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ARMAND BOREL

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STABLE REAL COHOMOLOGY OF ARITHMETIC GROUPS

BY ARMAND BOREL

A Henri Cartan,
à l'occasion de son
70e *anniversaire*

Let Γ be an arithmetic subgroup of a semi-simple group G defined over the field of rational numbers $\mathbf{Q}$. The real cohomology $H^*(\Gamma)$ of Γ may be identified with the cohomology of the complex Ω_X^Γ of Γ-invariant smooth differential forms on the symmetric space X of maximal compact subgroups of the group $G(\mathbf{R})$ of real points of G. Let I_G be the space of differential forms on X which are invariant under the identity component $G(\mathbf{R})^0$ of $G(\mathbf{R})$. It is well-known to consist of closed (in fact harmonic) forms, whence a natural homomorphism $j^* : I_G^\Gamma \to H^*(\Gamma)$. Our main result (7.5) gives a range, computable from the algebraic group structure of G, up to which j^* is an isomorphism. For a given group, it is rather small, about $n/4$ for $\mathbf{SL}_n(\mathbf{Z})$ for instance; however, we may consider sequence (G_n, Γ_n), where G_n and Γ_n are as G and Γ, for which our range tends to infinity, and of injective morphisms $f_n : (G_n, \Gamma_n) \to (G_{n+1}, \Gamma_{n+1})$. Since much is known about I_G, this allows us to determine the " stable real cohomology " $H^q(\Gamma_n)$, n large, and $H^*(\varinjlim \Gamma_n)$ for sequences of classical groups (§ 11). For example, $H^*(\varinjlim \mathbf{SL}_n(\mathbf{Z}))$ [resp. $H^*(\varinjlim \mathbf{Sp}_{2n}(\mathbf{Z}))$] is an exterior (resp. polynomial) algebra over infinitely many generators, one for each dimension of the form $4i+1$ (resp. $4i-2$) $(i = 1, 2, \ldots)$. The result for $\mathbf{SL}(\mathbf{Z})$ then implies that the dimension of $K_i(\mathbf{Z}) \otimes \mathbf{Q}$ is one if $i \equiv 1 \bmod 4$ and zero otherwise $(i \geqq 2)$.

The homomorphism j^* is defined for any discrete subgroup Γ of $\mathbf{G}(\mathbf{R})$. If $G(\mathbf{R})/\Gamma$ is compact, then, as a simple consequence of Hodge theory, j^* is injective in all dimensions. The results of Matsushima [19], recalled in paragraph 3, show the existence of a constant $m(G(\mathbf{R}))$, determined by the Lie algebra of $G(\mathbf{R})$, up to which j^* is surjective. Our theorem is thus the analogue for arithmetic subgroups of Matsushima's result. H. Garland had already shown in [9] that if $G(\mathbf{R})/\Gamma$ is only assumed to have finite invariant volume, and if every cohomology class in a given dimension $q \leqq m(G(\mathbf{R}))$ is representable by a square integrable form, then Matsushima's arguments carry over without change and show that j^* is surjective in dimension q. Moreover [10] gives, for Γ arithmetic, a range up to which this condition is fulfilled (a so-called " square integrability criterion "). Our main concern is therefore the injectivity of j^* in the arithmetic case, but our discussion will also include the surjectivity and the square integrability criterion.

As a first step, assuming only that $G(\mathbf{R})/\Gamma$ has finite invariant volume, we shall see (3.6) that, given an integer $m'(G)$, j^* is bijective at least up to min $(m(G(\mathbf{R})), m'(G))$ if there exists a subcomplex C of Ω_X^Γ such that, up to $m'(G)$: (i) $C \to \Omega_X^\Gamma$ induces an isomorphism in cohomology; (ii) $I_G^\Gamma \subset C$; (iii) C^q consists of square integrable forms. The surjectivity (3.5) is essentially Garland's argument, while injectivity follows from the fact that on a complete Riemannian manifold, a non-zero square integrable harmonic form is not the differential of a square integrable form (2.5).

There remains then to find such a C when Γ is arithmetic, and it suffices to do so when Γ is moreover torsion-free. Its definition (7.1), and the preliminaries in paragraphs 4 and 5, involve only X/Γ but are better understood if X/Γ is viewed as the interior of the compact manifold with corners $\overline{X}/\Gamma$ of [4], (*see* also § 6). From that point of view, the main use of paragraph 4 is to give a decomposition of the invariant Riemannian metric near a boundary point y, with respect to a frame adapted to the corner structure around y, and paragraph 5 gives a criterion for the square integrability of a differential form σ near y, in terms of a growth condition of the coefficients of σ with respect to that frame. C is then defined by means of such growth conditions near the boundary points, i. e., on Siegel sets, for forms and their differentials. It follows immediately from paragraph 5 that C satisfies (ii) in all dimensions, and (iii) at least up to a constant $c(G)$ defined in terms of the $\mathbf{Q}$-roots of G. To prove (i) we introduce a presheaf $\mathcal{F}$ on $\overline{X}/\Gamma$. For y on the boundary and U a neighborhood of y in $\overline{X}/\Gamma$, $\mathcal{F}(U)$ is the space of forms on $U \cap (X/\Gamma)$ satisfying the growth condition which is part of the definition of C. For $y \in X/\Gamma$, $\mathcal{F}(U)$ is simply the space of differential forms defined in U. This presheaf is a sheaf whose space of sections on $\overline{X}/\Gamma$ and X/Γ are C and Ω_X^Γ respectively. Since the inclusion $X/\Gamma \subsetneq \overline{X}/\Gamma$ is a homotopy equivalence, it suffices then to prove that $\mathcal{F}$ is a fine resolution of $\mathbf{R}$ on $\overline{X}/\Gamma$. The " fineness " is easy, so that the main point is to check a local Poincaré lemma near a boundary point, i. e., on a Siegel set. This is done by showing that the usual homotopy operator used to prove the Poincaré lemma on euclidean space does not alter our growth condition.

Theorem 7.5 yields a square integrability criterion which is analogous to, although somewhat weaker than, that of [10]. It can also be obtained more simply by proving that the cohomology of $\overline{X}/\Gamma$ can be computed by means of differential forms which, locally around the boundary, are lifted from forms on the boundary. This is done in paragraph 8, but is not used in the rest of the paper.

Paragraph 9 contains some remarks on $m(G(\mathbf{R}))$ and $c(G)$. In particular, they are both $\geqq (rk_{\mathbf{Q}}(G)/4)-1$, hence tend to infinity with $rk_{\mathbf{Q}}\, G$. Paragraph 10 reviews some known facts about the stable cohomology of classical compact symmetric spaces, which all occur in the Bott periodicity [7]. Modulo well-known isomorphisms, this gives $\varinjlim I_G^q$ when G runs through suitable sequences of classical groups, and allows us in paragraph 11 to describe explicitly $H^*(\varinjlim \Gamma_n)$ for sequences of classical arithmetic groups which occur in K-theory or hermitian K-theory. This yields the ranks of the corresponding K_i or ${}_\varepsilon L_i$ groups (§ 12).

The main results of this paper have been announced in [2]. In fact, [2] states them more generally for S-arithmetic groups. Those will be considered in another paper.

The growth condition used in paragraph 7 is not the one underlying [2], and yields somewhat sharper results. I thank G. Harder very much for having suggested it. I would also like to thank G. Prasad, who read the manuscript and pointed out a considerable number of misprints and corrections.

0. Notation and conventions

0.1. For $x \in \mathbf{R}_+ = \{ x \in \mathbf{R} \mid x > 0 \}$, we let $[x]'$ be the greatest integer strictly smaller than x.

0.2. Let u, v be functions on a set X, with values in $\mathbf{R}_+$. We write $u \prec v$ if there exists a constant $c > 0$ such that $u(x) \leqq c\, v(x)$ for all $x \in X$, $u \succ v$ if $v \prec u$ and $u \asymp v$ if $u \prec v$ and $v \prec u$.

0.3. Manifolds are smooth, i. e., C^∞ and oriented. Let M be a manifold. $C^\infty(M)$ is the space of real valued smooth functions and Ω_M the space of smooth exterior differential forms on M. For $x \in M$, $T_x(M)$ is the tangent space to M at x. If A is a tensor field on M, then A_x is the value of A at $x \in M$. If M is a Riemannian manifold, (A_x, B_x) is the scalar product defined by the metric of two tensors of the same type at x, and $| A_x | = (A_x, A_x)^{1/2}$; moreover $(A, B)_M$ denotes the integral of (A_x, B_x) over M, and $L^2(M)$ or $L^2(M, dv)$ [resp. $L^1(M)$ or $L^1(M, dv)$] is the space of square integrable (resp. integrable) functions on M, with respect to the Riemannian volume element dv.

0.4. The connected component of the identity of an algebraic group (in the Zariski topology), or of a topological group G, H is denoted G^0, H^0, ... The Lie algebra of an algebraic group or of a real Lie group G, H, ... is denoted L(G), L(H), ...

0.5. Our algebraic groups are linear and may be identified with algebraic subgroups of some $\mathbf{GL}_n(\mathbf{C})$. If we wish to emphasize a field of definition k for a given algebraic group H, we shall write H/k.

If k' is a finite separable extension of a field k, then $R_{k'/k}$ is the functor of restriction of scalars from k' to k ([23], Chap. I).

0.6. If a group G operates on a set A, then A^G is the set of fixed points of G in A.

1. A Stokes formula for complete Riemannian manifolds

In this section, M is a connected Riemannian manifold, g the metric tensor, $d(\ ,\ )$ and dv the associated distance function and volume element.

Our main aim in paragraphs 1 and 2 is to prove 1.3, 1.4 and 2.3, which are essentially known, although not exactly stated in the form we need them. The arguments are slight variations of the known ones.

1.1. We recall first that if M is *complete*, there exists a proper smooth function $\mu : M \to [0, \infty)$ and a constant $c > 0$ such that $|\operatorname{grad} \mu(x)| \leqq c$ for all $x \in M$ ([22], § 35).

Such a function can be obtained by regularization ([22], § 15) from the function $r : x \to d(o, x)$, where o is some chosen point of M. (The latter function is proper and satisfies our second condition wherever it is differentiable, i. e., almost everywhere.)

Although this is not needed in this paper, we remark that the converse is true : assume μ to exist. Let $\gamma : s \mapsto x(s)$ be a geodesic segment parametrized by arc length, where s varies over some bounded interval J of the real line. Assume the length of γ to be finite. Then, since $|d\gamma/ds| = 1$ identically, the variation of $\mu \circ \gamma$ on J is bounded and Im γ is contained in a bounded set (μ being proper). But then Im γ can be extended (at both ends) to a longer geodesic segment. Hence M is complete.

1.2. Lemma. — *Assume* M *to be complete. There exist compact sets* $C_r \subset D_r$ $(r \in \mathbf{R}_+^*)$ *such that* C_r *contains the interior of* $C_{r'}$ *if* $r > r'$, *and* M *is the union of the* C_r, *a family of smooth functions* σ_r $(r > 0)$ *and a constant* d *such that* $0 \leqq \sigma_r(x) \leqq 1$ $(x \in M)$, $\sigma_r(x) = 1$ *for* $x \in C_r$, $\sigma_r(x) = 0$ *for* $x \notin D_r$, and

$$|\operatorname{grad} \sigma_r(x)| \leqq d\, r^{-1}, \qquad (r > 0).$$

Let $m : [0, \infty) \to [0, 1]$ be a smooth function equal to 1 on $[0, 1]$ and 0 on $[2, \infty)$. Then put $\sigma_r(x) = m(\mu(x)/r)$, where μ is as in 1.1.

1.3. Proposition. — *Assume* M *to be complete. Let* X *be a continuous vector field on* M. *Assume that* $|X_x|$ *is bounded on* M *and that* $L_X(dv) = 0$. *Let* E *be a finite dimensional Hilbert space and* f *be a* C^1 E-*valued function on* M *such that* f, Xf *are integrable. Then*

$$\int_M Xf\,dv = 0.$$

Since $L_X(dv) = 0$, we have $Xf\,dv = L_X(f\,dv)$ hence, by H. Cartan's formula

(1) $$Xf\,dv = d(i(X)f\,dv) + i(X)\,d(f\,dv) = d(i(X)f\,dv).$$

Assume first that f has compact support. Then if B is a bounded open subset of M containing the support of f and having a sufficiently regular boundary ∂B, we have, by Stokes' formula

(2) $$\int_M Xf\,dv = \int_B d(i(X)f\,dv) = \int_{\partial B} i(X)f\,dv = 0.$$

Assume now simply f, Xf to be integrable and let σ_r be as in 1.2. Then, by (2) :

(3) $$0 = \int X(\sigma_r f)\,dv = \int f X\sigma_r\,dv + \int \sigma_r Xf\,dv.$$

Clearly

(4) $$\lim_{r \to \infty} \int_M \sigma_r Xf\,dv = \int_M Xf\,dv.$$

On the other hand

$$|X\sigma_r(x)| = |(X_x, \text{grad}\ \sigma_r(x))| \leqq |X_x|\, d/r, \qquad (x \in M),$$

is bounded on M, hence

$$\left|\int_M f X \sigma_r\, dv\right| \leqq (d/r) \operatorname{Max}_x |X_x| \int_M |f(x)|_E\, dv \to 0$$

as $r \to \infty$; together with (3) and (4), this proves the proposition.

1.4. COROLLARY. — *Let u, v be* E*-valued functions on* M *of class* C^1. *Assume that*

$$h(x) = (u(x), v(x))_E \quad \textit{and} \quad x \mapsto (Xu(x), v(x))_E,\ x \mapsto (u(x), Xv(x))_E$$

are in $L^1(M, dv)$. *Then* $(Xu, v)_M + (u, Xv)_M = 0$.

Since

$$Xh(x) = (Xu(x), v(x))_E + (u(x), Xv(x))_E,$$

the function h satisfies the conditions imposed on f in 1.3, hence $\int X\, h(x)\, dv = 0$, which proves 1.4.

1.5. COROLLARY. — *Let u, v be* E*-valued functions on* M *of class* C^1 *such that u, v, Xu, Xv are all square integrable on* M. *Then*

$$(Xu, v)_M + (u, Xv)_M = 0.$$

All the scalar products mentioned in the assumptions of 1.4 are integrable, hence 1.5 is a special case of 1.4.

2. Square integrable forms

2.1. Let M be a connected oriented complete Riemannian manifold, n its dimension, Ω^p_M the space of smooth real-valued exterior differential p-forms on M, $\Omega^* = \Omega^*_M$ the direct sum of the Ω^p. As usual, we have the exterior differentiation $d : \Omega^p_M \to \Omega^{p+1}_M$, the star operator $\bigstar : \Omega^p_M \xrightarrow{\sim} \Omega^{n-p}_M$, the operator $\partial = (-1)^{n(p+1)+1} \bigstar d \bigstar : \Omega^p \to \Omega^{p-1}$ and the Laplace-Beltrami operator $\Delta = d\partial + \partial d$.

For $\alpha, \beta \in \Omega^p$, the scalar product $(\alpha, \beta) = (\alpha, \beta)_M$ is defined by

$$(a, \beta)_M = \int_M (\alpha_x, \beta_x)\, dv$$

and we put $\|\alpha\| = (\alpha, \alpha)^{1/2}$. This scalar product is extended to Ω^*_M by decreeing that Ω^p and Ω^q are orthogonal if $p \neq q$. We let $\Omega^*_{M,(2)} = \Omega^p_{(2)}$ be the space of square integrable p-forms, and $\Omega^*_{(2)} = \Omega^*_{M,(2)}$ the direct sum of the $\Omega^p_{(2)}$.

2.2. PROPOSITION. – *Let* $\alpha \in \Omega^p_M$, $\beta \in \Omega^{p+1}_M$. *Assume that the functions*

$$x \to |\alpha_x| . |\beta_x|, \quad x \mapsto ((d\alpha)_x, \beta_x) \quad \textit{and} \quad x \mapsto (\alpha_x, (\partial\beta)_x)$$

are integrable. Then

$$(\alpha, \partial\beta)_M = (d\alpha, \beta)_M.$$

If one of α, β has compact support, this is standard. Therefore, with σ_r as in 1.2, we have

$$(\sigma_r \alpha, \partial\beta) = (d(\sigma_r \alpha), \beta). \tag{1}$$

We have $d(\sigma_r \alpha) = d\sigma_r \wedge \alpha + \sigma_r d\alpha$. The scalar products $(\sigma_r \alpha, \partial\beta)$ and $(\sigma_r d\alpha, \beta)$ tend respectively to $(\alpha, \partial\beta)$ and $(d\alpha, \beta)$ as $r \to \infty$. It suffices therefore to prove

$$\lim_{r\to\infty} (d\sigma_r \wedge \alpha, \beta) = 0. \tag{2}$$

By elementary algebra, there exists a constant $c > 0$ independent of r, σ, α such that $|(d\sigma_r \wedge \alpha)_x| \leqq c\,|(d\sigma_r)_x|.|\alpha_x|$. Taking 1.2 into account, we have then

$$|d\sigma_r(x) \wedge \alpha_x| \leqq c\,|d\sigma_r(x)|.|\alpha_x| \leqq c(d/r)|\alpha_x|$$

whence,

$$|(d\sigma_r \wedge \alpha, \beta)| \leqq c(d/r)\int |\alpha_x||\beta_x|\,dv$$

and (2) follows.

2.3. COROLLARY. — *Assume that* α, $d\alpha$, β, $\partial\beta$ *are square integrable. Then*

$$(d\alpha, \beta)_M = (\alpha, \partial\beta)_M.$$

By the Schwarz inequality, all the assumptions of 2.2 are fulfilled.

2.4. A form ω is *harmonic* if $\Delta\omega = 0$. We let $\mathscr{H}^p_{(2)}$ or $\mathscr{H}^p_{M,(2)}$ denote the space of square integrable harmonic p-forms. Since M is complete, $\omega \in \mathscr{H}^p_{(2)}$ if and only if $\omega \in \Omega^p_{(2)}$ and $d\omega = \partial\omega = 0$, by a theorem of Andreotti-Vesentini (*see* e. g. [22], § 35).

We recall that, by a result of Kodaira, if $\omega \in \Omega^p_{(2)}$ and $d\omega = 0$, then there exists $\sigma \in \Omega^{p-1}$ and $H\omega \in \mathscr{H}^p_{(2)}$ such that $\omega = H\omega + d\sigma$ (in [22], this follows from Theorems 14 and 24).

In fact, the result is also valid in the non-complete case, with " harmonic " being defined by the conditions $d\omega = \partial\omega = 0$, *loc cit.*

Let us denote by $H^p_{(2)}(M)$ the space of real p-dimensional cohomology classes of M which may be represented by a closed square integrable p-form. We have then natural maps

$$\mathscr{H}^p_{(2)} \xrightarrow{\mu} H^p_{(2)}(M) \xrightarrow{\nu} H^p(M).$$

By the result of Kodaira quoted above, μ is surjective. If M is compact then μ and ν are bijective by Hodge theory. In general, μ need not be injective, nor ν surjective. There is however a weaker form of injectivity which will be useful later :

2.5. PROPOSITION. — *Let* $\omega \in \mathscr{H}^p_{(2)}$ *and assume that* $\omega = d\sigma$, *for some* $\sigma \in \Omega^{p-1}_{(2)}$. *Then* $\omega = 0$.

In fact, $\partial\omega = 0$ by 2.4. Hence ω, $\partial\omega$, σ, and $d\sigma = \omega$ are all square integrable. By 2.3, we have then

$$(d\sigma, d\sigma) = (\omega, d\sigma) = (\partial\omega, \sigma) = 0,$$

whence $d\sigma = 0$ and $\omega = 0$.

2.6. REMARKS. – The proposition holds true if the assumption $\sigma \in \Omega_{(2)}^{p-1}$ is replaced by : the function $x \mapsto |\sigma_x| . |\omega_x|$ is integrable. The proof is the same, except for the use of 2.2 instead of 2.3.

3. The map $j^q : I_G^\Gamma \to H^*(\Gamma)$

3.1. In this section, G is a real semi-simple Lie group, whose identity component has finite index and finite center, K is a maximal compact subgroup of G and $X = K \backslash G$ the space of maximal compact subgroups of G. Endowed with a G-invariant Riemannian metric, X is a complete symmetric Riemannian manifold with negative curvature, diffeomorphic to euclidean space.

For a subgroup H of G, we let Ω_X^H be the algebra of differential forms on X which are invariant under H. If $H = G^0$, we write I_G instead of Ω_X^H. By well-known facts, which go back to E. Cartan, I_G consists of harmonic forms, and we have

$$I_G = H^*(L(G), L(K)), \tag{1}$$

where the second term denotes relative Lie algebra cohomology.

Let Γ be a discrete subgroup of G and $\pi : X \to X/\Gamma$ the natural projection. Γ acts properly on X, and

$$H^*(\Gamma) = H^*(\Omega_X^\Gamma) = H^*(X/\Gamma). \tag{2}$$

This is well-known if Γ is torsion free, because it then acts freely on X, X/Γ is a $K(\Gamma, 1)$, and π induces an isomorphism : $\Omega_{X/\Gamma} \xrightarrow{\sim} \Omega_X^\Gamma$. In the general case, this follows since the isotropy groups of Γ on X are finite, and therefore have trivial real cohomology (for a more general result, *see* e. g. [12])

Similarly, if $\Gamma \to GL(E)$ is a finite dimensional real or complex linear representation of Γ, then

$$H^*(\Gamma; E) = H^*((\Omega_X \otimes E)^\Gamma). \tag{3}$$

Since the elements of I_G are closed, the inclusion $I_G^\Gamma \subsetneq \Omega_X^\Gamma$ induces a homomorphism

$$j^* : \quad I_G^\Gamma \to H^*(\Gamma) \tag{4}$$

whose restriction to I_G^q will be denoted j^q, or j_Γ^q if necessary.

If $G = G^0$, there is a canonical isomorphism

$$H^*(\Gamma) = H^*(L(G), L(K); C^\infty(\Gamma\backslash G)) \quad ([20], \S\ 3). \tag{5}$$

In view of (5), we may then identify j^* with the homomorphism of Lie algebra cohomology induced by the inclusion $\mathbf{R} \to C^\infty(\Gamma\backslash G)$ of $\mathbf{R}$ onto the space of constant functions. If $\Gamma\backslash G$ is compact, then $\mathbf{R}$ has an *invariant* supplement, hence j^* is injective. This is then also true for G not connected by a standard argument (*see* proof of 3.4 below).

For other interpretations of j^*, at any rate when G is linear, we refer to 10.2.

3.2. Paragraphs 3.4 and 3.5 are devoted to some known results of Y. Matsushima and H. Garland. However, we shall state them in a slightly more general manner, and shall have accordingly to make some remarks on the proofs. To this end we need some notation and facts recalled here. The reader willing to take 3.4 and 3.5 for granted may skip this, since 3.2 will not be referred to elsewhere in this paper.

Let $L(G) = L(K)+\mathfrak{p}$ be the Cartan decomposition of $L(G)$ and

$$\theta : \ (x+y \mapsto x-y) \qquad (x \in L(K),\ y \in \mathfrak{p})$$

the corresponding Cartan involution. On G we consider the right invariant metric equal to $g_0(X, Y) = -B(X, \theta(Y))$. $(X, Y \in L(G))$ on $L(G)$, where B is the Killing form, whence also a metric on G/Γ. Let $(X_i)_{1 \leqq i \leqq N}$ be an orthonormal basis of $\mathfrak{p}$ and $(X_a)_{N < a \leqq \dim G}$ one of $L(K)$. Identify the $(X_j)_{1 \leqq j \leqq \dim G}$ to right invariant vector fields on G, hence to vector fields on G/Γ. Since the metric is right invariant, and G is unimodular, we have

$$(1) \qquad |X_{i,x}| = 1, \qquad (1 \leqq i \leqq \dim G;\ x \in G/\Gamma),$$

$$(2) \qquad L_{X_i}(dv) = 0, \qquad (1 \leqq i \leqq \dim G).$$

Let (ω^j) be the basis of Maurer-Cartan forms dual to (X_j). For a strictly increasing sequence $I = \{ i_1, \ldots, i_q \}$ of indices between 1 and $\dim G$, let $\omega^I = \omega^{i_1} \wedge \ldots \wedge \omega^{i_q}$ and $|I| = \text{Card}\, I$. For simplicity, we shall assume Γ torsion-free. Let

$$\pi : \ G/\Gamma \to X/\Gamma = K\backslash G/\Gamma$$

be the canonical projection. Then $\sigma \mapsto \sigma_0 = \sigma \circ \pi$ defines an isomorphism of $\Omega_{X/\Gamma} = \Omega_X^\Gamma$ onto the complex Ω_0 of forms on G/Γ which are left invariant under K and annihilated by the interior products $i(X)$ $(X \in L(K))$. Any such form can be written uniquely

$$(3) \qquad \sigma_0 = \sum_{|I|=q} \sigma_I \omega^I,$$

with $\sigma_I \in C^\infty(G/\Gamma)$, and where I runs through the strictly increasing subsequences of $\Lambda = \{ 1, 2, \ldots, N \}$. There is a constant c such that for $\sigma, \tau \in \Omega_{X/\Gamma}$:

$$(4) \qquad (\sigma, \tau)_{X/\Gamma} = c \sum_I \int_{G/\Gamma} (\sigma_I, \tau_I)\, dv.$$

We have, for σ of degree $\geqq 1$:

$$(5) \qquad (\Delta\sigma)_0 = -\sum_I C\, \sigma_I \omega^I,$$

where C is the Casimir operator for G.

3.3. Assume G to be connected, simple non-compact. In [19], Y. Matsushima has attached to G a constant, depending in fact only on L (G), to be denoted m (G). We sketch its definition. Let $\mathfrak{m}$ be the second symmetric power of $\mathfrak{p}$. On $\mathfrak{m}$ let (,) be the scalar product defined by the Killing form and let P (,) be the quadratic form

$$P(\xi, \eta) = \sum R_{iklj} \xi_{kl} \eta_{ij},$$

where R_{iklj} denotes the curvature tensor. Then

$$m(G) = \max q \,|\, (A/q)(\xi, \xi) + P(\xi, \xi) > 0 \qquad (\xi \in \mathfrak{m} - \{0\}),$$

where A is a certain constant, defined in terms of the restriction of the Killing form to L (K).

DEFINITION. — For G as in 3.1, let $m(G) = \min m(G_i)$, where G_i runs through the simple connected non-compact normal subgroups of G^0.

3.4. THEOREM (Matsushima [19]). — *Let* Γ *be a discrete subgroup of* G *and assume* G/Γ *to be compact. Then*

(i) j^q *is injective for all* q*' s*;

(ii) j^q *is surjective for* $q \leqq m(G)$.

In fact, in [19], Theorem 1, G is connected, simple non-compact, and Γ is torsion-free. We discuss briefly how to go from there to the general case. Γ is linear, finitely generated. By a well-known result of A. Selberg, it has a torsion-free subgroup Γ′ of finite index, which may then be assumed to be normal, and contained in G^0. We have then

$$H^*(\Gamma) = H^*(\Gamma')^{\Gamma/\Gamma'}, \qquad I_G^{\Gamma} = (I_G^{\Gamma'})^{\Gamma/\Gamma'}, \qquad \Omega_X^{\Gamma} = (\Omega_X^{\Gamma'})^{\Gamma/\Gamma'},$$

from which it is clear that the result for Γ′ implies the result for Γ. Thus we may assume that G is connected and Γ torsion-free.

We already saw (3.1) that j^* is injective. This also follows from Hodge theory, since I_G consists of harmonic forms.

By examining Matsushima's proof of (ii), it is not difficult to see that the argument is valid for a non-simple group, provided the constant A is replaced by the minimum of the corresponding constants assigned to the various non-compact factors. This, which would suffice for our purposes, may yield a constant m (G) somewhat smaller than the one defined above. That one can use m (G) as defined above was pointed out to me by H. Garland. I sketch his argument. Let G_t ($1 \leqq t \leqq s$) be the simple non-compact factors of G. Then X is the Riemannian product of the symmetric spaces X_t corresponding to the G_t' s, and Δ is the sum of the corresponding Laplacian Δ_t. A form η on X/Γ is harmonic if and only if $(\Delta\eta, \eta) = 0$. We have $(\Delta\eta, \eta) = \sum (\Delta_t \eta, \eta)$ and, since the Δ_t are positive,

$$(1) \qquad \Delta\eta = 0 \iff \Delta_t \eta = 0 \qquad (1 \leqq t \leqq s).$$

The proof of 3.2 (5) also yields

$$(\Delta_t \eta)_0 = \sum -C_t \eta_I \omega^I$$

hence

$$\Delta\eta = 0 \quad \Leftrightarrow \quad -C_t \eta_I = 0 \qquad (1 \leqq t \leqq s,\ I \subset [1, N]). \tag{2}$$

Let η be a harmonic q-form and $\eta_0 = \sum \eta_I \omega^I$ the corresponding form on G/Γ. Let $q \leqq m(G)$. We have to prove it has constant coefficients. For this it is enough to show

$$X_j \eta_I = 0 \qquad (1 \leqq j \leqq N), \tag{3}$$

for all I' s in Λ. We may assume that $\mathfrak{p}_t = L(G_t) \cap \mathfrak{p}$, is spanned by some of the X_i' s $(1 \leqq t \leqq s)$. Let then J_t be the set of indices for which $X_i \in \mathfrak{p}_t$ and J_t' its complement. We can write

$$\eta_0 = \sum_{J \subset J_t'} \tau_J \wedge \omega^J, \tag{4}$$

where τ_J is homogeneous of degree $d(J) = q - |J|$ and of the form

$$\tau_J = \sum_{\substack{I \subset J_t \\ |I| = d(J)}} \eta_{J,I} \omega^I. \tag{5}$$

τ_J is a $d(J)$-form on G/Γ which is left-invariant under $K_t = K \cap G_t$, annihilated by the interior products $i(X)$ $(X \in L(K_t))$. Thus, if we put

$$\varphi(x) = (1/2q) \sum_{\substack{j, k \in J_t \\ I \subset J_t}} ([X_j, X_k]\eta_{J,I})^2 \tag{6}$$

and consider its integral over G/Γ, all the formal computations of Matsushima [19] go through and yield (3) for $j \in J_t$ since $d(J) = q - |J| \leqq m(G) \leqq m(G_t)$.

3.5. THEOREM (Garland [9]). — *Let Γ be a discrete torsion-free subgroup of* G. *Let $q \leqq m(G)$. Assume that every class of $H^q(\Omega_0)$ is representable by a square integrable form. Then j^q is surjective.*

In fact, Theorem 3.5 of [9] is a special case of our 3.5, but the argument is general. We sketch a slight variation of it. Let $q \leqq m(G)$. We have to prove that $\mathscr{H}^q_{(2)} \subset I_G^{q,\Gamma}$, or, equivalently

$$(\mathscr{H}^q_{(2)})_0 \subset (I_G^{q,\Gamma})_0. \tag{1}$$

Fix a Haar measure dy on G. As usual, the convolution product $u \bigstar v$ of two functions on G is defined by

$$u \bigstar v(x) = \int_G u(xy^{-1}) v(y)\, dy, \qquad (x \in G). \tag{2}$$

If $u \in C_c^\infty(G)$ and $v \in L^2(G/\Gamma)$, then $u \bigstar v \in C^\infty(G/\Gamma) \cap L^2(G/\Gamma)$, and for any element X in the universal enveloping algebra $U(L(G))$ of $L(G)$, we have

$$X(u \bigstar v) = (Xu) \bigstar v \in C^\infty(G/\Gamma) \cap L^2(G/\Gamma). \tag{3}$$

Let $\alpha \in C_c^\infty(G)$ be K-invariant [i. e. $\alpha(kxk^{-1}) = \alpha(x)\,(k \in K;\ x \in G)$]. Then $\alpha \bigstar$ commutes with the Casimir operator and K acting on G/Γ in the natural way. Therefore, if $\eta = \sum \eta_I \omega^I$ is in Ω_0 [resp. $(\mathscr{H}^q_{(2)})_0$], then so is

$$\eta_{(\alpha)} = \sum_I (\alpha \bigstar \eta_I)\,\omega^I.$$

Let now $\eta \in (\mathscr{H}^q_{(2)})_0$. Matsushima's proof of (1) in the case where G/Γ is compact starts with the function

$$\varphi(x) = \sum_{a,b,I} ([X_a, X_b]\,\eta_I)^2, \tag{4}$$

$(1 \leqq a, b \leqq N)$, studies its integral over G/Γ, and uses repeatedly Stokes' formula (1.5), which makes no difficulty when G/Γ is compact. Replace now η by $\eta_{(\alpha)}$. Then for any $Y \in U(L(G))$, $Y(\alpha \bigstar \eta_i)$ is square integrable; therefore by 3.2 (1), (2) we can apply 1.5 to $X = X_i$ $(1 \leqq i \leqq \dim G)$ and $u = Y(\alpha \bigstar \eta_I)$, $v = Y'(\alpha \bigstar \eta_{I'})$ $(Y, Y' \in U(L(G))$. Then all the steps in Matsushima's argument go through. This shows that $\eta_{(\alpha)} \in (I_G^{q,\Gamma})_0$ for any K-invariant $\alpha \in C_c^\infty(G)$. But we may choose a sequence α_n of such functions forming a Dirac sequence ([14], § 2), and then η_I is the L^2-limit of the $\alpha_n \bigstar \eta_I$. Therefore η belongs to the L^2-closure of the space of square integrable elements in $(I_G^{q,\Gamma})_0$. Since $I_G^{q,\Gamma}$ is finite-dimensional, the latter space is closed, whence (1).

Remark. — If $\eta \in (I_G^{q,\Gamma})_0$, then its coefficients η_I are constant on G/Γ (and conversely). Thus, if the volume of G/Γ is infinite, no non-zero element of $I_G^{q,\Gamma}$ is square integrable, and the theorem asserts simply that no element of $H^q(\Gamma)$ is representable by a square integrable form for $q \leqq m(G)$.

3.6. PROPOSITION. — *Let Γ be a discrete subgroup of* G *and Γ' a torsion-free normal subgroup of finite index of Γ. Assume that X/Γ has finite volume.*

(i) *Let C' be a subcomplex of $\Omega_X^{\Gamma'}$ and m' a positive constant such that*

(*a*) *the inclusion $C' \to \Omega_X^{\Gamma'}$ induces an isomorphism in cohomology up to dimension m';*

(*b*) *C'^q consists of square integrable forms for $q \leqq m'$;*

(*c*) *$I_G^{q,\Gamma'} \subset C'^q$ for $q \leqq m'$.*

Then $j^q : I_G^{q,\Gamma'} \to H^q(\Gamma')$ is injective for $q \leqq m'(G)$ and bijective for $q \leqq \min(m(G), m')$.

(ii) *Assume C' to be stable under the natural action of Γ/Γ' on $\Omega_X^{\Gamma'}$ and let $C = (C')^{\Gamma/\Gamma'}$. Then* (i) *is valid with C' and Γ' replaced by C and Γ.*

(i) Let $q \leqq m'(G)$ and $\omega \in I_G^{q,\Gamma'}$. Assume that $j^q(\omega) = 0$. Let $\sigma \in \Omega_X^{q-1,\Gamma'}$ be such that $\omega = d\sigma$. In view of (*c*) and (*a*), we may (and do) assume $\sigma \in C'^{q-1}$. But then σ is square integrable by (*b*), and $\omega = 0$ by 2.5. Thus j^q is injective. For $q \leqq \min(m', m(G))$ every class is representable by a square integrable form by (*a*), (*b*), and j^q is surjective (3.5).

(ii) This follows from (i) and the elementary relations

$$H^*(C) = H^*(C')^{\Gamma/\Gamma'}, \qquad H(\Omega_X^\Gamma) = H(\Omega_X^{\Gamma'})^{\Gamma/\Gamma'}, \qquad I_G^\Gamma = (I_G^{\Gamma'})^{\Gamma/\Gamma'}.$$

3.7. REMARK. — The proof of injectivity shows more generally that for $q \leqq m'$, the space $\mathscr{H}^q_{(2)} \cap C^q$ of harmonic forms contained in C^q injects into the cohomology. In particular, an element of $H^q(\Omega^\Gamma_X)$ has at most one harmonic representative in C^q.

4. Decomposition of the metric on X, associated to a parabolic subgroup

4.1. In this section, k is a subfield of **R** and G a connected semi-simple k-group. The groupe $G(\mathbf{R})$ of real points of G plays the role of our former G.

As before, K is a maximal compact subgroup of $G(\mathbf{R})$, $X = K\backslash G(\mathbf{R})$, θ the Cartan involution of $L(G(\mathbf{R}))$, or of $G(\mathbf{R})$, with respect to K, B(,) the Killing form of $L(G(\mathbf{R}))$ and g_0 the scalar product on $L(G(\mathbf{R}))$ defined by

$$g_0(\xi, \eta) = -B(\xi, \theta(\eta)), \qquad [\xi, \eta \in L(G(\mathbf{R}))].$$

Let $\sigma : G(\mathbf{R}) \to X$ be the canonical projection and $o = \sigma(K)$. Then $d\sigma_e$ identifies the orthogonal complement $\mathfrak{p}$ of $L(K)$ with respect to B (or, equivalently, to g_0) with $T_0(X)$. The restriction of g_0 to $\mathfrak{p}$ then defines a metric on $T_0(X)$, invariant under the natural action of K, and we let dx^2 be the corresponding $G(\mathbf{R})$-invariant metric on X.

4.2. Let P be a parabolic k-subgroup of G, M the Levi k-subgroup of P stable under θ ([4], 1.9), $S_P = M \cap R_d P$ the greatest torus of the split radical $R_d P$ of P stable under θ and $A = S_P(\mathbf{R})^0$. With ${}^0P$ defined as in ([4], 1.1) we have

$$(1)\qquad \begin{cases} P(\mathbf{R}) = A \ltimes {}^0P(\mathbf{R}), \qquad {}^0P = {}^0M \ltimes U, \\ K \cap P = K \cap {}^0M \quad \text{maximal compact in } P(\mathbf{R}). \end{cases}$$

We put

$$Z = (K \cap P)\backslash {}^0M(\mathbf{R})$$

and have then

$$(2)\qquad (K \cap P)\backslash {}^0P(\mathbf{R}) \cong Z \times U(\mathbf{R}),$$

which should be kept in mind to make the transition from [4] to this paper. The map σ induces, by passage to the quotient, an isomorphism

$$(3)\qquad \mu_0 : \; Y = A \times Z \times U(\mathbf{R}) \tilde{\to} X.$$

The map commutes with $P(\mathbf{R})$, where the action of $P(\mathbf{R})$ on Y is defined by

$$(4)\qquad \begin{cases} (b, z, u)\,p = (ab, zm, m^{-1}a^{-1}uamv), \\ [p = amv;\ a \in A;\ m \in {}^0M(\mathbf{R});\ v \in U(\mathbf{R})] \quad ([3], 1.5). \end{cases}$$

If ω is a differential form on X or Y, we shall also denote by $\omega.p$ its transform under the action of p. As in [3], we let $dy^2 = \mu_0^*(dx^2)$ and da^2 (resp. du^2) be the right-invariant metric on A [resp. $U(\mathbf{R})$] which is equal to the restriction of g_0 on $L(A)$ [resp. $L(U(\mathbf{R}))$]. We want to write da^2 and du^2 more explicitly.

Let Δ be a basis of the set of k-roots of G with respect to some maximal k-split torus S. The conjugacy classes over k of parabolic k-subgroups are parametrized by the subsets of Δ. Let $I = I(P)$ be the type of P ([4] 4.1). There is then a canonical isomorphism

$$\text{(5)} \qquad A \tilde{\to} (\mathbf{R}_+^*)^{\Delta - I}, \quad ([4], 4.2);$$

we can identify Δ-I with a basis of $X(S_P) \otimes \mathbf{Q}$ and write

$$\text{(6)} \qquad da^2 = \sum_{\alpha, \beta \in \Delta - I} c_{\alpha\beta}\, \alpha^{-1} \beta^{-1} d\alpha\, d\beta,$$

where $\sum c_{\alpha\beta}\, d\alpha\, d\beta$ is the restriction of the Killing form to L (A).

Let Φ_P be the set of roots of P with respect to S_P. For $\beta \in \Phi_P$, let

$$\text{(7)} \qquad \mathfrak{u}_\beta = \{ X \in L(G(\mathbf{R})) \mid \operatorname{Ad} a X = a^\beta X, \ (a \in A) \}.$$

Then

$$\text{(8)} \qquad L(U(\mathbf{R})) = \bigoplus_\beta \mathfrak{u}_\beta,$$

and each u_β is stable under $\operatorname{Ad}_{L(G)}(A\, {}^0M(\mathbf{R}))$. For $\beta \in \Phi_P$, let h_β be the right invariant scalar product on U (R) which is zero on $\mathfrak{u}_\alpha$ $(\alpha \neq \beta)$, and equal to g_0 on $\mathfrak{u}_\beta$. By [3], 1.4 (6), the spaces $\mathfrak{u}_\beta$ $(\beta \in \Phi_P)$ are mutually orthogonal. For $a \in A$, $m \in {}^0M(\mathbf{R})$, we have then

$$\text{(9)} \qquad (\operatorname{Int} am)^*(du^2) = \bigoplus_\beta a^{2\beta} (\operatorname{Int} m)^*\, h_\beta.$$

For $k \in K$, Ad k leaves g_0 invariant; therefore, if $k \in K \cap M$, then Int k leaves h_β invariant, hence $(\operatorname{Int} m)^* h_\beta$ depends only on the image z of m in Z under the natural projection. Let us then put

$$\text{(10)} \qquad (\operatorname{Int} m)^* h_\beta = h_\beta(z), \qquad [m \in {}^0M(\mathbf{R});\ z = om].$$

4.3. PROPOSITION. — *Let* $y = (a, z, u) \in Y$. *The spaces* L (A) a, T_z (Z), *and* $\mathfrak{u}_\beta\, u$ $(\beta \in \Phi_P)$ *are mutually orthogonal, and we have*

$$\text{(1)} \qquad (dy^2)_y = (da^2)_a \oplus (dz^2)_z \oplus \bigoplus_\beta 2^{-1} a^{2\beta} h_\beta(z).$$

By [3], 1.6, we have (1) with the last sum replaced by $(\operatorname{Int} am)^*\,(du^2)_u$, where $m \in {}^0M(\mathbf{R})$ is such that $om = z$. The proposition then follows from 4.2 (9), (10).

4.4. COROLLARY. — *Let* dv_Y, dv_A, dv_Z *and* dv_U *be the volume elements of the metrics* dy^2, da^2, dz^2 *and* du^2. *Then*

$$\text{(1)} \qquad \begin{cases} 2^e\, dv_Y = a^{2\rho}\, dv_A \wedge dv_Z \wedge dv_U, & [e = (\dim U)/2], \\ dv_A = c \bigwedge_{\alpha \in \Delta - I} \dfrac{d\alpha}{\alpha}, & \end{cases}$$

where $2\rho = 2\rho_P = \sum_{\beta \in \Phi_P} \beta \dim \mathfrak{u}_\beta$ and $c = (\det c_{\alpha\beta})^{1/2}$.

By definition, any rational **Q**-character of M is equal to ± 1 on ^{0}M, hence Ad $m \mid$ L (U (**R**)) is unimodular, and (Int m)* du^2 and du^2 have the same volume element. The corollary is then an obvious consequence of 4.3 and 4.2 (6).

5. Differential forms on Siegel sets

5.1. We keep the assumptions of paragraph 4 and first fix some convention and notation to describe differential forms on certain subsets of Y. Let $\Delta - \mathrm{I} = \{ \alpha_1, \ldots, \alpha_s \}$. We let $m = \dim \mathrm{X}$, fix a moving frame $(\omega^i)_{1 \leqq i \leqq m}$ on Y, where ω^i is lifted, under one of the natural projections, from $d \log \alpha_i$ on A if $i \leqq s$, from an orthonormal frame on Z if $s < i \leqq t$, and from a set of right invariant one-forms on U (**R**) which, at the origin, span the various subspaces $\mathfrak{u}^*_\beta$ $(\beta \in \Phi_{\mathrm{P}})$ if $t < i \leqq m$.

Let $\mathrm{I}_m = \{ 1, \ldots, m \}$. For $i \in \mathrm{I}_m$, we put

$$(1) \qquad \alpha(i) = \begin{cases} 0 & \text{if } 1 \leqq i \leqq t, \\ \beta & \text{if } \omega^i_e \in \mathfrak{u}^*_\beta. \end{cases}$$

For a subset J of I_m, we let

$$(2) \qquad \omega^{\mathrm{J}} = \Lambda_{i \in \mathrm{J}} \omega^i, \qquad \alpha(\mathrm{J}) = \sum_{i \in \mathrm{J}} \alpha(i).$$

A differential form τ of degree q on Y, or on an open subset of Y, will then be written

$$\tau = \sum_{\mathrm{J}} f_{\mathrm{J}} \omega^{\mathrm{J}}$$

where J runs through the subsequences of I_m whose number of elements $|\mathrm{J}|$ equals q.

5.2. X (A) will denote the group of continuous homomorphisms of A into $\mathbf{R}^*_+$. We have $\mathrm{X}(\mathrm{A}) \cong \mathrm{X}(\mathrm{S}_{\mathrm{P}}) \otimes_{\mathbf{Z}} \mathbf{R}$, and, by restriction, X (S_{P}) may be identified to a lattice in X (A). Any $\lambda \in \mathrm{X}(\mathrm{A})$ is a linear combinaison of the $\alpha_i' s$ and can be written

$$\lambda = \sum c_i \alpha_i.$$

We write $\lambda \gg 0$ if $c_i > 0$ for *all* $i' s$, and $\lambda \underline{\gg} 0$ if $c_i \geqq 0$ for all $i' s$.

5.3. For $t > 0$, let $\{ \mathrm{A}_t = a \in \mathrm{A} \mid a^\alpha \leqq t \, (\alpha \in \Delta - \mathrm{I}) \}$. A Siegel *set* $\mathfrak{S}_{t, \omega}$ (resp. a *cylindrical set* $\mathfrak{C}_{t, \omega}$) in X, with respect to P, o, is a set of the form

$$\mathfrak{S}_{t, \omega} = \mu_0 (\mathrm{A}_t \times \omega), \qquad [\text{resp. } \mathfrak{C}_{t, \omega} = \mu_0 (\mathrm{A}_t \times \omega \times \mathrm{U}(\mathbf{R})],$$

where ω is relatively compact in $\mathrm{Z} \times \mathrm{U}(\mathbf{R})$ (resp. Z).

In pratice, it is understood that ω is such that the complement of the set of interior points has measure zero; by definition a (smooth) differential form of such a set U is the restriction of a (smooth) form defined in some open neighborhood of U.

5.4. LEMMA. — *Let $\mathfrak{S}$ be a Siegel set with respect to* P, *o and f a positive measurable function on* $\mathfrak{S}$. *Let* $\lambda \in X(A)$; *assume that* $(f \circ \mu_0)(a, q) \prec a^\lambda$ $(a \in A_t,\ q \in \omega)$, *and that* $2\rho+\lambda \gg 0$. *Then* $f \in L^1(\mathfrak{S}, dv_X)$. *In particular,* $\mathfrak{S}$ *has finite volume.*

This amounts to showing that $a^\lambda \in L^1(A_t \times \omega, dv_Y)$. Write

$$2\rho+\lambda = \sum_{1 \leq i \leq s} m_i \alpha_i.$$

Then, by 4.4, there is a constant $d > 0$ such that

$$\int_{A_t \times \omega} a^\lambda \, dv_Y = d \prod_{i \leq s} \int_0^t \alpha_i^{m_i - 1} \, d\alpha_i.$$

Since $m_i > 0$ for all i by assumption, each factor on the right hand side is finite, whence the first assertion. Since $\rho \gg 0$, this shows that any constant function is integrable, whence the second assertion.

5.5. PROPOSITION. — *Let* $\mathfrak{S}$ *be a Siegel set with respect to* P, *o,* σ *a continuous q-form on* $\mathfrak{S}$, *and* $\tau = \mu_0^*(\sigma) = \sum f_J \omega^J$ *Assume there exists* $\lambda \in X(A)$ *such that* $\rho+\lambda-\nu \gg 0$ *whenever* ν *is a weight of* A *in* $\bigotimes_{i \leq q} \Lambda^i L(U(\mathbf{R}))$, *and that* $|f_J| \prec a^\lambda$ *on* $\mu_0^{-1}(\mathfrak{S})$ *for all* J. *Then* σ *is square integrable on* $\mathfrak{S}$.

We have to show

(1) $$\int_{\mathfrak{S}} (\sigma_x, \sigma_x) \, dv_X < \infty,$$

where (,) denotes the scalar product on $\Lambda^q T^*(X)$ associated to dx^2. This amounts to proving

(2) $$\int_{A_t \times \omega} (\tau_y, \tau_y) \, dv_Y < \infty,$$

where (,) now denotes the scalar product on $\Lambda^q T^*(Y)$ associated to dy^2. With (ω^i) as in 5.1, write

(3) $$dy^2 = \sum_{1 \leq i, j \leq m} g_{ij} \omega^i \omega^j.$$

As usual, let (g^{ij}) be the inverse matrix to (g_{ij}). For two strictly increasing sequences $J = \{j_1, \ldots, j_q\}$, $J' = \{j'_1, \ldots, j'_q\}$ in I_m, put

(4) $$g^{J, J'} = \det(g^{j_i, j'_k})$$

and let

(5) $$f^J = \sum_{J'} g^{J, J'} f_{J'}.$$

Then

(6) $$(\tau_y, \tau_y) = \sum_{J, J'} g^{J, J'} f_J f_{J'} = \sum_J f^J f_J.$$

We have therefore to prove that $f^J f_J$ is integrable on $A_t \times \omega$. Write

$$\mathrm{I}_m = \mathrm{I}_{-1} \cup \mathrm{I}_0 \cup \bigcup_\beta \mathrm{I}_\beta, \tag{7}$$

where $\mathrm{I}_{-1} = \{1, \ldots, s\}$, $\mathrm{I}_0 = \{s+1, \ldots, t\}$ and $\mathrm{I}_\beta = \{ i \in \mathrm{I}_m \mid \omega_e^i \in \mathfrak{u}_\beta^* \}$. Let, using the notation of 4.2 (10) :

$$h_{\beta, z} = \sum_{i, j \in \mathrm{I}_\beta} h_{\beta, ij}(z)\, \omega^i \omega^j. \tag{8}$$

Let $h_\beta^{ij}(z)$ be the inverse matrix to $(h_{\beta, ij}(z))$ and (c^{ij}) the inverse matrix to (c_{ij}). We have

$$g^{ij} = \begin{cases} c^{ij} & (i, j \in \mathrm{I}_{-1}), \\ \delta_{ij} & (i, j \in \mathrm{I}_0), \\ 2a^{-2\beta} h_\beta^{ij}(z) & (i, j \in \mathrm{I}_\beta), \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

It follows that $g^{\mathrm{J}, \mathrm{J}'} = 0$ unless J and J′ have the same number of elements in each of I_{-1}, I_0, I_β ($\beta \in \Phi_{\mathrm{P}}$). Assume this is the case. Then

$$\alpha(\mathrm{J}) = \alpha(\mathrm{J}') \quad \textit{and} \quad |g^{\mathrm{J}, \mathrm{J}'}(y)| \prec a^{-2\alpha(\mathrm{J})}, \quad [y \in \mathrm{A}_t \times \mathrm{C} \times \mathrm{U}(\mathbf{R}),\ a = \mathrm{pr}_{\mathrm{A}}\, y], \tag{10}$$

where C is relatively compact in Z.
It follows then from the assumption on the f_J' s and from (10) that we have

$$|f^{\mathrm{J}}| \leqq \sum_{\mathrm{J}'} |g^{\mathrm{J}, \mathrm{J}'}| . |f_{\mathrm{J}'}| \prec a^{-2\alpha(\mathrm{J}) + \lambda}, \tag{11}$$

$$|f^{\mathrm{J}} . f_{\mathrm{J}}| \prec a^{-2\alpha(\mathrm{J}) + 2\lambda}, \tag{12}$$

on $\mathrm{A}_t \times \mathrm{C} \times \mathrm{U}(\mathbf{R})$. By 5.4, $f^J f_J$ is integrable on $\mathrm{A}_t \times \omega$ if

$$2\rho + 2\lambda - 2\alpha(\mathrm{J}) \gg 0. \tag{13}$$

But $\alpha(\mathrm{J})$ is the sum of at most q elements of Φ_{P}, and an element β occurs at most $|\mathrm{I}_\beta| = \dim \mathfrak{u}_\beta$ times. Thus our assumption on λ implies that (13) is fulfilled for all J with q elements, and the proposition is proved.

5.6. PROPOSITION. — *Let* $\sigma = \sum_{\mathrm{J}} f_{\mathrm{J}} \omega^{\mathrm{J}}$ *be a q-form on* Y *and* $b \in \mathrm{A}$. *Then*

$$(\sigma b)_y = \sum_{\mathrm{J}} f_{\mathrm{J}}(yb)\, b^{-\alpha(\mathrm{J})} \omega^{\mathrm{J}} \qquad (y \in \mathrm{Y}). \tag{1}$$

In particular, if $\sigma b = \sigma$, *we have*

$$f_{\mathrm{J}}((a, q)\, b) = b^{\alpha(\mathrm{J})} f_{\mathrm{J}}(a, q), \qquad [y = (a, q);\ a \in \mathrm{A};\ q \in \mathrm{Z} \times \mathrm{U}(\mathbf{R})]. \tag{2}$$

We have

$$(\sigma b)_y = \sum_{\mathrm{J}} f_{\mathrm{J}}(yb)(\omega^{\mathrm{J}} b).$$

It suffices therefore to prove, using the notation of 5.1 (1) :

(3) $$\omega^j b = b^{-\alpha(j)} \omega^j,$$

for any $j \, (1 \leqq j \leqq m)$. By 4.2 (4), we have

$$(a, z, u)\, b = (ab, z, b^{-1} ub), \qquad [a \in \mathrm{A};\ z \in \mathrm{Z};\ u \in \mathrm{U}(\mathbf{R})].$$

If $j \leqq s$, then $\omega^j = d\alpha_j/\alpha_j$, $\alpha_j(i) = 0$; and (3) follows from the invariance of $d\alpha_j/\alpha_j$ under translations on A. For $s < j \leqq t$, $\omega^j b = \omega^j$ and $\alpha(j) = 0$. If now $j > t$ and $\alpha(j) = \beta \in \Phi_{\mathrm{P}}$, then $\omega^j b$ is the right-invariant form on U (**R**) whose value at e is equal to $\mathrm{Ad}\, b^{-1}(\omega_e^j)$. Since $\omega_e^j \in \mathfrak{u}_\beta^*$ and $\mathrm{Ad}\, b^{-1}$ is the dilatation by $b^{-\beta}$ on $\mathfrak{u}^*$, this proves (3) also in that case.

5.7. Corollary. — *Assume* σ *to be invariant under* A, *and let* $a_0 \in \mathrm{A}$. *Then*

(1) $$f_{\mathrm{J}}(a, z, u) = a^{\alpha(\mathrm{J})} f_{\mathrm{J}}(1, z, aua^{-1}) \qquad [a \in \mathrm{A},\ z \in \mathrm{Z},\ u \in \mathrm{U}(\mathbf{R})].$$

In particular, on $\mathfrak{S}' = \mathrm{A}_t \times \omega$ [ω *relatively compact in* $\mathrm{Z} \times \mathrm{U}(\mathbf{R})$], *we have*

(2) $$|f_{\mathrm{J}}(a, q)| \prec a^{\alpha(\mathrm{J})} \qquad (a \in \mathrm{A}_t;\ q \in \omega)$$

and σ *is square integrable.*

(1) is a special case of 5.6 (2). It is elementary and well known that if C is relatively compact in U (**R**), then $\{ aua^{-1} \mid a \in \mathrm{A}_t;\ u \in \mathrm{C} \}$ is relatively compact. Hence (1) ⇒ (2). By 5.5 (10) we have

$$|f^{\mathrm{J}}(a, q)| \leqq \sum_{\mathrm{J}'} |g^{\mathrm{J},\mathrm{J}'}| |f_{\mathrm{J}'}| \prec a^{-\alpha(\mathrm{J})},$$

whence $|f^{\mathrm{J}} f_{\mathrm{J}}| \prec 1$. The latter is then integrable by (5.4).

6. The manifold with corners $\overline{\mathrm{X}}/\Gamma$

In this section, G is a connected semi-simple **Q**-group and Γ a torsion-free arithmetic subgroup. We shall recall first some results of [4], and then introduce neighborhoods and partitions of unity which reflect the structure of manifold with corners of $\overline{\mathrm{X}}/\Gamma$.

6.1. As in [4], $\mathfrak{P}$ is the set of parabolic **Q**-subgroups of G and $\pi : \mathrm{X} \to \mathrm{X}/\Gamma$ the natural projection. Our former A will be denoted A_{P}. This is in slight conflict with [4], where A_{P} denotes a subgroup of $\mathrm{P}/\mathrm{R}_u \mathrm{P}$. Our present A_{P} is then the unique lifting in P (**R**) of the previous one which is stable under the Cartan involution associated to K. It depends then on K, or o; if more precision appears to be desirable we then shall also write $\mathrm{A}_{o,\mathrm{P}}$ instead of A_{P}. By convention, the geodesic action of $a \in \mathrm{A}_{o,\mathrm{P}}$ is the geodesic action, as defined in [4], paragraph 3, of the canonical image of a in the center in $\mathrm{P}/\mathrm{R}_u \mathrm{P}$. Consequently, modulo the canonical identification of $\mathrm{A}_{o,\mathrm{P}}$ with the $\mathrm{A}_{\mathrm{P}} \subset \mathrm{P}/\mathrm{R}_u \mathrm{P}$ of [4], the maps μ_0 of this paper and of ([4], 5.4) are the same, and it follows that the results of [4] involving A_{P} to be used here are valid with the present interpretation of A_{P} and μ_0.

Given $P \in \mathfrak{P}$, there exists, by [4], paragraph 10, a number $t(o, P) > 0$ such that if $t \leqq t(o, P)$, the equivalence relations defined by Γ and $\Gamma_P = \Gamma \cap P$ on

$$U_{o,P,t} = \mu_0 (A_{P,t} \times Z \times U(\mathbf{R}))$$

are the same and consequently $\mu_0' = \pi \circ \mu_0$ is an isomorphism

$$\mu_0' : \quad A_{P,t} \times (Z \times U(\mathbf{R}))/\Gamma_P = A_{o,P,t} \times (Z \times U(\mathbf{R}))/\Gamma_P \xrightarrow{\sim} \pi(U_{o,P,t}).$$

Moreover, the action of the semi-group $A_{o,P,1}$ on A_t by translations gives rise *via* μ_0' to an invariantly defined action stemming from the geodesic action ([4], 10.3). In our present set up it can be characterized as follows : given $x \in \pi(U_{o,P,t})$ let $y \in \pi^{-1}(x)$. Then $x \circ a = \pi(y \circ a)$. If $y' \in \pi^{-1}(x)$, then $y' = y\,\sigma$, with $\sigma \in \Gamma_P$, whence $y' \circ a = (y \circ a)\,\sigma$ and $\pi(y' \circ a) = \pi(y \circ a)$, as it should be. This description does not involve the origin o. We recall that $U_{o,P,t}$ does indeed depend on o, however not in a drastic manner. In particular, it follows from ([4], 6.1) that, given $o' \in X$ and $t \leqq t(o, P)$, $t(o', P)$, there exists $t' \leqq t$ such that

$$\pi(U_{o,P,t}) \cap \pi(U_{o',P,t}) \supset \pi(U_{o,P,t'}) \cup \pi(U_{o',P,t'}).$$

Let $p \in P(\mathbf{R})$ be such that $o' = o\,p$. Then $\operatorname{Int} p^{-1} : A_{o,P} \xrightarrow{\sim} A_{o',P}$ commutes with the roots. If $x \in \pi(U_{o,P,t'} \cap U_{o',P,t'})$, then we have $x \circ a = x \circ a^p$ for $a \in A_{o,P,1}$. In the sequel, whenever we write $U_{o,P,t'}$ it is understood that $t \leqq t(o, P)$. Let $\gamma \in \Gamma$ and $P' = P^\gamma$. Then there exists $t' \leqq t$ such that

$$\pi(U_{o,P,t}) \cap \pi(U_{o,P',t}) \supset \pi(U_{o,P,t'}) \cup \pi(U_{o,P',t'})$$

and the geodesic actions of $A_{o,P,1}$ and $A_{o,P',1}$ are compatible with the canonical isomorphisms

$$A_{o,P,1} \xrightarrow{\sim} A_{P,1} \xrightarrow{\sim} A_{P',1} \xrightarrow{\sim} A_{o,P',1},$$

where $A_{P,1}$ and $A_{P',1}$ are as in [4]. This follows from the above and 10.3, 5.6 (2) of [4].

6.2. In [4], paragraph 7, X is enlarged to a manifold with corners $\overline{X}$, which is the disjoint union of faces $e(P)$ $(P \in \mathfrak{P})$, where $e(G) = X$. The face $e(P)$ may be identified with $Z \times U(\mathbf{R})$. The group Γ operates freely and properly on $\overline{X}$, and $\overline{X}/\Gamma$ is a compact manifold with corners, disjoint union of faces

$$e'(P) = (Z \times U(\mathbf{R}))/\Gamma_P \cong e(P)/\Gamma_P,$$

where P runs through a set of representatives of $\mathfrak{P}/\Gamma$ ([4], § 9). The above map μ_0' extends to an isomorphism μ_0'' of manifolds with corners of $\overline{A}_{P,t} \times (\overline{e(P)/\Gamma_P})$ onto a neighborhood $\pi(\tilde{U}_{o,P,t})$ of $\overline{e'(P)}$. We let pr_P denote the projection of $\pi(\tilde{U}_{o,P,t})$ onto $\overline{e'(P)}$, carried over from the projection of $\overline{A}_{P,t} \times (\overline{e(P)/\Gamma_P})$ onto its second factor. Its fibers are the orbits of $A_{P,1}$ under the geodesic action. If P is replaced by a conjugate $P' = P^\gamma$

under an element $\gamma \in \Gamma$, then, by the end remarks of 6.1, the induced geodesic actions of $A_{o, P, 1}$ and $A_{o, P', 1}$ are defined in neighborhoods of $\overline{e'(P)} = \overline{e'(P')}$, and are the same, modulo the canonical isomorphism $\operatorname{Int} \gamma^{-1} : A_{o, P, 1} \xrightarrow{\sim} A_{o, P', 1}$. Thus $pr_P = pr_{P'}$ sufficiently close to $\overline{e'(P)}$, and the germ of projection pr'_P defined by pr_P depends only on $e'(P)$, and not on the choice of P in its Γ-conjugacy class.

Let $Q \in \mathfrak{P}$, containing P. We have a canonical inclusion $A_Q \subset A_P$ and in fact a canonical factorization $A_P = A_Q \times A_{P, Q}$, and this inclusion is compatible with geodesic action (*see* 3.11, 4.3, 5.5 in [4]). From this it follows that $pr_Q \circ \mu'_0$ induces an isomorphism of $\overline{A}_{P, Q, t} \times \overline{e'(P)}$ onto a neighborhood of $\overline{e'(P)}$ in $\overline{e'(Q)}$; if $pr_{P, Q}$ is the canonical projection of the latter on $\overline{e'(P)}$, then $pr_P = pr_{P, Q} \circ pr'_Q$.

6.3. Special neighborhoods. — It will be slightly more convenient here to deal with open neighborhoods, and to use subsets of A_P defined by strict inequalities. Let then, for $t > 0$, $P \in \mathfrak{P}$ and $I = I(P)$:

$$A_{P, (t)} = \{ a \in A_P \mid a^\alpha < t \ (\alpha \in \Delta - I) \},$$

$$\overline{A}_{P, (t)} = \{ a \in \overline{A}_P \mid a^\alpha < t \ (\alpha \in \Delta - I) \}.$$

Thus $A_{P, (t)}$ is open in the corner $\overline{A}_P$. If ω is open relatively compact in $e(P)$, then $\mu_0 (A_{P, (t)} \times \omega)$ is the interior of the Siegel set $\mathfrak{S}_{t, \omega}$ and will be called an open Siegel set.

Let $y \in e(P)$ and y' its image in $e'(P)$. Let ω be an open relatively compact neighborhood of y in $e(P)$ on which $e(P) \to e'(P)$ is injective. Then $\mu_0 (A_{P, (t)} \times \omega)$ is an open Siegel set in X on which π is injective, and μ'_0 extends to an isomorphism of manifolds with corners of $\overline{A}_{P, (t)} \times \omega$ onto an open neighborhood U of y' in $\overline{X}/\Gamma$, to be called a *special neighborhood;* U is the isomorphic image under π (extended to $\overline{X}$) of a neighborhood of y in the corner X(P). Clearly, y' has a fundamental system of special neighborhoods. If $Q \in \mathfrak{P}$ is such that $e'(Q) \cap U \neq \varnothing$, then a Γ-conjugate of Q contains P, and $pr_Q U$ contains a special neighborhood of y' in $\overline{e'(Q)}$. We note that the Γ-conjugacy class of P is completely determined by U; in fact if s is the maximum of $|\Delta - I(Q)|$, $(Q \in \mathfrak{P})$ for Q such that $e'(Q) \cap U \neq \varnothing$, then s is attained only on the Γ-conjugates of P. We shall say that P *is associated to* U.

If $P = G$, i. e. if $y' \in X/\Gamma$, a neighborhood of y' is special if it is the isomorphic image under π of an open relatively compact neighborhood of y in X.

6.4. Special covers and partitions of unity. — A special cover of $\overline{X}/\Gamma$ is a finite cover by special neighborhoods. Any cover of $\overline{X}/\Gamma$ has a special refinement.

Lemma. — *Let $\mathcal{U} = (U_i)_{i \in L}$ be a finite cover of $\overline{X}/\Gamma$. There exists a smooth partition of unity (λ_i) subordinated to $\mathcal{U}$ with the following property : given $P \in \mathfrak{P}$, $x \in e'(P)$, there exists a special neighborhood U of x in $\overline{X}/\Gamma$ such that λ_i is constant along the fibres of $pr_P \mid U$ for all $i \in L$.*

We may assume $\mathscr{U}$ to be special. For $i \in L$, let P_i be associated to U_i. We have then an isomorphism

(1) $$\mu'_0 : \bar{A}_{P_i, (t_i)} \times \omega_i \tilde{\to} U_i$$

with ω_i open relatively compact in $e'(P_i)$.

There exist covers $\mathscr{V} = (V_i)_{i \in L}$ and $\mathscr{W} = (W_i)_{i \in L}$ such that $\bar{W}_i \subset V$, $\bar{V}_i \subset U$ and V_i (resp. W_i) is isomorphic under ${\mu'_0}^{-1}$ to $\bar{A}_{P_i, (t'_i)} \times \omega'_i$ (resp. $\bar{A}_{P_i, (t''_i)} \times \omega''_i$), where $(0 < t'' < t' < t$, $\bar{\omega}''_i \subset \omega'_i$, $\bar{\omega}'_i \subset \omega_i$; $i \in L)$. Let ρ_{i0} be a smooth function on $e'(P)$ with support in ω'_i, equal to one on ω''_i. On $\bar{A}_{P_i}$ we can construct a smooth function ρ_{i1} with support in $\bar{A}_{P_i, (t'_i)}$ equal to one on $\bar{A}_{P_i, (t''_i)}$ $(i \in L)$, which, for each face F of $\bar{A}_P$, is independent of the coordinates transversal to F sufficiently close to F. Otherwise said, if $Q \supset P$ is such that $\mu_0 : \bar{A}_P \times e(P) \tilde{\to} X(P)$ maps $F \times e(P)$ onto $e(Q)$ (*see* [4], 5.3, 5.4), then, sufficiently close to F, ρ_{i1} is constant along the sets $A_Q \times \{ b \}$ $(b \in A_{P,Q})$. On U_i, identified with $\bar{A}_{P_i, (t_i)} \times \omega_i$, we then put $\tau_i = \rho_{i0}\,\rho_{i1}$, and let τ be the sum of the τ_i. We claim that $\lambda_i = \tau_i/\tau$ defines the sought for partition of unity.

Let $P \in \mathfrak{P}$ and $y \in e'(P)$. Take a special neighborhood U of y such that for any $i \in L$, $U \cap U_i \neq \varnothing$ implies $U \subset U_i$. Let $J \subset L$ be the set of $i \in L$ for which $U \subset U_i$. Then $\tau \mid U$ is the sum of the $\lambda'_i s$ with $i \in J$ and if λ_i is not identically zero on U, then $i \in J$. It suffices therefore to see that for $i \in J$, λ_i is constant along the fibres of pr_P on some special neighborhood V_1 of y in V. If $i \in J$, then P is Γ-conjugate to a subgroup P' containing P_i. The relation $pr_{P_i} = pr_{P_i P'} \circ pr_{P'}$ (*see* 6.2) implies that λ_i is constant along the fiber of $pr_{P'}$. But (6.2) $pr_{P'}$ and pr_P coincide sufficiently close to $\overline{e'(P)}$, whence our assertion.

7. A square integrability criterion and the injectivity of j^q

In this section, G *is a semi-simple* **Q**-*group and* Γ *an arithmetic subgroup of* G.

We use the notation of paragraph 5, except that we write ρ_P and A_P instead of ρ and A.

7.1. For $q \in N$, $\lambda \in X(A_P)$, we let $c(P, q, \lambda)$ denote the condition

(★) $$\rho_P + \lambda - \nu \gg 0,$$

for any weight ν of A_P in $\underset{i \leqq q}{\otimes} \Lambda^i L(R_u P(\mathbf{R}))$ (*see* 5.2 for $\gg$).

Let $Q \in \mathfrak{P}$ and assume P to be conjugate to a subgroup of Q. There is then a canonical monomorphism $A_Q \to A_P$ (an inclusion if $Q \supset P$), whence a canonical homomorphism $r_{QP} : X(A_P) \to X(A_Q)$. We have

(1) $$r_{QP}(\rho_P) = \rho_Q, \qquad \Phi_Q \subset r_{QP}(\Phi_P) \subset \Phi_Q \cup \{0\},$$

(2) $$\Delta - I(Q) \subset r_{QP}(\Delta - I(P)) \subset (\Delta - I(Q)) \cup \{0\},$$

and, for $\beta \in \Phi_Q$:

(3) $$\dim \mathfrak{u}_\beta = \sum_{\delta \in \Phi_P \cap r_{QP}^{-1}(\beta)} \dim \mathfrak{u}_\delta,$$

therefore

(4) $$c(\mathrm{P}, q, \lambda) \Rightarrow c(\mathrm{Q}, q, r_{\mathrm{QP}}(\lambda)) \qquad [\lambda \in \mathrm{X}(\mathrm{A}_{\mathrm{P}}),\ q \in \mathbf{N}].$$

Let P be *minimal*, and $\lambda \in \mathrm{X}(\mathrm{A}_{\mathrm{P}})$. We let then

(5) $$c(\mathrm{G}, \lambda) = \max q \mid c(\mathrm{P}, q, \lambda) \textit{ holds true}.$$

By (4), $c(\mathrm{Q}, q, r_{\mathrm{QP}}(\lambda))$ is then true for any $\mathrm{Q} \in \mathfrak{P}$ and any $q \leqq c(\mathrm{G}, \lambda)$. By definition, we have $c(\mathrm{G}, \lambda) = c(\mathrm{G}^0, \lambda)$.

If G is anisotropic over **Q**, then $\mathrm{P} = \mathrm{G}$, $\mathrm{A}_{\mathrm{P}} = \{e\}$, $\lambda = 0$, and we agree to put $c(\mathrm{G}, \lambda) = \infty$.

Assume G to be an almost direct product of **Q**-subgroups G_i $(1 \leqq i \leqq s)$. Then P and A_{P} are accordingly decomposed and λ is a sum of elements $\lambda_i \in \mathrm{X}(\mathrm{A}_{\mathrm{P}} \cap \mathrm{G}_i)$. It is then clear that $c(\mathrm{G}, \lambda) = \min_i c(\mathrm{G}_i, \lambda_i)$.

In the sequel, we write $c(\mathrm{G})$ for $c(\mathrm{G}, 0)$.

7.2. As in 5.5, let σ be a form defined on a Siegel set $\mathfrak{S}_{t,\omega}$ and

(1) $$\tau = \mu_0^*(\sigma) = \sum_{\mathrm{J}} f_{\mathrm{J}}\, \omega^{\mathrm{J}}.$$

We shall say that σ (or τ) has *logarithmic growth* if there exists a polynomial P in s variables, with real coefficients, such that

(2) $$|f_{\mathrm{J}}(a, q)| \prec |\mathrm{P}(\log a^{\alpha_1}, \ldots, \log a^{\alpha_s})| \qquad (a \in \mathrm{A}_t,\ q \in \omega, \text{ all J}).$$

We have then, for any $\varepsilon > 0$:

(3) $$|f_{\mathrm{J}}(a, q)| \prec a^{-\varepsilon d}, \qquad \text{where} \quad d = \sum_{1 \leqq i \leqq s} \alpha_i.$$

A form σ' on $\pi(\mathfrak{S}_{t,\omega})$ is said to have logarithmic growth if $\sigma' \circ \pi$ has logarithmic growth on $\mathfrak{S}_{t,\omega}$.

Let V be an open subset of $\overline{\mathrm{X}/\Gamma}$. A form σ defined on $\mathrm{V} \cap \mathrm{X}/\Gamma$ has *logarithmic growth near the boundary* if any $y \in (\partial\overline{\mathrm{X}/\Gamma}) \cap \mathrm{V}$ has a special neighborhood W in V such that σ has logarithmic growth on $\mathrm{W} \cap (\mathrm{X}/\Gamma)$.

7.3. LEMMA. — *Let* $\mathrm{P} \in \mathfrak{P}$, $x \in e'(\mathrm{P})$, U *be a special neighborhood of* x *and* σ *a form with logarithmic growth on* $\mathrm{U} \cap (\mathrm{X}/\Gamma)$. *Then* σ *has logarithmic growth near the boundary of* U.

Let $y \in (\partial\overline{\mathrm{X}/\Gamma}) \cap \mathrm{U}$. Then there exists $\mathrm{Q} \in \mathfrak{P}$ containing P such that $y \in e'(\mathrm{Q})$. We have to show that σ has logarithmic growth on the intersection of X/Γ with some special neighborhood of y.

After renumbering, if needed, we may assume that, for some $s' < s$, we have

(1) $$\mathrm{A}_{\mathrm{Q}} = \{a \in \mathrm{A}_{\mathrm{P}} \mid a^{\alpha_i} = 1\ (s' < i \leqq s)\}.$$

We let $A_Q^\perp$ be the connected closed subgroup of A_P whose Lie algebra is orthogonal to $L(A_Q)$ with respect to the restriction of the Killing form. It is equal to $A_P \cap {}^0Q$, and plays the role of A_P, if P is viewed as a subgroup of Q rather than G.

Let $U' = R_u Q$, ${}^0M'$ the analogue of ${}^0M$ in 4.1 (1) and $Z' = (K \cap Q)\backslash {}^0M'(\mathbf{R})$. We have the canonical isomorphisms

$$(2) \qquad Y = A_P \times Z \times U(\mathbf{R}) \xrightarrow{\mu_{oP}} X \xleftarrow{\mu_{oQ}} Y' = A_Q \times Z' \times U'(\mathbf{R});$$

moreover, a similar result, applied to P and Q, gives an isomorphism

$$(3) \qquad A_Q^\perp \times Z \times U(\mathbf{R}) \tilde{\to} Z' \times U'(\mathbf{R}).$$

From this it follows that the special neighborhoods of y are boxed between sets of the form $\overline{A}_{Q,t} \times \omega_1 \times \omega_2$ with ω_1 (resp. ω_2) open relatively compact in $A_Q^\perp$ [resp. $Z \times U(\mathbf{R})$]. It suffices therefore to study the behavior of τ on $A_{Q,t} \times \omega_1 \times \omega_2$.

There is a unique choice of constants b_{ij} $(1 \leqq i \leqq s' < j \leqq s)$ such that

$$(4) \qquad \tau^i = d \log \alpha_i + \sum_j b_{ij}\, d \log \alpha_j$$

is zero on $L(A_Q^\perp)$. Let

$$(5) \qquad \omega'^i = \omega^i + \sum_j b_{ij}\, \omega^j, \qquad (1 \leqq i \leqq s' < j \leqq s).$$

Let ω'^i be defined by (5) for $i \leqq s'$, by $\omega'^i = \omega^i$ for $i > s'$. Write

$$(6) \qquad \tau = \sum_J h_J\, \omega'^J.$$

Let $\nu = \mu_{o,P}^{-1} \circ \mu_{0,Q}$. Then

$$(7) \qquad \mu_{o,Q}^*(\sigma) = \nu^*(\tau) = \sum_J (h_J \circ \nu)\, \nu^*(\omega'^J).$$

It is immediate from the above that $(\nu^*(\omega'^i))$ is a moving frame which can be used to study the behavior of $\mu_{0,Q}^*(\sigma)$ on special neighborhoods of points of $e'(Q)$. Going back to Y, we see that we have to study the $h'_J s$ on $A_{Q,t} \times \omega_1 \times \omega_2$. But the $h'_J s$ are linear combinations with constant coefficients of the $f'_J s$. Since $|\log a^{\alpha_i}|$ is bounded on ω_1, there exists a polynomial P′ in s' variables such that

$$(8) \qquad |P(\log a^{\alpha_1}, \ldots, \log a^{\alpha_s})| \leqq |P'(\log a^{\alpha_1}, \ldots, \log a^{\alpha_{s'}})| \quad \text{on } A_{Q,t} \times \omega_1.$$

We have then

$$(9) \qquad |h_J(a, q)| \prec |P'(a)|, \qquad (a \in A_{Q,t};\ q \in \omega_1 \times \omega_2).$$

7.4. THEOREM. — *Let* G *be connected and* Γ *be a torsion-free arithmetic subgroup of* G. *Let* C *be the subcomplex of the elements in* $\Omega^*_{X/\Gamma}$ *which, together with their exterior derivatives, have logarithmic growth near the boundary of* X/Γ. *Then :*

(*a*) *The inclusion map* $C \to \Omega^*_{X/\Gamma}$ *induces an isomorphism in cohomology.*

(*b*) *For* $q \leqq c(G)$, C^q *consists of square integrable forms.*

(*c*) $I_G^\Gamma \subset C$.

Assume that the form σ in 7.2 is invariant under A_P. Then, by 5.7, the coefficients f_J are bounded in absolute value on $A_{P,t}\times\omega$, hence σ has logarithmic growth. Since $d\sigma=0$ if $\sigma\in I_G^\Gamma$, this proves (*c*).

Let $q\leqq c(G)$ and $\sigma\in C^q$. To prove that σ is square integrable it suffices to show that if $P\in\mathfrak{P}$, $y\in e'(P)$ and U is a special neighborhood of y in $\overline{X}/\Gamma$, then σ is square integrable on $\mathfrak{S}=U\cap(X/\Gamma)$. The latter is a Siegel set and we are in the situation of 7.2. By assumption and definition (7.1), the condition $c(P, q, 0)$ holds true. But then, we also have $c(P, q, -\varepsilon d)$ for a suitably small $\varepsilon>0$. By 7.2 (3) and 5.5, it follows that σ is square integrable on $\mathfrak{S}$, whence (*b*).

There remains to prove (*a*). For an open set $V\subset\overline{X}/\Gamma$, let $\mathscr{F}(V)$ be the space of smooth forms σ on $V\cap(X/\Gamma)$ such that σ and $d\sigma$ have logarithmic growth near the boundary (7.2). Clearly if V′ is open in V, the restriction $\Omega_V\to\Omega_{V'}$ maps $\mathscr{F}(V)$ into $\mathscr{F}(V')$; hence $\mathscr{F}: V\mapsto\mathscr{F}(V)$ is a presheaf on $\overline{X}/\Gamma$. Obviously, it is a sheaf, and in fact a differential sheaf of algebras, the differential being defined by the exterior differentiation of forms, and we have

$$(1)\qquad \mathscr{F}(\overline{X}/\Gamma)=C,\qquad \mathscr{F}(X/\Gamma)=\Omega^*_{X/\Gamma}.$$

We now want to prove that $\mathscr{F}$ is a fine resolution of **R** on $\overline{X}/\Gamma$.

$\mathscr{F}$ IS FINE. — Let f be a smooth function on $\overline{X}/\Gamma$. On $A_{P,t}\times\omega$ (notation of 7.2), express df as a linear combination of the ω^i $(i>s)$ and the $d\alpha_i$. Then the coefficients are bounded in absolute value since $\mu_0^*(df)$ extends smoothly to $\overline{A}_{P,t}\times\omega$. This is then *a fortiori* true if df is expressed as linear combination of the ω^i and $d\log\alpha_i$. Thus f and df have logarithmic growth near the boundary; then, so have $f.\sigma$ and $d(f.\sigma)$ for $\sigma\in\mathscr{F}(V)$, V open in $\overline{X}/\Gamma$.

Let $\mathscr{U}=(U_i)_{i\in I}$ be a finite open cover of $\overline{X}/\Gamma$. We choose a smooth partition of unity (λ_i) subordinated to $\mathscr{U}$. Let $\sigma\in C$. We have then $\sigma=\sum\lambda_i\sigma$, with $\operatorname{supp}(\lambda_i\sigma)\subset U_i$, and $\lambda_i\sigma\in C$ by the above, hence $\mathscr{F}$ is fine.

F IS A RESOLUTION OF **R**. — Let $\mathscr{H}^*(\mathscr{F})$ be the derived sheaf of $\mathscr{F}$ and $\mathscr{H}^*_x(\mathscr{F})$ its stalk at $x\in\overline{X}/\Gamma$. Since C contains the constant functions, it is clear that

$$\mathscr{H}^0(\mathscr{F})\cong(\overline{X}/\Gamma)\times\mathbf{R}.$$

There remains to show that $\mathscr{H}^q_x(\mathscr{F})=0$ for all $x\in\overline{X}/\Gamma$ and $q>0$. If $x\in X/\Gamma$, this is just the Poincaré lemma. Let now $x\in\partial\overline{X}/\Gamma$ and $P\in\mathfrak{P}$ be such that $x\in e'(P)$. We have to prove that if V is an (arbitrarily small) special neighborhood of x and σ a closed form in $\mathscr{F}(V)$, then there exists a special neighborhood V′ of x in V and $\varphi\in\mathscr{F}(V')$ such that $\sigma=d\varphi$ on V′.

We let $\mathfrak{S}=V\cap(X/\Gamma)=\mathfrak{S}_{t,\omega}$ and use the notation of 7.2. We may assume that ω is an open ball in a coordinate patch of $Z\times U(\mathbf{R})$, with local coordinates $(x^i)_{s<i\leqq m}$ centered on a point q_0 mapping onto x. Fix t_0 $(0<t_0<t)$. On A_P we takes as coordinates

$$x^i=\log a^{\alpha_i}-\log t_0.$$

In these coordinates, $U_t = \mu_0^{-1}(\mathfrak{S})$ is then given by

$$U_t = \{x \mid x^i < \log t/t_0, (1 \leqq i \leqq s); |x^j| < 1, (s < j \leqq m)\}.$$

In particular, U_t is star-shaped with respect to the origin. The dx^j $(s < j \leqq m)$ are linear combinations with bounded coefficients of the ω^j $(s < j \leqq m)$ on ω, and conversely; moreover $dx^i = \omega^i$ for $1 \leqq i \leqq s$. Therefore, if

$$\varphi = \sum_J c_J \omega^J = \sum d_J dx^J \tag{2}$$

is a q-form on U_t, then φ has logarithmic growth if and only if there exists a polynomial $Q(x^1, \ldots, x^s)$ such that

$$|d_J(x)| \prec |Q(x)| \qquad (x \in U_t;\ \text{all } J). \tag{3}$$

Let now

$$\mu_0^*(\sigma) = \tau = \sum f_J \omega^J = \sum h_J dx^J \tag{4}$$

be a closed q-form on U_t $(q > 0)$. In view of the remark just made and of 7.3, it suffices to prove that on some set $A_{P,t'} \times \omega'$, where $0 < t' < t$ and ω' is an open neighborhood of q_0 in ω, there exists a $(q-1)$-form

$$\varphi = \sum c_I dx^I,$$

whose coefficients satisfy a polynomial growth condition, such that $\tau = d\varphi$. To check this, it is enough to see that the usual homotopy operator A in euclidean space is compatible with polynomial growth conditions, which is immediate from its definition. We recall it briefly.

On the form $f\,dx^J$ $(J = \{j_1, \ldots, j_q\})$, the operator A is defined by

$$A(f\,dx^J) = \sum_{I \subset J} c_I dx^I,$$

where, for I equal to J with j_i erased :

$$c_I = (-1)^{i-1} x^{j_i} \int_0^1 f(xt)\, t^{q-1} dt. \tag{5}$$

It leaves Ω_U^* stable if U is open, star-shaped with respect to the origin, and satisfies $dA + Ad = \mathrm{Id}$. Since τ is closed, we have then $\tau = d\varphi$, with

$$\varphi = A\tau = \sum c_I dx^I, \tag{6}$$

$$c_I(x) = \sum_{j \notin I} \pm x^j \int_0^1 h_{I \cup \{j\}}(xt)\, t^{q-1} dt, \tag{7}$$

which clearly satisfies a polynomial growth condition in $U_{t'}$ $(0 < t' < t_0)$ if the h_J do so.

This proves that $\mathscr{F}$ is a fine resolution of **R**. By one of the main theorems of sheaf theory ([11], 4.6.1) we have then a canonical isomorphism

$$\alpha : \ H^*(C) \to H^*(\overline{X}/\Gamma). \tag{8}$$

We now consider the diagram

$$\begin{array}{ccc} H^*(C) & \xrightarrow{\alpha} & H^*(\overline{X}/\Gamma) \\ \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle r^*} \\ H^*(\Omega^*_{X/\Gamma}) & \xrightarrow{\beta} & H^*(X/\Gamma) \end{array} \tag{9}$$

where β is the isomorphism of the de Rham theorem, i^* is defined by inclusion and r^* by restriction. The isomorphisms of (1) carry the inclusion $C \to \Omega^*_{X/\Gamma}$ over to the restriction map of cross-sections. Therefore, the diagram (9) is commutative. The maps α and β are isomorphisms; since $X/\Gamma \subsetneq \overline{X}/\Gamma$ is a homotopy equivalence, so is r^*. Consequently i^* is an isomorphism, which ends the proof of (*a*) and of the theorem.

7.5. THEOREM. — *Let* G *be a semi-simple* **Q**-*group and* Γ *an arithmetic subgroup of* G. *The homomorphism* $j^q : I_G^\Gamma \to H^q(\Gamma)$ *is injective for* $q \leqq c(G)$, *and surjective for* $q \leqq \min(c(G), m(G(\mathbf{R})))$.

Let Γ' be a normal torsion-free subgroup of finite index of Γ, contained in G^0. Let C' be the complex of elements in $\Omega_X^{\Gamma'}$ which have logarithmic growth at the boundary. Since Γ/Γ' acts as a group of automorphisms of $\overline{X}/\Gamma'$, it is clear that C' is stable under the natural action of Γ/Γ' on $\Omega_X^{\Gamma'}$. The theorem then follows from 7.4 and 3.6, and this also shows :

7.6. COROLLARY. — *For* $q \leqq c(G)$, *any cohomology class of* $H^q(\Omega_X^\Gamma) = H^q(\Gamma)$ *is representable by a closed q-form which is square integrable modulo* Γ. *More precisely the real cohomology of* Γ *is that of a subcomplex* C *of* Ω_X^Γ *all of whose elements of degree* $q \leqq c(G)$ *are square integrable modulo* Γ.

Remark. – The square integrability criterion given by 7.6 is quite analogous to, but slightly weaker than one announced by H. Garland-W. C. Hsiang [10]. For scalar valued forms, their bound is the maximum of q for which $\rho - \nu \gg 0$ when ν runs through *some* of the weights of A in $\otimes_{i \leqq q} \Lambda^i L(U(\mathbf{R}))$.

7.7. EXAMPLE. — Let k be an algebraic number field, $n = r_1 + 2r_2$ its degree, where r_1 (resp. r_2) is the number of real (resp. complex) places of k. Let $G = R_{k/\mathbf{Q}} \mathbf{SL}_2$, and P a minimal parabolic **Q**-subgroup. Then Φ_P consists of one root α with multiplicity n, and $\rho_P = (r_2 + (r_1/2))\alpha$. We get therefore

$$c(G) = \begin{cases} [r_1/2 + r_2] & \text{if } r_1 \text{ is odd}, \\ (r_1/2) + r_2 - 1 & \text{if } r_1 \text{ is even}. \end{cases}$$

Comparison with the results of G. Harder [13] shows that this bound for square integrability is sharp if $r_1 \leqq 1$, but not otherwise.

8. Another subcomplex of Ω_X^Γ

8.1. We keep the assumptions of 7.4. Let V be an open subset of $\overline{X}/\Gamma$. A differential form σ on $V \cap (X/\Gamma)$ is said to be locally lifted from the boundary if for any $x \in V \cap (\partial\overline{X}/\Gamma)$

and $P \in \mathfrak{P}$ such that $x \in e'(P)$, there exists a special neighborhood U of x in V and a differential form σ' on $U' = \mathrm{pr}_P U$ such that $\sigma = (\mathrm{pr}_P)^* \sigma'$ on $U \cap (X/\Gamma)$. If, in the notation of 7.2, we write

$$\mu_0^*(\sigma) = \sum f_J \omega^J,$$

then $f_J(a, q)$ is independent of a, and is identically zero if J contains an index $i \leqq s$. In particular, the coefficients f_J are bounded in absolute value, and σ has *a fortiori* logarithmic growth.

The forms on V of this type form a subalgebra closed under exterior differentiation, whose elements, together with their exterior differentials, have logarithmic growth near the boundary.

8.2. PROPOSITION. — *Let* C_0 *be the subcomplex of elements of* Ω_X^Γ *which are locally lifted from the boundary. Then*

(*a*) *The inclusion map* $C_0 \to \Omega_X^{\Gamma'}$ *induces an isomorphism in cohomology.*

(*b*) C_0 *is contained in the complex* C *of* 7.2. *In particular* C^q *consists of square integrable forms* mod Γ *for* $q \leqq c(G)$.

The first assertion of (*b*) follows from 8.1, and the second then from 7.4.

The proof of (*a*) is quite similar to, only a bit simpler than, that of 7.4 (*a*). For V open in $\overline{X}/\Gamma$, let $\mathscr{F}(V)$ be the space of forms on $V \cap (X/\Gamma)$ which are locally lifted from the boundary. The functor $V \mapsto \mathscr{F}(V)$ again defines a differential sheaf of algebras on $\overline{X}/\Gamma$, and we have

$$(1) \qquad \mathscr{F}(\overline{X}/\Gamma) = C_0, \qquad \mathscr{F}(X/\Gamma) = \Omega^*_{X/\Gamma}.$$

As in 7.4, it suffices then to show that $\mathscr{F}$ is a fine resolution of **R**.

$\mathscr{F}$ IS FINE. — Let $\mathscr{U} = (U)_{i \in I}$ be a finite open cover of $\overline{X}/\Gamma$ and $\sigma \in C_0$. We have to show that we can write

$$\sigma = \sum_{i \in I} \sigma_i \quad \textit{with} \quad \sigma_i \in C_0 \quad \textit{and} \quad \mathrm{supp}(\sigma_i) \subset U_i \quad (i \in I).$$

By 6.4 there exists a partition of unity $(\lambda_i)_{i \in I}$ which is subordinated to U and is *special*. But then $\lambda_i \in C_0^0$, and $\sigma_i = \lambda_i \sigma$ satisfies our conditions.

$\mathscr{F}$ IS A RESOLUTION OF **R**. — Let $\mathscr{H}^*_x(\mathscr{F})$ be the derived sheaf of $\mathscr{F}$ and $\mathscr{H}^*_x(\mathscr{F})$ its stalk at $x \in \overline{X}/\Gamma$. Since C_0 contains the constant functions, we have $\mathscr{H}^0(\mathscr{F}) \cong \overline{X}/\Gamma \times \mathbf{R}$. There remains to show that $H^q_x(\mathscr{F}) = 0$ for $x \in \overline{X}/\Gamma$ and $q > 0$. If $x \in X/\Gamma$, this is just Poincaré lemma. Let now x be on the boundary and $P \in \mathfrak{P}$ be such that $x \in e'(P)$. It suffices to show that if W is a special neighborhood of x, and σ is a closed q-form on W which can be written $\sigma = (\mathrm{pr}_P)^*(\sigma')$, with $\sigma' \in \Omega^q_{W'}$ ($W' = \mathrm{pr}_P W$), necessarily closed, then σ is the coboundary near x of a similar form. This follows from the Poincaré lemma on W'.

Remark. — The complex C_0 gives the same square integrability criterion as C, and is more natural from the differential geometric point of view. Its drawback for our purposes is that it does not contain I_G^Γ.

9. The constants $c(\mathrm{G})$ and $m(\mathrm{G}(\mathbf{R}))$

We content ourselves with rough estimates, sufficient for the needs of paragraph 11. Only 9.5 (3) will be used there.

9.1. Let Φ be a root system ([6], VI, § 1) in a real vector space V. Assume first V to be irreducible, and let Φ^0 be the subsystem consisting of the non-multipliable elements of Φ (i. e. $a \in \Phi$, $m\,a \in \Phi$, $m \in \mathbf{Z} \Rightarrow m = \pm 1$). It is an irreducible reduced root system. Fix an order on Φ and let Δ be the corresponding basis of simple roots of Φ. Any $v \in \mathrm{V}$ can be written uniquely as a linear combination $v = \sum_{\alpha \in \Delta} c_\alpha\, \alpha$. We put $v \gg 0$ if $c_\alpha > 0$ for all $\alpha \in \Delta$, $v \geqq 0$ if $c_\alpha \geqq 0$ for all $\alpha \in \Delta$, and $v \geqq v'$ ($v' \in \mathrm{V}$) if $v - v' \geqq 0$.

Let $2r$ be the sum of the positive roots of Φ and d_0 the highest root of Φ^0. Let

(1) $$c(\Phi) = \max p \mid r - p\,d_0 \gg 0.$$

If Φ is not irreducible, it is a sum of irreducible ones, and we put

(2) $$c(\Phi) = \min c(\Phi')$$

where Φ' runs through the irreducible factors of Φ.

The value of $c(\Phi)$ for the various irreducible root systems can be readily computed from the tables in [6], p. 250-270. The result is for the classical systems

(3)

$\mathbf{A}_n$	$\mathbf{B}_n$	$\mathbf{C}_n$	$\mathbf{D}_4$	$\mathbf{D}_n$ $(n \geqq 5)$	$\mathbf{BC}_n$
$[n/2]'$	$n-2$	$[n/2]'$	3	n	$[(n+1)/2]'$

while for the exceptional types $\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$, $\mathbf{G}_2$, $\mathbf{F}_4$, $c(\Phi)$ is equal to 4, 8, 14, 3, 1 respectively.

9.2. Let k be a field, H a connected semi-simple k-group of strictly positive k-rank ([5], § 5), S a maximal k-split torus of H and P a minimal parabolic k-subgroup containing S. Replacing $\mathrm{X}(\mathrm{A}_\mathrm{P})$ by $\mathrm{X}(\mathrm{S}) \otimes \mathbf{R}$, we define, in analogy with 7.1 :

(1) $$c(\mathrm{H}/k) = \max q \mid \rho_\mathrm{P} - \nu \gg 0,$$

where ν runs through the weights of S in $\bigotimes_{i \leqq q} \Lambda^i \mathrm{L}(\mathrm{R}_u(\mathrm{P}))$.

Replacing P by a parabolic k-subgroup Q, S by a maximal k-split torus of the radical of Q, we can also consider the condition $c(\mathrm{Q}, q, 0)$; again, the inequality

(2) $$c(\mathrm{H}/k) \leqq \max q \mid c(\mathrm{Q}, q, 0)$$

holds true.

9.3. The relative root system ${}_k\Phi(\mathrm{H})$ of H, also to be denoted $\Phi(\mathrm{H}/k)$, decomposes into the direct sum of the relative root systems of the almost k-simple factors of H whose k-rank is > 0 and those are irreducible ([5], § 5). We claim

(1) $$c(\mathrm{H}/k) \geqq c(\Phi(\mathrm{H}/k)).$$

In view of the definitions, it suffices to prove this for H almost simple over k. In this case $\rho - r \geqq 0$ because 2ρ contains each term occurring in $2r$ with a coefficient $\geqq 1$, and, on the other hand, $p\,d_0 - \nu \geqq 0$ for any sum ν of p roots. More precisely, the coefficient α occurs in 2ρ a number of times equal to its multiplicity, hence if all the multiplicities are $\geqq m$, we have

$$\rho - pd_0 \geqq mr + pd_0 \geqq m(r - pd_0),$$

and therefore

$$c(\mathrm{H}/k) \geqq mc(\Phi(\mathrm{H}/k)), \qquad (1 \leqq m \leqq \min_\alpha \text{ multiplicity } \alpha). \tag{2}$$

9.1 (3) and 9.3 (1) imply in particular that if H is almost simple over k, then

$$c(\mathrm{H}/k) \geqq [rk_k(\mathrm{H})/2]'. \tag{3}$$

If k' is a finite separable extension and $\mathrm{H} = \mathrm{R}_{k'/k}\,\mathrm{H}'$, where H′ is an almost k'-simple k'-group, then

$$c(\mathrm{H}/k) \geqq [k' : k]\,c(\mathrm{H}'/k'). \tag{4}$$

In fact, by [5], 6.21, the root systems $\Phi(\mathrm{H}'/k')$ and $\varphi(\mathrm{H}/k)$ are canonically isomorphic and the passage from the former to the latter multiplies the multiplicities by $[k' : k]$.

We note also that if k'' is any extension of k, then

$$c(\mathrm{H}/k) \geqq c(\mathrm{H}/k''), \tag{5}$$

because, if P is a minimal parabolic k-subgroup of H, we have, using 9.2 (2) :

$$c(\mathrm{H}/k) = c(\mathrm{H}/k,\,\mathrm{P}) \geqq c(\mathrm{H}/k'').$$

9.4. Let now $k = \mathbf{R}$ and H be almost simple over $\mathbf{R}$, of strictly positive $\mathbf{R}$-rank. The tables of [16] and [19] then show immediately that the constant $m(\mathrm{H}(\mathbf{R}))$ defined in 3.3 satisfies

$$m(\mathrm{H}(\mathbf{R})) \geqq [(rk_{\mathbf{R}}\,\mathrm{H})/4]'. \tag{1}$$

9.5. Let now G be a connected almost $\mathbf{Q}$-simple $\mathbf{Q}$-group. Up to isogeny, which does not affect the constants in consideration here, $\mathrm{G} = \mathrm{R}_{k/\mathbf{Q}}\,\mathrm{G}'$, where k is a finite extension of $\mathbf{Q}$ and G′ is an absolutely simple k-group. We have then ([23], 1.3.1) :

$$\mathrm{G}/\mathbf{R} = \prod_{v \in \mathrm{S}(k)} \mathrm{R}_{k_v/\mathbf{R}}(\mathrm{G}'/k_v), \tag{1}$$

where S (k) denotes the set of infinite places of k, and k_v the completion of k at $v \in \mathrm{S}(k)$, whence also

$$\mathrm{G}'(\mathbf{R}) = \prod_{v \in \mathrm{S}(k)} \mathrm{G}'(k_v) = \prod_{v \in \mathrm{S}(k)} (\mathrm{R}_{k_v/\mathbf{R}}\,\mathrm{G}')(\mathbf{R}). \tag{2}$$

The k_v-rank of (G'/k_v) is at least equal to the k-rank of G′, i. e. to the $\mathbf{Q}$-rank of G. Therefore, 9.3 (3) and 9.4 imply

$$\min(c(\mathrm{G}/\mathbf{Q}),\, m(\mathrm{G}(\mathbf{R}))) \geqq [(rk_m\,\mathrm{G})/4]'. \tag{3}$$

10. Some sequences of compact homogeneous spaces

In this section, we review some known facts about the projective limits of certain sequences of cohomology rings of compact symmetric spaces.

10.1. Given a sequence of graded modules $A_n (n \geqq 1)$ over a ring R and of homomorphisms $f_n : A_{n+1} \to A_n$ preserving the degrees, we denote by $\varprojlim^* A_n$ the graded module $A = \otimes_j A^j$, where $A^j = \varprojlim (A^j_n, f_n)$. If the A_n are graded algebras, and the $f'_n s$ are graded algebra homomorphisms, then A is an algebra in the obvious way. $\varprojlim^*$ is thus the projective limit in the graded category; the ordinary $\varprojlim$ would be the direct product of the $A^{j'} s$ rather than their direct sum. In all cases to be considered here, $R = \mathbf{R}$, $\mathbf{C}$ the $A'_n s$ are finite dimensional and, for fixed j, the sequence (A^j_n, f_n) is stationary.

10.2. Let G be a semi-simple algebraic group over **R**, and K a maximal compact subgroup of $G(\mathbf{R})$. We have already mentioned the isomorphism

$$I_G = I_{G(\mathbf{R})} \cong H^*(L(G(\mathbf{R})), L(K)) \quad (3.1). \tag{1}$$

Although we shall not need this, we also recall that

$$H^*(L(G(\mathbf{R})), L(K)) \cong \tilde{H}^*(G(\mathbf{R})^0) \tag{2}$$

where $\tilde{H}$ denotes the Eilenberg-Mac Lane cohomology, computed with continuous or smooth real valued cochains ([8], [15]). From that point of view, j^*_Γ [Γ discrete in $G(\mathbf{R})^0$] is just the restriction map in Eilenberg-Mac Lane cohomology.

Let G_u be a maximal compact subgroup of $G(\mathbf{C})$ containing K and $X_u = K^0 \backslash G^0_u$. The space X_u is a simply connected compact symmetric space, the " compact dual " or " compact twin " to $X = K \backslash G(\mathbf{R})$. We also have, by E. Cartan :

$$H^*(L(G(\mathbf{R})), L(K)) \cong H^*(L(G_u), L(K)) \cong H^*(X_u). \tag{3}$$

Since $G(\mathbf{R})/G(\mathbf{R})^0 \cong K/K^0$:

$$(K \backslash G_u)^0 \cong (K \cap G^0_u) \backslash G^0_u, \tag{4}$$

hence

$$H^*((K \backslash G_u)^0) = H^*((K \cap G^0_u) \backslash G^0_u) = H^*(X_u)^{(K \cap G^0_u)/K^0}. \tag{5}$$

10.3. Let G′ be another semi-simple **R**-group and $f : G \to G'$ an injective **R**-morphism. There is then a canonical morphism

$$f^* : I_{G'(\mathbf{R})} \to I_{G(\mathbf{R})}.$$

In fact, we can choose a maximal compact subgroup K′ of $G'(\mathbf{R})$ containing $f(K)$. Then f induces an embedding of $K \backslash G(\mathbf{R})$ into $K' \backslash G'(\mathbf{R})$ (whose image is a totally geodesic closed submanifold), and f^* is just given by the restriction of differential forms. Since any two maximal compact subgroups of G′ (**R**) are conjugate by an inner automosphism leaving their intersection pointwise fixed, f^* is independent, up to unique isomorphisms, of the choices made. The homomorphism can of course also be defined in a natural way from the other interpretations of I_G given in 10.2. For example, we may take a maximal compact subgroup G'_u of G′ (**C**) containing $f(G_u)$, and a conjugate K″ of K′, whence an embedding $f : X_u \subsetneq X'_u = (K'' \backslash G'_u)^0$, and f^* is the induced map on cohomology.

10.4. In this paragraph and the next, we fix some notation for the classical groups.

$\mathbf{SL}_n$ and $\mathbf{Sp}_{2n}$ are as usual the special linear group and the symplectic group.

$\mathbf{O}_n$ [resp. O_n (**C**)] is the automorphism group of $\mathbf{R}^n$ (resp. $\mathbf{C}^n$) endowed with the standard quadratic form $\sum x_i^2$.

$\mathbf{O}_{n,n}$ is the **Q**-group whose group of k points (k field of characteristic zero) is the automorphism group of k^{2n} endowed with the quadratic form $\sum_{1 \leqq i \leqq n} x_i x_{n+i}$.

$\mathbf{U}_n$ is the unitary group on $\mathbf{C}^n$, $\mathbf{U}_{n,n}$ the automorphism group of $\mathbf{C}^{2n}$ endowed with the hermitian form $\sum_{1 \leqq i \leqq n} (x_i \bar{x}_{n+1} + x_{n+i} \bar{x}_i)$.

If G is one of the above orthogonal or unitary groups, SG is the subgroup of elements of determinant one in G.

10.5. We let **H** be the field of quaternion numbers over the reals, i, j, k the standard anticommuting units of square -1 of **H**.

$\alpha : x \mapsto \bar{x}$ is the conjugation, which maps i, j, k onto $-i$, $-j$, $-k$ respectively and $\beta : x \mapsto -k \bar{x} k$ is the involution which fixes i, j and sends k onto $-k$.

$\mathbf{USp}_n$ (resp. $\mathbf{Sp}_{n,n}$) is the automorphism group of $\mathbf{H}^n$ (resp. $\mathbf{H}^{2n}$) endowed with the hermitian form

$$\sum_{1 \leqq i \leqq n} x_i \bar{x}_i \quad \Big[\text{resp. } \sum_{1 \leqq i \leqq n} (x_i \bar{x}_{n+i} + x_{n+i} \bar{x}_i)\Big].$$

$\mathbf{SO}^*_{2n}$ is the group of real points of the **R**-form of $\mathbf{SO}_{2n}$ of type D III in E. Cartan's notation. *A priori*, since this classification is one of Lie algebras, this means that the maximal compact subgroups of the identity component of $\mathbf{SO}^*_{2n}$ are isomorphic to $\mathbf{U}_n$, but, in fact, $\mathbf{SO}^*_{2n}$ is connected. To see this, it is enough to check that the non-trivial involutive automorphism of $\mathbf{SO}_{2n}$ which fixes $\mathbf{U}_n$ has $\mathbf{U}_n$ as its full fixed point set, and this is immediate. $\mathbf{SO}^*_{2n}$ may be identified with the group of automorphisms of $\mathbf{H}^n$ endowed either with the β-hermitian form

(1) $$\Phi_0(x, y) = \sum_{1 \leqq i \leqq n} x_i \beta(y_i),$$

or with the anti-hermitian form

(2) $$\Phi'_0(x, y) = \sum_{1 \leqq i \leqq n} x_i k \bar{y}_i.$$

We recall that α is the unique involution μ of **H** such that $\mu(x) \in \mathbf{R}\, x^{-1}$ ($x \in \mathbf{H}$, $x \neq 0$), and that any other involution is conjugate to β by an inner automorphism. Moreover, any non-degenerate β-hermitian (resp. α-antihermitian) form on $\mathbf{H}^n$ is equivalent to Φ_0 (resp. Φ'_0). If $\varepsilon = \pm 1$, and Φ is an ε-α-hermitian form on $\mathbf{H}^n$, then $\Phi'(x, y) = \Phi(x, y).k$ is -ε-β-hermitian.

10.6. We now consider sequences of non-compact classical real Lie groups (H_n, f_n) where $f_n : H_n \to H_{n+1}$ is injective. In the following table we give the n-th term of the sequence, the corresponding compact twin to the symmetric space of maximal compact subgroups of H_n, and the value of $\varprojlim^* I_{H_n}$. In all cases, f_n is the obvious map.

$E(\{x_i, d^0 x_i\})$ [resp. $P(\{x_i, d^0 x_i\})$] is the graded exterior (resp. polynomial) algebra over $\mathbf{R}$, with generators x_i ($i = 1, 2, \ldots$), where $d^0 x_i$ is the degree of x_i.

H_n	$X_{n,u}$	$\varprojlim^* I_{H_n}$
$\mathbf{SL}_n(\mathbf{C})$	$\mathbf{SU}_n$	$E(\{x_i, 2i+1\})$
$\mathbf{Sp}_{2n}(\mathbf{C})$	$\mathbf{USp}_n$	$E(\{x_i, 4i-1\})$
$\mathbf{SO}_n(\mathbf{C})$	$\mathbf{SO}_n$	$E(\{x_i, 4i-1\})$
$\mathbf{SL}_n(\mathbf{R})$	$\mathbf{SU}_n/\mathbf{SO}_n$	$E(\{x_i, 4i+1\})$
$\mathbf{SL}_n(\mathbf{H})$	$\mathbf{SU}_{2n}/\mathbf{USp}_n$	$E(\{x_i, 4i+1\})$
$\mathbf{Sp}_{2n}(\mathbf{R})$	$\mathbf{USp}_n/\mathbf{U}_n$	$P(\{x_i, 4i-2\})$
$\mathbf{SO}_{n,n}(\mathbf{R})$	$\mathbf{SO}_{2n}/((\mathbf{O}_n \times \mathbf{O}_n) \cap \mathbf{SO}_{2n})$	$P(\{x_i, 4i\})$
$\mathbf{O}_{n,n}(\mathbf{R})$	$\mathbf{SO}_{2n}/((\mathbf{O}_n \times \mathbf{O}_n) \cap \mathbf{SO}_{2n})$	$P(\{x_i, 4i\})$
$\mathbf{U}_{n,n}$	$\mathbf{U}_{2n}/\mathbf{U}_n \times \mathbf{U}_n$	$P(\{x_i, 2i\})$
$\mathbf{SU}_{n,n}$	$\mathbf{SU}_{2n}/((\mathbf{U}_n \times \mathbf{U}_n) \cap \mathbf{SU}_{2n})$	$P(\{x_i, 2i\})$
$\mathbf{Sp}_{n,n}$	$\mathbf{USp}_{2n}/\mathbf{USp}_n \times \mathbf{USp}_n$	$P(\{x_i, 4i\})$
$\mathbf{SO}^*_{2n}$	$\mathbf{SO}_{2n}/\mathbf{U}_n$	$P(\{x_i, 4i-2\})$

These spaces all occur in Bott periodicity, and the $\varprojlim^*$ on the right hand side are computed in [7], in particular in Exp. 16, by H. Cartan. More precisely, for given series $(X_{n,u})$, let $X = \varprojlim X_{n,u}$ be the inductive limit of the $X_{n,u}$, endowed with the inductive limit topology. Then

(1) $$H^*(X) \cong \varprojlim^* H^*(X_{n,u}) \cong \varprojlim{}^* I_{H_n}$$

is determined in [7]. In fact, it is shown that, in a given dimension, the sequence is stationary, i. e., given j, there exists $n(j)$ such that

(2) $$I^j_{H_m} = H^j(X_{m,u}) \cong H^j(X) = \varprojlim^* I^j_{H_n} \qquad [m \geqq n(j)].$$

[In particular, there is no problem about the first equality in (1).] The space X is a « weak » H-space ([7], Exp. 11), which explains why its real cohomology algebra is free anticommutative. The homotopy groups of X are finitely generated, periodic, of period dividing 8, and we have

(3) $$\pi_j(X) \otimes \mathbf{R} \cong (\operatorname{Im} \pi_j(X)) \otimes \mathbf{R} \subset H_j(X),$$

under the Hurewicz map. The second term is the space $P_j(X)$ of primitive elements (for the diagonal map) in $H_j(X)$. It is the dual to the quotient of $H^j(X)$ by the space of decomposable elements, the space of indecomposable elements of degree j. As the above table shows, its dimension is periodic, of period 2 or 4.

11. Stable cohomology of classical arithmetic groups

11.1. THEOREM. — *Let* G_n *be a semi-simple* **Q**-*group*, Γ_n *an arithmetic subgroup of* G_n, *and* $f_n : G_n \to G_{n+1}$ *an injective morphism which maps* Γ_n *into* Γ_{n+1} (i = 1, 2, ...). *Assume:* (i) *Given* $q \in \mathbf{N}$, *there exists* $n(q)$ *such that* $(I^q_{G_n})^{\Gamma_n} = I^q_{G_n}$ *for* $n \geqq n(q)$; (ii) $m(G_n(\mathbf{R}))$ *and* $c(G_n)$ *tend to infinity with* n. *Then*

$$H^*(\varinjlim \Gamma_n) = \varprojlim{}^* H^*(\Gamma_n) = \varprojlim{}^* I_{G_n(\mathbf{R})}. \tag{1}$$

By 7.5, for every n :

$$j^q : (I^q_{G_n})^{\Gamma_n} \to H^q(\Gamma_n)$$

is an isomorphism for $q \leqq \min(c(G_n), m(G_n(\mathbf{R}))$. Therefore, given q, j^q_n yields an isomorphism of $I^q_{G_n}$ onto $H^q(\Gamma_n)$ for every big enough n. Since the j^q_n are compatible with the homomorphisms f^*_n defining the two $\varprojlim{}^*$, this proves the second equality of (1).

We have obviously $H_*(\varinjlim \Gamma_n) = \varinjlim H_*(\Gamma_n)$. The first equality follows then from the fact that the dual of a countable inductive limit of finite dimensional vector spaces is the projective limit of the dual spaces. In fact, in all examples below, the sequences are stationary in each dimension and the $\varprojlim{}^*$ will be finite dimensional in each degree.

11.2. REMARKS. — The condition (i) is of course satisfied if $\Gamma_n \subset G_n(\mathbf{R})^0$. In that respect, we recall that $G_n(\mathbf{R})$ is connected (as a Lie group), if G_n is connected and simply connected as an algebraic group; in the present case, the latter condition is equivalent to : $G_n(\mathbf{C})$ or $G_{n,u}$ is connected and simply connected.

By 9.5, (ii) is fulfilled if $rk_{\mathbf{Q}}(G_n) \to \infty$.

11.3. Let k be a number field, S the set of archimedian places of k, $d = r_1 + 2r_2$ the degree of k, where r_1 (resp. r_2) is the number of real (resp. complex) places of k.

Let G'_n be an absolutely almost simple k-group, Γ'_n an arithmetic subgroup of G'_n, and $f'_n : G'_n \to G'_{n+1}$ an injective k-morphism which maps Γ'_n into Γ'_{n+1}. Assume that $rk_k G'_n \to \infty$. Then

$$G_n = R_{k/\mathbf{Q}} G'_n, \qquad f_n = R_{k/\mathbf{Q}} f'_n, \tag{1}$$

and the image Γ_n of Γ'_n under the canonical isomorphism $G'_n(k) \tilde{\to} G_n(\mathbf{Q})$ satisfy the assumptions of 11.1 [except possibly for (i)].

For $v \in S$, let k_v be the completion of k at v. Thus $k_v = \mathbf{R}$ or $\mathbf{C}$ depending on whether v is real or complex. Let $G'_{n,v}$ be the k_v-group obtained from G'_n by extension of the groundfield $k \to k_v$. We have then ([23], Th. 1.3.1) :

$$(2) \qquad G_n(\mathbf{R}) = \prod_{v \in S} G'_{n,v}(k_v).$$

If v is complex, then $G'_{n,v}(k_v) = G'_n(\mathbf{C})$. If v is real, $G'_{n,v}(k_v)$ is a real form of G' which may depend on v. At any rate, the symmetric space of maximal compact subgroups of $G_n(\mathbf{R})$ is the product of the corresponding symmetric spaces for the groups $G'_{n,v}(k_v)$ and $f_n : G_n(\mathbf{R}) \to G_{n+1}(\mathbf{R})$ is also compatible with that decomposition. As a result

$$(3) \qquad \underset{\longleftarrow}{\lim}{}^* I_{G_n(\mathbf{R})} = \underset{v \in S}{\otimes} I_v,$$

where

$$(4) \qquad \begin{cases} I_v = \underset{\longleftarrow}{\lim}{}^* I_{G'_{n,v}(\mathbf{R})}, & (v \text{ real}), \\ I_v = \underset{\longleftarrow}{\lim}{}^* I_{G'_n(\mathbf{C})} & (v \text{ complex}). \end{cases}$$

If, in addition, (i) is fulfilled (and this is a rather harmless condition in practice), then we have

$$(5) \qquad H^*(\underset{\longrightarrow}{\lim}\, \Gamma'_n) = \underset{\longleftarrow}{\lim}{}^* H^*(\Gamma'_n) = \underset{v \in S}{\otimes} I_v.$$

We now proceed to give a series of examples. In all of them, the sequences $G'_{n,v}(k_v)$ are among those listed in 10.5, and, in a given degree, the corresponding sequence of cohomology spaces is always stationary. Thus we have more precisely, given $q \in \mathbf{N}$:

$$(6) \qquad H^q(\Gamma'_n) = (\underset{v \in S}{\otimes} I_v)^q \quad \text{for } n \text{ large enough.}$$

11.4. If $G'_n = \mathbf{SL}_n/k$, $\mathbf{Sp}_{2n}/k$ or $\mathbf{O}_{n,n}/k$, then I_v is given by the following table

G'_n	I_v (v real)	I_v (v complex)
$\mathbf{SL}_n/k$	$E(\{x_i, 4i+1\})$	$E(\{x_i, 2i+1\})$
$\mathbf{Sp}_{2n}/k$	$P(\{x_i, 4i-2\})$	$E(\{x_i, 4i-1\})$
$\mathbf{O}_{n,n}/k$	$P(\{x_i, 4i\})$	$E(\{x_i, 4i-1\})$

In the first two cases, $G_n(\mathbf{R})$ is connected (*see* 11.2), 11.1 (i) is always fulfilled, and 11.3 (5), (6) hold. As group Γ'_n, we may in particular take $\mathbf{SL}_n(\mathfrak{o})$ or $\mathbf{Sp}_{2n}(\mathfrak{o})$, where $\mathfrak{o}$ is the ring of integers of k.

Let now $G'_n = \mathbf{O}_{n,n}/k$. Then $G'_{n,v}(k_v) = \mathbf{O}_{n,n}(\mathbf{R})$, $\mathbf{O}_{n,n}(\mathbf{C})$, depending on whether v is real or complex. It is not connected, and in fact for v real, the $2n$-form corresponding to the Euler class, which is in $I_{\mathbf{SO}_{n,n}}$, is not invariant under $\mathbf{O}_{n,n}(\mathbf{R})$. However, it disappears in the stable range, so that (i) is also fulfilled for any choice of Γ'_n, and we again have 11.3 (5), (6).

11.5. Let D be a central division algebra over k, and s^2 its degree over k. Then $D \underset{k}{\otimes} k_v$ is a matrix algebra over a division algebra L_v with center k_v. There are three possibilities

$$(1)\qquad \begin{cases} (a) & k_v = \mathbf{C}, & L_v = \mathbf{C}, & D \underset{k}{\otimes} k_v = \mathbf{M}_s(\mathbf{C}); \\ (c) & k_v = \mathbf{R}, & L_v = \mathbf{R}, & D \underset{k}{\otimes} k_v = \mathbf{M}_s(\mathbf{R}); \\ (d) & k_v = \mathbf{R}, & L_v = \mathbf{H}, & D \underset{k}{\otimes} k_v = \mathbf{M}_{s/2}(\mathbf{H}). \end{cases}$$

Take $G'_n = \mathbf{SL}_n(D)$, and for Γ'_n the arithmetic subgroup consisting of elements with entries in some fixed order of D. We have then

$$(2)\qquad G'_{n,v}(k_v) = \mathbf{SL}_{ns}(\mathbf{C}), \quad \mathbf{SL}_{ns}(\mathbf{R}), \quad \mathbf{SL}_{ns/2}(\mathbf{H})$$

corresponding to the cases (a), (b) and (c). Since G'_n is simply connected, (i) is fulfilled and 11.5 obtains. Moreover, we see from 10.6 that the factors I_v associated to the cases (b) and (c) are the same. Hence

$$(3)\qquad H^*(\varinjlim \Gamma'_n) = \overset{r_1}{\otimes} E(\{x_i, 4i+1\}) \overset{r_2}{\otimes} E(\{x_i, 2i+1\}).$$

In particular, we always have

$$(4)\qquad H^2(\varinjlim \Gamma_n) = 0.$$

In fact, since $rk_{\mathbf{Q}}\, G_n = n-1$ in this case, 11.1 gives more precisely

$$H^2(\Gamma_n) = 0 \qquad \text{for} \quad n > 9.$$

This applies in particular to the case where $D = k$, $\Gamma_n = \mathbf{SL}_n(\mathfrak{o})$, and we get Garland's result [9].

11.6. The following examples will be associated to algebras with involution and we first fix some notation.

D is a division algebra over k, and j an involution of D over k. Let k' be the center of D. If j is of the first kind, i. e. is trivial on k', then we assume $k' = k$. It is known that D is then a quaternion algebra over k.

If j is of the second kind, i. e. non trivial on k', then k' is a quadratic extension of k, and j restricts to k' to the non-trivial automorphism of k' over k.

Let $\varepsilon = \pm 1$. On D^{2n}, we consider the ε-j-hermitian form

$$(1)\qquad {}_{\varepsilon}\Phi(x, y) = \sum_{1 \leq i \leq n} (x_i j(y_{n+i}) + \varepsilon x_{n+1} j(y_i)).$$

There is a k-group ${}_{\varepsilon}\mathbf{O}_{n,n,D}$ whose group of points over D is the automorphism group of D^{2n} endowed with the form (1). We let ${}_{\varepsilon}G'_n$ be the derived group of the identity component of ${}_{\varepsilon}\mathbf{O}_{n,n,D}$ and Γ_n be the arithmetic subgroup consisting of the elements of ${}_{\varepsilon}G'_n$ with coefficients in some order $\mathfrak{o}$ of D.

The groups ${}_{\varepsilon}G'_n$, together with those considered in 11.4, are k-forms of the classical groups. The isomorphism class of ${}_{\varepsilon}G'_{n,v}(k_v)$ depends only on that of $(D \otimes_k k_v, j)$. It will be described without further justification, but the latter can be easily extracted from [18], Chapter II, and from the elementary remarks on quaternions made in 10.5 and 11.7.

11.7. We now consider the case where D is a quaternion algebra with center k. Assume first that j is the " standard " involution, such that $x+j(x)$ and $x\,j(x)$ are respectively the reduced trace and the reduced norm of x ($x \in D$). Let us denote by t' the involution of $\mathbf{M}_2(F)$, (F any field) which is given by

$$t'(x) = s\,{}^t x\, s^{-1} \quad \text{with} \quad s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{1}$$

There are, up to isomorphism, three possibilities for $(D \otimes k_v, j)$:

$$\left\{ \begin{array}{lll} (a) & v \text{ complex}, & (D \otimes_k k_v, j) \cong (\mathbf{M}_2(\mathbf{C}), t'); \\ (b) & v \text{ real} & (D \otimes_k k_v, j) \cong (\mathbf{M}_2(\mathbf{R}), t'); \\ (c) & v \text{ real} & (D \otimes_k k_v, j) \cong (\mathbf{H}, \alpha). \end{array} \right. \tag{2}$$

The following table gives the corresponding values of ${}_{\varepsilon}G'_{n,v}(k_v)$ and I_v :

	(a)	(b)	(c)
${}_1G'_{n,v}(k_v)$	$\mathbf{Sp}_{4n}(\mathbf{C})$	$\mathbf{Sp}_{4n}(\mathbf{R})$	$\mathbf{Sp}_{n,n}$
I_v	$E(\{x_i, 4i-1\})$	$P(\{x_i, 4i-2\})$	$P(\{x_i, 4i\})$
${}_{-1}G'_{n,v}(k_v)$	$\mathbf{SO}_{4n}(\mathbf{C})$	$\mathbf{SO}_{2n,2n}(\mathbf{R})$	$\mathbf{SO}^*_{4n}$
I_v	$E(\{x_i, 4i-1\})$	$P(\{x_i, 4i\})$	$P(\{x_i, 4i-2\})$

If j is not equivalent to the standard involution, then the three possibilities for $(D \otimes_k k_v, j)$ are respectively

$$(\mathbf{M}_2(\mathbf{C}), t), \quad (\mathbf{M}_2(\mathbf{R}), t), \quad (\mathbf{H}, \beta) \tag{3}$$

and we get the table corresponding to the above one just by interchanging the cases $\varepsilon = 1$ and $\varepsilon = -1$.

The groups involved are connected, except for $\mathbf{SO}_{2n,2n}(\mathbf{R})$ which has two components (*see* 10.5 for $\mathbf{SO}^*_{2n}$). As in 11.4, the non-connectedness of $\mathbf{SO}_{2n,2n}(\mathbf{R})$ does not matter in the stable range, so that 11.1 (i) is fulfilled and 11.3 (5), (6) hold for any coherent choice of the Γ''_n.

11.8. Let now j be of the second kind, and denote $x \mapsto \bar{x}$ the non-trivial automorphism of k' over k. Let m^2 be the degree of D over k'. For $(D \otimes k_v, j)$ the possibilities are the following, up to isomorphism :

(*a*) v complex. Then $k' \otimes_k k_v = \mathbf{C} \oplus \mathbf{C}$, the algebra $D \otimes_k \mathbf{C}$ is isomorphic to $\mathbf{M}_m(\mathbf{C}) \oplus \mathbf{M}_m(\mathbf{C})$, and j is $(x, y) \mapsto ({}^t y, {}^t x)$.

(*b*) v real, extends to two real places of k'. Then $D \underset{k}{\otimes} k_v = \mathbf{M}_m(\mathbf{R}) \oplus \mathbf{M}_m(\mathbf{R})$ and j is as in (*a*).

(*c*) v real, but extends to a complex place of k'. Then $k' \otimes k_v = \mathbf{C}$, j induces the complex conjugation on $\mathbf{C}$, $D \otimes k_v \cong \mathbf{M}_m(\mathbf{C})$ and $j(x) = s\,{}^t\bar{x}\,s^{-1}$, with s hermitian. We have then, for both values of ε,

	(*a*)	(*b*)	(*c*)
${}_\varepsilon G'_{n,v}(k_v)$	$\mathbf{SL}_{2nm}(\mathbf{C})$	$\mathbf{SL}_{2nm}(\mathbf{R})$	$\mathbf{SU}_{nm,nm}$
I_v	$E(\{x_i, 2i+1\})$	$E(\{x_i, 4i+1\})$	$E(\{x_i, 2i\})$

Thus G'_n is simply connected, and 11.3 (5), (6) obtain.

11.9. We have considered only " hyperbolic forms ", for convenience, and because these cases are those of interest in K-theory. But 11.1 can be applied to other sequences. As a simple example, fix a non-degenerate quadratic form F_0 on k^m, say anisotropic, let G_n be the orthogonal group of

$$F_0 + \sum_{1 \leqq i \leqq n} x_i x_{n+i}$$

in k^{m+2n}, and $\Gamma_n = G_n(\mathfrak{o})$. We have then again 11.3 (5), (6), and in fact the values of I_v are *the same* as in the $\mathbf{O}_{n,n}$ case. This is clear for v complex. For v real, we have

$$I_v = \varprojlim{}^* H^*(\mathbf{O}_{m+2n}(\mathbf{R})/(\mathbf{O}_{m+n}(\mathbf{R}) \times \mathbf{O}_n(\mathbf{R})).$$

But the homogeneous space on the right hand side is as good an approximation to the classifying space of $\mathbf{O}_n(\mathbf{R})$ as $\mathbf{O}_{2n}(\mathbf{R})/(\mathbf{O}_n(\mathbf{R}) \times \mathbf{O}_n(\mathbf{R}))$ hence the $\varprojlim{}^*$ does not depend on m.

The same is true in the hermitian case.

11.10. Remark. — In a sequel to this paper, we shall see that 11.1 is also true (with the same range) for S-arithmetic groups. It will follow the that $H^*(\varinjlim G_n(\mathbf{Q})) = H^*(\varinjlim \Gamma_n)$ or, in the set up of 11.3, that we also have

$$\text{(1)} \qquad H^*(\varinjlim G'_n(k)) = \varprojlim{}^* H^*(G'_n(k)) = \bigotimes_{v \in S} I_v.$$

12. Applications to K and L-theory

12.1. Let D be a division algebra over k and $\mathfrak{o}$ an order of D. For $i \geqq 2$, Quillen's group $K_i\,\mathfrak{o}$, as defined in [21], is the i-th homotopy group of an H-space which has the same homology as $\varinjlim \mathbf{SL}_n\,\mathfrak{o}$. Therefore, by the fact recalled in 10.6, $\dim(K_i \otimes \mathbf{R})$ is equal to the dimension of the space of indecomposable elements in $\varprojlim{}^* H^*(\mathbf{SL}_n\,\mathfrak{o})$ and 11.5 (3) implies :

12.2. PROPOSITION. — *Let* $\mathfrak{o}$ *be an order in a central division algebra* D *over* k. *Then, for* $i \geqq 2$, $\dim(K_i\,\mathfrak{o} \otimes \mathbf{R})$ *is periodic of period four and is equal to* 0, r_1+r_2, 0, r_2 *depending on whether* $i \equiv 0, 1, 2, 3 \bmod 4$, *where* r_1 (*resp.* r_2) *is the number of real* (*resp. complex*) *places of* k.

12.3. We now revert to the notation of 11.6. A result of L. I. Wasserstein (*see* [1], Chap. II, § 5, p. 156) implies that $M = \varinjlim {}_{\varepsilon}G'_n(\mathfrak{o})$ and $\varinjlim {}_{\varepsilon}\mathbf{O}_{n,n,D}(\mathfrak{o})$ have the same derived group, which is perfect (and generated by " elementary " matrices). Quillen's +-construction, applied to the classifying space B_M of M, yields a space B_M^+ with fundamental group M/(M, M) and the same homology as M, and we let ${}_{\varepsilon}L_i\,\mathfrak{o} = \pi_i(B_M^+)$ $(i \geqq 2)$. These are the groups introduced and so denoted by M. Karoubi [17] (for more general rings A, except for the blanket assumption that 2 is invertible in A, which plays no role here). The space B_M^+ is also and H-space, and as in 10.6, the dimension if ${}_{\varepsilon}L_i \otimes \mathbf{R}$ is equal to the dimension of the space of indecomposable elements in $H^i(M)$. The results of paragraph 11 then also give the dimension of ${}_{\varepsilon}L_i(\mathfrak{o}) \otimes \mathbf{R}$ $(i \geqq 2)$ in the various cases considered there. It is again periodic of period 4. Corresponding to 11.4, 11.7 and 11.8, we have the following cases :

(1) $D = k$, $j = \mathrm{Id}$, and $\mathfrak{o}$ is the ring of integers of k. Then the period giving $\dim({}_{\varepsilon}L_i(\mathfrak{o}) \otimes \mathbf{R})$ starting with $i \equiv 2 \bmod 4$ is

$$\varepsilon = 1 \quad : \quad 0 \ \ r_2 \ \ r_1 \ \ 0,$$

$$\varepsilon = -1 : \quad r_1 \ \ r_2 \ \ 0 \ \ 0.$$

(2) D is quaternion algebra and j the standard involution. We let r_b (resp. r_c) be the number of real places for which (*b*) [resp. (*c*)] of 11.7 holds. The period is then,

$$\varepsilon = 1 \quad : \quad r_b \ \ r_2 \ \ r_c \ \ 0,$$

$$\varepsilon = -1 : \quad r_c \ \ r_2 \ \ r_b \ \ 0.$$

If j is not the standard involution, then we have only to interchange the cases $\varepsilon = 1$ and $\varepsilon = -1$.

(3) In the case of 11.8, let again r_b (resp. r_c) be the number of real places for which (*b*) [resp. (*c*)] holds. Let $\mathfrak{o}$ be an order of D stable under j. The period for $\dim({}_{\varepsilon}L_i(\mathfrak{o}) \otimes R)$, always starting with $i \equiv 2\,(4)$, is now, for both values of ε :

$$r_c \ \ r_2 \ \ r_c \ \ r_b+r_2.$$

In view of [2], the above results are also valid if $\mathfrak{o}$ is replaced by $\mathfrak{o}_S$ (S finite set of primes in k, stable under the involution) or by k. In particular, if $D = k$, they can be compared with those of [17] paragraph 5, p. 96. This shows that the part of the homology explicity determined there in terms of the Witt group of k, corresponds of the one associated here to the real places of k.

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A. Borel,
The Institute for Advanced Study,
Princeton, N. J. 08540,
U. S. A.

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