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DUALITY IN THE FLAT COHOMOLOGY OF A SURFACE

By J. S. Milne (*)

Let $\pi : X \to S$ be a smooth proper morphism whose fibres are of dimension m, let μ_{l^n} be the sheaf of l^nth roots of unity on X, and let $\mu_{l^n}(r) = \mu_{l^n}^{\otimes r}$. The Poincaré duality theorem for étale cohomology states that, for any prime l different from the residue characteristics of S, the sheaves $R^i \pi_* \mu_{l^n}(r)$ and $R^{2m-i} \pi_* \mu_{l^n}(m-r)$ are dual on S. In this paper we show that, if the étale cohomology is replaced by flat cohomology, it is possible to prove analogous results for p-torsion sheaves, where p is the characteristic of S. The motivation for such theorems comes from the study of both the arithmetic, and the geometry, of surfaces. In proving that Tate's conjecture for a surface over a finite field implied the Artin-Tate conjecture, it was necessary to use such a duality theorem [13]. In his study of the geometry of families of K 3-surfaces in characteristic p, Artin found it necessary to conjecture, and assume, such theorems [2].

For a surface over a finite field, the flat duality theorem has precisely the same form as the étale duality theorem, viz. it states that the group $H^i(X_{f1}, \mu_{p^n})$ is finite, and is dual to $H^{5-i}(X_{f1}, \mu_{p^n})$. The main difficulty in extending this result to higher dimensional varieties is in finding a satisfactory definition of $\mu_{p^n}(r)$. It seems unlikely that the obvious definition, $\mu_{p^n}(r) = \mu_{p^n}^{\otimes r}$, is the correct one. However in paragraph 1 below we define sheaves $\nu(r)$ of F_p-modules for the étale topology on X with the properties that, $H^i(X_{et}, \nu(r))$ is finite, $H^i(X_{et}, \nu(r))$ is dual to $H^{m+1-i}(X_{et}, \nu(m-r))$, and $H^i(X_{et}, \nu(1)) \approx H^{i-1}(X_{f1}, \mu_p)$. Thus, for a surface, the étale duality theorem for $\nu(1)$ corresponds to the flat duality theorem for μ_p. For a higher dimensional variety, the étale duality theorem for $\nu(r)$ should be thought of as the étale image of a flat duality theorem (which may not exist).

For a surface over an algebracally closed field, the flat duality theorem takes on a very different form from its étale counterpart. In this case the cohomology groups $H^i(X_{f1}, \mu_p)$ are finite for $i = 0, 1, 4$ but for $i = 2, 3$ there may be a subgroup of $H^i(X_{f1}, \mu_p)$ which is in a natural way a vector space over the ground field. Thus H^2 and H^3 may be infinite, which, in particular, prevents H^3 from being dual to H^1. The correct result, which was conjectured by M. Artin, is that there is a mixed finite group-vector space duality, under which the vector space part of H^3 is the linear dual of H^2 and the finite part of H^i is the dual of the finite part of H^{4-i}. To state this more precisely, recall that a scheme is said to be perfect if its absolute Frobenius is an isomorphism. The perfect p-power-

torsion group schemes over a perfect field form an abelian category in which every object is an extension of finite étale groups and vector groups. The functor

$$G \mapsto G^{\vee} = R\,Hom\,(G,\,\mathbf{Q}_p/\mathbf{Z}_p)$$

is an auto-duality of the derived category, under which a vector group is taken to its linear dual and a finite group to its Pontryagin dual. For a surface proper and smooth over a perfect field we show that $R\,\pi_*\,\mu_{p^n}$ is representable (on perfect schemes over k) by a complex of perfect groups and that $R\,\pi_*\,\mu_{p^n} \approx (R\,\pi_*\,\mu_{p^n})^{\vee}\,[-4]$. More generally, we prove such a result for a family of surfaces over a perfect base scheme.

The steps in the proof are as follows. (1) Grothendieck's duality theorem for the Zariski cohomology of a coherent O_X-module may be regarded as a duality theorem for the étale cohomology; this simply says that the Zariski and étale cohomologies of such a sheaf agree, and that the duality theorem behaves well with respect to étale localization. (2) By using a theorem of Breen on the vanishing of sheaf Exts, it is possible to interpret the theorem in (1) as giving a duality of $\mathbf{F}_p$-sheaves. (3) The étale sheaves $v\,(r)$ are defined as sub-sheaves of the sheaves of differentials, and (2) is used to prove a duality for them. (4) By replacing the differentials by Bloch's sheaves of typical curves on K-groups, it is possible to extend the duality in (3) to a duality of $\mathbf{Z}/p^n\,\mathbf{Z}$-sheaves. (5) Finally we inter-prete $H^i\,(X_{fl},\,\mu_{p^n})$ as the étale cohomology group $H^{i-1}\,(X_{et},\,v_n\,(1))$ and read off the required result from (4).

The duality when the ground field is finite is much more elementary than the general case, and is proved in paragraphs 1, 3, and 4. The results of paragraph 5 (duality with $\mathbf{Z}/p^n\,\mathbf{Z}$ coefficients, $n > 1$, over an arbitrary perfect base scheme) depend upon an as yet unproved axiom (5.1) which would follow from an extension of some of Bloch's results from perfect fields to perfect base rings.

This paper owes much to M. Artin, who was the first person to understand what the structure of the groups $H^i\,(X_{fl},\,\mu_{p^n})$ should be when the ground field is not finite. It is also a pleasure to thank S. Bloch, L. Breen and P. Deligne for conversations on some of these questions.

1. Duality Modulo p: Case of a Finite Base Field

Let S be a perfect scheme of characteristic $p \neq 0$ and let $\pi : X \to S$ be a smooth morphism whose fibres are all of dimension m. As S is perfect, the S-scheme $\pi^{(p/S)} : X^{(p/S)} \to S$ may be (and will be) identified with $F_{abs}^{-1} \circ \pi : X \to S$ where F_{abs} denotes the absolute Frobenius on S. Once this identification has been made, the relative Frobenius F of X/S becomes identified with the absolute Frobenius on X. Thus $F : X \to X^{(p)} (= X)$ is the identity map on the underlying topological spaces and is the map $f \mapsto f^p$ on sheaves. It makes O_X into a locally-free $O_{X^{(p)}} (= O_X)$-module of rank p^m. The functor F_* is exact on étale sheaves and takes locally-free O_X-modules to locally-free $O_{X^{(p)}}$-modules. [If M is an O_X-module, then $F_*\,M$ is the O_X-module with the same underlying sheaf of abelian groups but on which $f \in O_X$ acts as $f^p = F\,(f)$].

We shall need the notion of a Cartier operator acting in a slightly more general situation than usual.

LEMMA 1.1. — *There is a unique family of additive maps* $C : \Omega^r_{X/S,\, d=0} \to \Omega^r_{X/S}$ *with the properties:*

(a) $C(1) = 1$;

(b) $C(f^p \omega) = f\, C(\omega)$, $f \in O_X$, $\omega \in \Omega^r_{X/S,\, d=0}$;

(c) $C(\omega \wedge \omega') = C(\omega) \wedge C(\omega')$, ω, ω' *closed*;

(d) $C(\omega) = 0$ *if and only if* ω *is exact*;

(e) $C(f^{p-1}\, df) = df$.

Proof. — (a) and (b) imply that $C(f^p) = f$ for all $f^p \in O_X^p = \Omega^0_{X/S,\, d=0}$. Every closed differential 1-form is locally a sum of exact differentials and differentials of the form $f^{p-1}\, df$, and so (d) and (e) imply that C is uniquely determined on $\Omega^1_{X/S,\, d=0}$. Now (c) implies that C is uniquely determined on all $\Omega^r_{X/S,\, d=0}$.

For the existence, we define

$$C = W^{-1} \circ C_l \qquad \text{where} \quad C_l : \ F_*(\Omega^r_{X/S,\, d=0}) \to \Omega^r_{X^{(p)}/S}$$

is the $O_{X^{(p)}}$-linear Cartier operator defined, for example, in [12] (7.2) and

$$W : F_*(\Omega^r_{X/S}) \to \Omega^r_{X^{(p)}/S}$$

is the identity map (it is p-linear as a map of $O_{X^{(p)}}$-modules). It is easy to check that C has all the required properties.

REMARK 1.2. — As S is perfect, $\Omega^r_{X/S} = \Omega^r_{X/F_p}$.

For any integer $r \geq 0$ we define $v(r) = \text{Ker}\,(\Omega^r_{X,\, d=0} \xrightarrow{\ 1-C\ } \Omega^r_X)$. It is to be regarded as a sheaf on the small étale site of X. Clearly $v(0)$ is the constant sheaf $Z/p\,Z$, and $v(r) = 0$ for $r > m$.

LEMMA 1.3. — *The sequence*

$$0 \to v(r) \xrightarrow{\ i\ } \Omega^r_{X/S,\, d=0} \xrightarrow{\ 1-C\ } \Omega^r_{X/S} \to 0$$

is exact (relative to the étale topology).

Proof. — Let $x_1, \ldots, x_m$ be a system of local coordinates for X/S in a neighborhood U of some point P on X, and let $u_i = x_i - 1$. Then any $\omega \in \Gamma(U, \Omega^r_{X/S})$ can be written in the form

$$\omega = \sum f_{(j)} \frac{du_{j_1}}{u_{j_1}} \wedge \ldots \wedge \frac{du_{j_r}}{u_{j_r}}$$

near P. As

$$(1-C)\left(g^p \frac{du_1}{u_1} \wedge \ldots \wedge \frac{du_r}{u_r} \right) = (g^p - g)\frac{du_1}{u_1} \wedge \ldots \wedge \frac{du_r}{u_r},$$

in order to get ω in the image of $1-C$ one only has to be able to solve the Artin-Schreier equations $T^p - T = f_{(j)}$. This can be done by passing to the étale covering defined by these equations, which shows that $1-C$ is surjective, and that the sequence is exact.

REMARK 1.4. — It is probable that $v(r)$ is the additive subsheaf of $\Omega^r_{X/S, d=0}$ generated locally (for the étale topology) by differentials of the form $df_1/f_1 \wedge \ldots \wedge df_r/f_r$, although I have only written out a proof of this in the case that $r = 2$ and S is the spectrum of a field. (For $r = 1$ it is due to Cartier.) This would imply the existence of an exact sequence of étale sheaves

$$K_r \, O_X \xrightarrow{d \log \wedge \, d \log \wedge \, \ldots} \Omega^r_{X, d=0} \xrightarrow{C-1} \Omega^r_X \to 0,$$

which one hopes extends to an exact sequence

$$0 \to K_r \, O_X \xrightarrow{p} K_r \, O_X \to \Omega^r_{X, d=0} \to \Omega^r_X \to 0.$$

LEMMA 1.5. — *There is an exact commutative diagram,*

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
d\Omega^{r-1} \xrightarrow{\;\;1\;\;} d\Omega^{r-1} \\
\downarrow & & \downarrow \\
0 \to v(r) \xrightarrow{i} \Omega^r_{d=0} \xrightarrow{1-C} \Omega^r \longrightarrow 0 \\
\| & \quad \downarrow{c} & \downarrow \\
0 \to v(r) \xrightarrow{i} \Omega^r \xrightarrow{C^{-1}-1} \Omega^r/d\Omega^{r-1} \to 0 \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

Proof. — $C : \Omega^r_{d=0} \to \Omega^r$ is surjective with kernel $d\Omega^{r-1}$ (this may be proved directly, or by using the corresponding facts for C_l [12] (7.2)). Thus it induces an isomorphism $\Omega^r_{d=0}/d\,\Omega^{r-1} \to \Omega^r$ and we write C^{-1} for the inverse map $\Omega^r \to \Omega^r/d\Omega^{r-1}$. The commutativity of the top square follows from (d) of (1.1), and that of the other two squares is obvious (the map $\Omega^r \to \Omega^r/d\Omega^{r-1}$ at right is the canonical map onto the quotient). The exactness of the two columns is now clear, and that of the middle row is proved in (1.3). The exactness of the last row follows from the snake lemma.

LEMMA 1.6. — *If* $M \times N \to \Omega^m_{X/S}$ *is a bilinear non-degenerate pairing of locally-free* O_X*-modules of finite rank, then*

$$F_* M \times F_* N \to F_* \Omega^m_{X/S} \xrightarrow{C_l} \Omega^m_{X^{(P)}/S}$$

is a bilinear non-degenerate pairing of $O_{X^{(P)}}$*-modules.*

Proof. — This is easy to check directly. Alternatively it may be interpreted as a statement of Grothendieck duality for the finite morphism $F : X \to X^{(p)}$.

LEMMA 1.7. — *The pairing*

$$\langle \omega, \tau \rangle = C_l(\omega \wedge \tau) : F_*(\Omega^r_{X/S,\, d=0}) \times F_*(\Omega^{m-r}_{X/S}/d\Omega^{m-r}_{X/S}) \to \Omega^m_{X^{(p)}/S}$$

is a bilinear non-degenerate pairing of locally-free $O_{X^{(p)}}$-modules of finite rank.

Proof. — The freeness assertion follows from the fact that locally, the map $F_* \Omega^r \xrightarrow{d} F_* \Omega^{r+1}$ is of the form $(V^r \xrightarrow{d} V^{r+1}) \otimes_{F_p} O_{X^{(p)}}$ where $d : V^r \to V^{r+1}$ is a linear map of finite-dimensional F_p-vector spaces. Consider the diagram,

$$
\begin{array}{ccc}
F_*(\Omega^r_{X/S}) & \times & F_*(\Omega^{m-r}_{X/S}) \\
\downarrow{\scriptstyle(-1)^{r-1}d} & \uparrow{\scriptstyle d} & \searrow \\
& & \Omega^m_{X^{(p)}/S} \\
F_*(\Omega^{r+1}_{X/S}) & \times & F_*(\Omega^{m-r-1}_{X/S}) \nearrow
\end{array}
$$

in which both pairings send (ω, τ) to $C_l(\omega \wedge \tau)$. The diagram commutes because

$$\langle \omega, d\tau \rangle - \langle (-1)^{r-1} d\omega, \tau \rangle = C_l(d(\omega \wedge \tau)) = 0.$$

As d is $O_{X^{(p)}}$-linear, and the two pairings are non-degenerate (1.6), there is a non-degenerate pairing induced on the kernel and cokernel, which is exactly the required pairing.

Let

$$X^\bullet = (X^0 \xrightarrow{d_X} X^1), \qquad Y^\bullet = (Y^0 \xrightarrow{d_Y} Y^1)$$

and

$$Z^\bullet = (Z^0 \xrightarrow{d_Z} Z^1)$$

be complexes. A pairing

$$X^\bullet \times Y^\bullet \to Z^\bullet$$

is a system of pairings

$$\langle \,,\, \rangle^0_{0,0} : X^0 \times Y^0 \to Z^0,$$
$$\langle \,,\, \rangle^1_{0,1} : X^0 \times Y^1 \to Z^1,$$
$$\langle \,,\, \rangle^1_{1,0} : X^1 \times Y^0 \to Z^1,$$

such that

$$d_Z \langle x, y \rangle^0_{0,0} = \langle x, d_Y y \rangle^1_{0,1} + \langle d_X x, y \rangle^1_{1,0}$$

for all $x \in X^0$, $y \in Y^0$. Such a pairing is the same thing as a mapping

$$X^\bullet \otimes Y^\bullet \to Z^\bullet$$

and induces a mapping

$$\varphi^\bullet : X^\bullet \to \mathrm{Hom}^\bullet(Y^\bullet, Z^\bullet),$$

with

$$\varphi^0(x) = \langle x, - \rangle^0_{0,0} + \langle x, - \rangle^1_{1,0},$$
$$\varphi^1(x) = -\langle x, - \rangle^1_{0,1}.$$

LEMMA 1.8. — *The pairings*

$$\langle \omega, \tau \rangle^0_{0,0} = \omega \wedge \tau : \Omega^r_{d=0} \times \Omega^{m-r} \to \Omega^m,$$

$$\langle \omega, \tau \rangle^1_{1,0} = \omega \wedge \tau : \Omega^r \times \Omega^{m-r} \to \Omega^m,$$

$$\langle \omega, \tau \rangle^1_{0,1} = C(\omega \wedge \tau) : \Omega^r_{d=0} \times \Omega^{m-r}/d\Omega^{m-r-1} \to \Omega^m$$

define a pairing of complexes

$$X^\cdot \times Y^\cdot \to Z^\cdot,$$

with

$$X^\cdot = (\Omega^r_{d=0} \xrightarrow{1-C} \Omega^r),$$

$$Y^\cdot = (\Omega^{m-r} \xrightarrow{C^{-1}-1} \Omega^{m-r}/d\Omega^{m-r-1}),$$

$$Z^\cdot = (\Omega^m \xrightarrow{1-C} \Omega^m).$$

Proof. — This is trivial to verify, using (1.1 c).

Now restrict to the case that X is proper and smooth over a perfect field k (so S = spec k). From Grothendieck duality theory [8] we get a trace map $t : H^m(X, \Omega^m) \xrightarrow{\approx} k$, and the map $k \to k$ induced by $C : H^m(X, \Omega^m) \to H^m(X, \Omega^m)$ is $F^{-1} = (a \mapsto a^{1/p})$ [because $C : H^m(X, \Omega^m) \to H^m(X, \Omega^m)$ is dual to $F : H^0(X, O_X) \to H^0(X, O_X)$, see (1.16)]. Thus, if k is finite, there is a commutative diagram,

$$
\begin{array}{ccccccc}
H^m(X, \Omega^m) & \xrightarrow{1-C} & H^m(X, \Omega^m) & \longrightarrow & H^{m+1}(X, \nu(m)) & \to & 0 \\
\downarrow t & & \downarrow t & & \downarrow \eta & & \\
k & \xrightarrow{1-F^{-1}} & k & \xrightarrow{tr_{k/F_p}} & Z/p\,Z & \longrightarrow & 0
\end{array}
$$

in which the top row is the cohomology sequence of the sequence in (1.3), tr_{k/F_p} denotes the trace map from k to F_p, and η is the unique isomorphism making the diagram commute.

THEOREM 1.9. — *Let* X *be a proper smooth variety of dimension m over a finite field k. The pairing*

$$H^i(X, \nu(r)) \times H^{m+1-i}(X, \nu(m-r)) \to Z/p\,Z,$$

defined by η *and the pairing*

$$(\omega, \tau) \mapsto \omega \wedge \tau : \nu(r) \times \nu(m-r) \to \nu(m)$$

is a non-degenerate pairing of finite groups for all i.

Proof. — One should observe first that, because of (1.1 c), the pairing

$$(\omega, \tau) \mapsto \omega \wedge \tau : \Omega^r \times \Omega^{m-r} \to \Omega^m$$

does map $\nu(r) \times \nu(m-r)$ into $\nu(m)$. For definiteness, we shall take the pairing to be defined using the Yoneda product (for a discussion of such things, see [7]).

Consider the diagram,

$$\ldots \longrightarrow H^{i-1}(\Omega^r) \longrightarrow H^i(v(r)) \longrightarrow H^i(\Omega^r_{d=0}) \longrightarrow H^i(\Omega^r) \longrightarrow \ldots$$
$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$
$$\ldots \to H^{m+1-i}(\Omega^{m-r})^* \to H^{m+1-i}(v(m-r))^* \to H^{m-i}(\Omega^{m-r}/d\Omega^{m-r-1})^* \to H^{m-i}(\Omega^{m-r})^* \to \ldots$$

The top row is the cohomology sequence of the sequence in (1.3). As Ω^r is a coherent O_X-module and $\Omega^r_{d=0}$ (or, rather $F_* \, \Omega^r_{d=0}$) is a coherent $O_{X(p)}$-module, two out of three terms in the top row are finite abelian groups. It follows that $H^i(X, v(r))$ is finite for all i and r.

The stars on the lower sequence mean that we have taken (Pontryagin) duals of the finite abelian groups. This sequence is the dual of the cohomology sequence of the bottom row of (1.5), and so is exact. The maps

$$H^i(\Omega^r) \to H^{m-i}(\Omega^{m-r})^* \qquad \text{and} \qquad H^i(\Omega^r_{d=0}) \to H^{m-i}(\Omega^{m-r}/d\Omega^{m-r-1})^*$$

come from the pairings $\langle ., . \rangle^1_{1,0}$ and $\langle ., . \rangle^1_{0,1}$ of (1.8) and the map

$$H^m(\Omega^m) \xrightarrow{t} k \xrightarrow{tr} \mathbf{Z}/p\,\mathbf{Z}.$$

The first map is an isomorphism because $\Omega^r \times \Omega^{m-r} \to \Omega^m$ is a perfect pairing for Grothendieck duality. The second is an isomorphism because of (1.6) and the commutativity of

$$\langle ., . \rangle^1_{0,1} : \Omega^r_{d=0} \quad \times \quad \Omega^{m-r}/d\Omega^{m-r-1} \longrightarrow \Omega^m_{X/S}$$
$$\| \qquad\qquad\qquad \| \qquad\qquad \approx \downarrow w$$
$$C_l(. \wedge .) : F_*(\Omega^r_{d=0}) \times F_*(\Omega^{m-r}/d\Omega^{m-r}) \to \Omega^m_{X(p)/S}$$

(The vertical maps are isomorphisms, or equalities, of the sheaves as sheaves of abelian groups.) The map $H^i(v(r)) \to H^{m+1-i}(v(m-r))^*$ is that induced by the pairing in the statement of the theorem. To prove the theorem, it remains to show that the diagram commutes. The commutativity of the two left-hand squares may be checked, for example, using (2.2) and (2.3) of [7], and the right-hand square may be checked directly from (1.8).

Alternatively, one may identify the maps

$$H^i(\Omega^r) \to H^{m-i}(\Omega^{m-r})^* \qquad \text{and} \qquad H^i(\Omega^r_{d=0}) \to H^{m-i}(\Omega^{m-r}/d\Omega^{m-r+1})^*$$

with the isomorphisms given by Grothendieck duality for $X \to \text{spec } F_p$. Then $H^i(v(r)) = H^i(X^\cdot)$, $H^j(v(m-r)) = H^j(Y^\cdot)$, and η is a map $H^m(Z^\cdot) \to \mathbf{Z}/p\,\mathbf{Z}$, where $X^\cdot$, $Y^\cdot$ and $Z^\cdot$ are the complexes in (1.8). The theorem now follows immediately from the usual formalism.

COROLLARY 1.10. — *Let* X *be a projective smooth surface over a finite field* k. *There are canonical pairings*

$$H^i(X_{fl}, \mu_p) \times H^{5-i}(X_{fl}, \mu_p) \to \mathbf{Z}/p\,\mathbf{Z},$$

which are non-degenerate pairings of finite groups for all i.

Proof. — Let $f : X_{fl} \to X_{et}$ be the obvious morphism from the flat site over X (say, the large f. p. p. f. site) to the étale site over X. From the exact sequence on X_{fl},

$$0 \to \mu_p \to G_m \xrightarrow{F} G_m \to 0,$$

we get can exact sequence

$$0 \to G_m \xrightarrow{F} G_m \to R^1 f_* \mu_p \to 0.$$

For any U étale over X, there is an exact sequence of Zariski sheaves on U,

$$0 \to O_U^* \xrightarrow{p} O_U^* \xrightarrow{d \log} \Omega^1_{U/k, d=0} \xrightarrow{1-C} \Omega^1_{U/k}.$$

Thus we get an exact sequence of étale sheaves,

$$0 \to G_m \xrightarrow{F} G_m \to \Omega^1_{X/k, d=0} \xrightarrow{1-C} \Omega^1_{X/k} \to 0$$

and hence an isomorphism $R^1 f_* \mu_p \xrightarrow{\approx} \nu(1)$. Thus

$$H^i(X_{et}, \nu(1)) \xrightarrow{\approx} H^{i+1}(X_{fl}, \mu_p)$$

and the corollary follows.

COMPATIBILITIES 1.11. — (*a*) For any étale map $\pi : X' \to X$, $\pi^* \nu(r)_X \approx \nu(r)_{X'}$, and hence there is a map

$$\pi^* : \quad H^i(X, \nu(r)) \to H^i(X', \nu(r)).$$

Also $\pi_* \pi^* (\nu(r)) \to \nu(r)$ induces a "trace" map

$$\pi_* : \quad H^i(X', \nu(r)) \to H^i(X, \nu(r)).$$

If we write $((a, b) \mapsto a.b)$ for the pairing of the theorem, then the usual formula, $d(\pi^* a.b) = a.\pi_* b$ holds, where $d = \deg(\pi)$,

$$a \in H^i(X, \nu(r)) \quad \text{and} \quad b \in H^{m+1-i}(X', \nu(m-r)).$$

This applies, in particular, when $X' = X \otimes_k k'$, k'/k finite.

(*b*) Let $\bar{k}$ be the algebraic closure of k, and let $\Gamma = \mathrm{Gal}(\bar{k}/k)$. Define

$$\eta_{\bar{k}} : H^m(X, \nu(m)) \xrightarrow{\approx} \mathbf{Z}/p\,\mathbf{Z}$$

so that the following diagram commutes:

$$H^{m-1}(\bar{X}, \Omega^m) \xrightarrow{0} H^m(\bar{X}, \nu(m)) \to H^m(\bar{X}, \Omega^m) \xrightarrow{1-C} H^m(\bar{X}, \Omega^m) \to 0$$
$$\approx \downarrow \eta_{\bar{k}} \qquad \approx \downarrow t \qquad \approx \downarrow t$$
$$0 \longrightarrow \mathbf{Z}/p\,\mathbf{Z} \longrightarrow k \xrightarrow{1-F^{-1}} k \longrightarrow 0.$$

The Hochschild-Serre spectral sequence for $\bar{X} = X \otimes \bar{k}/X$ gives an isomorphism $\varphi : H^m(\bar{X}, \nu(m))_\Gamma \to H^{m+1}(X, \nu(m))$ where, for any Γ-module M, $M_\Gamma = M/(\sigma-1)\,M$,

σ = generator of Γ. If X is projective, this map can be explicitly described as follows :
if $\alpha \in H^{m+1}(X, \nu(m))$ is represented by a Čech cocycle $(a_{i_0 \dots i_{m+1}})$ then, as

$$H^{m+1}(\overline{X}, \nu(m)) = 0,$$

$(a_{i_0 \dots i_{m+1}})$ can be written as a coboundary $\delta(b_{i_0 \dots i_m})$ over $\overline{X}$; the cocycle $(b^{\sigma}_{i_0 \dots i_m} - b_{i_0 \dots i_m})$ represents $\varphi^{-1}(\alpha)$ in $H^m(\overline{X}, \nu(m))_\Gamma$.

The following diagram commutes:

$$\begin{array}{ccc}
H^m(\overline{X}, \nu(m))_\Gamma & \xrightarrow{\varphi} & H^{m+1}(X, \nu(m)) \\
\downarrow{\eta_{\overline{k}}} & & \downarrow{\eta} \\
\mathbf{Z}/p\,\mathbf{Z} & = = = = = & \mathbf{Z}/p\,\mathbf{Z}
\end{array}$$

(c) Let $\overline{k}$ be any algebraically closed field, and let X be a projective smooth surface over $\overline{k}$. There is a commutative diagram:

$$\begin{array}{ccc}
CH^2(X) & \xrightarrow{Hg^2} & H^2(X, \Omega^2) \\
& {}^{B^2}\searrow \quad \nearrow {}_{d\log \wedge d\log} & \\
& H^2(X_{Zar}, K_2 O_X) &
\end{array}$$

where $CH^2(X)$ is the group of zero-cycles modulo rational equivalence, Hg^2 is the cycle map defined, for example, in [8], and $B^2 : CH^2(X) \to H^2(X_{Zar}, K_2 O_X)$ is the isomorphism defined in [3]. It follows that the image of Hg^2 is contained in $H^2(X, \nu(2))$. The map $\eta_{\overline{k}} \circ Hg^2 : CH^2(X) \to \mathbf{Z}/p\,\mathbf{Z}$ is the degree map (mod p).

REMARK 1.12. — It would be interesting to know if, for all schemes X of characteristic p, there exist canonical sheaves $\mu(r)$ on X_{fl} and canonical pairings

$$\mu(r) \times \mu(s) \to \mu(r+s)$$

such that:

(a) $\mu(0) = \mathbf{Z}/p\,\mathbf{Z}, \qquad \mu(1) = \mu_p ;$

(b) $R^i f_* \mu(r) = 0, \qquad i \neq r,$

$$= \nu(r), \qquad i = r,$$

where $f : X_{fl} \to X_{et}$ is the obvious morphism of sites;

(c) the pairing $\nu(r) \times \nu(s) \to \nu(r+s)$ defined by $\mu(r) \times \mu(s) \to \mu(r+s)$ and (b) is that defined in (1.9). One might even hope that $\mu(r) = \text{Ker}(K_r O_X \xrightarrow{p} K_r O_X)$ will have these properties, although the experts assure me that such a hope is naive.

REMARK 1.13. — It would be interesting to verify that the pairing in (1.10) is the same as that obtained by fibring X over a curve, $X \to C$, and using the auto-duality of the Jacobian of X/C (see [1]). If this could be done, it would be possible to extend (1.10) to a duality for the sheaves μ_{p^n} without using Bloch's theory [4].

2. Duality Modulo p: Case of a Perfect Base Scheme

Throughout this section S will be a perfect scheme of characteristic p. For any S-scheme T, we get a sequence of S-morphisms,

$$T \xleftarrow{F_{abs}} T^{(1/p)} \xleftarrow{F_{abs}} T^{(1/p^2)} \longleftarrow \ldots$$

where $T^{(1/p^i)}$ is the S-scheme $F^i_{abs} \circ g : T \to S$. We define T^{pf} to be the S-scheme which is the inverse limit of this system i. e. $T^{pf} = \varprojlim T^{(1/p^i)}$. This limit exists because F_{abs} is an affine morphism, and if T is affine then so also is T^{pf}. We define Pf/S to be the category of all perfect S-schemes. For any S-schemes T and U with U perfect, the canonical map $T^{pf} \to T$ defines an isomorphism $\mathrm{Hom}_S (U, T^{pf}) \xrightarrow{\approx} \mathrm{Hom}_S (U, T)$. Thus, for any commutative group scheme G over S, the functor $U \mapsto G (U) : Pf/S \to Ab$ is represented by the object G^{pf} of Pf/S. If L is a locally-free sheaf of O_S-modules of finite rank, then we also use L to denote the vector group which represents the functor of S-schemes, $T \mapsto \Gamma (T, L \otimes_{O_S} O_T)$. Thus $L^{pf} \in \mathrm{Ob} (Pf/S)$ represents the same functor on perfect S-schemes. If $L^{\cdot}$ is a complex of locally-free O_S-modules of finite rank then $L^{\check{\;}}$ denotes the linear dual $\mathrm{Hom}_{O_S} (L^{\cdot}, O_S)$ of the complex.

$(Pf/S)_{et}$ will mean the site whose underlying category is Pf/S and which has the étale topology. We write $\mathscr{S}(p)$ for the category of sheaves of F_p-modules on $(Pf/S)_{et}$, and $\mathscr{S}$ for the category of all sheaves. Thus, for example, a locally-free sheaf L of O_S-modules of finite rank defines an element of $\mathscr{S}(p)$, which we still denote by L.

PROPOSITION 2.1. — *For any bounded above complex $L^{\cdot}$ of locally-free O_S-modules of finite rank, there is a canonical isomorphism,*

$$L^{\check{\;}} \to R \, Hom_{\mathscr{S}(p)}(L^{\cdot}, Z/p \, Z)[1],$$

in the derived category $D (\mathscr{S}(p))$ *of* $\mathscr{S}(p)$.

Proof. — The exact sequence $0 \to Z/p \, Z \to O_S \xrightarrow{F-1} O_S \to 0$ defines a quasi-isomorphism of complexes $Z/p \, Z \to (O_S \xrightarrow{F-1} O_S)$. Let $(O_S \xrightarrow{F-1} O_S) \to I^{\cdot}$ be a quasi-isomorphism into a complex whose objects are injective and let $O_S \to (O_S \to O_S)$ be the map of complexes which sends O_S to the second O_S of $(O_S \to O_S)$ by the identity map. These maps induce the maps in the following diagram:

$$R \, Hom_{\mathscr{S}(p)}(L^{\cdot}, Z/p \, Z)[1] \to R \, Hom_{\mathscr{S}(p)}(L^{\cdot}, O_S \to O_S)[1] \to Hom_{\mathscr{S}(p)}(L^{\cdot}, I^{\cdot})[1]$$
$$\uparrow$$
$$L^{\check{\;}} = Hom_{O_S}(L^{\cdot}, O_S).$$

As the horizontal maps are isomorphisms in $D (\mathscr{S}(p))$, we get a functorial map

$$L^{\check{\;}} \to R \, Hom_{\mathscr{S}(p)}(L^{\cdot}, Z/p \, Z)[1].$$

In showing that the map is an isomorphism, we may assume $L^{\cdot} = L$ is a complex with only one non-zero object [11] (I, 7.1). Thus it remains to show:

(a) $$\mathrm{Ext}^i_{\mathscr{S}(p)}(L, \mathbf{Z}/p\,\mathbf{Z}) = 0, \qquad i \neq 1;$$

(b) $$L^{\vee} \xrightarrow{\approx} \mathrm{Ext}^1_{\mathscr{S}(p)}(L, \mathbf{Z}/p\,\mathbf{Z}).$$

Since these statements are local, we may assume that L is free, then that $L = O_S$, and finally that S is affine. In this case, we will prove the slightly stronger statement:

(a') $$\mathrm{Ext}^i_{\mathscr{S}(p)}(O_S, \mathbf{Z}/p\,\mathbf{Z}) = 0, \qquad i \neq 1;$$

(b') $$\Gamma(S, O_S^{\vee}) \xrightarrow{\approx} \mathrm{Ext}^1_{\mathscr{S}(p)}(O_S, \mathbf{Z}/p\,\mathbf{Z}).$$

LEMMA 2.2 (L. Breen). — *Let S be a perfect affine scheme. For any* $i \neq 0$,

$$\mathrm{Ext}^i_{\mathscr{S}(p)}(O_S, O_S) = 0.$$

Proof [6]. — (One may also refer to [5], where it is shown that $\mathrm{Ext}^i(\mathbf{G}_a, \mathbf{G}_a)$, when computed in flat $\mathbf{F}_p$-sheaves over S, is killed by a power of F for $i > 0$. This implies (2.2), esentially because, on Pf/S, F is invertible).

On using this, and the sequence at the start of this proof, one reduces to showing that the vertical map in

$$\mathrm{Hom}_{O_S}(O_S, O_S)$$
$$\downarrow$$
$$(\mathrm{Hom}_{\mathscr{S}(p)}(O_S, O_S) \xrightarrow{\ F-1\ } \mathrm{Hom}_{\mathscr{S}(p)}(O_S, O_S))$$

is a quasi-isomorphism of complexes. But $\mathrm{Hom}_{\mathscr{S}(p)}(O_S, O_S) = \mathrm{Hom}_S(\mathbf{G}_a^{pf}, \mathbf{G}_a)$, and an easy calculation shows that, if $S = \mathrm{spec}\, A$, then $\mathrm{Hom}_S(\mathbf{G}_a^{pf}, \mathbf{G}_a) = A\,[F, F^{-1}]$ where $\varphi = \sum a_i\, F^i$ is the map (on points) $\varphi(t) = \sum a_i\, t^{p^i}$. Moreover φ is linear if an only if $a_i = 0$, $i \neq 0$. The map $(a \mapsto a) : A \to A\,[F, F^{-1}]$ has a section $\sum a_i\, F^i \mapsto \sum a_i^{p^{-i}}$, and another easy calculation shows that this gives a splitting

$$A[F, F^{-1}] = A \oplus (F-1)\,A[F, F^{-1}]$$

i. e. any homomorphism $\varphi : \mathbf{G}_a^{pf} \to \mathbf{G}_a$ can be written as $\varphi = \varphi_0 + (F-1)\,\varphi_1$ with φ_0 linear and φ_0 and φ_1 uniquely determined. This completes the proof.

REMARK 2.3. — We will say that a perfect group scheme is algebraic if it is of the form G^{pf} where G is a group scheme of finite-type over S. The perfect algebraic group schemes over a field k form an abelian category $\mathscr{G}$, which is isomorphic to the category of quasi-algebraic groups over k in the sense of Serre [15]. $\mathscr{G}$ is an abelian subcategory of the category of sheaves on $(Pf/S)_{\mathrm{et}}$. Let $\mathscr{G}(p^n)$ be the subcategory of $\mathscr{G}$ whose objects are killed by p^n. Then $\mathscr{G}(p^\infty) = \bigcup \mathscr{G}(p^n)$ consists of the unipotent perfect group schemes. The functor $G^{\cdot} \mapsto G^{\cdot\vee} = R\,\mathrm{Hom}_{\mathscr{S}(p)}(G^{\cdot}, \mathbf{Z}/p\mathbf{Z})$ defines an autoduality of the derived category $D^b(\mathscr{G}(p))$ of $\mathscr{G}(p)$ i. e. $G^{\cdot\vee}$ is again in $D^b(\mathscr{G}(p))$, and the canonical morphism $G^{\cdot} \to G^{\cdot\vee\vee}$ is a quasi-isomorphism, for any bounded complex whose objects are in $\mathscr{G}(p)$. Since, locally for the étale topology, every object in $\mathscr{G}(p)$ has a composition

series whose quotients are G_a or $Z/p\,Z$, it suffices to prove this for a complex $G^\cdot$ such that $G^i = 0$, $i \neq 0$, and $G^0 = G_a$ or $Z/p\,Z$. The case $G^0 = G_a$ follows from (2.1); the case $G^0 = Z/p\,Z$ is trivial using (2.2). For any

$$G^\cdot \in D^b(\mathscr{G}\,(p)), \qquad G^{\cdot\,\vee} = \varinjlim_n R\,Hom_{\mathscr{G}}(G^\cdot, Z/p^n Z).$$

Thus, if for any $G^\cdot \in D^b(\mathscr{G}\,(p^\infty))$, we define $G^{\cdot\,\vee} = \varinjlim_n R\,Hom_{\mathscr{G}}(G^\cdot, Z/p^n\,Z)$, this functor $G^\cdot \mapsto G^{\cdot\,\vee}$ defines an autoduality of $\mathscr{G}\,(p^\infty)$ (cf. [15], p. 55, and [2], § 3).

Let $\pi : X \to S$ be a proper smooth morphism whose fibres have dimension m. We write $Pf\,X/S$ for the category whose objects are pairs (Y, T) with T in Pf/S and Y an étale X_T-scheme. Also $(Pf\,X/S)_{et}$ denotes this category together with its étale topology. We write $\mathscr{X}\,(p)$ for the category of sheaves of F_p-modules on $(Pf\,X/S)_{et}$ and $\mathscr{X}$ for the category of all sheaves. Note that $\Omega^r_{X/S} \otimes_{O_X} O_{X_T} = \Omega^r_{X_T/T}$, and that all of the exact commutative diagrams of sheaves in paragraph 1 extend to exact commutative diagrams of sheaves on $(Pf\,X/S)_{et}$. The map $\pi : X \to S$ defines a morphism of sites

$$(Pf\,X/S)_{et} \to (Pf/S)_{et},$$

and hence functors $\pi\,(p)_* : \mathscr{X}\,(p) \to \mathscr{S}\,(p)$ and $\pi_* : \mathscr{X} \to \mathscr{S}$. I claim that $R\,\pi\,(p)_* = R\,\pi_*$ on $\mathscr{X}\,(p)$ i. e. that $R\,\pi\,(p)_*\,(F^\cdot)$, when regarded as an element of $D\,(\mathscr{S})$, represents $R\,\pi_*\,(F^\cdot)$, for any $F^\cdot$ in $D\,(\mathscr{X}\,(p))$. This follows from [10] (V. 3.5). In the future, we will not distinguish between $R\,\pi_*$ and $R\,\pi_*\,(p)$.

The trace map in [11] gives an isomorphism $R^m\,\pi_*\,\Omega^m_{X/S} \xrightarrow{\approx} O_S$. From this, and (1.3), we get a map

$$\eta : \quad R\,\pi_* v(m) \xrightarrow{\approx} R\,\pi_*(\Omega^m_{X/S} \xrightarrow{1-C} \Omega^m_{X/S}) \xrightarrow{t} (O_S, \xrightarrow{1-F^{-1}} O_S) \approx Z/p\,Z.$$

in $D\,(\mathscr{S}\,(p))$.

THEOREM 2.4. — *Let* $\pi : X \to S$ *be as above. The map*

$$R\,\pi_* v(r) \to R\,Hom_{\mathscr{S}\,(p)}(R\,\pi_* v(m-r), Z/p\,Z),$$

induced by the pairing

(1.9) $v(r) \times v(m-r) \to v(m)$

and the trace map

$$\eta : \quad R\,\pi_* v(m) \to Z/p\,Z \qquad (above)$$

is an isomorphism in $D\,(\mathscr{S}\,(p))$.

Proof. — In $D\,(\mathscr{X}\,(p))$, one may identify $v(r)$, $v(m-r)$ and the pairing

$$v(r) \times v(m-r) \to v(m)$$

with the complexes $X^\cdot$, $Y^\cdot$ and the pairing $X^\cdot \times Y^\cdot \to Z^\cdot$ of (1.8). To show that

$$R\,\pi_* X^\cdot \to R\,Hom(R\,\pi_* Y^\cdot, Z/p\,Z)$$

is an isomorphism, it is enough to check it on the individual terms of $X^{\cdot}$ and $Y^{\cdot}$ [*cf.* the proof of (1.9)]. This follows from the next lemma.

LEMMA 2.5. — *Let* L *be a locally free* O_X-*module of finite rank, and let*

$$L^{\vee} = Hom_{O_X}(L, O_X).$$

The pairing $L \times (L \otimes \Omega^m_{X/S}) \to (\Omega^m_{X/S} \xrightarrow{1-C} \Omega^m_{X/S})$ *which pairs in the obvious way into the second* Ω^m, *and the trace map* η, *define an isomorphism,*

$$R\pi_* L \to R Hom_{\mathscr{S}(p)}(R\pi_*(L^{\vee} \otimes \Omega^m_{X/S}), \mathbf{Z}/p\mathbf{Z})$$

in $D(\mathscr{S}(p))$.

Proof. — We may assume that S is affine. There is a noetherian affine scheme S_0 and a map $S \to S_0$ such that X and L arise from similar objects X_0 and L_0 over S_0. There is a complex $K^{\cdot}(L_0)$ of locally-free O_{S_0}-modules of finite rank, with $K^i(L_0) = 0$ except for $0 \le i \le m$, which represents $R\pi_{0*} L_0$ (*see* for example [14], § 5; π_{0*} can be interpreted either as the map from sheaves on the big étale site, or the big Zariski site, on X_0 to those on S_0). It follows that $K^{\cdot}(L) = K^{\cdot}(L_0) \otimes_{O_{S_0}} O_S$ represents $R\pi_* L$ in $D(\mathscr{S}(p))$. Similarly there is such a complex $K^{\cdot}(L^{\vee} \otimes \Omega^m)$ representing $R\pi_*(L^{\vee} \otimes \Omega^m)$ in $D(\mathscr{S}(p))$. The map

$$R\pi_* L \to R Hom_{\mathscr{S}(p)}(R\pi_*(L^{\vee} \otimes \Omega^m), \mathbf{Z}/p\mathbf{Z})$$

of the lemma is a composite of the following three isomorphisms: $R\pi_* L \xrightarrow{\approx} K^{\cdot}(L)$; the map

$$K^{\cdot}(L) \xrightarrow{\approx} K^{\cdot}(L \otimes \Omega^m)^{\vee}$$

of Grothendieck duality [11]; the map

$$K^{\cdot}(L^{\vee} \otimes \Omega^m)^{\vee} \to R Hom_{\mathscr{S}(p)}(R\pi_*(L^{\vee} \otimes \Omega^m), \mathbf{Z}/p\mathbf{Z})$$

induced by

$$K^{\cdot}(L^{\vee} \otimes \Omega^m) \approx R\pi_*(L^{\vee} \otimes \Omega^m)$$

and the map of (2.1).

COROLLARY 2.6. — *Let* $\pi : X \to S$ *be as in* (2.4) *with fibres of dimension* $m = 2$. *There is a canonical isomorphism,*

$$R(\pi f)_* \mu_p \to R Hom_{\mathscr{S}(p)}(R(\pi f)_* \mu_p, \mathbf{Z}/p\mathbf{Z}),$$

where f *is the canonical morphism of sites* $X_{fl} \to (Pf X/S)_{et}$.

Proof:

$$R(\pi f)_*(\mu_p) = (R\pi_*)(Rf_*)(\mu_p) = R\pi_*(\nu(1)).$$

COROLLARY 2.7. — *Let* $\pi : X \to S$ *be as in* (2.4) *with* S *the spectrum of a perfect field.*

(a) $R^i \pi_* \nu(r)$ *is representable by a perfect unipotent group scheme* $G^i(r)$.

(b) *Write* $U^i(r)$ *for the connected component of* $G^i(r)$; *then* $U^i(r)$ *is the linear dual of* $U^{m+1-i}(m-r)$.

(c) *Write* $D^i(r) = G^i(r)/U^i(r)$; *then* $D^i(r)$ *is an étale group scheme, and the pairing* $R^i \pi_* \nu(r) \times R^{m-i} \pi_* \nu(m-r) \to R^m \pi_* \nu(m) \approx \mathbf{Z}/p\,\mathbf{Z}$ *defines a non-degenerate pairing* $D^i(r) \times D^{m-i}(m-r) \to \mathbf{Z}/p\,\mathbf{Z}$.

Proof. — $R^i \pi_* \nu(r)$ is representable because both the terms $R^i \pi_* \Omega^r$ and $R^i \pi_* (\Omega^r/d\Omega^{r-1})$ in the exact sequence

$$\ldots \to R^i \pi_* \nu(r) \to R^i \pi_* \Omega^r \to R^i \pi_* (\Omega^r/d\Omega^{r-1}) \to \ldots$$

are representable by vector groups. Both (b) and (c) follow immediately from the theorem on making explicit the auto-duality of $D(\mathscr{G}(p))$.

REMARK 2.8. — Let X be a complete smooth surface with structure map $\pi : X \to S = \operatorname{spec} k$, where k is a perfect field, and consider the diagram of sites:

$$\begin{array}{ccc} X_{\mathrm{fl}} & \xrightarrow{f} & (PfX/S)_{\mathrm{et}} \\ \pi^{\mathrm{fl}} \downarrow & \searrow{\scriptstyle \pi f} & \downarrow \pi \\ S_{\mathrm{fl}} & \xrightarrow{f} & (Pf/S)_{\mathrm{et}} \end{array}$$

Write $G^i(\mu_p) = G^{i-1}(1)$, $U^i(\mu_p) = U^{i-1}(1)$, and $D^i(\mu_p) = D^{i-1}(1)$ [cf. (2.7)].

(a) For any algebraically closed field $\bar{k} \supset k$, $G^i(\mu_p)(\bar{k}) = H^i(X \otimes_k \bar{k}, \mu_p)$.

(b) For any perfect scheme T over S, $G^i(\mu_p)(T) = R^i \pi_*^{\mathrm{fl}}(\mu_p)(T)$.

(c) $U^i(\mu_p)$ is the linear dual of $U^{5-i}(\mu_p)$, and $D^i(\mu_p)$ is the Pontryagin dual of $D^{4-i}(\mu_p)$.

Indeed, (a) follows from the spectral sequence

$$H^i(\operatorname{spec}(\bar{k}), R^j(\pi f)_*(\mu_p)) \quad \Rightarrow \quad H^{i+j}(X \otimes_k \bar{k}, \mu_p),$$

since $R^j(\pi f)_*(\mu_p) = G^j(\mu_p)$ and cohomology over $\operatorname{spec}(\bar{k})_{\mathrm{et}}$ is trivial. For (b) one has to use the result of Artin (statement in [2]) that $R^j \pi_*^{\mathrm{fl}} \mu_p$ is representable by an algebraic group scheme over k. For any algebraic group scheme G over k, one can show that $R^0 f_* G$ is representable by G^{pf} on $(Pf/S)_{\mathrm{et}}$, and $R^i f_* G = 0$ for $i > 0$. The latter assertion follows from [9] (11.7) if G is smooth; otherwise it only has to be checked for $G = \mu_p$ or α_p, and this is easy using the usual smooth resolutions of these two groups. Now (b) follows from the spectral sequence

$$(R^i f_*)(R^j \pi_*^{\mathrm{fl}})(\mu_p) \quad \Rightarrow \quad R^{i+j}(\pi f)_* \mu_p.$$

Finally, (c) is simply a re-statement of (2.7).

REMARK 2.9. — As is explained in [2], the above results can be used to obtain a filtration on the base scheme of any smooth proper family $\pi : X \to S$ of surfaces. For example, consider such a family of K 3-surfaces, and assume that S is of characteristic $p \neq 2$. We may replace S by its perfection, since this neither changes the underlying topological space of S, nor the geometric fibres of X/S. Assume that $H^0(X_s, \Omega^1) = 0$ for all $s \in S$, which implies that $H^1(X_s, \Omega^1)$ has dimension 20 for all $s \in S$, and that $R^1 \pi_* \Omega^1$ is

locally-free. (No example is known of a K 3-surface with $H^0(\Omega^1) \neq 0$.) Consider the open subscheme S' of S containing those points s for which

$$F : H^2(X_s, O_{X_s}) \to H^2(X_s, O_{X_s})$$

is non-zero. On S' we have $R^1 \pi_*(dO_X) = 0 = R^2 \pi_*(dO_X)$, and the obvious map $\Omega^1 \to \Omega^1/dO_X$ defines a linear isomorphism $\varphi : R^1 \pi_* \Omega^1 \to R^1 \pi_*(\Omega^1/dO_X)$. Write ψ for the p^{-1}-linear map $\psi : R^1 \pi_* \Omega^1 \to R^1 \pi_*(\Omega^1/dO_X)$ defined by the inverse Cartier operator. Then $\mathrm{Ker}(\varphi - \psi) = R^1 \pi_* \nu(1)$ is an étale group scheme of rank $\leq p^{20}$. Note that $R^1 \pi_* \nu(1)_s = G^2(X_s, \mu_p) = D^2(X_s, \mu_p)$. $D^2(X_{\bar{s}}, \mu_p)(\bar{s}) = H^2(X_{\bar{s}}, \mu_p)$ has even dimension over F_p, for any geometric point $\bar{s}$ of S. Write

$$S'_h = \{ s \in S' \,|\, \mathrm{rank}(R^1 \pi_* \nu(1)_s) \leq p^{20-2h} \}, \qquad 1 \leq h \leq 10.$$

Then S'_h is a closed subspace of S', and the filtration $S' = S'_1 \supset \ldots \supset S'_{10}$ agrees with that defined by Artin in [2, § 7] using the height h of the formal Brauer group of X_s. (This follows from [4].)

At the opposite extreme, consider a family of supersingular K 3 surfaces $\pi : X \to S$ with S perfect. The fact that X_s is supersingular means that $U^2(X_s, \mu_p)$ is one-dimensional. $R^1 \pi_* \nu(1)$ is represented by a perfect group scheme $G^2(X/S, \mu_p)$ and $U^2(X/S, \mu_p) = R^1 \pi_*(O_X/dO_X)$ is a one-dimensional vector group for all $s \in S$. Consider the diagram of sheaves on $(Pf/S)_{\mathrm{et}}$,

$$
\begin{array}{c}
0 \\
\downarrow \\
U^2(X/S, \mu_p) \\
\downarrow \\
\mathrm{Pic}(X/S) \to G^2(X/S, \mu_p) \to R^2 \pi_* G_m \\
\downarrow \\
D^2(X/S, \mu_p) \\
\downarrow \\
0
\end{array}
$$

in which the row comes from the Kummer sequence. Note that $\mathrm{Pic}(X/S)$ is discrete, and so $\mathrm{Pic}(X/S) = \mathrm{NS}(X/S) \approx Z^{22}$. One can show [see (5.3) below] that the kernel of $\mathrm{NS}(X/S) \to D^2(X/S, \mu_p)$ is $p\,\mathrm{NS}(X/S)^*$, where $\mathrm{NS}(X/S)^* = \mathrm{Hom}(\mathrm{NS}(X/S), Z)$ and $\mathrm{NS}(X/S)$ is regarded as a subsheaf of $\mathrm{NS}(X/S)^*$ via intersection product. Thus, we get a map $p\,\mathrm{NS}(X/S)^* \to U^2(X/S, \mu_p)$. Since $\mathrm{NS}(X/S)^*$ is torsion-free, one can write this map uniquely in the form $\alpha\, p$ where α is a map $\mathrm{NS}(X/S)^* \to U^2(X/S, \mu_p)$. This is Artin's period map [2]. It is possible to explicitly describe α in terms of the differentials. On each fibre there is an exact sequence

$$\mathrm{NS}(X_s) \to \mathrm{NS}(X_s)^* \overset{\alpha}{\to} U^2(X_s, \mu_p) \to \mathrm{Br}(X_s) \to 0$$

and $U^2(X_s, \mu_p) \approx G_a$. Artin defines a filtration of S using the dimension of $\mathrm{Ker}(\alpha) = \mathrm{NS}(X_s)^*/\mathrm{NS}(X_s)$.

3. Trace Maps

Throughout this section X will be a projective smooth variety of dimension m over a perfect field k. The characteristic p of k is greater than 2 and m. The symbol X_{et} will denote the small étale site on X i. e. the étale site whose underlying category has as objects all étale X-schemes of finite-type. X_{Zar} will denote the Zariski site which has the same underlying category, but which is endowed with the Zariski topology. A sheaf F on X_{Zar} defines an associated sheaf a F on X_{et}, and the map F $\to$ a F induces a map

$$H^i(X_{Zar}, F) \to H^i(X_{et}, a F).$$

If F is the sheaf defined by a coherent O_X-module, then

$$F \to a F \qquad \text{and} \qquad H^i(X_{Zar}, F) \to H^i(X_{et}, a F)$$

are isomorphisms. We will repeatedly use the fact that if $0 \to F' \to F \to F'' \to 0$ is an exact sequence of sheaves on X_{Zar}, then $0 \to a F' \to a F \to a F'' \to 0$ is an exact sequence of sheaves on X_{et}, and if the maps $H^i(X_{Zar}, -) \to H^i(X_{et}, a(-))$ are isomorphisms for two out of three of the sheaves for all i, then they are isomorphisms for the third also. We shall usually denote a F simply by F, even when a F and F do not define the same functors.

In [4] there is defined a projective system $\{ C_n^{\cdot} \}_{n \in N}$ of complexes of sheaves on X such that C_n^q is a certain subsheaf of the sheaf of typical curves of length n on K_{q+1}. Since C_n^q has a composition series whose quotients are coherent modules over O_X or $O_{X_{(p)}}$, its cohomology groups on X_{Zar} and on X_{et} agree. Write $C^{\cdot}$ for the pro-system $\{ C_n^{\cdot} \}$. The stalk C_x^q is generated by symbols of the form

$$\{ E(a T^{p^n}), r_1, \ldots, r_q \}, \qquad a \in O_{X,x}, \quad r_i \in O_{X,x}^*, \quad n \geq 0,$$
$$\{ E(a T^{p^n}), r_1, \ldots, r_{q-1}, T \}, \qquad a \in O_{X,x}, \quad r_i \in O_{X,x}^*, \quad n \geq 0,$$

where E denotes the Artin-Hasse exponential. $C_{n_0}^q$ is generated by symbols with $n < n_0$, and those with $n \geq n_0$ are zero in $C_{n_0}^q$. The Frobenius map $F : \{ C_n^q \} \to \{ C_n^q \}$ acts on symbols by

$$F \{ E(a T^{p^n}), r_1, \ldots, r_q \} = \{ E(a^p T^{p^n}), r_1, \ldots, r_q \},$$

$$F \{ E(a T^{p^n}), r_1, \ldots, r_{q-1}, T \} = \begin{cases} \{ E(a T^{p^{n-1}}), r_1, \ldots, r_{q-1}, T \}, & n \geq 1, \\ -\{ E(a^p T), r_1, \ldots, r_{q-1}, -a \}, & n = 0. \end{cases}$$

(In fact the symbol $\{ E(a^p T, r_1, \ldots, r_{q-1}, -a \}$ makes sense only when a is invertible. However that restriction is not serious; for, $O_{X,x}$ being local hence additively generated by invertible elements, one can work in the family of generators only with those for which a is invertible.)

LEMMA 3.1. — F *defines maps* $C_n^q \to C_n^q / d C_n^{q-1}$, $n \geq 0$.

Proof. — d here denotes the boundary map (written δ in [4]) which acts on symbols by

$$d \{ E(a T^{p^n}), r_1, \ldots, r_{q-1} \} = \{ E(a T^{p^n}), r_1, \ldots, r_{q-1}, T \},$$
$$d \{ E(a T^{p^n}), r_1, \ldots, r_{q-2}, T \} = 0.$$

Thus, in the notation of [4] (II,8.1), $\mathrm{filt}^n\,(\widehat{\mathrm{TCK}}_{q+1}\,(\mathrm{R}))$ maps to zero under

$$\widehat{\mathrm{TCK}}_{q+1}(\mathrm{R}) \xrightarrow{\ \mathrm{F}\ } \widehat{\mathrm{TCK}}_{q+1}(\mathrm{R}) \to \mathrm{TC}_n\,\mathrm{K}_{q+1}(\mathrm{R})/d\,(\mathrm{TC}_n\,\mathrm{K}_q(\mathrm{R})),$$

and we get the required map.

We will also write F for the map $\mathrm{C}_n^q \to \mathrm{C}_n^q/d\mathrm{C}_n^{q-1}$ defined by F. For any $q \geqq 0$, $n \geqq 0$ we define a sheaf $v_n\,(q)$ on the small étale site on X by $v_n\,(q) = \mathrm{Ker}\,(\mathrm{C}_n^q \xrightarrow{\ \mathrm{F}-1\ } \mathrm{C}_n^q/d\mathrm{C}_n^{q-1})$. For example,

$$v_n(0) = \mathrm{Ker}\,(\mathrm{W}_n \xrightarrow{\ \mathrm{F}-1\ } \mathrm{W}_n) = \mathbf{Z}/p^n\,\mathbf{Z}.$$

LEMMA 3.2. — *There are commutative diagrams for any q,*

$$
\begin{array}{ccccccccc}
0 & \to & v(q) & \to & \Omega^q & \xrightarrow{\ \mathrm{C}^{-1}-1\ } & \Omega^q/d\Omega^{q-1} & \to & 0 \\
 & & \downarrow{\scriptstyle\approx} & & {\scriptstyle\rho}\downarrow{\scriptstyle\approx} & & \downarrow{\scriptstyle\approx} & & \\
0 & \to & v_1(q) & \to & \mathrm{C}_1^q & \xrightarrow{\ \mathrm{F}-1\ } & \mathrm{C}_1^q/d\mathrm{C}_1^{q-1} & \to & 0
\end{array}
$$

Thus, in particular, $v\,(q) \approx v_1\,(q)$ canonically.

Proof. — The top row is as in (1.5) and ρ is the map such that

$$\rho\left(a\,\frac{dr_1}{r_1}\wedge\,\ldots\,\wedge\,\frac{dr_q}{r_q}\right) = \{\,\mathrm{E}(-a\,\mathrm{T}),\,r_1,\,\ldots,\,r_q\,\}.$$

The maps ρ define an isomorphism of complexes $\Omega^{\boldsymbol{\cdot}} \to \mathrm{C}_1^{\boldsymbol{\cdot}}$ [4] (II,8.3.1), and so ρ also induces the isomorphism $\Omega^q/d\Omega^{2-1} \to \mathrm{C}_1^q/d\mathrm{C}_1^{q-1}$. The right-hand square commutes because

$$\mathrm{C}^{-1}\left(a\,\frac{dr_1}{r_1}\wedge\,\ldots\,\wedge\,\frac{dr_q}{r_q}\right) = a^p\,\frac{dr_1}{r_1}\wedge\,\ldots\,\wedge\,\frac{dr_q}{r_q}.$$

The isomorphism $v\,(q) \to v_1\,(q)$ is the unique map making the diagram commute.

LEMMA 3.3. — *For any $n' \leqq n$ there is an exact commutative diagram of étale sheaves,*

$$
\begin{array}{ccccccccc}
0 & \to & v_n(q) & \to & \mathrm{C}_n^q & \xrightarrow{\ \mathrm{F}-1\ } & \mathrm{C}_n^q/d\mathrm{C}_n^{q-1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & v_{n'}(q) & \to & \mathrm{C}_{n'}^q & \xrightarrow{\ \mathrm{F}-1\ } & \mathrm{C}_{n'}^q/d\mathrm{C}_{n'}^{q-1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Proof. — The vertical arrows are the obvious projection maps. To show that the rows are exact it suffices to show that $\mathrm{F}-1$ is surjective, but this is obvious by looking at symbols (*cf.* the proof of [4] (II, 8.5.1)). The maps

$$\mathrm{C}_n^q \to \mathrm{C}_{n'}^q \qquad \text{and} \qquad \mathrm{C}_n^q/d\mathrm{C}_n^{q-1} \to \mathrm{C}_{n'}^q/d\mathrm{C}_{n'}^{q-1}$$

are obviously surjective (on X_{et} or X_{Zar}) and the diagram certainly commutes, and so it only remains to show that $v_n(q) \to v_{n'}(q)$ is surjective. It suffices to do this with $n' = n-1$. Then the diagram extends to an exact commutative diagram,

(3.4)

$$
\begin{array}{ccccc}
& & 0 & & 0 \\
& & \downarrow & & \downarrow \\
0 \longrightarrow \varphi^q_{n-1} \longrightarrow & & \Phi^q_{n-1} \xrightarrow{F-1} & \Phi^q_{n-1}/\varphi^q_{n-1} & \\
& & \downarrow & \downarrow & \downarrow \\
0 \longrightarrow v_n(q) \longrightarrow & & C^q_n \xrightarrow{F-1} & C^q_n/dC^{q-1}_n \longrightarrow 0 \\
& & \downarrow & \downarrow & \downarrow \\
0 \to v_{n-1}(q) \to & & C^q_{n-1} \xrightarrow{F-1} & C^q_{n-1}/dC^{q-1}_{n-1} \to 0 \\
& & \downarrow & & \downarrow \\
& & 0 & & 0
\end{array}
$$

where $\Phi^q_{n-1} = \mathrm{Ker}\,(C^q_n \to C^q_{n-1})$ (this sheaf is written as $T\,\Phi_{n-1}\,K_{q+1}$ in [4]) and $\varphi^q_{n-1} = \Phi^q_{n-1} \cap dC^{q-1}_n \subset C^q_n$. Thus we must show that the map $\Phi^q_n \to \Phi^q_n/\varphi^q_n$ induced by $F-1$ is surjective for all n, q. There is an exact sequence [4] (II, 8.2.1):

$$
0 \to \Omega^q/D_n \xrightarrow{\rho_1} \Phi^q_n \xrightarrow{\rho_2} \Omega^{q-1}/E_n \to 0,
$$

where

$$
\rho_1\left(a\,\frac{dr_1}{r_1} \wedge \dots \wedge \frac{dr_q}{r_q} \right) = \{ E(-a\,T^{p^n}),\, r_1,\, \dots,\, r_q \},
$$

$$
\rho_2\{ E(a\,T^{p^n}),\, r_1,\, \dots,\, r_{q-1},\, T \} = a\,\frac{dr_1}{r_1} \wedge \dots \wedge \frac{dr_{q-1}}{r_{q-1}}.
$$

On dividing the middle term of this sequence by φ^q_n we get an exact sequence

$$
0 \to \Omega^q/D \xrightarrow[\approx]{\rho_1} \Phi^q_n/\varphi^q_n \to 0,
$$

where $D = \mathrm{Ker}\,(\Omega^q \to \Phi^q_n \to \Phi^q_n/\varphi^q_n)$. Note that $D \supset D_{n+1}$ because,

$$
\begin{aligned}
\rho_1\left(-a_1^{p^n}\frac{da_1}{a_1} \wedge \dots \wedge \frac{da_q}{a_q} \right) &= \{ E(a_1^{p^n}\,T^{p^n}),\, a_1,\, \dots,\, a_q \} \\
&= p^n\{ E(a_1\,T),\, a_1,\, \dots,\, a_q \} \\
&= \pm\,p^n\{ E(a_1\,T),\, a_2,\, \dots,\, a_q,\, T \} \in dC^{q-1}_{n+1}.
\end{aligned}
$$

From the exact commutative diagram,

(3.5)

$$
\begin{array}{ccccc}
0 \to \Omega^q/D_n \longrightarrow & \Phi^q_n \longrightarrow & \Omega^{q-1}/E_n \to 0 \\
\downarrow{\scriptstyle C^{-1}-1} & \downarrow{\scriptstyle F-1} & \downarrow \\
0 \to \Omega^q/D \to & \Phi^q_n/\varphi^q_n \longrightarrow & 0
\end{array}
$$

we see that it suffices to prove that $\Omega^q/D_n \xrightarrow{C^{-1}-1} \Omega^q/D$ is surjective, but this follows from the next lemma and the fact that $D \supset D_{n+1}$.

LEMMA 3.6. — *For every q and n there is an exact commutative diagram of étale sheaves*

$$
\begin{array}{ccccccc}
0 \to v(q) \to & \Omega^q & \xrightarrow{\ C^{-1}-1\ } & \Omega^q/D_1 & \longrightarrow 0 \\
& \| & \downarrow & \downarrow \\
0 \to v(q) \to & \Omega^q/D_n & \xrightarrow{\ C^{-1}-1\ } & \Omega^q/D_{n+1} & \to 0
\end{array}
$$

Proof. — As $D_1 = d\Omega^{q-1}$, the first row is as in (1.5). To prove the lemma, we must show that the map $D_n \xrightarrow{\ C^{-1}-1\ } D_{n+1}/D_1$ is an isomorphism, but this may be done by induction using that C^{-1} (and hence $C^{-1}-1$) induces an isomorphism $D_n/D_{n-1} \to D_{n+1}/D_n$.

LEMMA 3.7. — *The canonical map $H^m(X_{Zar}, C_n^m/dC_n^{m-1}) \to H^m(X_{et}, C_n^m/dC_n^{m-1})$ is an isomorphism.*

Proof. — As each C_n^q, as a Zariski sheaf, is built up out of coherent modules, the maps $H^r(X_{Zar}, C_n^q) \to H^r(X_{et}, C_n^q)$ are isomorphisms. Thus $H^i(X_{Zar}, C_n^\cdot) \xrightarrow{\sim} H^i(X_{et}, C_n^\cdot)$ for all i. For $i = 2m$, this is the required isomorphism.

LEMMA 3.8. (a). — *If k is algebraically closed, then*

$$H^m(X, v_n(m)) \to H^m(X, v_{n'}(m))$$

is surjective for all $n' \leqq n$.

(b) *If k is finite, then there is an exact sequence*

$$H^{m+1}(X, v_1(m)) \to H^{m+1}(X, v_n(m)) \to H^{m+1}(X, v_{n-1}(m)) \to 0$$

for every n.

Proof. — There is an exact commutative diagram [*see* (3.5)]:

$$
\begin{array}{ccccccc}
& 0 & & 0 & & 0 \\
& \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & \alpha & \longrightarrow & \varphi_{n-1}^m & \longrightarrow & \Omega^{m-1}/E_{n-1} \to 0 \\
& \downarrow & & \downarrow & & \downarrow \\
0 \to & \Omega^m/D_{n-1} & \longrightarrow & \Phi_{n-1}^m & \longrightarrow & \Omega^{m-1}/E_{n-1} \to 0 \\
& \downarrow{\scriptstyle C^{-1}-1} & & \downarrow{\scriptstyle F-1} & & \downarrow \\
0 \longrightarrow & \Omega^m/D & \longrightarrow & \Phi_{n-1}^m/\varphi_{n-1}^m & \longrightarrow & 0 \\
& \downarrow & & \downarrow & & \\
& 0 & & 0 & &
\end{array}
$$

in which $\varphi_{n-1}^m = \Phi_{n-1}^m \cap dC_n^{m-1} = \text{Ker}\,(v_n(m) \to v_{n-1}(m))$ and α is defined to make the diagram exact. Note that $H^i(\Omega^{m-1}/E_{n-1}) = 0$, $i > m$, for any perfect base field k, because there is a Zariski-exact sequence of sheaves

$$0 \to \Omega^{m-1} \xrightarrow{\ C^{-n+1}\ } \Omega^{m-1}/D_{n-1} \to \Omega^{m-1}/E_{n-1} \to 0$$

in which Ω^{m-1} and Ω^{m-1}/D_{n-1} are built up out of coherent modules [4] (II, 8.2).

From the diagram

$$\Omega^m/D_{n-1} \xrightarrow{\;C^{-1}-1\;} \Omega^m/D$$

$$C^{-1}-1 \downarrow \qquad \nearrow$$

$$\Omega^m/D_n$$

We get an exact sequence of kernels,

$$0 \to \nu_1(m) \to \alpha \to D/D_n \to 0.$$

Since the right-hand column of (3.4) is exact relative to the Zariski topology, (3.7) implies that $H^i(X_{et}, \Phi^m_{n-1}/\varphi^m_{n-1}) = 0$ for $i > m$. Now the cohomology sequence of

$$0 \to D \to \Omega^m \to \Phi^m_{n-1}/\varphi^m_{n-1} \to 0$$

gives that $H^i(X_{et}, D) = 0$ for $i > m$, and it follows that $H^i(D/D_n) = 0$ for $i > m$ (over any perfect base field).

Now take k to be algebraically closed. The above computations show that there are surjective maps $H^{m+1}(\nu_1(m)) \to H^{m+1}(\alpha) \to H^{m+1}(\varphi^m_{n-1})$. As $H^{m+1}(\nu_1(m)) = 0$ [see the diagram in (1.11 b)], this implies that $H^{m+1}(\varphi^m_{n-1}) = 0$, and now the cohomology sequence of

$$0 \to \varphi^m_{n-1} \to \nu_n(m) \to \nu_{n-1}(m) \to 0 \quad \text{gives} (a).$$

If k is finite, then again there are surjective maps

$$H^{m+1}(\nu_1(m)) \to H^{m+1}(\varphi^m_{n-1}) \quad \text{and} \quad 0 = H^{m+2}(\nu_1(m)) \to H^{m+2}(\varphi^m_{n-1}).$$

Now (b) follows from the same cohomology sequence as (a).

REMARK 3.9. — In the notation of [4] (II, 8.1), $\text{filt}^{n-n'}(\widehat{\text{TCK}}_{q+1}(R))$ maps to zero under

$$p^{n'}: \widehat{\text{TCK}}_{q+1}(R) \to TC_n K_{q+1}(R),$$

and hence multiplication by $p^{n'}$ on C^q_n factors

$$p^{n'} = (C^q_n \to C^q_{n-n'} \xrightarrow{j} C^q_n).$$

This induces a factoring of $p^{n'}$ on $\nu_n(q)$,

$$p^{n'} = (\nu_n(q) \to \nu_{n-n'}(q) \xrightarrow{j} \nu_n(q)).$$

As the map $\nu_n(q) \to \nu_{n-n'}(q)$ is surjective (3.3), j is uniquely determined by this last equality. For any n and n' with $n \geqq n'+1$, $n' \geqq 1$ there is a complex

$$\nu_{n'} \xrightarrow{j} \nu_n \to \nu_{n-n'} \to 0.$$

I claim that, for X/k as in (b) of the last lemma, the sequences

$$H^{m+1}(X, \nu_{n'}) \to H^{m+1}(X, \nu_n) \to H^{m+1}(X, \nu_{n-n'}) \to 0$$

are exact. Indeed, assume that this is true for a given n and all n' with $n \geqq n' + 1$, $n' \geqq 1$. Then a diagram chase in

$$
\begin{array}{ccc}
H^{m+1}(v_1) & = & H^{m+1}(v_1) \\
\downarrow & & \downarrow \\
H^{m+1}(v_{n'+1}) \to H^{m+1}(v_{n+1}) & \to & H^{m+1}(v_{n-n'}) \to 0 \\
\downarrow & \downarrow & \parallel \\
H^{m+1}(v_{n'}) \longrightarrow H^{m+1}(v_n) & \longrightarrow & H^{m+1}(v_{n-n'}) \to 0 \\
\downarrow & \downarrow & \\
0 & 0 &
\end{array}
$$

shows that the same is true for $n+1$.

For each q, we write $v_\infty(q)$ for the pro-system $\{v_n(q)\}$ of étale sheaves on X. The maps $v_n(q) \to v_1(q) = v(q)$ induce a map $v_\infty(q) \to v(q)$. We define $H^i(X_{et}, v_\infty(q))$ to be $\varprojlim H^i(X, v_n(q))$. For any perfect field $K \supset k$ there is a unique map $\eta_K : H^m(X_K, v(m)) \to \mathbf{Z}/p\,\mathbf{Z}$ such that

$$
\begin{array}{ccc}
H^m(X_K, v(m)) \to H^m(X_K, \Omega^m) & \xrightarrow{C^{-1}-1} & H^m(X_K, \Omega^m/d\Omega^{m-1}) \\
\downarrow{\eta_K} \qquad \approx \downarrow{t} & F-1 & \approx \downarrow{t} \\
0 \longrightarrow \mathbf{Z}/p\,\mathbf{Z} \longrightarrow k & \longrightarrow & k
\end{array}
$$

commutes (cf. § 1).

THEOREM 3.10. — For each perfect field $K \supset k$ there is a homomorphism

$$
\eta_{K,\infty} : H^m(X_K, v_\infty(m)) \to \mathbf{Z}_p
$$

such that:

(a)
$$
\begin{array}{ccc}
H^m(X_K, v_\infty(m)) & \xrightarrow{\eta_{K,\infty}} & \mathbf{Z}_p \\
\downarrow & & \downarrow{\text{canonical}} \\
H^m(X_K, v(m)) & \xrightarrow{\eta_K} & \mathbf{Z}/p\,\mathbf{Z}
\end{array}
$$

commutes;

(b) $\eta_{K,\infty}$ is functorial in K;

(c) $\eta_{K,\infty}$ is surjective if K is algebraically closed.

Proof. — The diagram

$$
\begin{array}{ccc}
C^{m-1} & \xrightarrow{F} & C^{m-1} \\
\downarrow{d} & & \downarrow{d} \\
C^m & \xrightarrow{pF} & C^m
\end{array}
$$

commutes, and hence $p\,F$ induces a map $(p\,F) : C^m/dC^{m-1} \to C^m/dC^{m-1}$. On cohomology, the map $(p\,F) : H^m(C^m/dC^{m-1}) \to H^m(C^m/dC^{m-1})$ becomes identified via

the isomorphism [4] (III, 1.2.1) and the trace map for crystalline cohomology, with $p \times$ (usual Frobenius) : $W_K \to W_K$, where $W_K =$ Witt vectors over K. Thus

$$p^{-1}(pF) : H^m(C^m/dC^{m-1}) \to H^m(C^m/dC^{m-1})$$

is defined and is the usual Frobenius. Consider the commutative diagram,

(3.11)

$$H^m(X_K, v_\infty(m)) \longrightarrow H^m(X_K, C^m) \xrightarrow{\ F-1\ } H^m(X_K, C^m/dC^{m-1})$$

with the vertical map 1 to $H^m(X_K, C^m/dC^{m-1})$ and the diagonal map $p^{-1}(pF)-1$.

in which the top row is the cohomology sequence of

$$0 \to v_\infty \to C^m \xrightarrow{\ F-1\ } C^m \to 0.$$

If K is algebraically closed, this cohomology sequence is exact, because $\{ H^m(X_K, v_n(m)) \}$ satisfies the Mittag-Leffler condition by (3.8 a). In any case, we get from the diagram a complex,

$$H^m(v_\infty(m)) \to \mathrm{Ker}(p^{-1}(pF)-1) \to \mathrm{Coker}(H^m(C^m) \xrightarrow{1} H^m(C^m/dC^{m-1})).$$

The map $H^m(C^m) \to H^m(C^m/dC^{m-1})$ is surjective since we have checked (3.7) that these groups may be computed using Zariski cohomology. But, as follows from the discussion above, $\mathrm{Ker}(p^{-1}(pF)-1)$ may be identified with $\mathrm{Ker}(F-1 : W \to W) = Z_p$. Thus we have a map $\eta_{K,\infty} : H^m(X_K, v_\infty(m)) \to Z_p$ which clearly satisfies (b) and (c) of the theorem.

For (a), consider the diagram

$$H^m(X_K, v(m)) \longrightarrow H^m(X_K, \Omega^m) \xrightarrow{\ C^{-1}-1\ } H^m(X_K, \Omega^m/d\Omega^{m-1})$$

with the vertical map $\approx$ to $H^m(X_K, \Omega^m/d\Omega^{m-1})$ and the diagonal map $(\mathrm{Frob})-1$.

There is a map from (3.11) to this diagram, and the definitions of η_K and $\eta_{K,\infty}$ are compatible. This proves (a).

Now let X be a variety over a finite field k, and write $\overline{X} = X \otimes \overline{k}$ where $\overline{k}$ is the algebraic closure of k, etc., as in paragraph 1. Recall that there are maps

(1.9) $\eta :\ H^{m+1}(X, v(m)) \xrightarrow{\approx} Z/pZ,$

(1.11) $\varphi :\ H^m(\overline{X}, v(m)) \longrightarrow H^{m+1}(X, v(m)).$

Similarly, for any $n \in N \cup \{ \infty \}$, there is such a map

$$\varphi_n :\ H^m(\overline{X}, v_n(m)) \to H^{m+1}(X, v_n(m))$$

arising from the Hochschild-Serre spectral sequence for $\overline{X}/X$. Also, when X is a surface, there is a map

(1.11) $B^2: \quad CH^2(\overline{X}) \to H^2(\overline{X}, K_2 O_{\overline{X}})$.

The map

$$\{r_1, r_2\} \mapsto \{E(T), r_1, r_2\} : K_2 O_{\overline{X}} \to C_n^2$$

defines a map $\psi_n : K_2 O_{\overline{X}} \to v_n(2)$ for every n and thus a map

$$\psi = (\psi_n) : K_2 O_{\overline{X}} \to \{v_n(2)\}.$$

COROLLARY 3.12. — *Assume that k is finite. There is an isomorphism*

$$\alpha : H^{m+1}(X, v_\infty(m)) \to Z_p$$

such that:

(a)

$$
\begin{array}{ccc}
H^{m+1}(X, v_\infty(m)) & \xrightarrow{\alpha} & Z_p \\
\downarrow & & \downarrow{\scriptstyle canonical} \\
H^{m+1}(X, v(m)) & \xrightarrow{\eta} & Z/pZ
\end{array}
$$

commutes;

(b)

$$
\begin{array}{ccc}
H^m(\overline{X}, v_\infty(m)) & \xrightarrow{\eta^*_{\infty}} & Z_p \\
\downarrow{\scriptstyle \varphi_\infty} & & \| \\
H^{m+1}(X, v_\infty(m)) & \xrightarrow{\alpha} & Z_p
\end{array}
$$

commutes.

Moreover, when X is a surface, for any zero-cycle Z on $\overline{X}$,

$$\alpha \, \varphi_\infty \, \psi \, B^2(Z) \equiv \deg(Z) \pmod{p},$$

where the left-hand term is the image of Z under the maps

$$CH^2(\overline{X}) \xrightarrow{B^2} H^2(\overline{X}, K_2) \xrightarrow{\psi} H^2(\overline{X}, v_\infty(2)) \xrightarrow{\varphi_\infty} H^3(X, v_\infty(2)) \xrightarrow{\alpha} Z_p.$$

Proof. — From (3.11) with $K = k$, we get an exact sequence

$$0 = \mathrm{Coker}(H^m(X, C^m) \xrightarrow{1} H^m(X, C^m/dC^{m-1}))$$
$$\to H^{m+1}(X, v_\infty(m)) \to \mathrm{Coker}(p^{-1}(pF)-1) \to 0.$$

But

$$\mathrm{Coker}(p^{-1}(pF)-1) \approx \mathrm{Coker}(W \xrightarrow{F-1} W) \xrightarrow[\mathrm{trace}]{\approx} Z_p.$$

Thus we have defined α.

The proof of (a) is similar to the proof of (3.10 a). The proof of (b) is straightforward, and the final statement follows from (a) and (1.11 c).

REMARK 3.13. — (a) It is highly likely that, in the notation of the corollary, $\alpha \varphi_\infty \psi B^2 (Z) = \deg(Z)$, but in any case, we may modify α so that this is true. Consider the diagram

$$
\begin{array}{ccc}
CH^2(\overline{X}) & \xrightarrow{\varphi_\infty \psi B^2} & H^3(X, v_\infty(2)) \\
\downarrow{\scriptstyle \deg} & \searrow & \downarrow{\scriptstyle \alpha} \\
\mathbf{Z} & & \mathbf{Z}_p
\end{array}
$$

I claim that $\varphi_\infty \psi B^2 \alpha$ factors through $\mathbf{Z}$. This is equivalent to saying that any cycle of degree zero maps to zero under $\varphi_\infty \psi B^2 \alpha$. But I claim that such a cycle is a torsion element, and hence must map to zero because $\mathbf{Z}_p$ is torsion-free. Any element of the form Z_0-Z_1, where Z_0 and Z_1 are simple points on an irreducible curve C of $\overline{X}$, is torsion in $CH^2(\overline{X})$ because it is torsion in the (generalized) Jacobian of C. But the group of cycles of degree zero in $CH^2(\overline{X})$ is generated by such cycles.

Thus we get a map $\beta : \mathbf{Z} \to \mathbf{Z}_p$ making the above diagram commute, and because of (3.12 a), β extends to an isomorphism $\overline{\beta} : \mathbf{Z}_p \to \mathbf{Z}_p$. We now define

$$\eta_\infty : H^3(X, v_\infty(2)) \to \mathbf{Z}_p$$

to be $\alpha \circ \overline{\beta}^{-1}$. Then η_∞ is an isomorphism, satisfies the condition (a) for α, and also has the property that $\eta_\infty \varphi_\infty \psi B^2 (Z) = \deg(Z)$ for any $Z \in CH^2(\overline{X})$.

(b) On passing to the inverse limit over n' in the exact sequences (3.9):

$$H^3(v_{n'}(2)) \xrightarrow{j} H^3(v_{n+n'}(2)) \to H^3(v_n(2)) \to 0,$$

we get an exact sequence

$$H^3(v_\infty(2)) \xrightarrow{p^n} H^3(v_\infty(2)) \to H^3(v_n(2)) \to 0.$$

Thus

$$\eta_\infty : \quad H^3(v_\infty(2)) \xrightarrow{\approx} \mathbf{Z}_p$$

induces a family of isomorphisms

$$\eta_n : \quad H^3(X, v_n(2)) \xrightarrow{\approx} \mathbf{Z}/p^n \mathbf{Z}.$$

(c) It follows now, by counting, that the sequences

$$0 \to H^3(X, v_{n'}(2)) \xrightarrow{j} H^3(X, v_{n+n'}(2)) \to H^3(v_n(2)) \to 0$$

are exact.

REMARK 3.14. — In order to have a completely satisfactory theory, one would like to be able to define the $v_n(q)$ so that

$$0 \to v_{n'}(q) \to v_{n+n'}(q) \to v_n(q) \to 0$$

is exact for all q, n and n'. It is likely that if $v_n(q)$ is defined to be the image of $K_q O_X$ in C_n^q this will be true, but it does not seem possible to prove it at present.

4. Duality for a Surface Over a Finite Field

Throughout this section X will be a projective smooth surface over a finite field k of characteristic $p \neq 2$. In mild disagreement with the notation in paragraph 3, we define $v_n(1)$ to be the sheaf $O_X^*/O_X^{*p^n}$ on X_{et}. Multiplication by p on O_X^* induces a map $j : v_{n(1)} \to v_{n+1(1)}$, and the identity on O_X^* induces a map $v_{n+1(1)} \to v_{n(1)}$.

$\{ v_n(2) \}$ will be the same pro-system of sheaves as in paragraph 3. Recall (3.13) that there are trace maps $\eta_n : H^3(X, v_n(2)) \xrightarrow{\approx} Z/p^n Z$, functorial in k, compatible with varying n, and agreeing with the degree map on zero-cycles.

The pairing

$$O_X^* \times O_X^* \to K_2 O_X \to C_n^2,$$

$$(r_1, r_2) \mapsto \{ r_1, r_2 \} \mapsto \{ E(T), r_1, r_2 \},$$

induces pairings $v_n(1) \times v_n(1) \to v_n(2)$ for all n. They are functorial in k, and for all n,

$$
\begin{array}{ccc}
v_n(1) & \times \quad v_n(1) & \longrightarrow v_n(2) \\
\downarrow j & \uparrow & \downarrow j \\
v_{n+1}(1) & \times v_{n+1}(1) & \to v_{n+1}(2)
\end{array}
$$

commutes.

THEOREM 4.1. — *Let X/k be as above. The pairing*

$$H^i(X, v_n(1)) \times H^{3-i}(X, v_n(1) \to H^3(X, v_n(2)) \xrightarrow{\eta_n} Z/p^n Z$$

defined by the above pairing of sheaves is a non-degenerate pairing of finite groups for all n.

Proof. — As X is projective, étale cohomology may be computed by Čech cohomology, and the above pairing of cohomology groups is most simply defined by cup-product. For $n = 1$, the theorem is a special case of (1.9). For $n > 1$ it may be proved by induction, using the exact sequence

$$0 \to v_1(1) \to v_n(1) \to v_{n-1}(1) \to 0$$

and the above compatibilities.

COROLLARY 4.2. — *There is a non-degenerate pairing*

$$\varinjlim_n H^i(X, v_n(1)) \times \varprojlim_n H^{3-i}(X, v_n(1)) \to Q_p/Z_p.$$

Proof. — This arises from passing to the direct limit in

$$H^i(X, v_n(1)) \to \mathrm{Hom}(H^{3-i}(X, v_n(1)), H^3(X, v_n(2))).$$

COROLLARY 4.3. — (*a*) *There are canonical pairings*

$$H^i(X_{fl}, \mu_{p^n}) \times H^{5-i}(X_{fl}, \mu_{p^n}) \to Z/p^n Z$$

which are non-degenerate pairings of finite groups for all i and n.

(b) *There is a non-degenerate pairing*

$$H^i(X, \mu_\infty(1)) \times H^{5-i}(X, T_p\mu) \to Q_p/Z_p$$

for all i.

Proof. — These follow from (4.1) and (4.2) exactly as (1.10) follows from (1.9).

REMARK 4.4. — (a) In the situation of (4.1), $x.y = (-1)^{ij+1} y.x$ if $x \in H^i(X, v_n(1))$, $y \in H^j(X, v_n(1))$.

(b) The following diagram commutes:

$$
\begin{array}{ccccc}
\text{Pic}(X) & \times & \text{NS}(X) & \longrightarrow & Z \\
\downarrow & & \downarrow & & \downarrow \text{canonical} \\
 & & H^2(\overline{X}, \mu_{p^n})_\Gamma & & \\
 & & \downarrow & & \\
H^2(X, \mu_{p^n}) & \times & H^3(X, \mu_{p^n}) & \to & Z/p^n Z
\end{array}
$$

The top pairing is intersection product and the botton pairing is as in (4.3 a). The upper two vertical maps are boundary maps arising from the exact sequence

$$0 \to \mu_{p^n} \to G_m \xrightarrow{p^n} G_m \to 0.$$

The map $H^2(\overline{X}, \mu_{p^n})_\Gamma \to H^3(X, \mu_{p^n})$ is that arising from the Hochschild-Serre spectral sequence for $\overline{X}/X$. [The notation is as in (1.11 b).]

This is a consequence of the commutativity of the following diagram

$$
\begin{array}{ccccccc}
\text{Pic}(\overline{X}) & \times & \text{Pic}(\overline{X}) & & \longrightarrow & & Z \\
\uparrow \approx & & \uparrow \approx & & & & \| \\
H^1(\overline{X}, O_{\overline{X}}^*) & \times & H^1(\overline{X}, O_{\overline{X}}^*) & \to H^2(\overline{X}, K_2 O_{\overline{X}}) & \longrightarrow & & Z \\
\downarrow & & \downarrow & & \downarrow \psi & & \downarrow \\
H^1(\overline{X}, v_n(1)) & \times & H^1(\overline{X}, v_n(1)) & \to H^2(\overline{X}, v_n(2)) & & & \text{canonical} \\
\uparrow & & \downarrow \varphi_n & & \downarrow \varphi_n & & \downarrow \\
H^1(X, v_n(1)) & \times & H^2(X, v_n(1)) & \to H^3(X, v_n(2)) & \xrightarrow{\eta_n} & Z/p^n Z & \downarrow \\
\| & & \| & & & & \| \\
H^2(X, \mu_{p^n}) & \times & H^3(X, \mu_{p^n}) & & \longrightarrow & & Z/p^n Z
\end{array}
$$

The top pairing is intersection product, and the second pairing is defined by the natural pairing $O_{\overline{X}}^* \times O_{\overline{X}}^* \to K_2 O_{\overline{X}}$. The map $H^2(\overline{X}, K_2 O_{\overline{X}}) \to Z$ is $\deg \circ (B^2)^{-1}$. The fact that the top rectangle commutes is an easy consequence of the results in [3]. The commutativity of the rectangle with one side (canonical) : $Z \to Z/p^n Z$ follows from the definition of η_n in (3.13). The commutativity of the rest of the diagram is obvious from the various definitions.

(c) If δ_n is the boundary map $\delta_n : H^r(X, \mu_n) \to H^{r+1}(X, \mu_n)$ arising from the exact sequence

$$0 \to \mu_{p^n} \to \mu_{p^{2n}} \to \mu_{p^n} \to 0,$$

then

$$\langle y, \delta_n x \rangle + (-1)^r \langle x, \delta_n y \rangle = 0,$$

where $x \in H^r(X_{f1}, \mu_{p^n})$, $y \in H^{4-r}(X_{f1}, \mu_{p^n})$, and $\langle\ ,\ \rangle$ denotes the pairing of (4.3).

Indeed, write $(w, z) \mapsto w.z$ for the pairing of (4.1), and let x' and y' be elements of $H^{r-1}(X, \nu_n(1))$ and $H^{3-r}(X, \nu_n(1))$ corresponding to x and y. A standard formula for cup-products states that

$$(\delta_n x').y' + (-1)^{r-1} x'.(\delta_n y') = \delta_n(x'.y'),$$

where δ_n now denotes the boundary map arising from the exact sequences

$$0 \to \nu_n^l(1)^l \to \nu_{2n}(1) \to \nu_n(1) \to 0,$$

or

$$0 \to \varphi \to \nu_{2n}(2) \to \nu_n(2) \to 0.$$

But (§ 3),

$$H^3(X, \varphi) \xrightarrow{\approx} H^3(X, \nu_n(2)),$$

and

$$H^3(X, \nu_n(2)) \to H^3(X, \nu_{2n}(2))$$

is injective. Thus

$$\delta_n(x'.y') = 0,$$

and

$$(\delta_n x').y' + (-1)^{r-1}(x'.\delta_n y') = 0.$$

By (4.4 a), this may also be written as

$$y'.(\delta_n x') + (-1)^r x'.(\delta_n y') = 0.$$

This is the required equation.

This completes the proof of all duality assertions required for [13].

By using an argument of M. Artin [2], we may deduce an amusing consequence.

THEOREM 4.5. — *Let X be a projective smooth surface over an algebraically closed field k of characteristic $p \neq 2$. Assume that the rank ρ of the Néron-Severi group NS(X) of X is equal to the second l-adic Betti number β_2 of X. Then the absolute value of the determinant of the intersection matrix $(D_i.D_j)$, for $\{D_i\}$ a basis of NS(X) mod torsion, is either a square or twice a square.*

Proof. — We may assume that X and the D_i are defined over a field which is finitely-generated over F_p, and then specialize to obtain a smooth variety X_0 over a finite field k_0. Since we have specialized smoothly, $\beta_2(X_0) = \beta_2(X)$ and the map $NS(X) \to NS(X_0)$ is injective. Since $\rho(X_0) \leqq \beta_2(X_0)$, we must have equalities

$$\rho(X) = \rho(X_0) = \beta_2(X_0) = \beta_2(X).$$

Thus NS (X) is of finite index in NS (X_0), and we are reduced to proving the same theorem for X_0. But $\rho(X_0) = \beta_2(X_0)$ implies that Tate's conjecture ((T) of [13]) holds for X_0 and hence, by the main theorem of [13], the Artin-Tate conjecture holds for X_0. This says that $\det(D_i . D_j) = [\text{NS}(X_0)_{\text{tors}}]^2 \, q^{\alpha(X_0)}/[\text{Br}(X_0)]$, where $k_0 = \mathbf{F}_q$. By taking k_0 sufficiently large we may ensure that q is a square, and it is known [13] that $[\text{Br}(X_0)]$ is either a square or twice a square.

REMARK 4.6. — If X is a projective smooth surface over a finite field k such that the rank of NS (X) is $\beta_2(X)$ [and so, in particular, a basis for NS (X) mod torsion is defined over k] then the degree of k over $\mathbf{F}_p$ is even. This follows from the theorem.

5. Duality for a Surface Over a Perfect Base Scheme

Throughout this section, X will be a proper smooth surface over a perfect affine scheme $S = \operatorname{spec} A$ of characteristic $p \neq 2$. The symbols $(Pf\,X/S)_{\text{et}}$, $(Pf/S)_{\text{et}}$, and $\pi : (Pf\,X/S)_{\text{et}} \to (Pf/S)_{\text{et}}$ will denote the same objects as in paragraph 2. We shall have to assume the following statement.

5.1. There exists a surjective morphism $\eta_\infty : R^2 \pi_* v_\infty(2) \to \mathbf{Z}_p$ of pro-sheaves on $(Pf/S)_{\text{et}}$ such that

$$\begin{array}{ccc} R^2 \pi_* v_\infty(2) & \xrightarrow{\eta_\infty} & \mathbf{Z}_p \\ \downarrow & & \downarrow \\ R^2 \pi_* v(2) & \xrightarrow{\eta} & \mathbf{Z}/p\,\mathbf{Z} \quad (\S 2) \end{array}$$

commutes.

If A is a field then we proved in (3.10) that there is such a morphism η_∞ defined on the restriction of $R^2 \pi_* v_\infty(2)$ to the category of perfect *fields* over A. The same proof will give (5.1) once it has been checked that Bloch's theory of typical curves on the K-functors works satisfactorily over any perfect base ring. We assume this. (Of course, we do not need the full theory; little more than the existence of the sheaves C_q^n will suffice.)

As in paragraph 4, we define $v_n(1)$ to be the sheaf $O_X^*/O_X^{*p^n}$ on $(Pf\,X/S)_{\text{et}}$. The pairing

$$O_X^* \times O_X^* \to K_2 \, O_X \to C_n^2,$$

$$(r_1, r_2) \mapsto \{r_1, r_2\} \mapsto \{E(T), r_1, r_2\},$$

induces a pairing $v_n(1) \times v_n(1) \to v_n(2)$ for all n. From its method of definition, we see that this pairing factors through

$$v_{n'}(2)/p^n v_{n'}(2) \to v_n(2) \quad \text{for all } n' > n.$$

Thus we obtain a pairing

$$v_n(1) \times v_n(1) \to v_{(n)}(2) = v_\infty(2)/p^n v_\infty(2).$$

There is an exact sequence of pro-sheaves on $(Pf/S)_{\text{et}}$,

$$R^2 \pi_* (v_\infty(2)) \xrightarrow{p^n} R^2 \pi_* (v_\infty(2)) \to R^2 \pi_* (v_{(n)}(2)) \to 0$$

and so η_∞ induces a surjective map $\eta'_{(n)} : R^2 \pi_* v_{(n)}(2) \to Z/p^n Z$. Moreover, if $n = 1$, this agrees with the map $\eta : R^2 \pi_* v(2) \overset{\approx}{\to} Z/p Z$ defined in paragraph 2. We define $\eta_{(n)} : R^2 \pi_* v_{(n)}(2) \to Q_p/Z_p$ to be the map $\eta'_{(n)}$ followed by the canonical embedding $Z/p^n Z \to Q_p/Z_p$.

THEOREM 5.2. — *Under the assumption* (5.1), *the map*

$$R \pi_* v_n(1) \to R Hom (R \pi_* v_n(1), Q_p/Z_p)$$

defined by the above pairing and trace map is an isomorphism in the derived category of sheaves on $(Pf/S)_{et}$.

Proof. — By using induction on n, and the exact sequence

$$0 \to v_n(1) \to v_{n+1}(1) \to v_1(1) \to 0,$$

one reduces to proving the theorem for the case $n = 1$. For this we only have to identify the above map with the map of (2.4). Since $\eta'_{(1)} = \eta$, there is a commutative diagram,

$$\begin{array}{ccc}
R \pi_* v(1) & \overset{\approx}{\to} & R Hom_{\mathscr{S}(p)}(R \pi_* v(1), Z/p Z) \\
\downarrow \approx & & \downarrow \\
R \pi_* v(1)' & \longrightarrow & R Hom_{\mathscr{S}}(R \pi_* v(1), Q_p/Z_p)
\end{array}$$

Thus we only have to prove that the right-hand vertical map is an isomorphism. Let $Q_p/Z_p \to I^\cdot$ be an injective resolution of Q_p/Z_p in $\mathscr{S}$, the category sheaves on $(Pf/S)_{et}$. There is a short exact sequence of complexes

$$0 \to I_p^\cdot \to I^\cdot \overset{p}{\to} I^\cdot \to 0 \quad \text{and} \quad (1/p) Z/Z \to I_p^\cdot$$

is an injective resolution of $(1/p) Z/Z$ in $\mathscr{S}(p)$. It follows that for any complex $F^\cdot$ in $\mathscr{S}(p)$,

$$R Hom_{\mathscr{S}(p)}(F^\cdot, Z/p Z) \overset{\approx}{\to} Hom_{\mathscr{S}(p)}(F^\cdot, I_p^\cdot) = Hom_{\mathscr{S}}(F^\cdot, I^\cdot) \overset{\approx}{\to} R Hom_{\mathscr{S}}(F^\cdot, Q_p/Z_p).$$

This completes the proof.

One may now read off analogues of the corollaries and remarks (2.6), (2.7), (2.8) and (2.9).

REMARK 5.3. — Let X be an elliptic supersingular K 3-surface (in the sense of [2]) over an algebraically closed field k of characteristic $p \neq 0$. Then Pic (X) = NS (X) is torsion-free [2] (§ 8) and the rank ρ of NS (X) is equal to β_2, the second l-adic Betti number [2] (1.7). It is proved in [4] that

$$\dim_{Q_p} (H^2(X, T_p \mu) \otimes Q_p) \leqq \dim(H^2_{crys}(X/W)_K),$$

and it is known that this last dimension is equal to β_2. On combining these statements and using the Kummer sequence on X, we get that $\rho = \dim_{Q_p} (H^2(X, T_p \mu) \otimes Q_p)$ and

that $NS(X) \to H^2(X, T_p \mu)$ has finite cokernel i. e. that $Br(X)(p)$ is finite. Consider the diagram

$$
\begin{array}{ccc}
 & 0 & \\
 & \downarrow & \\
 & U^2(\mu_{p^n})(k) & \\
 & \downarrow & \\
NS(X) \xrightarrow{p^n} NS(X) \to H^2(X, \mu_{p^n}) \to Br(X) \\
 \searrow & \downarrow & \\
 & D^2(\mu_{p^n})(k) & \\
 & \downarrow\downarrow & \\
 & 0 & \\
\end{array}
$$

[U^2 and D^2 have the same meanings as in (2.8)]. From this we get an exact sequence

$$0 \to p^n NS(X) \to Ker(NS(X) \to D^2_k) \to U^2(\mu_{p^n})(k).$$

The tangent space of $U^2(\mu_{p^n})$ is $H^2(X, O_X)$, and hence $U^2(\mu_{p^n})$ is one dimensional, and in particular is killed by p. Using the auto-duality of D^2, we may interprete

$$Ker(NS(X) \to D^2) \text{ as } p_k^n NS(X)^* \cap NS(X),$$

where $NS(X)^* = Hom(NS(X), Z)$ and $NS(X)$ is regarded as a subgroup of $NS(X)^*$ via the pairing induced by that on D^2. Thus the exact sequence gives

$$p(p^n NS(X)^* \cap NS(X)) \subset p^{n+1} NS(X)$$

for all n, which implies that $NS(X) \supset p NS(X)^*$. Since the pairing defined on $NS(X)$ by that on D^2 agrees modulo p with the intersection product, we also get that $NS(X) \supset p NS(X)^*$ when $NS(X)$ is regarded as a subgroup of $NS(X)^*$ *via* the intersection product, and $Ker(NS(X) \to D^2) = p NS(X)^*$. As has already been asserted in (2.9), Artin's definition of the period map is now justified and the main results of [2], notably those of paragraph 7, are now proved, even although we have not proved all assertions of the duality hypotheses [2] (4.1) assumed there.

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