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On convexity, the Weyl group and the Iwasawa decomposition

ON CONVEXITY,
THE WEYL GROUP AND THE IWASAWA DECOMPOSITION

By Bertram KOSTANT (*)

1. Introduction

1.1. If p is a positive definite matrix how does the spectrum of p change when p is multiplied by a unitary matrix ? C. Thompson in [14] (Theorem 1) and much earlier A. Horn in [16], proved the following theorem : Let p and q be any two positive definite $n \times n$ matrices and let $x_1 \geq x_2 \geq \ldots \geq x_n$ and $y_1 \geq y_2 \geq \ldots \geq y_n$ denote the respective sets of eigenvalues. Then there exists a unitary matrix v such that pv and q have the same spectrum if and only if $\det p = \det q$ and

$$(1.1.1) \qquad x_1 x_2 \ldots x_i \geq y_1 y_2 \ldots y_i \quad \text{for all } 1 \leq i \leq n.$$

This rather nice theorem may be cast in a form which makes sense for any semi-simple Lie group G. The point is that the theorem is then true for G. Let

$$(1.1.2) \qquad G = KAN$$

be an Iwasawa decomposition for G. *See* e. g. p. 234 in [2]. If $g \in G$ and $g = kan$ where $k \in K$, $a \in A$ and $n \in N$ then $a = a(g)$ is called the a-component of g.

Now let $\mathfrak{a}$, $\mathfrak{g}$ ($\mathfrak{a} \subseteq \mathfrak{g}$) be the Lie algebras of A and G and let W be the Weyl group of $(\mathfrak{a}, \mathfrak{g})$ regarded as operating in $\mathfrak{a}$ (and A). For each $x \in \mathfrak{a}$ let $\mathfrak{a}(x)$ be the convex hull of the Weyl group orbit $W.x$. Correspondingly for each $b \in A$ let

$$A(b) = \exp \mathfrak{a} (\log b)$$

[so that, multiplicatively, A (b) is the convex compact set having the Weyl group orbit $W.b$ as its extreme points].

Now let $b \in A$ be arbitrary. In doing analysis on semi-simple Lie groups or in representation theory, one is very often concerned with the a-component, $a(bv)$, of the product bv where $v \in K$ is arbitrary. (Recall, for example, Harish-Chandra's well known formula [see (4.1.2)] for the elementary spherical functions on G.) The generalization which we will prove here of the Horn-Thompson theorem determines the set of all $a(bv)$ as v runs through K. Not only is $a(bv) \in A(b)$ but in fact one has

THEOREM 4.1. — *Let* $b \in A$ *be arbitrary. Then*

$$A(b) = \{ a(bv) \mid v \in K \}.$$

The connection between the Weyl group convexity formulation in Theorem 4.1 and the Horn-Thompson theorem may be clarified by introducing a natural partial ordering in G. This partial ordering is different from (but was inspired by) a partial ordering defined by Thompson for GL $(n, \mathbf{C})$. Thompson's definition is based on the polar decomposition and hence depends on a particular choice of a Cartan decomposition of $\mathfrak{g}$. One obtains an invariant partial ordering by just noting that any element $g \in G$ may be uniquely written

(1.1.3) $g = ehu$

where e is elliptic, u is unipotent and h is hyperbolic and all three elements, e, h and u commute. Write $h = h(g)$. An element is hyperbolic if and only if it is conjugate to an element in A. Now for any element $g \in G$ let $A(g) \subseteq A$ be defined by putting $A(g) = A(b)$ where $b \in A$ is conjugate to the hyperbolic component, $h(g)$, occurring in (1.1.3). It is easy to see that $A(g)$ is independent of b. Given f, $g \in G$ define

(1.1.4) $g \geqq f$ in case $A(f) \subseteq A(g)$.

The ordering is independent of A since one has

THEOREM 3.1. — *For any finite dimensional representation* π *of* G *and any* $g \in G$ *let* $| \pi(g) |$ *denote the spectral radius of* $\pi(g)$. *Then if* g, $f \in G$ *one has* $g \geqq f$ *if and only if* $| \pi(g) | \geqq | \pi(f) |$ *for all finite dimensional representations* π *of* G.

REMARK 1.1. — In case say $G \subseteq SL(n, \mathbf{C})$ for some n the ordering in G induced by that in SL $(n, \mathbf{C})$ is not necessarily the given ordering in G. Thus it is not sufficient just to define the ordering in SL $(n, \mathbf{C})$ (*see* Remark 3.1.1).

Theorem 4.1 now says that $a(bv)$ runs through all elements $a \in A$ where $a \leqq b$ as v runs over K. That is, by Theorem 3.1, all $a \in A$ such that $| \pi(a) | \leqq | \pi(b) |$ for any finite dimensional representation π.

In case $G = SL(n, \mathbf{C})$ and $b = p$, $a = q$ are diagonal matrices, this just means (1.1.1) since it suffices to consider only the (fundamental) representations π_i, of G, $1 \leq i \leq n - 1$, on the space $\Lambda^i \mathbf{C}^n$ of alternating i-vectors. But one has $|\pi_i(b)| = x_1 \ldots x_i$ and $|\pi_i(a)| = y_1 \ldots y_i$. This in essence establishes the connection of Theorem 4.1 with the Horn-Thompson theorem. Not quite, since the Horn-Thompson theorem deals with eigenvalues and Theorem 4.1 deals with the Iwasawa decomposition.

Theorem 5.4 is the direct generalization of the Horn-Thompson theorem to the arbitrary semi-simple case. Indeed for $SL(n, \mathbf{C})$ it reduces to Horn's theorem which is stronger than Thompson's result in that Horn also proves the additional fact that if v is *any* unitary matrix and the eigenvalues $z_1, \ldots, z_n$, of pv, (no longer necessarily positive) are ordered so that $|z_1| \geq |z_2| \geq \ldots \geq |z_n|$ then one has

$$x_1 \ldots x_i \geq |z_1| \ldots |z_i| \quad \text{for all } i.$$

1.2. One may apply Theorem 4.1 to obtain some results on K-double cosets. For any $a \in A$ let

$$G_a = \{ g \in G \mid p(g) \geq a \}.$$

Here $p(g)$ is the " positive definite " component of g relative to the polar decomposition [see (4.2.6)].

Now one knows that

(1.2.1) $G = KAK.$

This fact is quite useful. It implies among other things that a spherical function on G is determined by its restriction to A. If we say that, f, $g \in G$ are congruent when $K f K = K g K$ then (1.2.1) of course says that any element in G is congruent to an element of A. Now if we replace A by a coset of N this is no longer true. Nevertheless one can now say exactly which elements in G are congruent to this coset.

THEOREM 5.3. — *Let* $a \in A$. *Then*

$$G_a = K \, a \, NK.$$

Since $p(g) \geq 1$ for all $g \in G$ the special case where $a = 1$ yields

THEOREM 5.1. — *One has*

$$G = KNK.$$

REMARK 1.2. — Theorem 5.1 says that a spherical function is determined by its restriction to N (as well as its restriction to A). It is suggestive therefore from Theorem 5.1 that it might be quite interesting to see

what the spherical functions look like on N.   Geometrically, Theorem 5.1 says that if X is the symmetric space G/K then any two points on X lie on some horocycle.   In case X is the unit disc this is of course clear since the horocycles are just the circles in the disc that are tangent to the boundary.

An element $a \in A$ is called regular if $\sigma\, a = a$, $\sigma \in W$, implies σ is the identity.

Another corollary of Theorem 5.3 is

THEOREM 5.5. — *Let $a \in A$ be regular and let D_a be the conjugacy class of a.   That is $D_a = \{ gag^{-1} \mid g \in G \}$.   Then*

$$G_a = K\, D_a\, K$$

(*See* Remark 5.5).

1.3. One of the main points of the paper [14] was to recast some generalizations of the Golden-Thompson inequality made by Lenard in [9].   The Golden-Thompson equality states that if x and y are two $n \times n$ Hermitian matrices then

(1.3.1) $\text{tr } e^x\, e^y \geq \text{tr } e^{x+y}$.

The generalization made in [9] replaces the trace by the character of any representation of GL $(n, \mathbf{C})$.   But now this generalizes to an arbitrary semi-simple Lie group in two steps, Theorem 6.1 and Theorem 6.3.

If $g \in G$ is hyperbolic and π is a finite dimensional representation of G then the eigenvalues of $\pi\,(g)$ are all positive.   Thus $\mid \pi\,(g) \mid$ then is just the maximal value of $\pi\,(g)$.   Hence if f, $g \in G$ are hyperbolic and $g \geq f$ then by Theorem 3.1 the maximal eigenvalue of $\pi\,(g)$ is greater or equal to the maximal eigenvalue of $\pi\,(f)$.   But the minimal eigenvalue of $\pi\,(g)$ is less or equal to the minimal eigenvalue of $\pi\,(f)$.   Thus one cannot immediately compare $\chi_\pi\,(f)$ and $\chi_\pi\,(g)$ if χ_π is the character of π.   However one has

THEOREM 6.1. — *Let f, $g \in G$ be hyperbolic and assume $g \geq f$.   Then if π is any finite dimensional representation one has*

$$\chi_\pi\,(g) \geq \chi_\pi\,(f).$$

Now let $\mathfrak{k}$ be the Lie algebra of K so that $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$ where $\mathfrak{p}$ is the (Killing form) orthocomplement of $\mathfrak{k}$ in $\mathfrak{g}$. If $P = \{ \exp x \mid x \in \mathfrak{p} \}$ it is easy to see that P^2 is the set of all hyperbolic elements in G (*see* Proposition 6. 2).   Thus if x, $y \in \mathfrak{p}$ then $e^x\, e^y$ and e^{x+}

are hyperbolic. As a dividend of the partial ordering based on (1.1.3) rather than on the polar decomposition one has

THEOREM 6.3. — *For any* x, $y \in \mathfrak{p}$,

$$(1.3.2) \qquad\qquad e^x\, e^y \succeq e^{x+y}$$

so that, by Theorem 6.1, $\chi_\pi\, (e^x\, e^y) \succeq \chi_\pi\, (e^{x+y})$ *for any finite dimensional representation* π *of* G.

1.4. The relation (1.3.2) has a nice geometric interpretation. In fact it is really a statement about geodesic triangles on Riemannian symmetric spaces of negative curvature.

An element $x \in \mathfrak{g}$ is called real semi-simple if ad x is diagonalizable and has real eigenvalues. Let $\mathfrak{l} \subseteq \mathfrak{g}$ denote the set of such elements. One notes that the exponential map sets up a bijection between $\mathfrak{l}$ and all hyperbolic elements in G. Furthermore the partial order in G defines a partial order in $\mathfrak{l}$. That is if x, $y \in \mathfrak{l}$ then $y \succeq x$ if exp $y \succeq$ exp x. This is equivalent to the condition that $\mathfrak{a}\,(x) \subseteq \mathfrak{a}\,(y)$ where for any $z \in \mathfrak{l}$, $\mathfrak{a}\,(z) = \mathfrak{a}\,(z')$ and $z' \in \mathfrak{a}$ is an element conjugate to z. The statement $y \succeq x$ is also equivalent to the condition that the maximal eigenvalue of $\pi\,(y)$ is greater or equal to the maximal eigenvalue of $\pi\,(x)$ for all finite dimensional representations π of G.

If $x \in \mathfrak{l}$ and (x, x) denotes the inner product defined by the Killing form then $(x, x) \succeq 0$ and one puts $|x| = (x, x)^{1/2}$. For x, $y \in \mathfrak{l}$ one easily has

$$(1.4.1) \qquad\qquad y \succeq x \quad \text{implies} \quad |y| \succeq |x|.$$

The converse is false in general. *See* Remark 7.1.

Now let X = G/K so that X has the structure (normalized by the Killing form on $\mathfrak{g}$) of a Riemannian symmetric of negative curvature. For any two points $r, s \in$ X let $d\,(r, s)$ be the distance of r to s., i. e. $d\,(r, s)$ is the length of the unique geodesic arc segment (r, s) joining r to s.

Now let $o, r,$ and s be any three points in X and consider the corresponding geodesic triangle. Let $a = d\,(o, r)$, $b = d\,(o, s)$ and $c = d\,(r, s)$. Also let ψ be the angle at o made by (o, r) and (o, s). Then the Law of Cosines in flat space is replaced by the inequality

$$(1.4.2) \qquad\qquad c^2 \succeq a^2 + b^2 - 2\,ab \cos \psi$$

on the space X. *See* Lemma 4 in [11]. But the point is that a geodesic arc segment (r, s) carries more information than just its lenght $d\,(r, s)$. In fact one can naturally associate to (r, s) a real semi-simple element

$x\,(r,\,s)\in \mathfrak{l}$. In the language of Elie Cartan the motion of X defined by $\exp x\,(r,\,s)$ is that transvection along the geodesic containing $(r,\,s)$ which carries r to s.

But now $d\,(r,\,s) = |\,x\,(r,\,s)\,|$ and (1.4.2) is just the statement that $x\,(r,\,o) + x\,(o,\,s)\in \mathfrak{l}$ and

(1.4.3) $|\,x\,(r,\,s)\,| \geqq |\,x\,(r,\,o) + x\,(o,\,s)\,|.$

What (1.3.2) amounts to (*see* the proof of Theorem 7.2) is that we can remove the absolute value signs in (1.4.3).

THEOREM 7.2. — *Let $o,\ r,\ s \in X = G/K$ be any three points. Then*

$$x\,(r,\,s) \geqq x\,(r,\,o) + x\,(o,\,s).$$

1.5. Now let $\lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_n$ be any n real numbers and let $\lambda = (\lambda_1,\,\lambda_2,\,\ldots,\,\lambda_n) \in \mathbf{R}^n = \mathfrak{a}_0$. For any permutation σ on $1,\,2,\,\ldots,\,n$ let $\sigma\lambda = (\lambda_{\sigma^{-1}(1)},\,\lambda_{\sigma^{-1}(2)},\,\ldots,\,\lambda_{\sigma^{-1}(n)}) \in \mathfrak{a}_0$ and let $\mathfrak{a}_0\,(\lambda) \subseteq \mathfrak{a}_0 = \mathbf{R}^n$ be the convex hull of all the vectors $\{\,\sigma\lambda\,\}$ for all permutations σ.

Now let $x = (x_{ij})$ be any $n \times n$ Hermitian matrix with eigenvalues $\lambda_1,\,\ldots,\,\lambda_n$. Then A. Horn in [3] proved that the " diagonal " $(x_{11},\,x_{22},\,\ldots,\,x_{nn})$ of x, regarded as a vector in $\mathbf{R}^n$, lies in $\mathfrak{a}_0\,(\lambda)$ and that one obtains all vectors in $\mathfrak{a}_0\,(\lambda)$ this way, by considering all Hermitian matrices x with the eigenvalues $\lambda_1,\,\ldots,\,\lambda_n$. This result may also be generalized to all semi-simple groups.

This time, however, the generalization, Theorem 8.2 is a statement about the Lie algebra $\mathfrak{g}$ and adjoint representations of K on $\mathfrak{p}$ rather than, as in the case of Theorem 4.1, a statement about G and double K-cosets. However the techniques, at least in one direction, of proving Theorem 4.1 and Theorem 8.2 are similar. Theorem 8.3 is an application of Theorem 8.2.

<div align="center">CONTENTS</div>

2. Preliminaries on elliptic, hyperbolic and unipotent elements

2.1. Let $\mathfrak{g}$ be any semi-simple Lie algebra over the real numbers $\mathbf{R}$. An element $x \in \mathfrak{g}$ is called real semi-simple (resp. nilpotent) if ad x is

diagonalizable over **R** (resp. ad x is nilpotent). In particular ad x has real eigenvalues if x is real semi-simple.

Now let G be any Lie group having $\mathfrak{g}$ as its Lie algebra and let exp : $\mathfrak{g} \to$ G denote the exponential map. An element $a \in$ G is called hyperbolic (resp. unipotent) if a is of the form $a = \exp x$ where x is real semi-simple (resp. nilpotent). In either case the element x is easily seen to be unique and we write $x = \log a$. Since only the identity $1 \in$ G is both hyperbolic and unipotent there is no ambiguity in this definition.

An element $e \in$ G is called elliptic if Ad e is diagonalizable over **C** with eigenvalues of norm 1. One has the following :

PROPOSITION 2.1. — *Let $g \in$ G be arbitrary. Then g may be uniquely written*

$$g = ehu$$

where e is elliptic, h is hyperbolic and u is unipotent and where the three elements e, h and u commute.

Proof. — Let $\mathfrak{g}_\mathbf{C}$ be the complexification of $\mathfrak{g}$ and let $G_c \subseteq$ Aut $\mathfrak{g}_\mathbf{C}$ be the adjoint group of $\mathfrak{g}_\mathbf{C}$. Thus the adjoint representation (Ad) of G maps G into G_c.

Now since G_c is algebraic one has the multiplicative Jordan decomposition Ad $g = s_1 u_1$ where s_1, $u_1 \in G_c$, s_1 is semi-simple (i. e. diagonalizable), u_1 is unipotent and s_1 and u_1 commute (*see* e. g. [12], p. 4-11). On the other hand by embedding s_1 in a complex torus of G_c it is clear that we uniquely write $s_1 = e_1 h_1$ where e_1, $h_1 \in G_c$ are two commuting semi-simple elements where the eigenvalues of e_1 have norm 1 and the eigenvalues of h_1 are positive. Thus one has the decomposition

(2.1.1)   Ad $g = e_1 h_1 u_1$.

The uniqueness of u_1 and s_1 imply (after conjugating by u_1) that all three elements, e_1, h_1 and u_1 commute.

Now since h_1 and u_1 define automorphisms of $\mathfrak{g}_\mathbf{C}$ and since all derivations are inner we may uniquely write $h_1 = \exp$ ad x, $u_1 = \exp$ ad z where x, $z \in \mathfrak{g}_\mathbf{C}$, ad z is nilpotent and ad x is diagonalizable with real eigenvalues. But then the commutativity of h_1 and u_1 implies the commutativity of h_1 and ad z and then the commutativity of ad x and ad z so that

(2.1.2) $[x, z] = 0$.

Now let σ be the automorphism of $\mathfrak{g}_\mathbf{C}$ defined by putting $\sigma = 1$ on $\mathfrak{g}$ and -1 on $i\,\mathfrak{g}$. Then $d \mapsto d^\sigma = \sigma\, d\sigma$ defines an automorphism of G_c. Clearly Ad g is fixed under this automorphism. But then by the unique-

ness of the decomposition (2.1.1) it follows that e_1, h_1 and u_1 are fixed by the automorphism so that $\mathfrak{g}$ is stable under e_1, h_1 and u_1. But then $\mathfrak{g}$ is stable under ad x and ad z so that x, $z \in \mathfrak{g}$. But then x is real semi-simple and z is nilpotent. Thus h, $u \in G$ are respectively hyperbolic and unipotent where

(2.1.3) $h = \exp x, \qquad u = \exp z.$

Moreover they commute by (2.1.2). Furthermore they commute with g since Ad g commutes with h_1 and u_1 and hence, by the uniqueness of x and z, fixes x and z. Thus if

(2.1.4) $e = gh^{-1} u^{-1}$

then e commutes with h and u. Also $g = ehu$. But clearly Ad $e = e_1$ and hence e is elliptic. Assume $g = e' h' u'$ is another decomposition satisfying the same conditions. Then by the uniqueness of (2.1.1) one has Ad $e' = e_1$, Ad $h' = h_1$ and Ad $u' = u_1$. But by taking the differential of Ad one has

$$\exp \text{ad} \log h' = \exp \text{ad} \, x = h_1 \quad \text{and} \quad \exp \text{ad} \log u' = \exp \text{ad} \, z = u_1.$$

Thus, by uniqueness, $x = \log h'$, $z = \log u'$ and hence $h' = h$, $u' = u$ and consequently $e' = e$.

<div style="text-align: right">Q. E D.</div>

Given $g \in G$ the components e, h and u of Proposition 2.1 will be written $e(g)$, $h(g)$ and $u(g)$ respectively and the decomposition

(2.1.5) $g = e(g) h(g) u(g)$

will be called the complete multiplicative Jordan decomposition of g.

REMARK 2.1. — Note that by uniqueness if $f \in G$ then $h(fgf^{-1}) = fh(g)f^{-1}$ (similarly for the elliptic and unipotent components). In particular f commutes with g if and only if it commutes with all three components. Furthermore it is then easy to see that if f, $g \in G$ commute

(2.1.6) $h(fg) = h(f) h(g), \qquad e(fg) = e(f) e(g) \qquad \text{and} \qquad u(fg) = u(f) u(g).$

2.2. Now let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a fixed Cartan decomposition of $\mathfrak{g}$. Thus if K is the connected subgroup of G corresponding to $\mathfrak{k}$ then Ad K is a maximal compact subgroup of Ad G. Also $\mathfrak{p}$ is the orthocomplement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form. Let $\mathfrak{a} \subseteq \mathfrak{p}$ be a maximal commutative subalgebra contained in $\mathfrak{p}$ so that $\mathfrak{a}$ is a maximal commutative of real semi-simple elements. Let $x_0 \in \mathfrak{a}$ be any fixed element for which

(2.2.1) $\mathfrak{p}^{x_0} = \mathfrak{a}$

where $\mathfrak{p}^{x_0}$ is the centralizer of x_0 in $\mathfrak{p}$ (see e. g. § 1.3, [8]), and let $\mathfrak{n} \subseteq \mathfrak{g}$ be the sum of all eigenspaces of ad x belonging to positive eigenvalues.

Then $\mathfrak{n}$ is a Lie subalgebra, all of whose elements are nilpotent, and one has the Iwasawa decomposition

$$(2.2.2) \qquad\qquad \mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$$

of $\mathfrak{g}$. If A and N are, respectively, the subgroups of G corresponding to $\mathfrak{a}$ and $\mathfrak{n}$ then the map $K \times A \times N \to G$, $(k, a, n) \mapsto kan$ is a diffeomorphism (*see* Theorem 5.1, p. 234, [2]) and one has the Iwasawa decomposition

$$(2.2.3) \qquad\qquad G = KAN$$

of G.

The elements in K are clearly elliptic since Ad K is compact. Also the elements in A are hyperbolic and the elements in N are unipotent (since the maps $\mathfrak{a} \to A$, $\mathfrak{n} \to N$ defined by exp are bijective).

If $g \in G$ is arbitrary then its components in K, A and N will be denoted by $k\,(g)$, $a\,(g)$ and $n\,(g)$ and

$$(2.2.4) \qquad\qquad g = k\,(g)\,a\,(g)\,n\,(g)$$

is called the Iwasawa decomposition of g.

REMARK 2.2. — Both the complete multiplicative Jordan decomposition and the Iwasawa decomposition write g as a product of an elliptic, hyperbolic and unipotent element. One notes, however, that the Iwasawa decomposition is not invariantly defined. It obviously depends on the choice of K, A and N.

2.3. If Z is the center of G then Z is the kernel of Ad and hence by definition the elements of Z are elliptic. We recall the following fact.

PROPOSITION 2.3. — *An element $e \in G$ is elliptic if and only if it is conjugate to an element in* K. *In particular* $Z \subseteq K$. *Furthermore any element $k \in K$ is of the form $k = \exp y$ for some $y \in \mathfrak{k}$.*

Proof. — Let $G_1 = \mathrm{Ad}\, G$ be the adjoint group of G so that $G_1 \subseteq G_c$. If $K_1 = \mathrm{Ad}\, K$, $A_1 = \mathrm{Ad}\, A$ and $N_1 = \mathrm{Ad}\, N$ then $G_1 = K_1\, A_1\, N_1$ is clearly an Iwasawa decomposition of G_1. In particular G_1/K_1 is simply connected since it is diffeomorphic to $A_1 \times N_1$ (*see* Theorem 5.1, p. 234, [2]). But then if $K_* = \mathrm{Ad}^{-1}\, K_1$ one has $G/K_* \cong G_1/K_1$ is simply connected. Thus K_* is connected. But the identity component of K_* is clearly K. Thus $K_* = K$.

We have already remarked that the elements of K are elliptic (since K_1 is compact) and hence any element conjugate to an element in K is elliptic. Now let $e \in G$ be any elliptic element. Then clearly the closure L_1 of the subgroup of G_1 generated by Ad e is compact. But then L_1

is conjugate to a subgroup of K_1. Thus e is conjugate to an element $k \in G$ such that $\operatorname{Ad} k \in K_1$. But then $k \in K_* = K$. Hence an element is elliptic if and only if it is conjugate to an element in K. Now $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{c}$ where $\mathfrak{c}$ is the center of $\mathfrak{k}$ and $\mathfrak{k}_0 = [\mathfrak{k}, \mathfrak{k}]$ is semi-simple. But if K_0 and C are the connected subgroups of G corresponding to $\mathfrak{k}_0$ and $\mathfrak{c}$ then $K = K_0 C$ since $K_0 C$ is clearly a Lie subgroup whose Lie algebra is $\mathfrak{k}$. Thus if $k \in K$ we may write $k = k_0 c$ where $k_0 \in K_0$ and $c \in C$. But K_0 is compact since $\mathfrak{k}_0$ is compact semi-simple. Thus $k_0 = \exp y_0$ for some $y_0 \in \mathfrak{k}_0$. Also $c = \exp z$ for some $z \in \mathfrak{c}_0$ since $\mathfrak{c}_0$ is abelian. Thus $k = \exp y_0 \exp z = \exp y$ where $y = y_0 + z \in \mathfrak{k}$, since $[y_0, z] = 0$.

<div align="right">Q. E. D.</div>

2.4. Now the Weyl group W associated with $(\mathfrak{a}, \mathfrak{g})$ is the finite group defined as the quotient of the normalizer of A in K modulo the centralizer of A in K. The Weyl group W naturally operates in $\mathfrak{a}$ and A and in such a manner that the isomorphism $\mathfrak{a} \to A$ defined by exp is a W-isomorphism.

Now for each real semi-simple element $x \in \mathfrak{g}$ (resp. hyperbolic element $h \in G$) let $w(x)$ [resp. $W(h)$] be the set of all elements in $\mathfrak{a}$ (resp. A) which are conjugate to x (resp. h).

PROPOSITION 2.4. — *An element $x \in \mathfrak{g}$ is real semi-simple if and only if it is conjugate to an element in $\mathfrak{a}$. Moreover in such a case $w(x)$ is a single W-orbit in $\mathfrak{a}$. Similarly an element $h \in G$ is hyperbolic if and only if it is conjugate to an element in A and in such a case $W(h)$ is a single W-orbit in A.*

Also if x is real semi-simple then $W(\exp x) = \exp(w(x))$.

Proof. — Since exp sets up a bijection between the set of real semi-simple elements in $\mathfrak{g}$ and hyperbolic elements in G and since exp commutes with conjugation it suffices only to prove that if x is hyperbolic then $w(x)$ is a single W-orbit.

Let x be real semi-simple. Since $\operatorname{ad} x$ is diagonalizable x lies in a Cartan subalgebra of $\mathfrak{g}$. But since the eigenvalues of $\operatorname{ad} x$ are real x lies in the vector part of the Cartan subalgebra. But the vector part of any Cartan subalgebra is conjugate to a subalgebra of $\mathfrak{a}$ (*see* e. g. Theorem 2 (2), p. 383, [13]). Thus x is conjugate to an element $y \in \mathfrak{a}$.

Now assume $z, y \in \mathfrak{a}$ are (Ad G) conjugate. We have only to show that they are W-conjugate. But for some $g \in G$, $\operatorname{Ad} g(z) = y$. Let $g = kan$ be the Iwasawa decomposition of g. Then $\operatorname{Ad} an(z) = z + u$ where $u \in \mathfrak{n}$ since A is abelian and A normalizes N. But then $\operatorname{Ad} k(z + u) = y$. But $z, y \in \mathfrak{p}$ and $\mathfrak{p}$ is invariant under Ad K, and $\mathfrak{p} \cap \mathfrak{n} = 0$. Thus $u = 0$. Hence $\operatorname{Ad} k(z) = y$ so that z and y are Ad K-conjugate. Thus any

polynomial function on $\mathfrak{p}$ which is invariant under Ad K takes the same values on z and y. But then any polynomial function on $\mathfrak{a}$ invariant under W also takes the same values on z and y (*see* Theorem 6.10, p. 430, [2]). Thus z and y are W-conjugate, say, by the argument in Lemma 9.2 (p. 1029 in [6]).

<div align="right">Q. E. D.</div>

Now extend the definition of W (h) from hyperbolic elements to all elements as follows : For any $g \in G$ let W $(g) =$ W $(h(g))$ where we recall, [*see* (2.1.5)], that $h(g)$ is the hyperbolic component of g. Thus we have associated to an arbitrary element $g \in G$ a single W-orbit in A.

REMARK 2.4. — Obviously W $(g) =$ W (f) if f, $g \in G$ are conjugate. *See* Remark 2.1.

2.5. Propositions 2.3 and 2.4 dealt respectively with elliptic and hyperbolic elements. An analogous statement for unipotent elements is that an element is unipotent if and only if it is conjugate to an element in N. More generally one has

PROPOSITION 2.5. — *Let* $g \in G$. *Then* g *is conjugate to an element in* AN *if and only if* $e(g) = 1$ [*see* (2.1.5) *for the definition of the elliptic part* $e(g)$ *of* g]. *Moreover in such a case there exists* $k \in K$ *such that* $kgk^{-1} \in AN$.

Furthermore if g *is conjugate to* $an \in AN$ *where* $a \in A$, $n \in N$ *then* W $(g) =$ W $(a) =$ W$.a$, *the Weyl group orbit of* a.

[REMARK 2.5. — *Note* a *is not necessarily equal to* $h(an)$.]

Proof. — Let S $=$ AN so that S is solvable and its Lie algebra is $\mathfrak{s} = \mathfrak{a} + \mathfrak{n}$.

Now let $f \in S$ so that $f =$ an where $a \in A$, $n \in N$. The decomposition $f =$ an is of course the Iwasawa decomposition of f. Let $f = ehu$ be the complete multiplicative Jordan decomposition of f. We wish to show that $e = 1$, $h \in S$ and $u \in N$. Towards this end define subspaces $\mathfrak{g}_i \subseteq \mathfrak{g}$, $i = 0, 1, \ldots$ inductively as follows : Let $\mathfrak{g}_0 = \mathfrak{g}$ and if $\mathfrak{g}_i$ has been defined let $\mathfrak{g}_{i+1} = [\mathfrak{n}, \mathfrak{g}_i]$ so that $\mathfrak{g}_{k+1} = 0$ for some minimal k. Now if $g \in G$ is in the normalizer of N then $\mathfrak{g}_i$ is clearly stable under Ad g for all i and Ad g defines an operator on $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ and hence an operator which we denote by $\widetilde{\text{Ad}}\ g$ on $\overset{k}{\underset{i=0}{\bigoplus}}\ \mathfrak{g}_i/\mathfrak{g}_{i+1} = \tilde{\mathfrak{g}}$. Furthermore the eigenvalues of $\widetilde{\text{Ad}}\ g$ on $\mathfrak{g}$ are the same as the eigenvalues of Ad g. But now f, a, n are in the normalizer of N and $\widetilde{\text{Ad}}\ n = 1$. Thus $\widetilde{\text{Ad}}\ f = \widetilde{\text{Ad}}\ a$. But the eigenvalues of Ad a are positive. Hence the eigenvalues of $\widetilde{\text{Ad}}\ f$ and $\widetilde{\text{Ad}}\ a$ are the same and consequently the eigenvalues of Ad f are positive. But now if Ad $f = e_1 h_1 u_1$ is the decomposition of Ad f defined as

in (2.1.1) (where $g = f$) it follows that $e_1 = 1$. Thus Ad $f = h_1 u_1$ is the multiplicative Jordan decomposition of Ad f. Hence h_1 and u_1 are polynomials in Ad f. But $\mathfrak{n}$ is stable under Ad f since $f = an$. Thus $\mathfrak{n}$ is stable under h_1 and u_1. But, if, as in the proof of Proposition 2.1, x and z are respectively the unique real and nilpotent elements in $\mathfrak{g}$ such that $h_1 = \exp \operatorname{ad} x$, $u_1 = \exp \operatorname{ad} z$ then clearly $\operatorname{ad} x$ and $\operatorname{ad} z$ are respectively polynomials in h_1 and u_1 and hence it follows that x and z are in the normalizer of $\mathfrak{n}$. But now we recall that $\mathfrak{n}$ (see § 2.2) is the sum of the eigenspaces of $\operatorname{ad} x_0$ belonging to the positive eigenvalues of $\operatorname{ad} x_0$. If $\mathfrak{n}^-$ is the corresponding sum for the negative eigenvalues then $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{n}^- \oplus \mathfrak{g}^{x_0}$ is a linear direct sum where $\mathfrak{g}^{x_0}$ is the centralizer of x_0. But $\mathfrak{g}^{x_0} = \mathfrak{a} + \mathfrak{m}$ where $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ (see e. g. Proposition 8, p. 771, in [8]). Clearly $\mathfrak{g}^{x_0}$ is in the normalizer of $\mathfrak{n}$. In fact one recalls that the normalizer $\mathfrak{b}$ of $\mathfrak{n}$ in $\mathfrak{g}$ is given by

(2.5.1) $$\mathfrak{b} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

For this it is enough to note that $\mathfrak{n}^- \cap \mathfrak{b} = 0$. But if we embed $\mathfrak{a}$ in a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{a} + \mathfrak{m}$ then $\mathfrak{n}$ is spanned by root vectors e_φ for some set R of roots and $\mathfrak{n}^-$ is spanned by root vectors for the roots in $-$ R. But since $0 \neq [e_\varphi, e_{-\varphi}] \in \mathfrak{h}$ and $[e_\varphi, e_{-\psi}] \in \mathfrak{h}^\perp$, where $\mathfrak{h}^\perp$ is the orthocomplement of $\mathfrak{h}$ in $\mathfrak{g}$ for ψ, $\varphi \in$ R, $\varphi \neq \psi$, it follows that $\mathfrak{n}^- \cap \mathfrak{b} = 0$.

Thus by (2.5.1) x, $z \in \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Let

$$x = x_1 + x_2 + x_3 \quad \text{and} \quad z = z_1 + z_2 + z_3$$

be the respective components in $\mathfrak{m}$, $\mathfrak{a}$ and $\mathfrak{n}$. Also for any $y \in \mathfrak{b}$ let $\widetilde{\operatorname{ad}}\, y$ be the operator on $\tilde{\mathfrak{g}} = \sum \mathfrak{g}_i/\mathfrak{g}_{i+1}$ defined by $\operatorname{ad} y$ in a manner similar to the definition of $\widetilde{\operatorname{Ad}}\, g$ for any $g \in$ G in the normalizer of N. Clearly the eigenvalues of $\widetilde{\operatorname{ad}}\, y$ are the same as the eigenvalues of $\operatorname{ad} y$. But now $\widetilde{\operatorname{ad}}\, \mathfrak{n} = 0$ so that $\widetilde{\operatorname{ad}}\, z = \widetilde{\operatorname{ad}}\, z_1 + \widetilde{\operatorname{ad}}\, z_2$. But $[\mathfrak{m}, \mathfrak{a}] = 0$ so that $\widetilde{\operatorname{ad}}\, z_1$ and $\widetilde{\operatorname{ad}}\, z_2$ commute. However $z_1 \in \mathfrak{m} \subseteq \mathfrak{k}$ so that the eigenvalues of $\widetilde{\operatorname{ad}}\, z_1$ are pure imaginary and $z_2 \in \mathfrak{a}$ so that the eigenvalues of $\widetilde{\operatorname{ad}}\, z_2$ are real. But the eigenvalues of $\operatorname{ad} z$ and hence of $\widetilde{\operatorname{ad}}\, z$ are zero. Thus

$$\widetilde{\operatorname{ad}}\, z_1 = \widetilde{\operatorname{ad}}\, z_2 = 0$$

since $\operatorname{ad} z_1$ and $\operatorname{ad} z_2$ are diagonalizable. Thus $z_1 = z_2 = 0$ and hence $z \in \mathfrak{n}$. But again since $\widetilde{\operatorname{ad}}\, \mathfrak{n} = 0$ one has $\widetilde{\operatorname{ad}}\, x = \widetilde{\operatorname{ad}}\, x_1 + \widetilde{\operatorname{ad}}\, x_2$. Also $\widetilde{\operatorname{ad}}\, x_1$ and $\widetilde{\operatorname{ad}}\, x_2$ commute, since $[\mathfrak{m}, \mathfrak{a}] = 0$, and the eigenvalues of $\widetilde{\operatorname{ad}}\, x_1$ and $\widetilde{\operatorname{ad}}\, x_2$ are, respectively, pure imaginary and real. Since the eigenvalues of $\widetilde{\operatorname{ad}}\, x$ are real this implies that $\widetilde{\operatorname{ad}}\, x_1 = 0$ and also $x_1 = 0$

since $\mathrm{ad}\, x_1$ is diagonalizable. Thus $x \in \mathfrak{a} + \mathfrak{n}$. But now the hyperbolic and unipotent components h and u of f are given by $h = \exp x$ and $u = \exp z$ [see (2.1.3)]. Thus $h \in AN$ and $u \in N$ and hence $hu = a'\, n'$ where $a' \in A$, $n' \in N$. But $f = ehu = ea'\, n'$. However $\mathrm{Ad}\, e = e_1 = 1$ as shown above. Thus $e \in Z$, the center of G. But then $e \in K$ by Proposition 2.3. Hence $f = ea'\, n'$ is the Iwasawa decomposition of $f = an$. By uniqueness one has $e = 1$ or

$$(2.5.2) \qquad\qquad\qquad e\,(f) = 1.$$

Thus $f = hu = \exp x \exp z = \exp(x + z)$ by (2.1.2).

But now $x \in \mathfrak{a} + \mathfrak{n} = \mathfrak{s}$ is a real semi-simple element. Thus x lies in a Cartan subalgebra of $\mathfrak{s}$. But $\mathfrak{a}$ is clearly a Cartan subalgebra of $\mathfrak{s}$ [since (see § 2.2) the existence of $x_0 \in \mathfrak{a}$ such that $\mathrm{ad}\, x_0$ is non-singular on $\mathfrak{n}$ shows that $\mathfrak{a}$ is its own normalizer in $\mathfrak{s}$]. However any two Cartan subalgebras of a solvable Lie algebra are conjugate (see e. g. [5], Theorem 1, p. 58). Thus there exists $b \in AN$ such that $\mathrm{Ad}\, b\,(x) = x_2 \in \mathfrak{a}$. Let $z_2 = \mathrm{Ad}\, b\,(z)$. But then $z_2 \in \mathfrak{n}$ since $\mathfrak{n}$ is an ideal in $\mathfrak{a} + \mathfrak{n}$. But $[x_2, z_2] = 0$ by (2.1.2) and hence if $a_2 = \exp x_2 \in A$ and $n_2 = \exp z_2$ then $bfb^{-1} = a_2\, n_2$ is both the Iwasawa and the complete multiplicative Jordan decomposition of bfb^{-1}. Thus (see Remark 2.4) one has $W\,(f) = W\,(a_2) = W.a_2$, the W-orbit of a_2. We assert that $a_2 = a$. Indeed $f = an$ so that $bfb^{-1} = bab^{-1}\, bnb^{-1}$. But $bab^{-1} = an_3$ where $n_3 \in N$ (since $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$). Thus $bfb^{-1} = an_4$ where $n_3\, bnb^{-1} = n_4 \in N$. Hence $a_2\, n_2 = an_4$ so that $a_2 = a$. Hence $W\,(f) = W.a$.

To finish the proof we have only to show that if $g \in G$ is any element such that $e\,(g) = 1$ then g is K-conjugate to an element in AN. Let $g = hu$ be the complete multiplicative Jordan decomposition of g and let $x = \log h$, $z = \log u$ so that $[x, z] = 0$ by (2.1.2). But then z is in the centralizer, $\mathfrak{g}^x$, of x. But $\mathfrak{g}^x$ is reductive (see e. g. § 3, p. 352 in [7]). Furthermore z is not only nilpotent in $\mathfrak{g}$ but also in $\mathfrak{g}^x$ which means that it is a nilpotent element in the semi-simple Lie algebra $[\mathfrak{g}^x, \mathfrak{g}^x]$ (see e. g. (3.1.3) p. 352, [1]). By the Jacobson-Morosov theorem there exists an S-triple (y, z, w) in $\mathfrak{g}^x$ in the notation of [6]. (See § 4, [6] p. 988.) In particular there exists a real semi-simple element y of $\mathfrak{g}$ contained in $\mathfrak{g}^x$ such that $[y, z] = z$. But x and y span a 2-dimensional abelian subalgebra $\mathfrak{d}$ whose elements are real semi-simple.

But then $\mathfrak{d}$ can be embedded in the vector part of a Cartan subalgebra of $\mathfrak{g}$ using the notation of [13]. However by Theorem 2 (2), p. 383 in [13] it follows that $\mathfrak{d}$ is conjugate, say by $\mathrm{Ad}\, f$, $f \in G$, to a subalgebra of $\mathfrak{a}$. Furthermore by applying an element of the Weyl group W, if necessary, we may assume that y is carried into the same Weyl chamber (see [1],

p. 5) as x_0 by Ad f.   But since $[y, z] = z$, i. e. z is an eigenvector for ad y corresponding to the eigenvalue 1 it follows that from the definition of $\mathfrak{n}$ that Ad $f(z) \in \mathfrak{n}$.   Thus Ad f carries both x and z into $\mathfrak{a} + \mathfrak{n}$.   Since $g = \exp(x + z)$ one has $fgf^{-1} \in$ AN.   However if $f^{-1} = kan$ is the Iwasawa decomposition of f^{-1} then clearly $k^{-1} gk \in$ AN since AN is obviously stable under Ad an.   Q. E. D.

3. The partial ordering in G and the spectral radius $| \pi_\lambda(g) |$, $g \in G$, $\lambda \in \hat{G}$.

3.1. For any real semi-simple element $x \in \mathfrak{g}$ let $\mathfrak{a}(x)$ be the convex hull of the Weyl group orbit $w(x) \subseteq \mathfrak{a}$.   Thus $\mathfrak{a}(x)$ is the compact convex subset of $\mathfrak{a}$ having $w(x)$ as its subset of extremal points.

Now, group-wise, for any $g \in G$ let A $(g) \subseteq$ A be the compact subset defined by putting

(3.1.1)   A $(g) = \exp \mathfrak{a} (\log h (g))$

where we recall $h(g)$ is the hyperbolic component of g [see (2.1.5)].   In particular note that W $(g) \subseteq$ A (g) [and in a multiplicative sense A (g) is the " convex hull " of W (g)].

REMARK 3.1. — Note the sets $\mathfrak{a}(x) \subseteq \mathfrak{a}$ for x real semi-simple or A $(g) \subseteq$ A for $g \in G$ arbitrary, are invariant under the Weyl group.

The following definition is different from, but was inspired by a definition made by Colin Thompson for GL $(n, \mathbf{C})$ in [14].   His definition simplified a previous one made by A. Lenard in [9].   Thompson's definition has nothing to do with convexity and is based on the polar decomposition.   For SL $(n, \mathbf{C})$ both agree on P.   See section 4.2.   This will be clearer as a consequence of Theorem 3.1 below and the Horn-Thompson Theorem, p. 470 in [14].   The definition here although defined in terms of A is easily seen to be independent of the maximal " $\mathbf{R}$-split torus " A.   The invariance of the definition is nevertheless more cogently illustrated by Theorem 3.1.

Given f, $g \in G$ we say that $g \geqslant f$ if A $(f) \subseteq$ A (g) [or equivalently if W $(f) \subseteq$ A (g)].   This obviously defines a partial ordering on G.

REMARK 3.1.1. — One should note that the partial order is not necessarily the same as the partial order on G that would be induced by a possible embedding of G in SL $(n, \mathbf{C})$.   Indeed it follows immediately from Remark 3.1.2 that if G $\subseteq$ G$'$ where G$'$ is also semi-simple and f, $g \in G$ then if $g \geqslant f$ in G one also has $g \geqslant f$ in G$'$.   It is the converse which may be false.   Indeed take for example the case where the $\mathbf{R}$-split rank (dim A) of G and G$'$ are the same, so that A is a maximal $\mathbf{R}$-split

torus in both G and G', but that the Weyl group W' of (A, G') is larger than W. If $\sigma \in W' - W$ and, say, $a = \exp x_0$ (*see* § 2.2) then a, $\sigma a \in A \subseteq G$ and $a \geqq \sigma a$ in G' but $a \neq \sigma a$ in G. It follows therefore that if $G \subseteq G' \subseteq SL(n, \mathbf{C})$ for some n then the partial ordering on G is not the one induced on G by that on $SL(n, \mathbf{C})$.

Now let $\hat{G}$ be an index set for the set of all equivalence classes of irreducible finite dimensional representations of G.

Also for any $\lambda \in \hat{G}$ let

(3.1.2) $$\pi_\lambda : \quad G \to \operatorname{Aut} V_\lambda$$

be a fixed representation in the class corresponding to λ.

Now for any $g \in G$ and let $\lambda \in \hat{G}$ let $|\pi_\lambda(g)|$ be the maximum of the absolute values of the eigenvalues of $\pi_\lambda(g)$. That is (as Helgason reminded me) $|\pi_\lambda(g)|$ is the spectral radius of $\pi_\lambda(g)$.

THEOREM 3.1. — *Let* f, $g \in G$. *Then* $g \geqq f$ *if and only if*

(3.1.3) $$|\pi_\lambda(g)| \geqq |\pi_\lambda(f)|$$

for all $\lambda \in \hat{G}$.

REMARK 3.1.2. — Using say, complete reducibility, note that one may substitute for the set of irreducible representations π_λ, $\lambda \in \hat{G}$, in Theorem 3.1 the collection of equivalence classes of *all* finite dimensional representations, irreducible or not.

3.2. Before proving Theorem 3.1 (*see* § 3.5) we will need notation which will be used in the proof and elsewhere. Let $\mathfrak{a}'$ be the real dual to $\mathfrak{a}$ and let $\Gamma \subseteq \mathfrak{a}'$ be the set of restricted roots. That is Γ is the set of non-zero weights for the adjoint action of $\mathfrak{a}$ on $\mathfrak{g}$. An account of the theory of restricted roots may be found in [1]. The set of positive restricted roots Γ_+ corresponding to $\mathfrak{u}$ may be given by

(3.2.1) $$\Gamma_+ = \{ \gamma \in \Gamma \mid \langle \gamma, x_0 \rangle \geqq 0 \}$$

where x_0 is fixed as in section 2.2. Let $\Sigma = \{ \beta_1, \ldots, \beta_k \} \subseteq \Gamma_+$ be the set of simple, positive restricted roots so that every $\gamma \in \Gamma$ is an integral combination of the β_i and Σ is a basis of $\mathfrak{a}'$.

Now the Killing form on $\mathfrak{g}$ induces a positive definite bilinear form on $\mathfrak{a}$ and by duality a positive definite bilinear form on $\mathfrak{a}'$. Then one knows that for any γ, $\delta \in \Gamma$, $\frac{2(\delta, \gamma)}{(\gamma, \gamma)}$ is an integer. (*See* § 2.4, [1]). That is, if $x_\gamma \in \mathfrak{a}$ is an element corresponding to $\frac{2\gamma}{(\gamma, \gamma)}$ under the isomorphism $\mathfrak{a} \to \mathfrak{a}'$ defined by the Killing form then $\langle \delta, x_\gamma \rangle \in \mathbf{Z}$ for all $\delta \in \Gamma$. In particular $\langle \gamma, x_\gamma \rangle = 2$.

Moreover for any $\gamma \in \Gamma$ there is an element $\tau(\gamma) \in W$ (the reflection defined by γ) such that for any $x \in \mathfrak{a}$,

(3.2.2) $$\tau(\gamma) x = x - \langle \gamma, x \rangle x_\gamma$$

and W is generated by these reflections.

Now let

(3.2.3) $$\mathfrak{a}_+ = \{ x \in \mathfrak{a} \mid \langle \beta_i, x \rangle \geqq 0 \text{ for } i = 1, 2, \ldots, r \}$$

so that $\mathfrak{a}_+$ is a Weyl chamber in $\mathfrak{a}$ (*see* p. 5 in [8]). One knows that any $x \in \mathfrak{a}$ is W-conjugate to a unique element in $\mathfrak{a}_+$ (*see* e. g. p. 52 in [4]).

REMARK 3.2. — Note that by Proposition 2.4 that any real semi-simple element $y \in \mathfrak{g}$ is conjugate to a unique element in $\mathfrak{a}_+$.

The subset $\mathfrak{a}_+$ is a cone in $\mathfrak{a}$. We define another cone $\mathfrak{a}_p$ in $\mathfrak{a}$ as follows. Let $x_i = x_{\beta_i}$, $i = i, 2, \ldots, k$ and put

(3.2.4) $$\mathfrak{a}_p = \left\{ x \in \mathfrak{a} \ \middle| \ x = \sum_{i=1}^{k} r_i x_i, \ r_i \geqq 0 \right\}$$

so that $\mathfrak{a}_p$ is the cone generated by the x_i. Since every positive root is a non-negative combination of positive simple roots one clearly has $x_\gamma \in \mathfrak{a}_p$ for any $\gamma \in \Gamma_+$.

Now by contragredience the Weyl group W also operates on the dual $\mathfrak{a}'$. Thus if $\mathfrak{a}'_+$ corresponds to $\mathfrak{a}_+$ under the Killing form induced isomorphism $\mathfrak{a} \to \mathfrak{a}'$ then $\mathfrak{a}'_+$ is a fundamental domain for the action of W in $\mathfrak{a}'$. The cone $\mathfrak{a}'_+$ may also be given by

(3.2.5) $$\mathfrak{a}'_+ = \{ \lambda \in \mathfrak{a}' \mid \langle \lambda, x \rangle \geqq 0 \text{ for all } x \in \mathfrak{a}_p \}.$$

The cone $\mathfrak{a}'_p \subseteq \mathfrak{a}'$ corresponding to $\mathfrak{a}_p$, on the other hand, is given by

(3.2.6) $$\mathfrak{a}'_p = \left\{ \lambda \in \mathfrak{a}' \ \middle| \ \lambda = \sum_{i=1}^{k} r_i \beta_i, \ r_i \geqq 0 \right\}$$

and one has $\gamma \in \mathfrak{a}'_p$ for any $\gamma \in \Gamma_+$.

Finally a sequence of positive restricted roots $\gamma_1, \gamma_2, \ldots, \gamma_n$ will be called a strongly positive sequence (of length n) if for any $x \in \mathfrak{a}_+$ one has

$$\langle \gamma_i, \tau(\gamma_j) \tau(\gamma_{j-1}) \ldots \tau(\gamma_1) x \rangle \geqq 0 \quad \text{for all } 1 \leqq j \leqq i \leqq n.$$

Any element $1 \neq \sigma \in W$ may be written as a product $\sigma = \tau(\gamma_n) \ldots \tau(\gamma_1)$ where $\gamma_1, \ldots, \gamma_n$ is a strongly positive sequence. (One joins the interior of $\mathfrak{a}_+$ with the interior of $\sigma \mathfrak{a}_+$ by a suitable line segment and considers the root hyperplanes that are crossed in going from $\mathfrak{a}_+$ to $\sigma \mathfrak{a}_+$ along the segment.) Now one has

LEMMA 3.2. — *Let* $x \in \mathfrak{a}_+$. *Then for any* $\sigma \in W$ *one has* $x - \sigma x \in \mathfrak{a}_p$. *Furthermore for any* $\lambda \in \mathfrak{a}'_+$ *one has*

$$\langle \lambda, x \rangle \geqq \langle \lambda, \sigma x \rangle.$$

Proof. — We may assume $\sigma \neq 1$. Let $\gamma_1, \ldots, \gamma_n$ be a strongly positive sequence such that $\sigma = \tau(\gamma_n) \ldots \tau(\gamma_1)$. Put

$$x_i = \tau(\gamma_i) \tau(\gamma_{i-1}) \ldots \tau(\gamma_1) x \quad \text{and} \quad x_0 = x.$$

Then $x_i = x_{i-1} - \langle \gamma_i, x_{i-1} \rangle x_{\gamma_i}$. But $\langle \gamma_i, x_{i-1} \rangle \geqq 0$. Thus $x_{i-1} - x_i \in \mathfrak{a}_p$. Since $\mathfrak{a}_p$ is a cone this implies $x - \sigma x \in \mathfrak{a}_p$. But now if $\lambda \in \mathfrak{a}'_+$ one has $\langle \lambda, x \rangle \geqq \langle \lambda, \sigma x \rangle$ by (3.2.5).

<div align="right">Q. E. D.</div>

3.3. Let ν_j, $j = 1, 2, \ldots, k$ be the basis of $\mathfrak{a}'$ such that $\langle \nu_j, x_i \rangle = \delta_{ij}$. Thus $\nu_j \in \mathfrak{a}'_+$ by (3.2.5) and in fact $\mathfrak{a}'_+$ is the cone generated by the ν_j. Now recall that $\mathfrak{a}(x)$, for any $x \in \mathfrak{a}$, is the convex hull of the W-orbit, $W.x$, of x.

LEMMA 3.3. — *Let* $x \in \mathfrak{a}_+$. *Then for any* $y \in \mathfrak{a}$:

(1) $y \in \mathfrak{a}(x)$ *if and only if* $x - \sigma y \in \mathfrak{a}_p$ *for all* $\sigma \in W$.

(2) *If* $y \in \mathfrak{a}_+$ *then* $y \in \mathfrak{a}(x)$ *if and only if* $x - y \in \mathfrak{a}_p$.

Proof. — Since the orbit $W.y$ always meets $\mathfrak{a}_+$ and since $y - \sigma y \in \mathfrak{a}_p$ for any $y \in \mathfrak{a}_+$ and $\sigma \in W$, by Lemma 3.2, and since $\mathfrak{a}(x)$ is stable under W it suffices only to prove (2). Assume $y \in \mathfrak{a}(x)$. Then, by a property of extreme points of convex sets, given any $\nu \in \mathfrak{a}'$ there exists $\sigma \in W$ such that $\langle \nu, \sigma x \rangle \geqq \langle \nu, y \rangle$. But if $\nu = \nu_j \in \mathfrak{a}'_+$ one has $\langle \nu_j, x \rangle \underset{h}{\geqq} \langle \nu_j, \sigma x \rangle$ by Lemma 3.2. Thus $\langle \nu_j, x - y \rangle \geqq 0$. But if $x - y = \sum_{i=1}^{h} r_i x_i$ then $\langle \nu_j, x - y \rangle = r_j$. Hence $r_j \geqq 0$ for all j so that $x - y \in \mathfrak{a}_p$.

Conversely assume $y \in \mathfrak{a}_+$ and $x - y \in \mathfrak{a}_p$. To prove that $y \in \mathfrak{a}(x)$ it suffices to show that y cannot be separated from $\mathfrak{a}(x)$ by a hyperplane or equivalently it suffices to show that for any $\nu \in \mathfrak{a}'$ there exists $\sigma \in W$ such that $\langle \nu, \sigma x \rangle \geqq \langle \nu, y \rangle$. But now given any $\nu \in \mathfrak{a}'$ there exists $\sigma \in W$ such that $\sigma^{-1} \nu \in \mathfrak{a}'_+$. But now if $x - y \in \mathfrak{a}_p$ then $x - \sigma^{-1} y \in \mathfrak{a}_p$ since $y - \sigma^{-1} y \in \mathfrak{a}_p$ by Lemma 3.2. Thus $\langle \sigma^{-1} \nu, x - \sigma^{-1} y \rangle \geqq 0$ by (3.2.5). Applying σ one has $\langle \nu, \sigma x - y \rangle \geqq 0$ or $\langle \nu, \sigma x \rangle \geqq \langle \nu, y \rangle$.

<div align="right">Q. E. D.</div>

3.4. The proof of Theorem 3.1 reduces to the case of hyperbolic elements because of the following Proposition. Proposition 3.4 asserts that the spectral radius $|\pi_\lambda(g)|$ of $\pi_\lambda(g)$ depends only on the hyperbolic part of g.

Proposition 3.4. — *Let* $\lambda \in \hat{G}$ *and let* $g \in G$. *Let* $h = h(g)$ *be the hyperbolic component of* g *[see* (2.1.5)*]. Then all the eigenvalues of* $\pi_\lambda(h)$ *are positive and* $|\pi_\lambda(g)|$ *is the maximal eigenvalue of* $\pi_\lambda(h)$.

Proof. — Let π_λ also denote the (differential) representation of $\mathfrak{g}$ and $\mathfrak{g}_{\mathbf{C}}$ defined by the group representation π_λ. If $x \in \mathfrak{g}$ is nilpotent then $\pi_\lambda(z)$ is a nilpotent operator. This follows, for example, from the representation theory of a TDS (*see* e. g. [5], § 2.1) and the Jacobson-Morosov theorem which asserts that any nilpotent element lies in a TDS of $\mathfrak{g}$. Thus if $u \in G$ is unipotent then all the eigenvalues of $\pi_\lambda(u)$ are equal to 1. If $x \in \mathfrak{g}$ is such that $\operatorname{ad} x$ is diagonalizable then $\pi_\lambda(x)$ is diagonalizable, using the theory of weights, since x can be embedded in a Cartan subalgebra $\mathfrak{h}_{\mathbf{C}}$ of $\mathfrak{g}_{\mathbf{C}}$. Thus $\pi_\lambda(g)$ is diagonalizable if g is either elliptic or hyperbolic by Proposition 2.3. But if x is real semi-simple then all the roots of $(\mathfrak{h}_{\mathbf{C}}, \mathfrak{g}_{\mathbf{C}})$ take real values on x. But since the weights of π_λ are rational combinations of the roots, it follows that the eigenvalues of $\pi_\lambda(x)$ are real and hence the eigenvalues of $\pi_\lambda(h)$ are positive for any hyperbolic element $h \in G$. Similarly if $x \in \mathfrak{k}$ then all the roots take pure imaginary eigenvalues on x and hence the same is true of the weights of π_λ. Thus $\pi_\lambda(k)$ has eigenvalues of norm 1 for any $k \in K$ by Proposition 2.3. But then, also by Proposition 2.3, $\pi_\lambda(e)$ has eigenvalues of norm 1 for all elliptic elements in G.

Now let $g = ehu$ be the complete multiplicative Jordan decomposition of g [*see* (2.1.5)], so that $\pi_\lambda(g) = \pi_\lambda(e)\,\pi_\lambda(h)\,\pi_\lambda(u)$. Since the three operators on the right commute the eigenvalues of $\pi_\lambda(g)$ use just products of the eigenvalues of $\pi_\lambda(e)$, $\pi_\lambda(h)$ and $\pi_\lambda(u)$ and hence it is clear that $|\pi_\lambda(g)|$ is the maximal eigenvalue of $\pi_\lambda(h)$.

<div align="right">Q. E. D.</div>

3.5. We now give the

Proof of Theorem 3.1. — Let g, $f \in G$ be arbitrary. Then there exists unique elements x, $y \in \mathfrak{a}_+$ such that x is conjugate to $\log h(g)$ and y is conjugate to $\log h(f)$. By Proposition 3.4 we have only to show that for any $\lambda \in \hat{G}$ the maximal eigenvalue of $\pi_\lambda(x)$ is not less than the maximal eigenvalue of $\pi_\lambda(y)$ if and only if $y \in \mathfrak{a}(x)$; or by Lemma 3.3 if and only if $x - y \in \mathfrak{a}_p$.

Now let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ and let $\mathfrak{h}_m$ be a maximal commutative subalgebra in $\mathfrak{m}$ so that $\mathfrak{h}_0 = \mathfrak{a} + \mathfrak{h}_m$ is a Cartan subalgebra of $\mathfrak{g}$ (*see* p. 221, [2]) and hence its complexification $\mathfrak{h}_{\mathbf{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbf{C}}$. Let $\mathfrak{h} = \mathfrak{a} + i\,\mathfrak{h}_m$ so that $\mathfrak{h}$ is the set of all real semi-simple elements (of $\mathfrak{g}_{\mathbf{C}}$) in $\mathfrak{h}_{\mathbf{C}}$. Let Δ be the set of roots of $(\mathfrak{h}_{\mathbf{C}}, \mathfrak{g}_{\mathbf{C}})$ so that $\Delta \subseteq \mathfrak{h}'$, the real dual of $\mathfrak{h}$. Furthermore if Δ_0 is the set of roots which vanish

on $\mathfrak{a}$ then the set of restrictions to $\mathfrak{a}$ of the roots in $\Delta - \Delta_0$ is just Γ (*see* § 2.4, p. 6, [1]). Furthermore we may choose a system of positive roots $\Delta_+ \subseteq \Delta$ so that x_0 [*see* (2.2.1)] is in the corresponding Weyl chamber, i. e. one has $\langle \varphi, x_0 \rangle \geqq 0$ for all $\varphi \in \Delta_+$. Thus any root in Δ_+ restricts either to zero or an element in Γ_+ as a linear functional on $\mathfrak{a}$.

Now let $\mathfrak{h}'_+ \subseteq \mathfrak{h}'$ be the co-Weyl chamber corresponding to Δ_+. Thus the elements in $\mathfrak{h}'_+$ are the dominant integral linear forms on $\mathfrak{h}$. In particular we may regard $\hat{G} \subseteq \mathfrak{h}'_+$ by using, as an index set for the classes of irreducible representations of G, the highest weights of the corresponding representation of $\mathfrak{g}_{\mathbf{C}}$.

But now if $\lambda \in \hat{G}$ and if $\Delta_\lambda \subseteq \mathfrak{h}'$ denotes the set of weights of the representation π_λ then the eigenvalues $\pi_\lambda(z)$ for any $z \in \mathfrak{a}$ are numbers of the form $\langle \mu, z \rangle$ where $\mu \in \Delta_\lambda$. However for any $\mu \in \Delta_\lambda$ one knows from representation theory that $\lambda - \mu$ is a sum of positive roots. Since positive roots restrict either to zero or elements of Γ_+ on $\mathfrak{a}$ it follows from (3.2.3) that

(3.5.1) $$\langle \lambda, z \rangle \geqq \langle \mu, z \rangle$$

for any $z \in \mathfrak{a}_+$. We assert first of all that (3.5.1) implies that

(3.5.2) $$\lambda \mid \mathfrak{a} \in \mathfrak{a}'_+.$$

Indeed the set of restrictions of Δ_λ to $\mathfrak{a}$ is clearly stable under the Weyl group W. Thus if $\lambda_1 = \lambda \mid \mathfrak{a}$ and $\sigma \in W$ is such that $\sigma \lambda_1 \in \mathfrak{a}'_+$ then there exists $\mu \in \Delta_\lambda$ such that $\mu \mid \mathfrak{a} = \sigma \lambda_1$. But for any $z \in \mathfrak{a}_+$,

$$\langle \mu, z \rangle = \langle \sigma \lambda_1, z \rangle \geqq \langle \lambda_1, z \rangle = \langle \lambda, z \rangle$$

by Lemma 3.2. But then one has $\langle \sigma \lambda_1, z \rangle = \langle \lambda_1, z \rangle$ by (3.5.1). Since this holds for all $z \in \mathfrak{a}_+$ this implies $\lambda_1 = \sigma \lambda_1 \in \mathfrak{a}'$ establishing (3.5.2).

Now to prove the theorem we have to show that the maximal eigenvalue of $\pi_\lambda(x)$ is not less than the maximal eigenvalue of $\pi_\lambda(y)$ if and only if $x - y \in \mathfrak{a}_p$. But by (3.5.1) the maximal eigenvalues involved are $\langle \lambda, x \rangle$ and $\langle \lambda, y \rangle$ respectively. Thus we have to show $\langle \lambda, x - y \rangle \geqq 0$ if and only if $x - y \in \mathfrak{a}_p$. But if $x - y \in \mathfrak{a}_p$ then $\langle \lambda, x - y \rangle \geqq 0$ by (3.2.5) and (3.5.2). Conversely assume

(3.5.3) $$\langle \lambda, x - y \rangle \geqq 0$$

for all $\lambda \in \hat{G}$.

We must prove that $x - y \in \mathfrak{a}_p$.

Let $\Pi = \{ \alpha_1, \ldots, \alpha_l \} \subseteq \Delta_+$ be the set of simple positive roots. Then the co-Weyl chamber $\mathfrak{h}'_+$ is the cone generated by $\lambda_1, \ldots, \lambda_l \in \mathfrak{h}'$ where

$\dfrac{2(\lambda_i,\,\alpha_j)}{(\alpha_j,\,\alpha_j)} = \delta_{ij}$, where the inner product on $\mathfrak{h}'$ is induced on $\mathfrak{h}'$ by the isomorphism $\mathfrak{h} \to \mathfrak{h}'$ defined by restricting the Killing form on $\mathfrak{g}_{\mathbf{C}}$ to h. Furthermore as one knows from representation theory the λ_i are the highest weights of the fundamental representations of a simply connected Lie group G_c^s having $\mathfrak{g}_{\mathbf{C}}$ as its Lie algebra (see e. g. [4], § 13.1, p. 67). But G_c^s has a finite center, say of order m. Thus the center is in the kernel of the irreducible representation of G_c^s having highest weight $m\,\lambda_i$. Hence $m\,\lambda_i$ is a highest weight of an irreducible representation of the adjoint group G_c. But Ad maps G into G_c. By taking the composition it follows therefore that $m\,\lambda_i \in \hat{G}$. Hence $\mathfrak{h}'_+$ is a cone generated by $\hat{G}$. Thus

$$(3.5.4) \qquad\qquad \langle \mu,\, x - y \rangle \geqq 0$$

for all $\mu \in \mathfrak{h}'_+$ by (3.5.3).

On the other hand an element $\mu \in \mathfrak{h}'$ clearly lies in $\mathfrak{h}'_+$ if and only if $(\mu,\,\alpha_i) \geqq 0$ for $i = 1, 2, \dots, l$. But now if $\nu_j \in \mathfrak{a}'$ is, as in section 3.3, the dual basis to the x_i then, in $\mathfrak{a}'$, one has $(\nu_j,\,\beta_i) \geqq 0$ for all $1 \leqq i$, $j \leqq k$ by definition of x_i. But now if we embed $\mathfrak{a}'$ in $\mathfrak{h}'$ by regarding the elements of $\mathfrak{a}'$ as having zero restrictions to $\mathfrak{h}_m$ then since $i\,\mathfrak{h}_m$ and $\mathfrak{a}$ are orthogonal with respect to the Killing form the inner product on $\mathfrak{a}'$ is just the restriction to $\mathfrak{a}'$ of the inner product on $\mathfrak{h}'$. Furthermore the map $\mathfrak{h}' \to \mathfrak{a}'$, defined by restriction to $\mathfrak{a}$, is just orthogonal projection. On the other hand one knows (see [1], § 2.8, p. 11) that any element in Π restricts either to zero or to an element in Σ. Thus $(\nu_j,\,\alpha_i) \geqq 0$ for $1 \leqq j \leqq k$, $1 \leqq i \leqq l$ and hence

$$(3.5.5) \qquad\qquad \nu_j \in \mathfrak{h}'_+ \qquad \text{or} \qquad \mathfrak{a}'_+ \subseteq \mathfrak{h}'_+.$$

But if $x - y = \displaystyle\sum_{i=1}^{k} r_i x_i$ then $r_j = (\nu_j,\, x - y) \geqq 0$ by (3.5.4). Thus $x - y \in \mathfrak{a}_p$ by (3.2.4). $\qquad$ Q. E. D.

4. The convexity theorem

4.1. Let $b \in A$. One of the questions that seems to arise quite often in doing analysis on semi-simple Lie groups or symmetric spaces (also in representation theory) is : what is the a-component (in the Iwasawa decomposition [see (2.2.4)]) when b is multiplied on the right by an element in K. That is, if $v \in K$ what is $a = a\,(bv)$ when we write

$$(4.1.1) \qquad\qquad bv = kan$$

for $k \in K$, $a \in A$, $n \in N$. In the notation of Harish-Chandra $\log a$ is written $H\,(bv)$. For example, the elementary spherical function φ_ν

corresponding to any $\nu \in \mathfrak{a}'_{\mathbb{C}}$ is determined by its restriction to A and its value at any $b \in A$ is given by

$$(4.1.2) \qquad\qquad \varphi_\nu(b) = \int_K e^{\langle \nu, \, \mathrm{H}(bv)\rangle} \, dv$$

(*see* e. g. p. 428, [2]).

Our main theorem here, Theorem 4.1, says that H (bv) lies in the convex hull of the Weyl group orbit of H (b) and that as v runs through K, H (bv) runs through *all* the elements in this convex set. That is, in the notation defined in section 3.1, one has

THEOREM 4.1. — *Let* $b \in A$ *be arbitrary. Then*

$$A(b) = \{\, a(bv) \mid v \in K \,\}.$$

The proof of Theorem 4.1 will be given after Lemma 4.8.

REMARK 4.1. — With regard to multiplication of b by elements in K, A, or N note that only for right multiplication by elements in K is there a difficulty in finding the a-component. (Because A normalizes N.)

4.2. As a first corollary we generalize Theorem 4.1 to the case where b is replaced by any element $g \in G$. Recall that $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a fixed Cartan decomposition of G. But since all the elements in $\mathfrak{p}$ are real semi-simple all the elements in

$$(4.2.1) \qquad\qquad P = \{\, p \in G \mid p = \exp x, x \in \mathfrak{p} \,\}$$

are hyperbolic. The set P is closed, diffeomorphic to $\mathfrak{p}$ by the exponential map and the map $P \times K \to G$, $(p, v) \mapsto pv$ is a diffeomorphism. In particular

$$(4.2.2) \qquad\qquad G = PK$$

and every element g can be uniquely written $g = pv$, the polar decomposition of g, and we write $p = p(g)$ and $v = v(g)$ (*see* e. g. [10], p. 155).

More explicitly, the Cartan involution θ of $\mathfrak{g}$ ($\theta = 1$ on $\mathfrak{k}$, $\theta = -1$ on $\mathfrak{p}$) induces an automorphism of G, $g \mapsto \theta(g)$. This is clear since θ induces an automorphism of a simply connected group G_s having $\mathfrak{g}$ as its Lie algebra. But $G = G_s/D$ where D is a central subgroup. However D is fixed by θ since by Proposition 2.3, D is in the subgroup corresponding to $\mathfrak{k}$. Thus θ induces an automorphism of G. For any $g \in G$ let

$$(4.2.3) \qquad\qquad g^* = \theta(g^{-1})$$

so that

$$(4.2.4) \qquad\qquad (fg)^* = g^* f^*.$$

In particular if $g = pv$ is the polar decomposition of g then

(4.2.5) $g^* = v^{-1} p.$

Now since the hyperbolic elements in G are in one-one correspondance with the real semi-simple elements in $\mathfrak{g}$ every hyperbolic element h has a unique hyperbolic square root $h^{1/2}$. In fact $h^{1/2} = \exp 1/2 \log h$. If $h \in P$ so is $h^{1/2}$. From (4.2.5) one has that for any $g \in G$, $gg^* \in P$ and

(4.2.6) $p(g) = (gg^*)^{1/2} \in P.$

Finally since the elements in $\mathfrak{p}$ are real semi-simple any element y in $\mathfrak{p}$ is conjugate to a unique element in $\mathfrak{a}_+$ by Proposition 2.4. Furthermore the argument in the proof of Proposition 2.4. shows in fact that y is K-conjugate to such an element. Thus if $A_+ = \exp \mathfrak{a}_+$ any element $p \in P$ can be

(4.2.7) $p = k \, dk^{-1}$

for a unique $d \in A_+$. But then by (4.2.2) one has the familiar product relation

(4.2.8) $G = KA_+ K$

so that any $g \in G$ can be written

(4.2.9) $g = k \, dv$

for $k \in K$, $d \in A_+$, $v \in K$.

REMARK 4.2. — The k and the v in (4.2.9) are not unique but the d is. In fact $g = k \, dk^{-1} kv$ so that $p(g) = k \, dk^{-1}$. Thus d is unique by (4.2.7). We write $d = d(g) \in A_+$.

Now as a corollary and a generalization of Theorem 4.1 one has the following. Note that the set in question $A(p) a(f)$ is still convex since it is just the translation of $A(p)$ by $a(f)$.

COROLLARY 4.2. — Let g, $f \in G$ be arbitrary. Then

$$A(p) . a(f) = \{ a(gvf) \mid v \in K \}$$

where $p = (gg^*)^{1/2}$, $A(p)$ is defined in section 3.1 and $a(g)$ is defined in section 2.2.

Proof. — Write $g = k_1 \, dk_2$ where $d = d(g) \in A_+$ and k_1, $k_2 \in K$. Then if $f = kan$ is the Iwasawa decomposition of f so that $a = a(f)$, one has $gvf = k_1 \, dk_2 \, vkan$. But as v runs over K so does $k_2 \, vk$ and hence if $dk_2 \, vk = k' \, a' \, n'$ is the Iwasawa decomposition then as v runs over K, a' runs over $A(d)$ by Theorem 4.1. But $dk_2 \, vka = k' \, a' \, an''$, where $n'' \in N$,

since A normalizes N. Thus $A(d) a = \{ a(gvf) \mid v \in K \}$. But $A(d) = A(p)$ by Remark 4.2. Also $a = a(f)$.

<div align="right">Q. E. D.</div>

4.3. Now since $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$ one knows that $\mathfrak{g}_u = \mathfrak{k} + i\mathfrak{p}$ is a compact real form of $\mathfrak{g}$. Thus for each $\lambda \in \hat{G}$ there exists a (by irreducibility) unique, up to scalar, Hilbert space structure on V_λ such that $\pi_\lambda(z)$ is skew-Hermitian for all $z \in \mathfrak{g}_u$. We will assume henceforth that V is given this structure. Thus for any $g \in G$ the operator norm $\| \pi_\lambda(g) \|$ is well defined. From standard properties of the operator norm one has that

(4.3.1) $$\| \pi_\lambda(g) \| \cdot \| \pi_\lambda(f) \| \geqq \| \pi_\lambda(gf) \|$$

for any $g, f \in G$. Also the operator norm is not less than the absolute value of any eigenvalue so that one always has the inequality

(4.3.2) $$\| \pi_\lambda(g) \| \geqq | \pi_\lambda(g) |$$

for any $g \in G$.

On the other hand since $\pi_\lambda(z)$ is skew-Hermitian for $z \in i\mathfrak{p}$ it follows that $\pi_\lambda(x)$ is Hermitian for $x \in \mathfrak{p}$ and hence $\pi_\lambda(p)$ is positive definite for any $p \in P$. For a positive definite operator, however, the spectral radius is the same as the operator norm so that one has

(4.3.3) $$| \pi_\lambda(p) | = \| \pi_\lambda(p) \|$$

for all $p \in P$. Thus under certain circumstances we can replace the spectral radius by the operator norm in Theorem 3.1 obtaining

PROPOSITION 4.3. — *Let $p \in P$ and $g \in G$ be arbitrary. If for every irreducible representation $\lambda \in \hat{G}$ one has*

$$\| \pi_\lambda(p) \| \geqq \| \pi_\lambda(g) \|$$

then $p \geqq g$ (see § 3.1).

Proof. — Under the assumption in the Proposition one has

$$| \pi_\lambda(p) | = \| \pi_\lambda(p) \| \geqq \| \pi_\lambda(g) \| \geqq | \pi_\lambda(g) |$$

for all $\lambda \in \hat{G}$ by (4.3.3) and (4.3.2). The result then follows from Theorem 3.1.

<div align="right">Q. E. D.</div>

But now $\pi_\lambda(z)$ is skew-Hermitian for any $z \in \mathfrak{k}$ and hence $\pi_\lambda(k)$ is unitary for any $k \in K$. Thus

(4.3.4) $$\| \pi_\lambda(k) \| = 1$$

for all $k \in K$.

Proposition 4.3 is applied to yield the following:

LEMMA 4.3. — *Let $p \in P$ and let $k,\ v \in K$ be arbitrary. Then, with respect to the partial ordering of section* 3.1,

$$p \geqq kpv.$$

Proof. — For any $\lambda \in \hat{G}$ one has

$$\| \pi_\lambda\ (p) \| \geqq \| \pi_\lambda\ (kpv) \|$$

by (4.3.1) since $\| \pi_\lambda\ (k) \| = \| \pi_\lambda\ (v) \| = 1$ by (4.3.4). The result then follows from Proposition 4.3.

We can now establish one half of Theorem 4.1, namely that

(4.3.5) $a\ (bv) \in A\ (b)$ for any $v \in K$.

Indeed we write the Iwasawa decomposition $bv = kan$. Then $k^{-1} bv = an$. But by Lemma 4.3 one has $b \geqq an$. That is $A\ (an) \subseteq A\ (b)$. But by Proposition 2.5 one has $A\ (an) = A\ (a)$. Thus $a\ (bv) = a \in A\ (b)$ proving (4.3.5).

4.4. We now use the Horn-Thompson result for $SL\ (2, \mathbf{R})$ (slightly modified with unipotent elements).

LEMMA 4.4 (Horn-Thompson). — *Assume $\mathfrak{g}$ is isomorphic to the Lie algebra of $SL\ (2, \mathbf{R})$. Let $b \in A,\ n \in N$. Then for any $a \in A\ (b)$ there exists k, $v \in K$ and $n' \in N$ such that*

$$kbnv = an'.$$

Proof. — Let G_1, as in section 2.3, be Ad G (the real adjoint group). We first observe the lemma is true for G if and only if it is true for G_1. Indeed AN maps bijectively onto its image by Ad (since $Z \subseteq K$). Thus, if the lemma is true in G, by applying Ad, it is true for G_1. Conversely if it is true for G_1 then in G it is immediate that k, v, n' exist so that $kbnv = zan'$ where z is in the center of G. But $z \in K$ by Proposition 2.3 so that one replaces k by $z^{-1} k$.

But now since G and $SL\ (2, \mathbf{R})$ having isomorphic adjoint groups it is enough to prove the result assuming that $G = SL\ (2, \mathbf{R})$. We may take $K = SO\ (2, \mathbf{R})$, A is the set of all 2×2 unimodular diagonal matrices with positive entries and N is the set of all lower triangular 2×2 matrices with 1's along the diagonal. Note that W is the group of order 2 where if $1 \neq \sigma \in W$ then $\sigma d = d^{-1}$ for any $d \in A$. It follows then that if $d \in A$ is such that the eigenvalues (on $\mathbf{R}^2$) of d are x and $\frac{1}{x}$ where $x \geqq 1 \geqq \frac{1}{x} > 0$ then A (d) is the set of all $c \in A$ with positive eigenvalues $y \geqq \frac{1}{y}$ such that $x \geqq y \geqq 1 \geqq \frac{1}{y} \geqq \frac{1}{x} > 0$. [Note that for c (resp. d) it is *not* speci-

fied that y (resp. x) should be in the upper left hand corner or in the lower right hand corner.]

Now write $bn = k_1 \, dk_2$ where $d \in A$ and $k_1, k_2 \in K$. Then $k_1^{-1} bn = dk_2$ so that $b = a \, (dk_2)$. By $(4.3.5)$ one has $b \in A \, (d)$. But now if $a \in A \, (b)$ then certainly $a \in A \, (d)$. However by Lemma 3, p. 471 in [14] where $x_2 = x_1^{-1}$ there exists $k_3, k_4 \in K$ [Thompson says unitary but in fact the matrices he constructs are in SO $(2, \mathbf{R})$] such that $k_3 \, dk_4 = an''$ for some $n'' \in N$. Now put $k = k_3 \, k_1^{-1} \in K$, $v = k_2^{-1} k_4 \in K$ then $kbnv = k_3 \, dk_4 = an''$.

<div align="right">Q. E. D.</div>

4.5. Now return to the general case. The real form $\mathfrak{g}$ of $\mathfrak{g_C}$ is called a normal real form (or $\mathbf{R}$-split form, or Chevalley form) in case $\mathfrak{a_C}$, the complexification of $\mathfrak{a}$, is a Cartan subalgebra of $\mathfrak{g}$. In this case the restricted roots are the roots in the usual sense, and W is the ordinary Weyl group, and the root spaces are 1-dimensional.

To finish the proof of Theorem 4.1 we assert it is enough to prove the theorem under the assumption that $\mathfrak{g}$ is a normal real form of $\mathfrak{g_C}$. Indeed let $\Gamma^* \subseteq \Gamma$ be the set of all restricted roots $\gamma \in \Gamma$ such that $\frac{\gamma}{2} \notin \Gamma$. For example $\Sigma \subseteq \Gamma^*$. Note that since γ and 2γ define the same hyperplane in $\mathfrak{a}$ the Weyl group W is generated by $\tau (\gamma)$ for $\gamma \in \Gamma^*$. But now there exists a semi-simple subalgebra $\mathfrak{g}^* \subseteq \mathfrak{g}$ such that $\mathfrak{a} \subseteq \mathfrak{g}^*$ and such that $\mathfrak{g}^*$ is a normal real form of its complexification of $\mathfrak{g_C^*}$ and $\mathfrak{a_C}$ is a Cartan subalgebra of $\mathfrak{g_C^*}$. Furthermore Γ^* is the set of roots of $(\mathfrak{a_C}, \mathfrak{g_C^*})$ and W is the usual Weyl group for $(\mathfrak{a_C}, \mathfrak{g_C^*})$. Finally $\mathfrak{g}^*$ is stable under θ so that

$$(4.5.1) \qquad\qquad \mathfrak{g}^* = \mathfrak{k}^* + \mathfrak{p}^*$$

is a Cartan decomposition where $\mathfrak{k}^* = \mathfrak{k} \cap \mathfrak{g}^*$ and $\mathfrak{p}^* = \mathfrak{p} \cap \mathfrak{g}^*$. For the proof of these statements see section 11.2, p. 786 in [8]. More specifically see Propositions 2.1 and 2.3 noting that now $\mathfrak{g}^* = \mathfrak{h}$ in the notation there since now center $\mathfrak{g} = 0$. *See* also Remark 14, p. 790.

Now let $K^* \subseteq G^*$ be the subgroup corresponding to $\mathfrak{k}^*$ and $\mathfrak{g}^*$ respectively. Since $\mathfrak{g}^*$ is semi-simple K^* and G^* are closed. Also $K^* \subseteq K$ and $A \subseteq G^*$. Also the definition of $A \, (b) \subseteq A$ for any $b \in A$ does not change if we replace G by G^* since W is not changed. Also using x_0 to define $\mathfrak{n}^* \subseteq \mathfrak{g}^*$ in the manner of section 2.2 one has $\mathfrak{n}^* = \mathfrak{n} \cap \mathfrak{g}^*$ so that $N^* \subseteq N$ where N^* is the subgroup corresponding to $\mathfrak{n}^*$. Thus for any $g \in G^*$ the a-component $a \, (g) \in A$ does not change if we replace G by G^*.

Now returning to the notation of Theorem 4.1 put

$$(4.5.2) \qquad\qquad C = \{ \, a \, (bv) \mid v \in K \, \}.$$

We have already shown [*see* $(4.3.5)$] that $C \subseteq A$ (b). To prove Theorem 4.1 we must prove that A $(b) \subseteq C$. But now if we put $C^* = \{\, a$ $(bv) \mid v \in K \,\}$ then $C^* \subseteq C$. However if we prove Theorem 4.1 under the assumption that $\mathfrak{g}$ is a normal real form of $\mathfrak{g}_C$ we can conclude that A $(b) \subseteq C^*$. But this implies A $(b) \subseteq C$ proving our assertion.

Henceforth until Theorem 4.1 is proved we will assume that $\mathfrak{g}$ is a real normal form of $\mathfrak{g}_C$.

4.6. Now for any root $\varphi \in \Gamma$ let $\mathfrak{a}_\varphi^\perp$ be the hyperplane in $\mathfrak{a}$ orthogonal to φ. Also for any $x \in \mathfrak{a}$ let $\{\, x,\, \tau\,(\varphi)\, x \,\} \subseteq \mathfrak{a}$ be the line segment joining x to its reflected image $\tau\,(\varphi)\, x$ through the hyperplane $\mathfrak{a}_\varphi^\perp$. Thus $\{\, x,\, \tau\,(\varphi)\, x \,\}$ is perpendicular to the hyperplane $\mathfrak{a}_\varphi^\perp$ unless of course it reduces to a point (i. e. if $x \in \mathfrak{a}_\varphi^\perp$). Let

$$(4.6.1) \qquad\qquad \mathfrak{c} = \log C = \{\, \log a\ (bv) \mid v \in K \,\} \subseteq \mathfrak{a}.$$

LEMMA 4.6. — $\mathfrak{c}$ *is stable under the Weyl group.* *In fact for any* $x \in \mathfrak{c}$ *and any root* $\gamma \in \Gamma$ *we claim*

$$\{\, x,\, \tau\,(\gamma)\, x \,\} \subseteq \mathfrak{c}.$$

Proof. — As in 3.2. let β_i, $i = 1,\, 2,\, \ldots,\, l$ be the simple positive roots. Fix $1 \leq i \leq l$ and write $\mathfrak{a}_i^\perp = \mathfrak{a}_{\beta_i}^\perp$ so that

$$(4.6.2) \qquad\qquad\qquad \mathfrak{a} = \mathfrak{a}_i \oplus \mathfrak{a}_i^\perp$$

where $\mathfrak{a}_i$ is the one-dimensional subspace spanned by $x_i = x_{\beta_i}$ (*see* § 3.2). Also let $e_i, f_i \in \mathfrak{g}$ be root vectors corresponding, respectively, to the roots β_i and $-\beta_i$. The real 3-dimensional space, $\mathfrak{g}_i$, spanned by x_i, e_i and f_i is a Lie subalgebra isomorphic to the Lie algebra of SL $(2, \mathbf{R})$ (a real TDS). Furthermore since $\theta\, x_i = -x_i$ and $\theta\,(\mathbf{R}\, e_i) = \mathbf{R} f_i$ (because $\theta = -1$ on $\mathfrak{a}$) it follows that $\mathfrak{g}_i$ is stable under θ and

$$(4.6.3) \qquad\qquad\qquad \mathfrak{g}_i = \mathfrak{k}_i \oplus \mathfrak{a}_i \oplus \mathfrak{n}_i$$

is an Iwasawa decomposition of $\mathfrak{g}_i$ where $\mathfrak{k}_i = \mathfrak{k} \cap \mathfrak{g}_i$, $\mathfrak{n}_i = \mathfrak{n} \cap \mathfrak{g}_i = \mathbf{R}\, e_i$ and $\mathfrak{a}_i$, defined above, is also given by $\mathfrak{a}_i = \mathfrak{a} \cap \mathfrak{g}_i$.

Thus if K_i, A_i and N_i are the subgroups of G corresponding to $\mathfrak{k}_i$, $\mathfrak{a}_i$, and $\mathfrak{n}_i$ respectively, then $G_i = K_i A_i N_i$ is an Iwasawa decomposition of G_i and if $g \in G_i$, then, by uniqueness, the Iwasawa decomposition of g is the same whether we regard g as an element of G_i or an element of G. In particular the a-component, a (g) is the same and a $(g) \in A_i$. However, the Weyl group of G_i operating on A_i or $\mathfrak{a}_i$ is of order 2 where the non-trivial element is defined by restricting $\tau_i = \tau$ (β_i) to A_i or $\mathfrak{a}_i$. Also for any $g \in G_i$ let A_i $(g) \subseteq A_i$ be defined as in section 3.1 but with respect to G_i not G.

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Now let $\mathfrak{n}_i^\perp \subseteq \mathfrak{g}$ be the space spanned by all root vectors belonging to all positive roots $\gamma \in \Gamma_+$ where $\gamma \neq \beta_i$. Thus

$$(4.6.4) \qquad\qquad \mathfrak{n} = \mathfrak{n}_i \oplus \mathfrak{n}_i^\perp.$$

Now $\mathfrak{n}_i$ is not only an ideal in $\mathfrak{n}$ but in fact one has

$$(4.6.5) \qquad\qquad [\mathfrak{g}_i, \mathfrak{n}_i^\perp] \subseteq \mathfrak{n}_i^\perp.$$

This is clear since if $\gamma \in \Gamma_+$ where $\gamma \neq \beta_i$ and $\gamma - \beta_i$ is a root then $\gamma - \beta_i \in \Gamma_+$ (otherwise β_i would not be simple). It follows therefore that if $N_i^\perp$ is the subgroup corresponding to $\mathfrak{n}_i$ then not only is

$$(4.6.6) \qquad\qquad N = N_i\, N_i^\perp$$

a semi-direct product but

$$(4.6.7) \qquad\qquad G_i \quad \text{normalizes } N_i^\perp.$$

Now let $y \in \mathfrak{c}$. Thus if $a = \exp y$ then there exists v, $k \in K$ and $n \in N$ such that

$$(4.6.8) \qquad\qquad bv = kan.$$

Now we want to show that $\{ y, \tau_i\, y \} \subseteq \mathfrak{c}$. But if we write $y = y_i + y_i^\perp$ where $y_i \in \mathfrak{a}_i$ and $y_i^\perp \in \mathfrak{a}_i^\perp$ then

$$(4.6.9) \qquad\qquad \{ y, \tau_i\, y \} = \{ ty_i + y_i^\perp \mid -1 \leq t \leq 1 \}.$$

But if $A_i^\perp$ is the subgroup of A corresponding to $\mathfrak{a}_i^\perp$ then $a = a_i\, a_i^\perp$ where $\exp y_i = a_i \in A_i$ and $\exp y_i^\perp = a_i^\perp \in A_i^\perp$. But then since

$$A_i\,(a_i) = \{ \exp ty_i \mid -1 \leq t \leq 1 \}$$

one has

$$\exp \{ y, \tau_i\, y \} = A_i\,(a_i)\, a_i^\perp$$

by $(4.6.9)$. Thus to show $\{ y, \tau_i\, y \} \subseteq \mathfrak{c}$ one must show $A_i\,(a_i)\, a_i^\perp \subseteq C$.

Now write $n = n_i\, n_i^\perp$ where $n_i \in N_i$, $n_i^\perp \in N_i^\perp$ according to the decomposition $(4.6.6)$. But then by $(4.6.8)$ one has

$$(4.6.10) \qquad\qquad k^{-1}\, bv = a_i\, a_i^\perp\, n_i\, n_i^\perp = a_i\, n_i\, a_i^\perp\, n_i^\perp.$$

The commutativity of n_i and $a_i^\perp$ follows from the relation

$$(4.6.11) \qquad\qquad [\mathfrak{a}_i^\perp, \mathfrak{g}_i] = 0.$$

The relation $(4.6.11)$ is immediate since β_i vanishes on $\mathfrak{a}_i^\perp$.

But if $c_i \in A_i\,(a_i)$ then by Lemma 4.4 there exists k_i, $v_i \in K_i$ and $n_i' \in N_i$ such that

$$(4.6.12) \qquad\qquad k_i\, a_i\, n_i\, v_i = c_i\, n_i'.$$

However by (4.6.10)

$$k_i \, k^{-1} \, b \, vv_i = k_i \, a_i \, n_i \, a_i^\perp \, n_i^\perp \, v_i.$$

But by (4.6.7) there exists $m_i^\perp \in \mathrm{N}_i^\perp$ such that $n_i^\perp \, v_i = v_i \, m_i^\perp$. However v_i commutes with $a_i^\perp$ by (4.6.11). Thus

$$(4.6.13) \qquad k_i \, k^{-1} \, bvv_i = k_i \, a_i \, n_i \, v_i \, a_i^\perp \, m_i^\perp$$
$$= c_i \, n_i' \, a_i^\perp \, m_i^\perp \qquad \text{by (4.6.12)}$$
$$= c_i \, a_i^\perp \, n_i' \, m_i^\perp \qquad \text{by (4.6.11)}$$
$$= c_i \, a_i^\perp \, m$$

where $m = n_i' \, m_i^\perp \in \mathrm{N}$. But if $kk_i^{-1} = k' \in \mathrm{K}$ then by (4.6.13) :

$$bvv_i = k' \, c_i \, a_i^\perp \, m$$

and hence

$$(4.6.14) \qquad a \, (bvv_i) = c_i \, a_i^\perp \in \mathrm{A}_i \, (a_i) \, a_i^\perp.$$

But c_i is arbitrary in $\mathrm{A}_i \, (a_i)$. Thus $\mathrm{A}_i \, (a_i) \, a_i^\perp \subseteq \mathrm{C}$ since $vv_i \in \mathrm{K}$ so that

$$(4.6.15) \qquad \{ y, \tau_i \, y \} \subseteq \mathfrak{c}$$

for any $y \in \mathfrak{c}$ and $1 \leq i \leq l$. In particular $\tau_i \, y \in \mathfrak{c}$. Since W is generated by the reflections τ_i, $1 \leq i \leq l$, this first of all implies that $\mathfrak{c}$ is stable under the action of W.

But now if $\gamma \in \Gamma$ is arbitrary (since γ can be embedded in some simple set of roots) there exists $1 \leq i \leq l$ and $\sigma \in \mathrm{W}$ such that $\sigma \tau_i \sigma^{-1} = \tau \, (\gamma)$. Now let $x \in \mathfrak{c}$ be arbitrary and put $y = \sigma^{-1} x$. Then by (4.6.15) one has $\{ y, \tau_i \, y \} \subseteq \mathfrak{c}$. However since $\mathfrak{c}$ is stable under W one has $\sigma \{ y, \tau_i \, y \} \subseteq \mathfrak{c}$. But by linearity $\sigma \{ y, \tau_i \, y \}$ is the line segment joining $\sigma \, y = x$ to $\sigma \tau_i \, y = \sigma \tau_i \, \sigma^{-1} \, \sigma \, y = \tau \, (\gamma) \, x$. Thus $\{ x, \tau \, (\gamma) \, x \} \subseteq \mathfrak{c}$.

Q. E. D.

4.7. Now for any root $\gamma \in \Gamma$ and any subset $\mathfrak{d} \subseteq \mathfrak{a}$ we define the subset

$$(4.7.1) \qquad \{ \mathfrak{d}, \tau \, (\gamma) \, \mathfrak{d} \} = \bigcup_{x \in \mathfrak{d}} \{ x, \tau \, (\gamma) \, x \}.$$

It is easy to give examples where, if $\mathfrak{d}$ is convex, then $\{ \mathfrak{d}, \tau \, (\gamma) \, \mathfrak{d} \}$ is not necessarily convex. However if $\mathfrak{d}$ is on one side of the hyperplane $\mathfrak{a}_\gamma^\perp$ then $\{ \mathfrak{d}, \tau \, (\gamma) \, \mathfrak{d} \}$ is indeed convex.

LEMMA 4.7. — *Assume $\mathfrak{d} \subseteq \mathfrak{a}$ is any convex subset such that $\langle \gamma, x \rangle \geq 0$ for all $x \in \mathfrak{d}$. Then $\{ \mathfrak{d}, \tau \, (\gamma) \, \mathfrak{d} \}$ is convex.*

Proof. — Let $y_i \in \{ \mathfrak{d}, \tau \, (\gamma) \, \mathfrak{d} \}$, $i = 1, 2$. If r_i, $i = 1, 2$, are positive numbers such that $r_1 + r_2 = 1$ we must show that

$$y = r_1 \, y_1 + r_2 \, y_2 \in \{ \mathfrak{d}, \tau \, (\gamma) \, \mathfrak{d} \}.$$

In fact if $y_i \in \{ x_i, \tau(\gamma) x_i \}$, $i = 1, 2$, where $x_i \in \mathfrak{d}$ it suffices to show that $y \in \{ x, \tau(\gamma) x \}$ where $x = r_1 x_1 + r_2 x_2$. This is clear since $x \in \mathfrak{d}$. But now $\tau(\gamma) x_i = x_i - \langle \gamma, x_i \rangle x_\gamma$, by (3.2.2) where by assumption $\langle \gamma, x_i \rangle \geq 0$. Thus

$$\{ x_i, \tau(\gamma) x_i \} = \{ x_i - t x_\gamma \mid 0 \leq t \leq \langle \gamma, x_i \rangle \}.$$

Hence there exists

(4.7.2)
$$0 \leq t_i \leq \langle \gamma, x_i \rangle$$

such that $y_i = x_i - t_i x_\gamma$. Thus $y = x - s x_\gamma$ where $s = r_1 t_1 + r_2 t_2$. But

$$\langle \gamma, x \rangle = \langle \gamma, r_1 x_1 + r_2 x_2 \rangle = r_1 \langle \gamma, x_1 \rangle + r_2 \langle \gamma, x_2 \rangle.$$

Thus $0 \leq s \leq \langle \gamma, x \rangle$ by (4.7.2) and hence

$$y \in \{ x, \tau(\gamma) x \} \subseteq \{ \mathfrak{d}, \tau(\gamma) \mathfrak{d} \}.$$

<div align="right">Q. E. D.</div>

4.8. The proof of Theorem 4.1 is now an immediate consequence of the following general fact about convexity and the Weyl group.

LEMMA 4.8. — *Let $\mathfrak{d} \subseteq \mathfrak{a}$ be any subset such that for any $z \in \mathfrak{d}$ and root $\gamma \in \Gamma$ the line segment $\{ z, \tau(\gamma) z \}$ lies in $\mathfrak{d}$. Then for any $x \in \mathfrak{d}$ the convex hull $\mathfrak{a}(x)$ of the Weyl group orbit $w(x) = W.x$ of x also lies in $\mathfrak{d}$.*

Proof. — Since $\tau(\gamma) z \in \mathfrak{d}$ for any $\gamma \in \Gamma$ and $z \in \mathfrak{d}$ it is clear that $\mathfrak{d}$ is stable under the action of W. Thus it suffices to show that $\mathfrak{a}(x) \subseteq \mathfrak{d}$ for any $x \in \mathfrak{d} \cap \mathfrak{a}_+$. Let $x \in \mathfrak{d} \cap \mathfrak{a}_+$ and assume, inductively, for any n elements, $\sigma_i \in W$, $i = 1, 2, \ldots, n$, the convex hull, $c(\sigma_1, \ldots, \sigma_n)$ of the vectors $x, \sigma_1 x, \ldots, \sigma_n x$ lies in $\mathfrak{d}$. The assumption is obviously true if $n = 0$. We now wish to prove that $c(\sigma_1, \ldots, \sigma_{n+1}) \subseteq \mathfrak{d}$ for any $n + 1$ elements $\sigma_i \in W$. We will prove this statement by induction in the following way : In section 3.2 we defined what we meant by a strongly positive sequence $\gamma_1, \ldots, \gamma_m$ (of positive roots). Assume inductively that for any such sequence of m positive roots and any n elements $\sigma_i \in W$ the convex hull $c(\sigma_1, \ldots, \sigma_n, \gamma_1, \ldots, \gamma_m)$ of $x, \sigma_1(x), \ldots, \sigma_n(x)$ and the additional element $\tau(\gamma_m) \tau(\gamma_{m-1}) \ldots \tau(\gamma_1) x = x_m$ lies in $\mathfrak{d}$. The statement is clearly true for $m = 0$ if $x_0 = x$. We will now prove that $c(\sigma_1, \ldots, \sigma_n, \gamma_1, \ldots, \gamma_{m+1}) \subseteq \mathfrak{d}$ for any n elements $\sigma_i \in W$ and a strongly positive sequence $\gamma_1, \ldots, \gamma_{m+1}$ of positive roots. Put $\gamma = \gamma_{m+1}$. Now by definition of a strongly positive sequence one has $\langle \gamma, x_m \rangle \geq 0$ since $x \in \mathfrak{a}_+$. One also has $\langle \gamma, x \rangle \geq 0$. On the other hand for any $1 \leq j \leq n$ one has $\langle \gamma, y \rangle \geq 0$ for either $y = \sigma_j x$ or for $y = \tau(\gamma) \sigma_j x$, the reflected image

of $\sigma_j x$. Put $\sigma_j' = \sigma_j$ or $\tau(\gamma)\sigma_j$, so that $\langle \gamma, \sigma_j' x \rangle \geq 0$. But now by induction $\mathfrak{d}_1 = c(\sigma_1', \ldots, \sigma_n', \gamma_1, \ldots, \gamma_m) \subseteq \mathfrak{d}$. On the other hand $\mathfrak{d}_1$ lies on one side of the hyperplane $\mathfrak{a}_\gamma^\perp$. However if $z \in \mathfrak{d}$ then by assumption $\{ z, \tau(\gamma) z \} \subseteq \mathfrak{d}$. Thus $\{ \mathfrak{d}_1, \tau(\gamma) \mathfrak{d}_1 \} \subseteq \mathfrak{d}$. But by Lemma 4.7 $\{ \mathfrak{d}_1, \tau(\gamma) \mathfrak{d}_1 \}$ is convex. However $\sigma_j x \in \{ \mathfrak{d}_1, \tau(\gamma) \mathfrak{d}_1 \}$ for any j since σ_j is one of the elements σ_j' or $\tau(\gamma) \sigma_j'$. But also

$$x_{m+1} = \tau(\gamma_{m+1}) x_m = \tau(\gamma) x_m \in \{ \mathfrak{d}_1, \tau(\gamma) \mathfrak{d}_1 \}.$$

Thus

$$c(\sigma_1, \ldots, \sigma_n, \gamma_1, \ldots, \gamma_{m+1}) \subseteq \{ \mathfrak{d}_1, \tau(\gamma) \mathfrak{d}_1 \}$$

since $\{ \mathfrak{d}_1, \tau(\gamma) \mathfrak{d}_1 \}$ is convex. Hence $c(\sigma_1, \ldots, \sigma_n, \gamma_1, \ldots, \gamma_{m+1}) \subseteq \mathfrak{d}$ proving the induction statement on m for all m. But now given $\sigma_{n+1} \in W$ there exists a strongly positive sequence $\gamma_1, \ldots, \gamma_m$ of positive roots such that $\sigma_{n+1} = \gamma_m \cdot \gamma_{m-1} \ldots \gamma_1$. Thus $c(\sigma_1, \ldots, \sigma_{n+1}) \subseteq \mathfrak{d}$ for any $n+1$ elements $\sigma_j \in W$ proving the induction assumption on n for all n. Putting n equal to the order of W one obviously has $\mathfrak{a}(x) \subseteq \mathfrak{d}$.

<div align="right">Q. E. D.</div>

REMARK 4.8. — Note that the argument at the end of the proof of Lemma 4.6 shows that one can weaken the assumption in Lemma 4.8 in that γ can be restricted to be simple.

We now give the

Proof of Theorem 4.1. — Recalling the notation of Theorem 4.1 we must prove that if $C \subseteq A$ is defined by (4.5.2) then $C = A(b)$. As remarked after (4.5.2) we have only to show that $A(b) \subseteq C$ or if $\mathfrak{c}$ is defined by (4.6.1) and $x = \log b$ we have only to show that $\mathfrak{a}(x) \subseteq \mathfrak{c}$. But $x \in \mathfrak{c}$ [by choosing $v = 1$ in (4.5.2)]. On the other hand for any $z \in \mathfrak{c}$ and $\gamma \in \Gamma$ one has $\{ z, \tau(\gamma) z \} \subseteq \mathfrak{c}$ by Lemma 4.6. But then by Lemma 4.8 one has $\mathfrak{a}(x) \subseteq \mathfrak{c}$.

<div align="right">Q. E. D.</div>

5. Applications : Some K-double coset theorems

5.1. Now by (4.2.8) one has that $G = KAK$. This fact is important in analysis on Lie groups since, among other things, it says that a spherical function on G is determined by its restriction to A. On the other hand a corollary to Corollary 4.2 is

THEOREM 5.1. — *For any semi-simple Lie group* G *one has*

$$G = KNK.$$

Proof. — If $x \in \mathfrak{a}$ then the sum $y = \sum \sigma x$, over all $\sigma \in W$ equals zero. This is clear since y is a W-invariant in $\mathfrak{a}$ and, since Γ spans $\mathfrak{a}'$, only 0 is a W-invariant. Thus $0 \in \mathfrak{a}\,(x)$ for any $x \in \mathfrak{a}$ and hence

(5.1.1) $$1 \in A\,(f)$$

for any $f \in G$. But if $g \in G$ is arbitrary then $1 \in A\,(p)$ where $p = p\,(g)$ and hence by Corollary 4.2 there exists $v \in K$ such that $a\,(gv^{-1}) = 1$. That is $gv^{-1} = kn$ where $k \in K$, $n \in N$. Hence $g = knv \in KNK$.

Q. E. D.

Theorem 5.1 says that a spherical function on G is now determined by its restriction to N.

REMARK 5.1.1. — It might be interesting to explore what the spherical functions look like on N. In the case of A the elementary spherical functions appear in exponential form [*see* e. g. (4.1.2)]. Although we have not investigated this it seems likely that for N these functions might have a natural power series expansion instead.

Of course just by dimension considerations it is clear that the spherical functions on N do not separate points; thereby creating " level surfaces ". This is also true for A but in that case the level surfaces are finite sets whose cardinality is at most the order of the Weyl group. One can approach this situation at least in the case where rank K = rank G (existence of the discrete series case, by Harish-Chandra's theorem). In fact if rank K = rank G and $k = \dim A$ then one can find a connected abelain subgroup $N_0 \subseteq N$ where $\dim N_0 = k$ such that

(5.1.2) $$G = KN_0 K.$$

In fact one can find k orthogonal restricted roots γ_i where $\gamma_i \mid \mathfrak{h}_m = 0$. These define k commuting TDS's and (5.1.2) follows already using (4.2.8) and the Horn-Thompson theorem on SL (2, **R**).

REMARK 5.1.2. — The relation (5.1.2) can be thought of as saying that N_0 carries the non-compact part of G. It should be easy to determine the measure decomposition of G defined by (5.1.2). It might then be interesting to investigate what the representative functions of, say, the discrete series look like on N_0.

5.2. Theorem 5.1 can be given a geometric interpretation. If X is the symmetric space

(5.2.1) $$X = G/K$$

then a horocycle is an orbit of a conjugate $g\,N\,g^{-1}$ [$= k\,N\,k^{-1}$ where $k = k\,(g) \in K$] of N on X. Theorem 5.1 says the same then as

THEOREM 5.2. — *Any two points in* X *can be embedded in a horocycle.*

Proof. — If $o \in X$ is the coset of K then the statement of Theorem 5.1 is the statement that every point of X lies on a horocycle through o. Theorem 5.2 is just an application of the transitivity of G on X.

<div align="right">Q. E. D.</div>

REMARK 5.2. — One sees Theorem 5.2 immediately in the case of SL $(2, \mathbf{R})$. Here X can be identified with the unit disc in $\mathbf{C}$ and the horocycles are just the circles in X which are tangent to the boundary.

5.3. Theorem 5.1 says that every double coset of K in G contains a point of N. This, no doubt can be proved in a much simpler way than the one given here. In fact if rank K = rank G we have already indicated such a proof [*see* the paragraph after (5.1.2)]. A stronger use of Corollary 4.2 is the following generalization of Theorem 5.1. In effect it says that if N is replaced by a coset f N, $f \in G$, then it is no longer true. More explicitly

THEOREM 5.3. — *Let* $a \in A$ *be arbitrary and let*

$$(5.3.1) \qquad\qquad G_a = \{ g \in G \mid p(g) \geqq a \}$$

where $p(g)$ *is defined by* (4.2.6) *and the order relation* $p(g) \geqq a$ *is defined (in terms of convexity) in section* 3.1. *Then for any* $f \in G$ *one has*

$$G_a = K f NK$$

where $a = a(f)$.

Proof. — If we write $f = k_1 a n_1$ where $k_1 \in K$, $n_1 \in N$ then clearly K f NK = K a NK. Let L denote this set. If $g \in L$ then there exists k, $v^{-1} \in K$, $n \in N$ such that $g = kanv^{-1}$ or $gv = kan$. Thus $a(gv) = a$. But then by Corollary 4.2 one has $a \in A(p(g))$ or $p(g) \geqq a$. Thus $g \in G_a$. Conversely if $g \in G$ and $p(g) \geqq a$ then by Corollary 4.2 there exists $v \in K$ such that $a(gv) = a$. That is $gv = kan$ for some $k \in K$, $n \in N$. Hence $g = kanv^{-1} \in L$.

<div align="right">Q. E. D.</div>

5.4. The statement of Theorem 4.1 involves the Iwasawa decomposition and is expressed in terms of the a-component. The Horn-Thompson theorem on the other hand is an invariant formulation involving the complete multiplicative Jordan decomposition (since it is a statement about eigenvalues) and hence is expressed in terms of the hyperbolic component. The generalization of the Horn-Thompson theorem to the arbitrary semi-simple case follows, however, from Theorem 4.1.

THEOREM 5.4. — *Let* $p \in P = \exp \mathfrak{p}$. *Then for any* k, $v \in K$ *one has* $kpv \leqq p$. *That is,* $h(kpv) \leqq p$ *using the notation of* (2.1.5) *and section* 3.1. *Conversely given any hyperbolic element* $h \leqq p$ *there exists* k,

$v \in K$ such that $h(kpv)$ is conjugate to h. Moreover k and v can be chosen so that the elliptic component $e(kpv) = 1$.

Proof. — The statement that $kpv \leqq p$ is just Lemma 4.3. Assume $h \leqq p$ where h is a hyperbolic element. Since the statement is only up to conjugacy and since any hyperbolic element is conjugate to an element in A we may assume $h = a \in A$. But now by Corollary 4.2 there exists $v \in K$ such that $a(pv) = a$. That is $pv = k^{-1}$ an for some $k \in K$, $n \in N$. Thus $kpv = an$. But, by Proposition 2.5, $h(an)$ is conjugate to a. That is, $h(kpv)$ is conjugate to a. Also $e(kpv) = e(an) = 1$ by Proposition 2.5.

<div align="right">Q. E. D.</div>

5.5. Theorem 5.4 amounts to a reformulation of Corollary 4.2, using Proposition 2.5. Similarly Theorem 5.3 can be reformulated using Proposition 2.5.

For any $a \in A$ let

(5.5.1) $D_a = \{ g \in G \mid h(g) \text{ is conjugate to } a \text{ and } e(g) = 1 \}$.

REMARK 5.5. — In case $a \in A$ is regular hyperbolic in the sense that the centralizer of $x = \log a$ is just $\mathfrak{a} + \mathfrak{m}$ note that D_a is exactly the conjugacy class of a. Indeed $\mathfrak{a} + \mathfrak{m}$ has no non-trivial nilpotent elements and hence no non-trivial unipotent element can commute with a. Thus in this case $u(g) = 1$ as well as $e(g) = 1$ for any $g \in D_a$.

THEOREM 5.5. — Let $a \in A$ be arbitrary and let D_a be defined by (5.5.1). Let G_a, as in (5.3.1), be defined by putting $G_a = \{ g \in G \mid p(g) \geqq a \}$. Then

$$G_a = K D_a K.$$

Proof. — By Proposition 2.5 one clearly has

$$D_a = \bigcup_{k \in K} ka N k^{-1}.$$

Thus $K D_a K = K a N k = G_a$ by Theorem 5.3.

<div align="right">Q. E. D.</div>

6. Characters values and the partial ordering; A generalized Golden–Thompson inequality

6.1. Now by Theorem 3.1 if g, $f \in G$ then $g \geqq f$ if and only if the spectral radius $|\pi_\lambda(g)|$ of $\pi_\lambda(g)$ is greater than or equal to the spectral radius $|\pi_\lambda(f)|$ of $\pi_\lambda(f)$ for every irreducible representation π_λ, $\lambda \in \hat{G}$, of G. Now if g and f are hyperbolic then by Proposition 3.4. the eigenvalues of $\pi_\lambda(g)$ and $\pi_\lambda(f)$ are positive so that $g \geqq f$ if and only if the maximal eigenvalues of $\pi_\lambda(g)$ is greater than or equal to the maximal eigenvalues of $\pi_\lambda(f)$ for all $\lambda \in G$. But in such a case the minimal eigen-

value of $\pi_\lambda (g)$ is less than or equal to the minimal eigenvalue of $\pi_\lambda (f)$. Thus if χ_λ is the character of the representation π_λ it is not immediately obvious whether or not $\chi_\lambda (g) \geq \chi_\lambda (f)$ in case $g \geq f$. However it is true. The question of comparing character values was inspired by the Golden-Thompson inequality and Thompson's proof of it (*see* § 3 [14]). But the point of view taken here is that a comparison of character values can be made whenever g and f are hyperbolic and $g \geq f$, not just in the special case where $g = e^x e^y$ and $f = e^{x+y}$ for x, $y \in \mathfrak{p}$.

THEOREM 6.1. — *Let f, $g \in G$ be any two hyperbolic elements. Then if $\lambda \in \hat{G}$ is arbitrary and χ_λ is the character of the irreducible representation π_λ one has*

$$\chi_\lambda (g) \geq \chi_\lambda (f)$$

in case $g \geq f$.

Proof. — Since character values and the order relation in G are independent of conjugacy we may assume g, $f \in A$ and $g \geq f$. Let $y = \log g$ and $x = \log f$ so that $x \in \mathfrak{a} (y)$.

But now since $t \to e^t$ is a convex function on $\mathbf{R}$ it is immediate that if $\nu \in \mathfrak{a}'$ the multiplicative character $z \mapsto e^{\langle \nu, z \rangle}$, $z \in \mathfrak{a}$, is a convex function on $\mathfrak{a}$. However if $\tilde{\chi}_\lambda$ is the function on $\mathfrak{a}$ defined by $\tilde{\chi}_\lambda (z) = \chi_\lambda (\exp z)$ then $\tilde{\chi}_\lambda$ is a finite sum of such multiplicative characters and hence χ_λ is a convex function on $\mathfrak{a}$. But since $x \in \mathfrak{a} (y)$ there exists for each $\sigma \in W$ a positive scalar c_σ such that $\sum_{\sigma \in W} c_\sigma = 1$ and $x = \sum_{\sigma \in W} c_\sigma \sigma y$. Thus

$$(6.1.1) \qquad \qquad \tilde{\chi}_\lambda (x) \leq \sum_{\sigma \in W} c_\sigma \tilde{\chi}_\lambda (\sigma y).$$

But $\tilde{\chi}_\lambda$ is clearly invariant under the Weyl group. Thus $\tilde{\chi}_\lambda (\sigma y) = \chi_\lambda (g)$ for any $\sigma \in W$. Also $\tilde{\chi}_\lambda (x) = \chi_\lambda (f)$. Thus $(6.1.1)$ becomes

$$\chi_\lambda (f) \leq \left(\sum_{\sigma \in W} c_\sigma \right) \chi_\lambda (g) \text{ or } \chi_\lambda (f) \leq \chi_\lambda (g).$$

<div align="right">Q. E. D.</div>

REMARK 6.1. — It is unknown to us whether or not the converse of Theorem 6.1 is true. That is whether or not $g \geq f$ if and only if $\chi_\lambda (g) \geq \chi_\lambda (f)$ for all $\lambda \in G$ where g and f are hyperbolic.

6.2. Now let $L \in G$ be the set of all hyperbolic elements in G. We know $P \subseteq L$ where P is defined by $(4.2.1)$. But also $P^2 \subseteq L$ and in fact

PROPOSITION 6.2. — *One has*

$$P^2 = L.$$

Proof. — Let $g \in G$ and $p \in P$. Then first note that

$$(6.2.1) \qquad\qquad gpg^* \in P$$

where $g^* \in G$ is defined by $(4.2.3)$. Indeed if $f = gp^{1/2}$ then $gpg^* = ff^*$. But $ff^* \in P$ as noted in $(4.2.6)$. Now if $q \in P$ then $q^{1/2} pq^{1/2} \in P$ by $(6.2.1)$ since $q^{1/2} = (q^{1/2})^* \in P$. But $pq = q^{-1/2} (q^{1/2} pq^{1/2}) q^{1/2}$ so that pq is conjugate to $q^{1/2} pq^{1/2}$. Hence pq is hyperbolic or $P^2 \subseteq L$. On the other hand if $h \in L$ then h is conjugate to an element of $A \subseteq P$. That is there exists $g \in G$ and $p \in P$ such that $h = gpg^{-1} = (gpg^*) (g^*)^{-1} g^{-1}$. But $gpg^* = p_1 \in P$ by $(6.2.1)$ and $q_1 = (g^*)^{-1} g^{-1} \in P$ by, say, $(6.2.1)$, since clearly $(g^*)^{-1} = (g^{-1})^*$. Thus $h = p_1 q_1 \in P^2$. Hence $L = P^2$.

<div align="right">Q. E. D.</div>

6.3. Now if $x,\, y \in \mathfrak{p}$ then by Proposition 6.2 one has $e^x e^y$ is hyperbolic, writing e^x for exp x. But e^{x+y} is also hyperbolic. The Golden-Thompson inequality states that in the case where $G = \mathrm{SL}\,(n,\, \mathbf{C})$ [actually $\mathrm{GL}\,(n,\, \mathbf{C})$ but the added scalars are trivially dealt with] and $\mathfrak{p}$ is the set of Hermitian matrices in $\mathfrak{g}$ then

$$(6.3.1) \qquad\qquad \mathrm{tr}\, e^x e^y \geq \mathrm{tr}\, e^{x+y} .$$

This has been generalized by Lenard for the case where trace is replaced by the character of representations of G. But this follows from Theorem 6.1 as soon as one knows that $e^x e^y \geq e^{x+y}$ in the order relation. In fact Thompson's proof of $(6.3.1)$ in effect proves $e^x e^y \geq e^{x+y}$. This very nice proof however, carries over to the case where G is an arbitrary semi-simple group and we repeat it word for word.

REMARK 6.3. — Even if we could embed G in $\mathrm{SL}\,(n,\, \mathbf{C})$ for some n where $\mathfrak{p}$ corresponds to a subspace of Hermitian matrices the statement that $e^x e^y \geq e^{x+y}$ in $\mathrm{SL}\,(n,\, \mathbf{C})$ does not imply this holds in G (*see* Remark 3.1.1). It is just that Thompson's proof works for G.

Theorem 6.3. — *Let* $x,\, y \in \mathfrak{p}$ *be arbitrary. Then* $e^x e^y$ *and* e^{x+y} *are hyperbolic. Moreover*

$$(6.3.2) \qquad\qquad e^x e^y \geq e^{x+y}$$

so that, by Theorem 6.1, *for any* $\lambda \in \hat{G}$ *one has*

$$(6.3.3) \qquad\qquad \chi_\lambda (e^x e^y) \geq \chi_\lambda (e^{x+y}).$$

Proof. — Let $g \in G$ be arbitrary. We first observe that

$$(6.3.4) \qquad\qquad gg^* \geq g^2.$$

Indeed since $gg^* \in P$ it is enough to show by Proposition 4.3 that $\| \pi_\lambda (gg^*) \| \geq \| \pi_\lambda (g^2) \| = \| (\pi_\lambda (g))^2 \|$ for any $\lambda \in \hat{G}$. But since $\pi_\lambda (g^*)$

is the Hermitian adjoint of $\pi_\lambda(g)$ by (4.2.5) one has the familiar operator norm relation $\| \pi_\lambda(gg^*) \| = \| \pi_\lambda(g) \|^2$. However $\| \pi_\lambda(g) \|^2 \geqq \| (\pi_\lambda(g))^2 \|$ by (4.3.1) establishing (6.3.4).

More generally for any positive integer n one has $(gg^*)^n \geqq g^{2n}$. Indeed this follows from a similar argument as that above or from the general fact that

$$(6.3.5) \qquad\qquad g \geqq f \text{ implies } g^n \geqq f^n.$$

The relation (6.3.5) is an immediate consequence of the fact that if x, $y \in \mathfrak{a}$ and $x \in \mathfrak{a}(y)$ then clearly $nx \in \mathfrak{a}(ny) = n \mathfrak{a}(y)$. We are also implicitly using the obvious fact that

$$h(g^n) = h(g)^n$$

[see (2.1.6)].

Now let a, $b \in P$ and put $g = ab$. Then $gg^* = ab^2 a$ and $g^2 = abab$. It is then clear that $(gg^*)^n = ab^2(a^2 b^2)^{n-1} a$. But then $a(gg^*)^n a^{-1} = (a^2 b^2)^n$. But since $(gg^*)^n \geqq g^{2n}$ one has $a(gg^*)^n a^{-1} \geqq g^{2n}$ so that $(a^2 b^2)^n \geqq (ab)^{2n}$. Now put $n = 2^{k-1}$ for some positive integer k. Thus $(a^2 b^2)^{2^{k-1}} \geqq (ab)^{2^k}$. Iterating this relation backwards on k one gets the relation

$$(6.3.6) \qquad\qquad a^{2^k} b^{2^k} \geqq (ab)^{2^k}.$$

But then if $a = e^{x/2^k}$, $b = e^{y/2^k}$ one has

$$(6.3.7) \qquad\qquad e^x e^y \geqq (e^{x/2^k} e^{y/2^k})^{2^k}.$$

But one knows $\lim (e^{x/t} e^{y/t})^t = e^{x+y}$ as $t \to \infty$ (see e. g. [15], p. 110, 112). But then by taking the limit in (6.3.7) as $k \to \infty$ one has $e^x e^y \geqq e^{x+y}$. The continuity of the order relation is an immediate consequence of Theorem 3.1 since the spectral radius is a continuous function on the space of operators.

Q. E. D.

7. Applications to the Geometry of Symmetric Spaces of Negative Curvature

7.1. The inequality (6.3.2) has a geometric interpretation which, I think, may be useful to elaborate upon. In effect (6.3.2) is a statement about geodesic triangles in a symmetric space of negative curvature.

In section 3.1 we defined a partial ordering in G. However it could have been in $\mathfrak{g}$ as well. In doing so now we limit the order relation to the set $\mathfrak{l}$ of all real semi-simple elements. If x and y are two such elements define $y \geqq x$ if $\exp y \geqq \exp x$. That is, if

$$(7.1.1) \qquad\qquad \mathfrak{a}(x) \subseteq \mathfrak{a}(y).$$

Note that if $y \succeq x$ then for any positive number r one has

(7.1.2) $$ry \succeq rx,$$

[since clearly $\mathfrak{a}\,(rx) = r\,\mathfrak{a}\,(x)$] for any $x \in s$.

Now since $t \to e^t$ is a monotonic increasing function on **R**, Theorem 3.1 implies

THEOREM 7.1. — *Let $x,\, y \in \mathfrak{l}$. Then $y \succeq x$ if and only if for every $\lambda \in \hat{G}$ the maximal eigenvalue of $\pi_\lambda\,(y)$ is greater or equal to the maximal eigenvalue of $\pi_\lambda\,(x)$.*

Now for $x \in \mathfrak{l}$ let $(x,\, x)$ denote the inner product defined by the Killing form. Clearly $(x,\, x) \succeq 0$. Put $|\,x\,| = (x,\, x)^{1/2}$. Since the Killing form is invariant under conjugation and since its restriction to $\mathfrak{a}$ is invariant under W it is obvious that

(7.1.3) $$|\,y\,| \succeq |\,x\,|$$

in case $y \succeq x$.

REMARK 7.1. — The converse is obviously false. That is, if, for example, $x,\, y \in \mathfrak{a}$ then saying that $y \succeq x$ is giving more information than saying $|\,y\,| \succeq |\,x\,|$. The latter refers only to the sphere generated by y while the former refers to the smaller W-invariant convex set generated by y. An implication of this additional information is given in Theorem 7.1. But it is this additional information that (6.3.2) will give concerning geodesic triangles in symmetric spaces of negative curvature.

7.2. Let $X = G/K$ so that X has a natural structure (normalized by using the Killing form) of a Riemannian symmetric space of negative curvature on which G operates as a group of motions.

Now to every two points $r,\, s \in X$ we may associate a real semi-simple element $x\,(r,\, s) \in \mathfrak{l}$ as follows : Let $\mathfrak{k}_r$ be the Lie algebra of an isotropy subgroup at r. Thus if $\mathfrak{p}_r$ is the orthogonal complement of $\mathfrak{k}_r$ in $\mathfrak{g}$ then $\mathfrak{g} = \mathfrak{k}_r + \mathfrak{p}_r$ is a Cartan decomposition. But the exponential map σ_r at r defines a bijection of $\mathfrak{p}_r$ onto X, recalling that any two points of X can be joined by a unique geodesic. Thus there exists a unique element $x\,(r,\, s) \in \mathfrak{p}_r \subset \mathfrak{l}$ such that $\sigma_r\,(x\,(r,\, s)) = s$. Since G operates as motions on X one clearly has

(7.2.1) $$\operatorname{Ad} g\,(x\,(r,\, s)) = x\,(g.r,\, g.s)$$

for any $g \in G$ where $g.r$ and $g.s$ are, respectively, the image of r and s under the action of g.

In the language of Elie Cartan, the one parameter group of motions
on X generated by $x(r, s)$ is the group of transvections associated with
the geodesic joining r and s. But then if $d(r, s)$ is the distance of r to s,
one has

(7.2.2) $d(r, s) = |x(r, s)|.$

Now let o, r, and s be any three points in X and consider the geodesic
triangle generated by these three points, and consider the corresponding
three real semi-simple elements $x(r, o)$, $x(o, s)$ and $x(r, s)$.

Now if instead of X we were dealing with ordinary Euclidean space
then we would have $x(r, o) + x(o, s) = x(r, s)$ so in particular

$$|x(r, o) + x(o, s)| = |x(r, s)|.$$

However since we are dealing with a symmetric space of negative curva-
ture then the geometry of such a space yields the fact that $d(r, s)$ is
bigger than it would be in a flat space, or

(7.2.3) $|x(r, o) + x(o, s)| \leq |x(r, s)| = d(r, s).$

REMARK 7.2. — This relation is proved in [11]. In fact if $a = d(o, r)$,
$b = d(o, s)$ and ψ is the angle between $x(o, r)$ and $x(o, s)$ then

$$|x(r, o) + x(o, s)|^2 = a^2 + b^2 - 2ab\cos\psi$$

since easily one has $x(r, o) = -x(o, r)$ [see e. g. (7.2.6)]. But then
if $c = d(r, s)$, so that a, b and c are the lengths of the sides of the geodesic
triangle made by o, r, and s, then (7.2.3) (by squaring) is just the sta-
tement that the law of cosines in flat spaces is replaced by the inequality

(7.2.4) $a^2 + b^2 - 2ab\cos\psi \leq c^2$

is a symmetric space of negative curvature. But (7.2.4) is exactly
Lemma 4, p. 35 in [11].

But the point now is that a geodesic arc in X carries more information
than just its length. Theorem 7.2 says that we may remove the absolute
signs in (7.2.3) (see Remark 7.1).

THEOREM 7.2. — Let o, r and s be as above. Then $x(r, o) + x(o, s)$
is real semi simple and

$$x(r, o) + x(o, s) \leq x(r, s).$$

Proof. — Let $\pi : P \to X = G/K$ be the map defined by $\pi(q) = q^{1/2} K$.
It is clear from (4.2.2) that π is a bijection. But now if $f \in G$ let $T_f : X \to X$

be the motion induced by the action of f on X. Thus $T_f (g \, K) = fg \, K$ for any $g \in G$. We now note that

$$(7.2.5) \qquad\qquad T_f \, \pi \, (q) = \pi \, (fqf^*)$$

for any $q \in P$. Indeed $T_f \, \pi \, (q) = f \, q^{1/2} \, K$. But for any $g \in G$ one has $g \, K = p \, (g) \, K$ by (4.2.2). Thus $T_f \, \pi \, (q) = p \, (fq^{1/2}) \, K = (fqf^*)^{1/2} \, K = \pi (fqf^*)$ using (4.2.6).

Now for any p, $q \in P$ we next assert that

$$(7.2.6) \qquad\qquad \exp x \, (\pi \, (p), \, \pi \, (q)) = (qp^{-1})^{1/2}$$

recalling that qp^{-1} is hyperbolic by Proposition 6.2. Indeed by (7.2.5) one has $T_{p^{1/2}} \, \pi \, (1) = \pi \, (p)$. Now let $d \in P$ be defined by $T_{p^{-1/2}} \, \pi \, (q) = \pi \, (d)$ so that $T_{p^{1/2}} \, \pi \, (d) = \pi \, (q)$. Thus $T_{p^{1/2}}$ carries the unique geodesic joining $\pi \, (1)$ and $\pi \, (d)$ onto the unique geodesic joining $\pi \, (p)$ to $\pi \, (q)$ and hence by (7.2.1) $\mathrm{Ad} \, p^{1/2} \, x \, (\pi \, (1), \, \pi \, (d)) = x \, (\pi \, (p), \, \pi \, (q))$ or

$$(7.2.7) \qquad p^{1/2} \exp x \, (\pi \, (1), \, \pi \, (d)) \, p^{-1/2} = \exp x \, (\pi \, (p), \, \pi \, (q)).$$

But $\mathfrak{p}_{\pi \, (1)} = \mathfrak{p}$ and hence $x \, (\pi \, (1), \, \pi \, (d)) = 1/2 \, \log d$ so that (7.2.7) implies

$$(7.2.8) \qquad\qquad \exp x \, (\pi \, (p), \, \pi \, (q)) = p^{1/2} \, d^{1/2} \, p^{-1/2}.$$

On the other hand since $T_{p^{1/2}} \, \pi \, (d) = \pi \, (q)$ one has $p^{1/2} \, dp^{1/2} = q$ by (7.2.5). Multiplying on the right by p^{-1} one has $p^{1/2} \, dp^{-1/2} = qp^{-1}$ and hence $p^{1/2} \, d^{1/2} \, p^{-1/2} = (qp^{-1})^{1/2}$. Comparing with (7.2.8) then yields (7.2.6).

Now return to the points o, r, $s \in X$. Without loss [using (7.2.1) and the fact that the partial order is preserved by conjugation] we may assume $\pi \, (1) = o$. Let p, $q \in P$ be defined so that $\pi \, (p) = r$ and $\pi \, (q) = s$. Now by (7.2.6) $\exp x \, (r, o) = p^{-1/2}$ and $\exp x \, (o, s) = q^{1/2}$. But then $x \, (r, o)$ and $x \, (o, s)$ lie in $\mathfrak{p}$ and hence $x \, (r, o) + x \, (o, s) \in \mathfrak{p}$ is real semi-simple. But by (6.3.2) :

$$\exp \, (2 \, x \, (r, o) + 2 \, x \, (o, s)) \leq \exp 2 \, x \, (o, s) \exp 2 \, x \, (r, o) = qp^{-1}.$$

Hence

$$2 \, (x \, (r, o) + x \, (o, s)) \leq \log qp^{-1} = 2 \, x \, (r, s),$$

using (7.2.6). But then $x \, (r, o) + x \, (o, s) \leq x \, (r, s)$ [see (7.1.2)].

<div align="right">Q. E. D.</div>

8. A Conjugation Convexity Theorem and a Generalization of a Theorem of A. Horn

8.1. Lemma 4.8 may be used also to prove sort of an additive version of Theorem 4.1. That is a theorem in $\mathfrak{g}$ rather than G where the adjoint action of K on $\mathfrak{p}$ replaces multiplication in G. The theorem we will

prove will bear a relation to a theorem of A. Horn (*see* [3], Theorems 4 and 5) that Theorem 5.4 bears to the Horn-Thompson theorem.

To state this result of Horn let $\mathfrak{p}_0$ denote the set of all $n \times n$ complex Hermitian matrices and let $\mathfrak{a}_0 \subseteq \mathfrak{p}_0$ denote the space of all diagonal Hermitian matrices in $\mathfrak{p}_0$. Consider the projection $\Gamma : \mathfrak{p}_0 \to \mathfrak{a}_0$ defined so that if $x \in \mathfrak{p}_0$ then Γx has the same diagonal entries as x. That is if x_{ij} is the i, j-component of x then $(\Gamma x)_{ij} = \delta_{ij} x_{ij}$.

Now let $x \in \mathfrak{p}_0$ and let $\lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_n$ be the eigenvalues of x and $y \in \mathfrak{a}_0$ be the diagonal matrix whose diagonal entries are the λ_i. That is $y_{ij} = \delta_{ij} \lambda_i$. Furthermore if σ is any permutation of $1, 2, \ldots, n$ let $\sigma y \in \mathfrak{a}_0$ be the diagonal matrix obtained by performing the permutation σ on the diagonal entries of y. Thus $(\sigma y)_{ij} = \delta_{ij} \lambda_{\sigma^{-1}i}$. Next let $\mathfrak{a}_0(y) \subseteq \mathfrak{a}_0$ denote the convex hull of the finite set $|\sigma y|$, over all permutations σ. Then not only does one have $x \in \mathfrak{a}_0(y)$ but in fact, the following is true : Let $O(y)$ denote the set of all Hermitian matrices with the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ [so that e. g. $x \in O(y)$], then Horn's theorem states that

(8.1.1) $$\Gamma O(y) = \mathfrak{a}_0(y).$$

8.2. We will generalize (8.1.1) [more precisely (8.1.1) for the traceless Hermitian matrices. However the addition or subtraction of the scalar matrices is trivially dealt with] to the arbitrary semi-simple case $\mathfrak{g}$. Using the notation we have established so far let $\mathfrak{q}$ be the orthocomplement with respect to the Killing form of $\mathfrak{a}$ in $\mathfrak{p}$ so that

(8.2.1) $$\mathfrak{p} = \mathfrak{a} + \mathfrak{q}$$

is an orthogonal direct sum. Let

(8.2.2) $$\Gamma : \mathfrak{p} \to \mathfrak{a}$$

be the orthogonal projection of $\mathfrak{p}$ on $\mathfrak{a}$.

Now for any $y \in \mathfrak{p}$ let $O(y) = \operatorname{Ad} K(y) \subseteq \mathfrak{p}$, the Ad K-orbit in $\mathfrak{p}$ defined by y.

THEOREM 8.2. — *For any $y \in \mathfrak{p}$ one has*

$$\Gamma O(y) = \mathfrak{a}(y)$$

where $\mathfrak{a}(y)$ is defined in section 3.1.

Proof. — Since any element in $\mathfrak{p}$ is Ad K conjugate to an element in $\mathfrak{a}$ we can assume $y \in \mathfrak{a}$. Now if there exists $z \in O(y)$ such that $\Gamma z \notin \mathfrak{a}(y)$ then by the separation theorem there would exist a linear functional $\nu \in \mathfrak{a}'$ such that for all $\sigma \in W$:

(8.2.3) $$\langle \nu, \Gamma z \rangle > \langle \nu, \sigma y \rangle.$$

Furthermore we can choose ν so that

$$(8.2.4) \qquad\qquad \langle \nu, x_\gamma \rangle \neq 0 \quad \text{for all } \gamma \in \Gamma$$

(*see* § 3.2) since such linear functionals form a dense set in $\mathfrak{a}'$.

Now assume such a z and, hence ν, exist. But then since $O(y)$ is compact the function $x \mapsto \varphi(x) = \langle \nu, \Gamma x \rangle$ on $O(y)$ has a maximal value. Hence we can choose z so that the maximal value of φ is taken at z, and (8.2.3) is satisfied. But now if ν corresponds to the element $w \in \mathfrak{a}$ under the isomorphism $\mathfrak{a} \to \mathfrak{a}'$ defined by the Killing form then

$$(8.2.5) \qquad\qquad (w, x) = \langle \nu, \Gamma x \rangle = \varphi(x)$$

for any $x \in O(y)$ where round brackets denote the Killing form inner product. On the other hand by (8.2.4) one has $\langle \gamma, w \rangle \neq 0$ for all $\gamma \in \Gamma$ so that [using, say (2.2.2)] the

$$(8.2.6) \qquad\qquad \text{centralizer of } w \text{ in } \mathfrak{p} \text{ is } \mathfrak{a}.$$

But now for any $x \in \mathfrak{k}$ the function ψ on $\mathbf{R}$ given by

$$\psi(t) = (w, \mathrm{Ad}\,(\exp tx)\,z)$$

has zero derivative at $t = 0$ since φ takes its maximal value at z. But $\psi'(0) = (w, [x, z]) = ([z, w], x)$ by the invariance of the Killing form. Thus $([z, w], x) = 0$ for all $x \in \mathfrak{k}$. But $[z, w] \in [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$ and the Killing form is non-singular on $\mathfrak{k}$. Thus $[z, w] = 0$ which implies $z \in \mathfrak{a}$ by (8.2.6). But since z is conjugate to y this implies $z = \sigma y$ for some $\sigma \in W$, by Proposition 2.4. This however contradicts (8.2.3). Thus we have proved

$$(8.2.7) \qquad\qquad \Gamma(O(y)) \subseteq \mathfrak{a}(y).$$

Now let $\mathfrak{c} = \Gamma(O(y))$. We must show that $\mathfrak{a}(y) \subseteq \mathfrak{c}$. To prove this, the argument of section 4.5 reducing the problem to the case where $\mathfrak{g}$ is a normal real form of $\mathfrak{g}_{\mathbf{C}}$ also applies here. One needs only to remark that if $\mathfrak{q}^*$ is the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{p}^*$, with respect to the Killing form on $\mathfrak{g}^*$ using the notation of section 4.5, then $\mathfrak{q}^* \subseteq \mathfrak{q}$ (so that the projection $\Gamma^* : \mathfrak{p}^* \to \mathfrak{a}$ is the restriction of Γ to $\mathfrak{p}^*$). But this is clear since one easily shows that, using the Killing form, that $\mathfrak{q}^* = [\mathfrak{k}^*, \mathfrak{a}] \subseteq [\mathfrak{k}, \mathfrak{a}] = \mathfrak{q}$. Thus we may assume $\mathfrak{g}$ is a normal real form of $\mathfrak{g}_{\mathbf{C}}$. Now clearly $y \in \mathfrak{c}$ since $\Gamma y = y$. Thus by Lemma 4.8 and Remark 4.8 $\mathfrak{a}(y) \subseteq \mathfrak{c}$ if it can be shown that one has $\{w, \tau(\beta_i) w\} \subseteq \mathfrak{c}$ for any $w \in \mathfrak{c}$ and any simple root β_i. (The reduction to the simple root case was a significant reduction in the group case. Here it is not necessary. We do so now only for notational convenience).

Let $w \in \mathfrak{k}$ and let $\mathfrak{g}_i$ be the real TDS defined as in the proof of Lemma 4.6. Using the notation of the proof of Lemma 4.6 one has

$$(8.2.8) \qquad \{ w, \tau(\beta_i) w \} = \{ t w_i + w_i^{\perp} \mid -1 \leq t \leq 1 \},$$

where w_i and $w_i^{\perp}$ are the components of w in $\mathfrak{a}_i$ and $\mathfrak{a}_i^{\perp}$ [see (4.6.9)]. But now $w = \Gamma(v.y)$ for some $v \in \mathrm{K}$, writing $v.y$ for $\mathrm{Ad}\, v\,(y)$. Now let $\mathrm{K}_i \subseteq \mathrm{G}_1$ be defined as in the proof of Lemma 4.6 and let the map

$$\sigma: \mathrm{K}_i \to \mathfrak{a}$$

be defined by putting $\sigma(k) = \Gamma(kv.y)$, so that

$$(8.2.9) \qquad \sigma(\mathrm{K}_i) \subseteq \mathfrak{k}.$$

Now if $w_i = 0$ then $\{ w, \tau(\beta_i) w \}$ reduces to the point w and one trivially has $\{ w, \tau(\beta_i) w \} \subseteq \mathfrak{k}$. Hence we may assume $w_i \neq 0$. But then there exists unique continuous functions $\rho: \mathrm{K}_i \to \mathbf{R}$ and $\psi: \mathrm{K}_i \to \mathfrak{a}_i^{\perp}$ such that

$$\sigma(k) = \rho(k) w_i + \psi(k).$$

We assert ψ is the constant function given by $\psi(k) = w_i^{\perp}$ for all $k \in \mathrm{K}_i$. Since $\psi(1) = w_i^{\perp}$ to prove this it suffices to show that the inner product $(z, \sigma(k))$ is constant as a function of k for any fixed $z \in \mathfrak{a}_i^{\perp}$. But

$$(z, \sigma(k)) = (z, \Gamma(kv.y)) = (z, kv.y) = (k^{-1}.z, v.y)$$

by the invariance of the Killing form. But $k^{-1}.z = z$ for any $k \in \mathrm{K}_i$ since [see (4.6.11)] $[\mathfrak{a}_i^{\perp}, \mathfrak{g}_i] = 0$. Thus $(z, \sigma(k))$ is constant as a function on K_i proving that $\sigma(k) = \rho(k) w_i + w_i^{\perp}$. But $\rho(1) = 1$. On the other hand since there is an element τ in the Weyl group W_i of $(\mathfrak{a}_i, \mathfrak{g}_i)$ such that $\tau w_i = -w_i$ there exists $k_i \in \mathrm{K}_i$ such that $k_i.w_i = -w_i$. But $\mathrm{Ad}\, k_i = 1$ on $\mathfrak{a}_i^{\perp}$. Thus $\mathfrak{a}$ is stable under $\mathrm{Ad}\, k_i$ and hence $\mathfrak{q}$ is stable under $\mathrm{Ad}\, k_i$. Hence

$$\sigma(k_i) = \Gamma(k_i v.y) = k_i.\Gamma(v.y) = -w_i + w_i^{\perp}.$$

Thus $\rho(k_i) = -1$. But then since K_i is connected $\rho(\mathrm{K}_i)$ take all values in $[-1, 1]$. Thus $\{ w, \tau(\beta_i) w \} \subseteq \sigma(\mathrm{K}_i)$ by (8.2.8). But then $\{ w, \tau(\beta_i) w \} \subseteq \mathfrak{k}$ by (8.2.9).

<div align="right">Q. E. D.</div>

8.3. Now one can prove a K-conjugation theorem which is analogous to the K-double coset result, Theorem 5.3. For any $x \in \mathfrak{a}$ let

$$(8.3.1) \qquad \mathfrak{p}_x = \{ y \in \mathfrak{p} \mid y \geq x \}$$

where the order relation is defined by (7.1.1).

THEOREM 8.3. — *For any* $x \in \mathfrak{a}$ *one has*

$$(8.3.2) \qquad \qquad \mathfrak{p}_x = \bigcup_{k \in K} \mathrm{Ad}\, k\, (x + \mathfrak{q})$$

where $\mathfrak{q}$ *is the orthocomplement of* $\mathfrak{a}$ *in* $\mathfrak{p}$.

Proof. — Let $\mathfrak{d}$ denote the right side of (8.3.2) and let $y \in \mathfrak{d}$. Then there exists $k \in K$ such that $y \in \mathrm{Ad}\, k\, (x + \mathfrak{q})$. Thus $k^{-1}.y \in x + \mathfrak{q}$ or $\Gamma\,(k^{-1}.y) = x$. But then $x \in \mathfrak{a}\,(y)$ by Theorem 8.2 and hence $y \geq x$ or $y \in \mathfrak{p}_x$. Conversely assume $y \in \mathfrak{p}_x$. Thus $x \in \mathfrak{a}\,(y)$ and hence there exists $k \in K$ such that $x = \Gamma\,(k^{-1}.y)$ by Theorem 8.2. That is, $k^{-1}y \in x + \mathfrak{q}$ or $y \in \mathrm{Ad}\, k\, (x + \mathfrak{q}) \subseteq \mathfrak{d}$.

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