frac{x}{2}\right).$$

If we assume (i) then $\theta(x) = 0(e^{\alpha x})$. Thus $\theta(x/2) = \theta(e^{(\alpha/2)x})$.

The proof that (iii) implies (iv) is similar.

In view of Proposition 18 we may write $\pi(x)$ as the Stieltjes integral

$$\pi(x) = \int_1^x \frac{1}{t}\, d\theta(t) + \text{const.}$$

If we integrate the right hand side by parts we obtain

$$(1) \qquad \pi(x) = \frac{\theta(x)}{x} + \int_1^x \frac{\theta(t)}{t^2}\, dt + \text{const.}$$

Similarly we obtain

$$\int_1^x \frac{e^{\alpha t}}{t}\, dt = \frac{e^{\alpha x}}{\alpha x} + \int_1^x \frac{e^{\alpha t}}{\alpha t^2}\, dt + \text{const.}$$

Proof of (ii) *implies* (iii). — Assuming (ii) we obtain from (1) that

$$\pi(x) = \frac{e^{\alpha x}}{\alpha x} + \frac{O(e^{\eta x})}{x} + \int_1^x \frac{e^{\alpha t}}{\alpha t^2}\, dt + \int_1^x \frac{O(e^{\eta t})}{t^2}\, dt + \text{const.}$$

Thus we have

$$\pi(x)-\int_1^x \frac{e^{\alpha t}}{t}\,dt = O(e^{\eta x})+\int_1^x \frac{O(e^{\eta t})}{t^2}\,dt = O(e^{\eta x}).$$

Thus we need only show (i), this will occupy most of the rest of this paper.

II. Setting the Stage

The basic reference for this section is Wallach [14 *a*], especially Chapter 7. We have tried to keep as close to his notation as possible.

Let $\mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition for $\mathfrak{g}$, the Lie algebra of G, with $\mathfrak{k}$ a compact Lie algebra. Let K be the connected subgroup of G corresponding to $\mathfrak{k}$, since we have assumed G has trivial center K is compact. We will identify the tangent space of G/K at *e*K with $\mathfrak{p}$. If B is the Killing form of G, then B is positive definite on $\mathfrak{p}$ and defines a G invariant Riemannian metric on G/K. With this metric the following facts are true: (*see* Helgason [6 *a*], p. 173, 205).

(1) The Levi-Civita connection has strictly negative curvature.

(2) Each pair of points in G/K is jointed by a unique geodesic.

(3) If $X \in \mathfrak{p}$ the geodesic in the direction of X through *e*K is given by $t \longrightarrow (\exp tX)K$.

Let $\Gamma \subseteq G$ be a discrete co-compact subgroup such that Γ acts without fixed points on G/K. Then $G/K \longrightarrow \Gamma\backslash G/K$ is the universal Riemannian covering (with the induced metric on $\Gamma\backslash G/K$). As was mentioned in Section I, if $\gamma \in \Gamma$ then

$$L(\gamma) = \inf_{x \in G/K} d(x, \gamma x).$$

Since d is G invariant, if $\tau = g\gamma g^{-1}$ for *any* $g \in G$ then

$$L(\gamma) = \inf_{x \in G/K} d(x, \tau x).$$

We choose a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$. Then $\dim_{\mathbf{R}} \mathfrak{a} = 1$. We choose a fixed set of positive roots Λ^+ for the pair $(\mathfrak{g}, \mathfrak{a})$. Let $H_0 \in \mathfrak{a}$ be such that $B(H_0, H_0) = 1$ and $\lambda(H_0) > 0$ for every $\lambda \in \Lambda^+$. We denote by A the subgroup corresponding to $\mathfrak{a}$. $\operatorname{Exp} | \mathfrak{a}$ is a diffeomorphism and its inverse we denote by log.

We let $\mathfrak{h}_-$ be a maximal abelian subalgebra of $\mathfrak{k}$ such that $[\mathfrak{h}_-, \mathfrak{a}] = \{0\}$, then $\mathfrak{h} = \mathfrak{h}_- \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. If $C = \{ g \mid g \in G \ \operatorname{Ad}(g)|_C = \mathrm{id} \}$ is the associated Cartan subgroup then it can be shown that $C = (C \cap K)A$. It is clear that C centralizes A.

Because G/Γ is compact, every element of Γ is semi-simple. Since G has real-rank one every semi-simple element of infinite order is conjugate to an element of C with non-trivial A component. Thus if $\gamma \in \Gamma$ then there are $u \in G$, $m_\gamma \in C \cap K$ and $a_\gamma \in A - \{e\}$, such that $m_\gamma a_\gamma = u \gamma u^{-1}$. Since there is an $m^* \in K$ such that $m^* a m^{*-1} = a^{-1}$ we may assume

$$\log a_\gamma = t H_0, \quad \text{with} \quad t > 0.$$

This is related to the geometry of $\Gamma \backslash G/K$ in the next proposition.

PROPOSITION 5. — *If* $\gamma \in \Gamma$ *and* $m_\gamma a_\gamma = u \gamma u^{-1}$ (*as above*) *then* $\log a_\gamma = L(\gamma) H_0$.

Before proving this proposition we state a lemma which can be found in Mostow ([12], p. 39).

LEMMA 6. — *Let* $g \in G$ *and* $Y \subseteq G/K$ *be such that:*

(1) $g Y = Y$.

(2) *If* $p, q \in Y$ *and* σ *is the geodesic joining* p *and* q, *then* $\sigma \subseteq Y$.

Then $\inf_{x \in G/K} d(x, gx) = \inf_{y \in Y} d(y, gy)$.

Proof of Proposition 5. — Let $Y = \{ (\exp t H_0) \mid t \in R \}$. Using our description of the geodesics though e K and the fact that $m_\gamma a_\gamma \in C$, it is easy to see that Y and $m_\gamma a_\gamma$ satisfy the hypothesis of Lemma 6. Thus we have

$$L(\gamma) = \inf_{t \in R} d((\exp t H_0) K, m_\gamma a_\gamma (\exp (t H_0)) K).$$

Using the invariance of d we obtain

$$L(\gamma) = d(e K, a_\gamma K).$$

The next proposition gives the structure of the conjugacy classes of Γ.

PROPOSITION 7. — *Let* P *be a maximal set of non-conjugate primitive elements, then:*

(i) $\bigcup_{\gamma \in P} \bigcup_{n=1}^{\infty} \{ \gamma^n \}$ *is a complete set of non-trivial conjugacy class representatives.*

(ii) *If* $\gamma \in P$ *then the centralizer of* γ (*in* Γ) *is equal to the centralizer of* γ^r *for every* r.

(iii) *If* $\gamma \in P$, *then* $L(\gamma^r) = r L(\gamma)$. $r \geqq 1$.

To prove Proposition 7 we need the following lemma of Preissmann [13].

LEMMA 8. — *Each* $\gamma \in \Gamma$ *leaves invariant a unique geodesic in* G/K.

We will use the following notation: If $h \in G$, *then*

$$G_h = \{ g : g \in G \ g h g^{-1} = h \}, \quad \text{if } \gamma \in \Gamma, \qquad \text{then} \quad \Gamma_\gamma = G_\gamma \cap \Gamma.$$

LEMMA 9:

(i) *For each* $\gamma \in \Gamma - \{ e \}$, Γ_γ *is infinite cyclic.*

(ii) *If* $\Gamma_\gamma = (\tau)$ *then* τ *is primitive.*

Proof:

(i) Imbed Γ_γ in the isometry group of R^1 via Lemma 8. By using the compactness of G/Γ it is easy to see that the image of Γ_γ in Iso (R^1) is discrete. Since Γ has no torsion this implies Γ_γ is infinite cyclic.

(ii) If $\Gamma_\gamma = (\tau)$ and $\tau = \xi^s$, then ξ commutes with γ thus $\xi = \tau^u$. Since Γ is torsion-free $u = \pm 1$.

Proof of Proposition 7:

(i) Clear.

(ii) Clear, since $a_\gamma^r = \exp r \, L(\gamma) \, H_0$, for every $r \geqq 1$.

(iii) *See* McKean ([11], Section 2.4).

III. Representation Theory and the Trace Formula

In this section we shall assume basic representation theory and facts which may be found in Wallach [14 *a*, *b*], and Borel [3]. For information about the trace formula see Wallach [14 *b*].

Most of the trace formula manipulations can be found in the preceding reference, however since we are obtaining a numerical result the normalizations chosen are important. Thus in this section we make explicit which facts and theorems we are using.

We denote by $C_c^\infty(G)$ the complex-valued C^∞ functions on G with compact support, and by $\hat{G}$ the set of equivalence classes of irreducible unitary representations of G. If $\omega \in \hat{G}$, we denote by $\theta_\omega(f)$ the number $\operatorname{tr} \pi(f)$, where $(\pi, H) \in \omega$ and

$$\pi(f) = \int_G f(g)\,\pi(g)\,dg, \qquad f \in C_c^\infty(G).$$

[$\pi(f)$ is known to be trace class.] We shall give G/Γ the invariant measure determined by Haar measure on G and counting measure on Γ. We will denote by μ the (left) regular representation of G on $L^2(G/\Gamma)$.

Since G/Γ is compact $\mu(f)$ is trace class for $f \in C_c^\infty(G)$ and moreover $L^2(G/\Gamma)$ decomposes over G as a direct sum of Hilbert spaces.

$$L^2(G/\Gamma) \cong \bigoplus_{\omega \in \hat{G}} N_\Gamma(\omega)\,\omega.$$

Each $N_\Gamma(\omega)$ is a non-negative integer. Thus we have

$$\operatorname{tr} \mu(f) = \sum_{\omega \in \hat{G}} N_\Gamma(\omega)\,\theta_\omega(f).$$

Furthermore following a standard manipulation (*see* Wallach [15 *b*]), and using Proposition 7 we may write

$$\operatorname{tr} \mu(f) = f(e)\operatorname{vol}(G/\Gamma) + \sum_{\gamma \in P} \sum_{k=1}^{\infty} \operatorname{vol}(G_{\gamma^k}/(\gamma)) \int_{G/G_{\gamma^k}} f(g\gamma^k g^{-1})\,dg.$$

To further analyze the terms in the trace formula we note that if $\gamma \in \Gamma$ is conjugate in G to $h_\gamma = m_\gamma a_\gamma$ $m_\gamma \in C \cap K$, $a_\gamma = \exp(L(\gamma) H_0)$, then we may transport the measure from G_{h_γ} to G_γ without affecting the product

$$\mathrm{vol}(G_{\gamma^k}/(\gamma)) \int_{G/G_{\gamma^k}} f(g\gamma^k g^{-1})\, dg.$$

Thus we obtain

$$\mathrm{tr}\,\mu(f) = f(e)\,\mathrm{vol}(G/\Gamma) + \sum_{\gamma \in P} \sum_{k=1}^{\infty} \mathrm{vol}(G_{h_\gamma^k}/(h_\gamma)) \int_{G/Gh_\gamma^k} f(gh_\gamma^k g^{-1})\, dg.$$

We have assumed G has trivial center and therefore is a linear group. Hence C is abelian (*see* Warner [15 *a*]). Thus we have $(h_\gamma) \subseteq C \subseteq G_{h_\gamma{}^k} \subseteq G$. Using standard techniques we obtain

$$(1) \qquad \int_{G/C} f(gh_\gamma^k g^{-1})\, dg = \mathrm{vol}(G_{h_\gamma^k}/C) \int_{G/G_{h_\gamma^k}} f(gh_\gamma^k g^{-1})\, dg,$$

and

$$(2) \qquad \mathrm{vol}(G_{h_\gamma^k}/(h_\gamma)) = \mathrm{vol}(G_{h_\gamma^k}/C) . \mathrm{vol}(C/(h_\gamma)).$$

Combining (1) and (2) we obtain

$$\mathrm{vol}(G_{h_\gamma^k}/(h_\gamma)) \int_{G/G_{h_\gamma^k}} f(gh_\gamma^k g^{-1})\, dg = \mathrm{vol}(C/(h_\gamma)) \int_{G/C} f(gh_\gamma^k g^{-1})\, dg.$$

This last equality is quite significant as we shall see shortly. However, we first make a digression concerning the Haar measure on G. As mentioned previously, the action of $\mathfrak{a}$ on $\mathfrak{g}$ may be diagonalized. For $\lambda \in \Lambda^+$ let $\mathfrak{g}_\lambda = \{ X \mid X \in \mathfrak{g}\ [H, X] = \lambda(H) X$ for every $H \in \mathfrak{a} \}$, let $\mathfrak{n} = \sum_{\lambda \in \Lambda} \mathfrak{g}_\lambda$, and let N be the connected subgroup corresponding to $\mathfrak{n}$. We denote by ρ that element of $\mathfrak{a}_{\mathbf{C}}^* = \mathrm{Hom}_{\mathbf{R}}(\mathfrak{a}, \mathbf{C})$ defined by $\rho(H) = 1/2\ \mathrm{tr}(\mathrm{ad}\, H \mid \mathfrak{n})$. The Iwasawa decomposition for G says that $G = KAN$. We shall take for Haar measure on G the measure dg such that for $f \in C_c^\infty(G)$,

$$\int_G f(g)\, dg = \iiint_{KAN} f(kan)\, e^{2\rho(\log a)}\, dn\, da\, dk.$$

Where dn is any Haar measure on N, the measure on K has volume 1, and da is such that $\int_A f(a)\, da = \int_{-\infty}^{\infty} f(\exp\ t H_0)\ dt$. ($dt$ = Lebesgue measure.) We shall give discrete groups counting measure and the measure on $C = (K \cap C) A$ will be the product of the Haar measure on $K \cap C$ with volume 1 and da. With these normalizations we have the next proposition.

PROPOSITION 10:

$$\mathrm{vol}(C/(h_\gamma)) = L(\gamma).$$

Proof. – Let U be a fundamental domain in A for a_γ ($h_\gamma = m_\gamma a_\gamma$), then it is easily seen that $C \cap K \times U$ is a fundamental domain for $m_\gamma a_\gamma$ in C. But

$$\mathrm{vol}(C \cap K \times U) = \mathrm{vol}(C \cap K).\mathrm{vol}(U)$$

and we may take for U $\exp([0, \log a_\gamma])$.

Combining this with (3) we obtain

$$\mathrm{tr}\,\mu(f) = \mathrm{vol}(G/\Gamma)f(e) + \sum_{\gamma \in P} L(\gamma) \sum_{k=1}^{\infty} \int_{G/C} f(gh_\gamma^k g^{-1})\, dg.$$

For our problem only a special kind of f need be considered. We denote by $C_c^\infty(G//K)$ the K bi-invariant functions i. e.

$$C_c^\infty(G//K) = \{f : f \in C_c^\infty(G)\ f(k_1 g k_2) = f(g), \text{ for } g \in G\ k_1, k_2 \in K\}.$$

If $f \in C_c^\infty(G//K)$, and (π, H) is a representation of G then $\pi(f)$ is particularly nice. Let

$$H^K = \{v : v \in H\ \pi(k)(v) = v, \text{ for every } k \in K\}.$$

PROPOSITION 11. – *Let $f \in C_c^\infty(G//K)$ and (π, H) be a unitary on representation of* G, *then*

(i) $\pi(f)(H) \subseteq H^K$ *and* $\pi(f)(H^{K\perp}) = (0)$.

(ii) *If* $\pi(f)$ *is of trace class then* $\mathrm{tr}\ \pi(f) = \mathrm{tr}(\pi(f)|_{H^K})$.

The proof is straightforward. Thus if $\hat{G}^K = \{\omega : (\pi, H) \in \omega \text{ and } H^K \neq (0)\}$ and if $f \in C_c^\infty(G//K)$ we have

$$\mathrm{tr}\,\mu(f) = \sum_{\omega \in \hat{G}^K} N_\Gamma(\omega)\theta_\omega(f).$$

Now for rank one groups, $\theta_\omega(f)$ (for $\omega \in \hat{G}^K$) have been classified and this is what we will discuss next.

Let $\nu \in \mathfrak{a}_{\mathbf{C}}^*$ then C^ν is the set of continuous functions f on G such that

$$f(gman) = e^{-(i\nu+\rho)(\log a)} f(g)$$

for every $g \in G$, $m \in M$, $a \in A$, and $n \in N$. Here ρ is as before and $i = \sqrt{-1}$. We define an inner product on C^ν by $\langle f_1, f_2 \rangle = \int_K f_1(k) f_2(k)\, dk$, and we let H^ν be the Hilbert space completion with respect to $\langle\, ,\, \rangle$. We let G act on C^ν by $\pi_\nu(g)f(x) = f(g^{-1}x)$, this action extends to a representation (π_ν, H^ν) which is unitary if $\mathrm{Im}\,\nu = 0$. The collection (π_ν, H^ν) is called the non-unitary principal series (corresponding to the trivial representation on M). It is easy to see that $(H^\nu)^K = \mathit{Span}_{\mathbf{C}}(\xi)_\nu$, where $\xi_\nu(kan) = e^{-(i\nu+\rho)\log(a)}$. It is known that if $f \in C_c^\infty(G//K)$ then $\pi_\nu(f)$ is trace class (*see* Wallach [14 *a*]) and one easily shows that

$$\mathrm{tr}\,\pi_\nu(f) = \int_G f(g) \int_K e^{-(i\nu+\rho)J(g^{-1}k)}\, dk\, dg = S(f)(\nu).$$

Here J is the function on G given by J $(kan) = \log a$. The right-hand side of the above equation defines a function on $\mathfrak{a}_{\mathbf{C}}^*$ known as the spherical transform of f, denoted by S (f). For this transform there is a Paley-Wiener theorem (*see* Helgason [6 *b*]).

PROPOSITION 12. — *Let* $\psi \in C^{\infty}(\mathfrak{a}_{\mathbf{C}}^*)$, *then a necessary and sufficient condition that there is an* $f \in C_c^{\infty}(G//K)$ *so that* $S(f) = \psi$ *is the following:*

There is an even C^{∞} *function with compact support* α, *such that* $\hat{\alpha} = \psi$, *where* $\hat{}$ *is the Euclidean Fourier transform.*

We can use S (f) to find tr $\mu(f)$ with the following theorem of Kostant.

PROPOSITION 13 (Kostant). — *Let* $\omega \in \hat{G}^K$, *then there is a* $\nu \in \mathfrak{a}_{\mathbf{C}}^*$ *such that*

(i) $\theta_{\omega}(f) = S(f)(\nu)$ *for every* $f \in C_c^{\infty}(G//K)$.

(ii) *If* ν *is as in* (i) *then either* $\operatorname{Im} \nu = 0$ *and* $\nu(H_0) > 0$ *or* $\nu = i\tau\rho$ *with* $0 < \tau \leqq 1$.

(iii) *The only representation with* $\tau = 1$ *is the trivial representation* (*see* Kostant [8]).

Thus there are $\{\nu_j\} \subseteq \mathfrak{a}_{\mathbf{C}}^*$, $\{\tau_l\} \subseteq (0, 1)$, *integers* N_j *and* M_l *so that*

$$\operatorname{tr} \mu(f) = \sum_{j=1}^{\infty} N_j S(f)(\nu_j) + \sum_{l=1}^{\infty} M_l S(f)(i\tau_l\rho) + S(f)(i\rho).$$

Here N_j and M_l have the obvious meaning, if ω corresponds to ν_j by Proposition 13 then $N_j = N_{\Gamma}(\omega)$.

The term S $(f)(i\rho)$ arises from the trivial representation of G which has multiplicity 1 in μ, and the fact that

$$\int_G f(g)\, dg = S(f)(i\rho).$$

The growth of the N_j and M_l will be important to us later so we record here a theorem of Gangolli.

PROPOSITION 14 (Gangolli). — (*see* Gangolli [5]):

(i) The set $\{ l : M_l \neq 0 \}$ *is finite.*

(ii) *If* $N(r) = \sum_{\| \nu_j \| \leq r} N_j$, *then* $N(r) = 0(r^n)$.

Here $n = \dim G/K$ *and* $\| \nu_j \| = | \nu_j(H_0) |$ (*see* Gangolli [5]).

The function f may be recovered from S (f) by the next proposition (the Plancherel Theorem).

PROPOSITION 15. — (*see* Helgason [6 *b*], p. 115).

There is an even continuous function C (λ) mapping $\mathfrak{a}^*$ into $\mathbf{R}^+$ such that if $f \in C_c^{\infty}(G//K)$ then

$$f(e) = \int_{\mathfrak{a}^{*+}} S(f)(\lambda)\, C(\lambda)\, d\lambda.$$

Moreover, there is a polynomial $p(s)$ of degree $n-1$ (without constant term), and an $m \in \{0, 1, -1\}$ such that $C(s\alpha) = \tanh^m s \; p(s)$. Here $\alpha \in \mathfrak{a}^*$ is such that $\alpha(H_0) = 1$, and $\mathfrak{a}^{*+} = \{\lambda : \lambda \in \mathfrak{a}^* \; \lambda(H_0) > 0\}$.

The trace formula can be simplified further with the aid of the next proposition (*see* Wallach [14*a*], p. 182).

PROPOSITION 16. — *Let* $D(ma) = e^{-\rho(\log a)} \left| \det(\mathrm{Ad}(ma) - I|_{\mathfrak{n}}) \right|$, $ma \in MA$, *then if* $D(ma) \neq 0$ *we have*

$$\int_{G/C} f(vmav^{-1})\, dv = \frac{e^{\rho(\log a)}}{D(ma)} \int_N f(an)\, dn.$$

Remark. — Wallach leaves the measure on C undetermined, however, it is easy to see that the correct measure to insure Proposition 16 is the product of the measure on A and a measure on K so that vol $(K \cap C) = 1/|Z|$, $|Z|$ = cardinality of the center. Thus our choice of measure is correct.

By diagonalizing $\mathrm{Ad}(m_\gamma a_\gamma)$ (on $\mathfrak{n}$) over **C** and using the fact that the eigenvalues of $\mathrm{Ad}(m_\gamma)$ must lie on the unit circle it is easy to see that $D(m_\gamma a_\gamma) \neq 0$, for $\gamma \in \Gamma - \{e\}$. The right hand side of the equation in Proposition 16 is related to $S(f)$ in the next proposition.

PROPOSITION 17. — (*See* Helgason [6*b*], p. 117):

$$S(f)(\nu) = \iint_{AN} e^{(\rho - i\nu)(\log a)} f(an)\, da\, dn, \qquad f \in C_c^\infty(G//K).$$

By using this proposition and the Fourier inversion theorem on $\mathbf{R}^1$ we obtain

$$e^{\rho(\log a)} \int_N f(an)\, dn = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ist}\, S(f)(s\alpha)\, ds.$$

Where $a = \exp tH_0$, $\alpha(H_0) = 1$.

Combining this equation, (4), and Proposition 15 we obtain

$$(5) \qquad \sum_{j=1}^{\infty} N_j S(f)(\nu_j) + \sum_{l=1}^{M} M_l S(f)(i\tau_l \rho) - \mathrm{vol}(G/\Gamma) \int_{\mathfrak{a}^{*+}} S(f)(\lambda) C(\lambda)\, d\lambda + S(f)(i\rho)$$
$$= \sum_{\gamma \in P} L(\gamma) \sum_{k=1}^{\infty} \frac{1}{D(m_\gamma^k a_\gamma^k)} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iskL(\gamma)}\, S(f)(s\alpha)\, ds.$$

We shall use this formula to solve our problem in the next section.

IV. Trace Computation

To facilitate our computation we use the various terms in the trace formula identifying letters. Let $f_T \in C_c^\infty(G//K)$. We define:

$$A_T = \sum_{j=1}^{\infty} N_j S(f_T)(v_j),$$

$$B_T = \sum_{l=1}^{M} M_l S(f_T)(i\tau_l \rho),$$

$$C_T = S(f_T)(i\rho),$$

$$D_T = \sum_{\gamma \in P} L(\gamma) \sum_{k=1}^{\infty} \frac{1}{2\pi D(m_\gamma^k a_\gamma^k)} \int_{-\infty}^{\infty} e^{iskL(\gamma)} S(f_T)(s\alpha)\, ds,$$

and

$$E_T = \operatorname{vol}(G/\Gamma) \int_0^\infty S(f_T)(s\alpha) C(s\alpha)\, ds.$$

Suppose for heuristic purposes that there was a function $f_T \in C_c^\infty(G//K)$ so that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iskL(\gamma)} S(f_T)(s\alpha)\, ds = D(m_\gamma^k a_\gamma^k) \chi_T(kL(\gamma)).$$

Here χ_T denotes the characteristic function of the interval $[-T, T]$. Then D_T would be $\Phi(T)$ (*see* Section I), thus we could estimate $\Phi(T)$ from A_T, B_T, C_T, and E_T. Unfortunately such a f_T does not exist since χ_T is not C^∞, moreover it is not clear how to include the dependence of $D(m_\gamma a_\gamma)$ on m_γ. We can approximate $\chi_T \cdot D(m_\gamma a_\gamma)$, and the difficulties with $D(m_\gamma a_\gamma)$ will "disappear asymptotically".

It is easy to see that

$$D(ma) = \prod_{\lambda \in \Lambda^+} \left| \det(e^{-\lambda/2 (\log a)} I - e^{\lambda/2 (\log a)} \rho_\lambda(m) \right|$$

where $\rho_\lambda(m) = \operatorname{Ad}(m)|\mathfrak{g}_\lambda$. Let $\lambda_0 \in \Lambda^+$ be such that $1/2\,\lambda_0 \notin \Lambda^+$, then it is well known that $\Lambda^+ \subseteq \{\lambda_0, 2\lambda_0\}$. Let $p = \dim \mathfrak{g}_{\lambda_0}$, $q = \dim \mathfrak{g}_{2\lambda_0}$. By diagonalizing $\operatorname{Ad}(m_\gamma)$ on $\mathfrak{g}_{\lambda_0}$ ($\mathfrak{g}_{2\lambda_0}$) we may find complex numbers $\{{}_jZ_\gamma\}$ $1 \leqq j \leqq p+q$ of modulus 1 so that

$$D(m_\gamma^k a_\gamma^k) = \beta_1(m_\gamma^k a_\gamma^k) \cdot \beta_2(m_\gamma^k a_\gamma^k),$$

with

$$\beta_1(m_\gamma^k a_\gamma^k) = \prod_{j=1}^{p} \left| e^{kL(\gamma)\lambda_0(H_0)/2} - {}_jZ_\gamma^k e^{-kL(\gamma)\lambda_0(H_0)/2} \right|$$

and

$$\beta_2(m_\gamma^k a_\gamma^k) = \prod_{j=p+1}^{p+q} \left| e^{kL(\gamma)\lambda_0(H_0)} - {}_jZ_\gamma^k e^{-kL(\gamma)\lambda_0(H_0)} \right|.$$

We now turn our attention to a point that was left hanging in the proof of Proposition 4.

PROPOSITION 18. — $\{L(\gamma)\}_{\gamma \in s.c.}$ *has no finite point of accumulation.*

Proof. — Suppose not and let φ be a C^∞ positive, even, compactly supported function which is 1 on a neighbourhood of the accumulation point. By the Paley-Wiener Theorem (Prop. 12) there is a function $f_\varphi \in C_c^\infty(G//K)$ such that

$$\varphi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ist} S(f_\varphi)(s\alpha)\,ds.$$

This implies D_{f_φ} is infinite, which is a contradiction.

We now begin our computation. Let φ be a non-negative C^∞ function of a real variable such that

(1) support of $\varphi \subseteq [-1, 1]$,

(2) φ is even,

(3) $\int_{-\infty}^{\infty} \varphi(x)\,dx = 1.$

Let $T > 0$ and fix ε, $0 < \varepsilon < 1$. (We shall choose ε precisely later.) Consider the family $\varphi_T(x) = e^{\varepsilon T}\varphi(e^{\varepsilon T}x)$, $T > 0$; it is easy to see that

$$(\varphi_T \bigstar \chi_T)(x) = \int_{-\infty}^{\infty} \varphi_T(y)\chi_T(x-y)\,dy$$

is an even C^∞ function with compact support. Let

$$g_T(t) = (\varphi_T \bigstar \chi_T)(t)\,\tilde{\beta}_1(t)\,\tilde{\beta}_2(t)$$

with

$$\tilde{\beta}_1(t) = 2^p \cos h^p\left(\frac{t\lambda_0(H_0)}{2}\right)$$

and

$$\tilde{\beta}_2(t) = 2^q \cos h^q(t\lambda_0(H_0)),$$

then $g_T(t)$ is also even, C^∞, and has compact support. Thus there is a function $f_T \in C_c^\infty(G//K)$ such that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ist} S(f_T)(s\alpha)\,ds = g_T(t),$$

and

$$S(f_T)(\nu) = \int_{-\infty}^{\infty} e^{-it\nu(H_0)} g_T(t)\,dt.$$

Our first task will be to estimate B_T and C_T. We will do this with the following lemma.

LEMMA 19:

$$\int_{-\infty}^{\infty} e^{at}(\varphi_T \bigstar \chi_T)(t)\,dt = \begin{cases} \dfrac{e^{|a|T}}{|a|} + O(e^{(|a|-\varepsilon)T}), & a \neq 0, \\ O(1), & a = 0. \end{cases}$$

The proof of the lemma is straightforward.

We estimate B_T first, recall

$$B_T = \sum_{l=1}^{M} S(f_T)(i\tau_l \rho), \qquad 0 < \tau_l < 1.$$

LEMMA 20. — *Let* $\tau = \max_l \{\tau_l\}$, *then* $B_T = O(e^{(\tau+1)\|\rho\|T})$.

Note. — We use the convention that if $\xi \in \mathfrak{a}_{\mathbf{C}}^*$ then $\|\xi\| = |\xi(H_0)|$.

Proof. — It clearly suffices to show that

$$S(f_T)(i\tau_l \rho) = O(e^{(\tau+1)\|\rho\|T}).$$

Now

$$S(f_T)(i\tau_l \rho) = \int_{-\infty}^{\infty} e^{-i\tau_l \rho(H_0)t}\, \tilde{\beta}_1(t)\,\tilde{\beta}_2(t)(\varphi_T \bigstar \chi_T)(t)\,dt.$$

Also $\tilde{\beta}_1(t)\cdot\tilde{\beta}_2(t)$ is a polynomial in e^t and e^{-t} say $\tilde{\beta}_1(t).\tilde{\beta}_2(t) = \sum_{j=1}^{r} c_j e^{\sigma_j t}$.

It is easy to see that

$$|\sigma_j| \leqq p\frac{\lambda_0(H_0)}{2} + q\lambda(H_0) = \frac{p+2q}{2}\lambda_0(H_0) = \|\rho\|.$$

Thus the lemma follows from Lemma 19.

LEMMA 21. — *If* $\varepsilon \leqq \lambda_0(H_0)/2$, *then*

$$C_T = \frac{e^{2\|\rho\|T}}{2\|\rho\|} + O(e^{(2\|\rho\|-\varepsilon)T}).$$

Proof. — If we expand $\tilde{\beta}_1(t)\cdot\tilde{\beta}_2(t)$ as before, the largest positive exponent of e^t is $\|\rho\|$ and the smallest negative exponent is $-\|\rho\|$. Moreover each of these occurs with co-efficient 1. Thus we have

$$C_T = \int_{-\infty}^{\infty} e^{2\|\rho\|t}(\varphi_T \bigstar \chi_T)(t)\,dt + \sum_{j=2}^{r} c_j \int_{-\infty}^{\infty} e^{(\|\rho\|+\sigma_j)t}\,dt.$$

Applying lemma 19 again we have

$$C_T = \frac{e^{2\|\rho\|T}}{2\|\rho\|} + O(e^{(2\|\rho\|-\varepsilon)T}) + O(e^{\theta T}).$$

Where $\theta = \max_j |\,\|\rho\| + \sigma_j|$.

Now it is easy to see that

$$\left| \|\rho\| + \sigma_j \right| \leqq \|\rho\| + \frac{(p+2q-1)\lambda_0(H_0)}{2} = 2\|\rho\| - \frac{\lambda_0(H_0)}{2}.$$

Thus if $\varepsilon \leqq \lambda_0(H_0)/2$:

$$O(e^{\theta T}) = O(e^{(2\|\rho\| - \varepsilon)T}).$$

Next we will estimate A_T and E_T; recalling Propositions 14 and 15, we see that both A_T and E_T are of the form $\int_0^\infty S(f_T)(s\alpha)\, dF(s)$. Where $F(s)$ is a non-negative function of bounded variation such that $F(0) = 0$ and $F(s) = O(s^n)$.

LEMMA 22. – *If* $L_T = \int_0^\infty S(f_T)(s\alpha)\, dF(s)$, $F(0) = 0$, $F(s) = O(s^n)$, F *a non-negative function of bounded variation,* $n = \dim G/K$, *then*

$$L_T = O(e^{(\|\rho\| + (n-1)\varepsilon)T}).$$

Proof. – A simple calculation shows that

$$S(f_T)(s\alpha) = \sum_{j=1}^{r} \frac{c_j}{\sqrt{2\pi}} \hat{\varphi}_T(i\sigma_j + s)\hat{\chi}_T(i\sigma_j + s),$$

with $\{c_j\}$, $\{\sigma_j\}$ as before, $\hat{}$ the Euclidean Fourier transform. Since

$$\hat{\chi}_T(u) = \begin{cases} \dfrac{\sin 2Tu}{u}, & u \neq 0, \\ 2T, & u = 0, \end{cases}$$

and $\hat{\varphi}_T(u) = \hat{\varphi}(e^{-\varepsilon T} u)$, obvious estimates show that there are constants M_1, and M_2 such that

$$|\hat{\chi}_T(i\sigma_j + s)| \leqq \frac{M_1 e^{|\sigma_j|T}}{1 + |s|}, \quad \text{for every } s, T > 0$$

and

$$|\hat{\varphi}_T(i\sigma_j + s)| \leqq M_2 |\hat{\varphi}(e^{-\varepsilon T} s)| \leqq M_2.$$

Moreover since φ is in $C_c^\infty(\mathbf{R})$ there is an M_3 so that $|\hat{\varphi}(u)| \leqq M_3(1 + |u|)^{-(n+1)}$. Thus there is an M_4 such that

$$|\hat{\varphi}_T(i\sigma_j + s)\hat{\chi}_T(i\sigma_j + s)| \leqq \frac{M_4 e^{|\sigma_j|T}}{(1+s)(1+e^{-\varepsilon T}s)^{n+1}}.$$

Uniformly in $|\sigma_j|$, T, and $s\ (s > 0)$. If we observe that $\max_j |\sigma_j| = \|\rho\|$, we obtain a constant M_5 so that

$$|L_T| \leqq M_5 e^{\|\rho\| T} \left\{ \int_0^{e^{\varepsilon T}} \frac{dF(s)}{1+s} + \int_{e^{\varepsilon T}}^\infty \frac{dF(s)}{(1+s)(1+e^{-\varepsilon T}s)^{n+1}} \right\}.$$

Integrating by parts and using the hypothesis on F we obtain

$$\int_0^{e^{\varepsilon T}} \frac{dF(s)}{1+s} = O(e^{(n-1)\varepsilon T}).$$

Using the fact that $e^{-\varepsilon T} \geqq (1+s)^{-1}$ for $s \geqq e^{\varepsilon T}$, integrating by parts, making the substitution $u = e^{-\varepsilon T} s$, and using the hypothesis on F, we obtain

$$0 \leqq \int_{e^{\varepsilon T}}^{\infty} \frac{dF(s)}{(1+e^{-\varepsilon T} s)^{n+1}(1+s)} \leqq O(e^{(n-1)\varepsilon T}) + e^{-\varepsilon T} \int_1^{\infty} \frac{e^{n\varepsilon T} u^n}{(1+u)^{n+2}} du,$$

thus $L_T = O(e^{(\| \rho \| + (n-1)\varepsilon) T})$.

A summary of our estimates is:

$$B_T = O(e^{(\tau+1)\| \rho \| T}), \qquad \tau = \max_l \{\tau_l\},$$

$$C_T = \frac{e^{2\| \rho \| T}}{2\| \rho \|} + O(e^{(2\| \rho \| - \varepsilon) T}), \qquad \text{if} \quad \varepsilon \leqq \frac{\| \lambda_0 \|}{2},$$

A_T and E_T are

$$O(e^{(\| \rho \| + (n-1)\varepsilon) T}).$$

Now $0 < \tau_l < 1$ and thus $(\tau+1)\| \rho \| < 2\| \rho \|$.

We next choose ε, the choice of ε depends on whether $q = 0$ or $q \neq 0$. If $q = 0$, a routine verification (*see* the curvature calculation) shows that if $\varepsilon = \| \rho \|/n$ then $\varepsilon \leqq \| \lambda_0 \|/2$. Moreover this choice minimizes both $2\| \rho \| - \varepsilon$ and $\| \rho \| + (n-1)\varepsilon$. If $q \neq 0$, we choose $\varepsilon \leqq \| \lambda_0 \|/2$. We need to verify that $(n-1)\varepsilon < \| \rho \|$. Now $\| \rho \| = (p+2q)/2 \| \lambda_0 \|$, and $n-1 = p+q$, thus $\| \rho \| + (n-1)\varepsilon < 2\| \rho \|$. Thus letting

$$\eta = \max\{\tau, 2\| \rho \| - \varepsilon, \| \rho \| + (n-1)\varepsilon\},$$

we obtain

$$D_T = \frac{e^{2\| \rho \| T}}{2\| \rho \|} + O(e^{\eta T})$$

and $\eta < 2\| \rho \|$.

We are now going to estimate D_T. Let

$$\xi(T) = \sum_{\gamma \in P} L(\gamma) \sum_{k=1}^{\infty} \frac{\tilde{\beta}_1(kL(\gamma))\tilde{\beta}_2(kL(\gamma))}{D(m_\gamma^k a_\gamma^k)} \chi_T(kL(\gamma)).$$

LEMMA 22:

$$\xi(T) = \frac{e^{2\| \rho \| T}}{2\| \rho \|} + O(e^{\eta T}).$$

Proof. — The idea is trap χ_T between $\varphi_{T_1} \bigstar \chi_{T_1}$ and $\varphi_{T_2} \bigstar \chi_{T_2}$ without changing the estimate. Elementary considerations show that if we choose T_1 such that $T_1 + e^{-\varepsilon T_1} = T$ then $\varphi_{T_1} \bigstar \chi_{T_1} \leqq \chi_T$, and T_2 such that $T_2 - e^{-\varepsilon T_2} = T$ then

$$\chi_T \leqq \varphi_{T_2} \bigstar \chi_{T_2}.$$

Thus $D_{T_1} \leqq \xi(T) \leqq D_{T_2}$. Hence it suffices to show that

$$D_{T_i} - \frac{e^{2\|\rho\| T}}{2\|\rho\|} = O(e^{\eta T}), \qquad i = 1, 2.$$

Now

$$D_{T_2} = O(e^{\eta T_2}) + \frac{e^{2\|\rho\| T_2}}{2\|\rho\|},$$

thus

$$D_{T_2} - \frac{e^{2\|\rho\| T}}{2\|\rho\|} = O(e^{\eta T_2}) + \frac{e^{2\|\rho\| T_2}}{2\|\rho\|} - e^{2\|\rho\|(T_2 - e^{-\varepsilon T_2})}.$$

It is clear that $1 - e^{2\|\rho\| e^{-\varepsilon T_2}} = O(e^{-\varepsilon T_2})$,
thus

$$D_{T_2} - \frac{e^{2\|\rho\| T}}{2\|\rho\|} = O(e^{\eta T_2}),$$

since $\eta \geqq 2\|\rho\| - \varepsilon$. Because $T_2 \leqq T+1$ we have $\theta(e^{\eta T_2}) = \theta(e^{\eta T})$. A similar argument works for D_{T_1}.

We next observe that if $z \in \mathbf{C}$ and $|z| = 1$, then

$$1 \leqq \frac{e^{\alpha} + e^{-\alpha}}{|e^{\alpha} - e^{-\alpha} z|}, \quad \text{for every real } \alpha > 0.$$

Thus $\xi(T) \geqq \Phi(T)$, (Φ as in Section I).

Let

$$\Delta(T) = \int_0^T tanh^p\left(\frac{\|\lambda_0\|}{2} t\right) tanh^q(\|\lambda_0\| t)\, d\xi(t).$$

Integrating by parts obtains

$$\Delta(T) = \xi(T)\, tanh^p\left(\frac{\|\lambda_0\|}{2} T\right) tanh^q(\|\lambda_0\| T) + \int_0^T \xi(t)\, W(t)\, dt.$$

Here we have set

$$W(t) = \frac{d}{dt}\left(tanh^p\left(\frac{\|\lambda_0\|}{2} t\right) tanh^q(\|\lambda_0\| t)\right).$$

An easy estimate shows that if $\alpha > 0$ and $k \in \mathbf{N}$, then $\tanh^k \alpha t = 1 + O(e^{-2\alpha t})$. Using this and the fact that $\eta \geqq 2\|\rho\| - (\|\lambda_0\|/2)$, we obtain

$$\Delta(T) = \frac{e^{2\|\rho\| T}}{2\|\rho\|} + O(e^{\eta T}) + \int_0^T \xi(t) W(t)\, dt.$$

Now it is easy to see that

$$\int_0^T \xi(t) W(t)\, dt = \int_0^T O(e^{(2\|\rho\| - \|\lambda_0\|)t})\, dt.$$

This uses the estimate for ξ and the fact that $\operatorname{sech}^2 \alpha t = O(e^{-2\alpha t})$. By our choice of η we obtain

$$\Delta(T) = \frac{e^{2\|\rho\| T}}{2\|\rho\|} + O(e^{\eta T}).$$

Computing $\Delta(T)$ from the definition and using the fact that if $|z| = 1$ and $\alpha > 0$ then $|e^{\alpha} - z e^{-\alpha}| \geqq e^{\alpha} - e^{-\alpha}$ we obtain

$$\Delta(T) \leqq \Phi(T) \leqq \xi(T).$$

This completes the proof of the estimate and the main theorem.

V. Some Calculations

In this section we calculate the sectional curvatures of $\Gamma \backslash G/K$. Clearly it suffices to calculate the sectional curvatures in $T_{eK}(G/K) = \mathfrak{p}$. Let $X, Y \in \mathfrak{p}$ be orthonormal with respect to B. If $S = \mathit{Span}\{X : Y\}$, then the sectional curvature of S, K(S) is given by $K(S) = B([X, Y], [X, Y])$ (*see* Helgason [6 *a*] p. 206). Let $\xi = \sup_{S \subseteq \mathfrak{p}} K(S)$, and let $\delta = \inf_{S \subseteq \mathfrak{p}} K(S)$.

Now $B([X, Y], [X, Y]) = -B(\operatorname{ad}(X)^2(Y), Y)$. Fix $X \in \mathfrak{a}$ so that $B(X, X) = 1$, since K is transitive on $\mathfrak{p}$ this is sufficient, then $\operatorname{ad}(X)^2$ leaves $\mathfrak{p}$ invariant and is semi-simple. Thus $\xi = \sup_{\substack{Y \in \mathfrak{a}\perp \\ \|Y\|=1}} -B(\operatorname{ad}(X)^2(Y), Y) = -T$ where T is the smallest non-zero eigenvalue of $\operatorname{ad}(X)^2$ on $\mathfrak{p}$, similarly $\delta = -L$ where L is the largest eigenvalue of $\operatorname{ad}(X)^2$ on $\mathfrak{p}$. By using the Cartan involution on $\mathfrak{g}$ it is easy to see

$$T = \min_{\lambda \in \Lambda^+} \lambda(X)^2 \quad \text{and} \quad L = \max_{\lambda \in \Lambda^+} \lambda(X)^2.$$

Case 1 ($q = 0$):

$$B(X, X) = 1 \quad \text{implies} \quad \lambda(X)^2 = \frac{1}{2} p.$$

Thus in this case the sectional curvature has constant value $-(1/2)p$.

Case 2 ($q \neq 0$). — In this case $2\lambda_0$ is the largest root, thus $\xi = -[1/(2p+8q)]$, and $\delta = -[4/(2p+8q)]$.

We next compute $2\|\rho\|$. If $q = 0$ then $\|\lambda_0\| = (2p)^{-1/2}$, thus $2\|\rho\| = (p/2)^{1/2}$. The dimension of $\Gamma\backslash G/K$ is $p+1$ thus putting things together

$$\alpha = (\dim \Gamma\backslash G/K - 1)\sqrt{-\text{curvature}},$$

(*see* Theorem 1).

When $q \neq 0$, then

$$2\|\rho\| = \frac{p+2q}{(2p+8q)^{1/2}}.$$

Now for a rank-one locally-symmetric space of negative curvature with Riemannian metric Q, we have $Q = \lambda B$ for $\lambda > 0$. If S_Q is the scalar curvature for Q, then $S_Q = n/\lambda$ where $n = \dim \Gamma\backslash G/K$. When $Q = B$ one checks directly that

$$2\|\rho\| = \frac{4(n-1)n(-\xi)+1}{6n(-\xi)^{1/2}}.$$

By using the fact that $\lambda = n/S_Q$ and the way α and ξ depend on λ one obtains the formulas for α in Theorem 1 and the remark following Corollary 3.

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