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ON p -ADIC DIFFERENTIAL EQUATIONS IV
GENERALIZED HYPERGEOMETRIC FUNCTIONS AS p -ADIC
ANALYTIC FUNCTIONS IN ONE VARIABLE

By B. DWORK

In a recent article, [1], we investigated the analytic continuation in the sense of Krasner of certain ratios of generalized hypergeometric series.

We are concerned with the series ($q \geq n$) :

$${}_nF_{q-1} \left(\begin{matrix} \theta_1, \dots, \theta_n; t \\ \sigma_1, \dots, \sigma_{q-1} \end{matrix} \right)$$

[*cf.* equation (2.2) below]. Our previous methods permitted a discussion of the case in which $q = n$ and $1 = \sigma_1 = \sigma_2 = \dots = \sigma_{q-1}$. The present article eliminates this restriction. Briefly, we show that subject to the hypotheses of Theorem 3.1 below, we have the formal congruence of equation (1.2). In [1, § 3] we showed how such a congruence leads to analytic continuation. The application of such functions to the theory of zeta functions of varieties is explained in [1, § 6].

It is well known that ${}_nF_{q-1}$ satisfies an q^{th} order ordinary differential equation with rational coefficients. In paragraph 4 of this article we obtain congruences (Theorem 4.1 below) satisfied by certain ratios of solutions of these equations. It seems quite likely that the results of this article can in the case $q = n$ be deduced from the action of Frobenius on p -adic cohomology ([2], [3]). However for $q > n$ the differential equation satisfied by ${}_nF_{q-1}$ has an irregular singularity and hence cannot be obtained by variation of cohomology of algebraic varieties. Thus the present article may provide new tests for conjectures [4] concerning the existence of Frobenius structures for ordinary linear differential equations.

We assume throughout that $q \geq n$. In the contrary case, $q < n$, the origin would be the irregular singularity and the series given by equation (2.2) below would be of asymptotic type. This would not deprive this situation of interest. From our point of view the difficulty is that

in this case $\beta = \frac{q-n}{p-1}$ would be negative and thus condition (iv) of Lemma 2.2 below could be satisfied only under extremely unusual circumstances. (In any case paragraph 2 remains valid.) At the suggestion of the referee some numerical examples are given in paragraph 5 to explain how conditions (iv) and condition (vi) of paragraph 3 below may be readily reduced to questions of primes in arithmetic progressions.

Notation :

- Ω = completion of algebraic closure of the p -adic rationals;
- \mathfrak{O} = ring of integers of Ω ;
- Ω^* = multiplicative group of Ω ;
- \mathbb{Z} = ring of ordinary integers;
- \mathbb{Z}_+ = the non-negative elements of \mathbb{Z} ;
- \mathfrak{C} = the set of all rational numbers which are p -integral but which are neither zero nor a negative rational integer;
- $\pi = (-p)^{\frac{1}{p-1}}$;
- $\beta = \frac{q-n}{p-1}$.

1. A FORMAL CONGRUENCE. — We first generalize Theorem 2 of [1]

THEOREM 1.1. — For $r = 0, 1, \dots$, let $A^{(r)}$ be a mapping of \mathbb{Z}_+ into Ω^* and let g_r be a mapping of \mathbb{Z}_+ into $\mathfrak{O} - \{0\}$ such that

- (i) $|A^{(r)}(0)| = 1$;
- (ii) $A^{(r)}(m) \in g_r(m) \mathfrak{O}$;
- (iii) for all $a, r, \mu, s \in \mathbb{Z}_+$ such that $a < p, \mu < p^s$ we have

$$(1.1) \quad \frac{A^{(r)}(a + p\mu + mp^{s+1})}{A^{(r)}(a + p\mu)} - \frac{A^{(r+1)}(\mu + mp^s)}{A^{(r+1)}(\mu)} \in p^{s+1} \frac{g_{r+s+1}(m)}{g_r(a + p\mu)} \mathfrak{O}.$$

Furthermore, let

$$F(X) = \sum_{j=0}^{\infty} A^{(0)}(j) X^j, \quad G(X) = \sum_{j=0}^{\infty} A^{(1)}(j) X^j$$

and let $F_{m,s}$ (respectively : $G_{m,s}$) denote the partial sum

$$\sum_{j=mp^s}^{(m+1)p^s-1} A^{(0)}(j) X^j \quad \left(\text{resp.} \quad \sum_{j=mp^s}^{(m+1)p^s-1} A^{(1)}(j) X^j \right).$$

Then

$$(1.2) \quad F(X) G_{m,s}(X^p) \equiv G(X^p) F_{m,s+1}(X) \pmod{g_s(m) p^{s+1} [[X]]}.$$

Note. — The functions g_r , may be viewed as mappings of Z_+ into the value group of Ω . Given the mappings $\{A^{(r)}\}_{r=0,1,\dots}$, the choice of g_r is to some extent arbitrary. In our earlier work we chose g_r to be the same as $|A^{(r)}|$.

Proof. — Let a, N be positive integers, $a < p$. We denote by $H_a(m, s, N)$, the coefficient of X^{a+pN} in the difference between the two sides of equation (1.2). Precisely as in our earlier work,

$$H_a(m, s, N) = \sum_{j=mp^s}^{(m+1)p^s-1} U_a(j, N),$$

where

$$U_a(j, N) = A^{(0)}(a + p(N - j)) A^{(1)}(j) - A^{(1)}(N - j) A^{(0)}(a + pj).$$

We must show

$$(1.3) \quad H_a(m, s, N) \in p^{s+1} g_{s+1}(m) \mathfrak{O}$$

for all $m, s, N \in Z_+$.

For $s \geq 1$ let α_s denote the statement,

$$\alpha_s: H_a(m, u, N) \equiv 0 \pmod{p^{u+1} g_{u+1}(m)} \quad \text{for } u \in [0, s]; \quad m, N \geq 0.$$

For $0 \leq t \leq s$, let $\beta_{t,s}$ denote the statement,

$$\beta_{t,s}: H_a(m, s, N + mp^s) \equiv \sum_{j=0}^{p^s-t-1} \frac{A^{(t+1)}(j + mp^{s-t})}{A^{(t+1)}(j)} H_a(j, t, N) \pmod{p^{s+1} g_{s+1}(m)}.$$

We now list three assertions whose validity imply equation (1.3).

ASSERTION 1. — $U_a(m, N) \in pg_1(m)$.

ASSERTION 2. — For $j < p^s$,

$$U_a(j + mp^s, N + mp^s) \equiv \frac{A^{(1)}(j + mp^s)}{A^{(1)}(j)} U_a(j, N) \pmod{p^{s+1} g_{s+1}(m)}.$$

ASSERTION 3. — For $t < s$, statements $\alpha_s, \beta_{t,s}$ imply $\beta_{t+1,s}$.

Before proving these assertions, we show that their validity implies equation (1.3). Since

$$(1.4) \quad U_a(m, N) = H_a(m, 0, N),$$

it is clear that Assertion 1 is equivalent to α_1 . We note that $\beta_{0,s}$ is the statement

$$\beta_{0,s}: H_a(m, s, N + mp^s) \equiv \sum_{j=0}^{p^s-1} \frac{A^{(1)}(j + mp^s)}{A^{(1)}(j)} H_a(j, 0, N) \pmod{p^{s+1} g_{s+1}(m)}.$$

It follows from equation (1.4) and Assertion 2 that the right side of statement $\beta_{0,s}$ coincides mod $p^{s+1} g_{s+1}(m)$ (since the sum is over $j < p^s$) with

$$\sum_{j=0}^{p^s-1} U_a(j + mp^s, N + mp^s),$$

which clearly coincides with the left side of $\beta_{0,s}$. Thus (assuming the validity of the three assertions) we have verified α_1 and using induction, assume α_s for fixed $s \geq 1$. Having verified $\beta_{0,s}$, we assume $\beta_{t_0,s}$ for fixed $t_0 \in [0, s]$. Assertion 3 now implies $\beta_{1+t_0,s}$ and hence we may assume $\beta_{t,s}$ for all $t \in [0, s]$. In particular this implies $\beta_{s,s}$. The remainder of this part of the proof follows [1, Theorem 2]. Briefly : Let γ_N be the statement (a, s both fixed) :

$$\gamma_N : H_a(0, s, N) \in p^{s+1}.$$

We know γ_N for $N \leq 0$. We let N' be minimal (if it exists) such that γ_N is false. By

$$\beta_{s,s} : H_a(m, s, N + mp^s) \equiv \frac{A^{(s+1)}(m)}{A^{(s+1)}(0)} H_a(0, s, N) \pmod{p^{s+1} g_{s+1}(m)},$$

and using hypotheses (i), (ii), we conclude that $\frac{A^{(s+1)}(m)}{A^{(s+1)}(0)} \in \mathfrak{O}$ and hence putting (for $m > 0$) $N = N' - mp^s < N'$, we conclude that for $m > 0$,

$$(1.5) \quad H_a(m, s, N') \equiv 0 \pmod{p^{s+1}}.$$

But for $(T+1)p^s > N'$, we have (cf. [1, equation (2.5)]) :

$$(1.6) \quad \sum_{m=0}^T H_a(m, s, N') = 0.$$

Equations (1.5) and (1.6) show that $H_a(0, s, N') \equiv 0 \pmod{p^{s+1}}$, contradicting the choice of N' . This proves (γ_N) and equation (1.3) now follows from $\beta_{s,s}$ and hypotheses (i), (iii).

The proof of the theorem has thus been reduced to the proof of these assertions.

Notation. — Let $A^{(0)}$ (resp. $A^{(1)}$) be denoted by A (resp. B).

Proof of Assertion 1. — By definition

$$(1.7) \quad U_a(m, N) = A(a) B(m) \left[\frac{A(a + p(N - m))}{A(a)} - \frac{B(N - m)}{B(0)} \right] \\ - B(N - m) A(a) \left[\frac{A(a + pm)}{A(a)} - \frac{B(m)}{B(0)} \right].$$

We apply hypothesis (iii) (with $0 = r = s = \mu$) and obtain

$$U_a(m, N) \in A(a) B(m) \frac{pg_1(N-m)}{g_0(a)} \mathfrak{O} + B(N-m) A(a) \frac{pg_1(m)}{g_0(a)}$$

which is

$$pg_1(m) g_1(N-m) \frac{A(a)}{g_0(a)} \left[\frac{B(m)}{g_1(m)} \mathfrak{O} + \frac{B(N-m)}{g_1(N-m)} \mathfrak{O} \right]$$

and Assertion 1 now follows hypothesis (ii).

Proof of Assertion 2. — It follows from the definitions that

$$(1.8) \quad U_a(j + mp^s, N + mp^s) - \frac{A^{(1)}(j + mp^s)}{A^{(1)}(j)} U_a(j, N) \\ = -B(N-j) A(a + pj) \left[\frac{A(a + pj + mp^{s+1})}{A(a + pj)} - \frac{B(j + mp^s)}{B(j)} \right].$$

Since $j < p^s$, we may apply hypothesis (iii) and deduce that the right side of the last equation lies in

$$p^{s+1} g_{s+1}(m) g_1(N-j) \frac{A(a + pj)}{g_0(a + pj)} \frac{B(N-j)}{g_1(N-j)},$$

which by hypothesis (ii) implies Assertion 2.

Proof of Assertion 3. — For $t < s$, we write $\beta_{t,s}$ in the form (putting $j = i + p\mu$, $i < p$, $\mu < p^{s-t-1}$):

$$\beta_{t,s} : H_a(m, s, N + mp^s) \equiv \sum_{i=0}^{p-1} \sum_{\mu=0}^{p^{s-t-1}-1} \frac{A^{(t+1)}(i + p\mu + mp^{s-t})}{A^{(t+1)}(i + p\mu)} H_a(i + p\mu, t, N) \\ [\text{mod } p^{s+1} g_{s+1}(m)].$$

We are to show that this statement together with α_s implies $\beta_{t+1,s}$. By a purely formal manipulation, we deduce from $\beta_{t,s}$, that if we define X by

$$(1.9) \quad H_a(m, s, N + mp^s) - \sum_{\mu=0}^{p^{s-t-1}-1} \frac{A^{(t+2)}(\mu + mp^{s-t-1})}{A^{(t+2)}(\mu)} \sum_{i=0}^{p-1} H_a(i + p\mu, t, N) = X$$

then

$$(1.10) \quad X \equiv \sum_{i=0}^{p-1} \sum_{\mu=0}^{p^{s-t-1}-1} H_a(i + p\mu, t, N) \\ \times \left[\frac{A^{(t+1)}(i + p\mu + mp^{s-t})}{A^{(t+1)}(i + p\mu)} - \frac{A^{(t+2)}(\mu + mp^{s-t-1})}{A^{(t+2)}(\mu)} \right] [\text{mod } p^{s+1} g_{s+1}(m)].$$

In the sum on the right side of this last equation, $\mu < p^{s-t-1}$ and hence by hypothesis (iii) the expression in the square brackets lies in $\mathfrak{O} p^{s-t} \frac{g_{s+1}(m)}{g_{t+1}(i + p\mu)}$. Furthermore $t < s$ and hence by α_s ,

$$H_a(i + p\mu, t, N) \in p^{t+1} g_{t+1}(i + p\mu) \mathfrak{O}.$$

These estimates, together with (1.10) show that

$$(1.11) \quad X \equiv 0 \pmod{p^{s+1} g_{s+1}(m)}.$$

We recall [1, equation (2.6)] that

$$(1.12) \quad H_a(\mu, t+1, N) = \sum_{i=0}^{p-1} H_a(i + p\mu, t, N).$$

The statement, $\beta_{t+1, s}$ follows from equations (1.11), (1.09), (1.12). This completes the proof of Assertion 3 and hence of the theorem.

2. BOUNDEDNESS OF HYPERGEOMETRIC SERIES. — To apply the preceding theorem to hypergeometric series, we must associate with each hypergeometric series, a sequence of such series and a definition of the auxiliary functions, g . In this section we give such definitions together with sufficient conditions for the applicability of hypotheses (i), (ii) of the theorem. (A corresponding discussion of hypothesis (iii) will be found in paragraph 3 below.)

Let p be a fixed prime and let \mathfrak{C} be the set of all rational numbers which are p -integral but are neither zero nor a negative rational integer. As in [1, § 1] we use the mapping $x \rightarrow x'$ of \mathfrak{C} into itself, defined by the condition that $px' - x$ be the minimal representative (in Z_+) of the class of $-x \pmod{p}$. By ν -fold iteration, we obtain

$$(2.1) \quad x \rightarrow x^{(\nu)}$$

a mapping of \mathfrak{C} into itself and we may use the same symbol to denote the component-wise application of this mapping to n -tuples whose components lie in \mathfrak{C} . If θ (resp. σ) is an n -tuple [resp. $(q-1)$ -tuple] with components in \mathfrak{C} then the (generalized) hypergeometric function is defined by

$$(2.2) \quad {}_nF_{q-1} \left[\begin{matrix} \theta \\ \sigma \end{matrix}; \pi^{q-n} X \right] = \sum_{m=0}^{\infty} A(m) X^m$$

where $\pi = (-p)^{\frac{1}{p-1}}$ and

$$\pi^{m(n-q)} A(m) = \frac{\prod_{i=1}^n (\theta_i)_m}{m! \prod_{j=1}^{q-1} (\sigma_j)_m},$$

it being recalled that

$$(x)_m = \begin{cases} 1 & \text{if } m = 0, \\ x(x+1) \dots (x+m-1) & \text{if } m > 0. \end{cases}$$

It is clear that for θ and σ with components in \mathfrak{C} , we have associated with ${}_nF_{n-1} \left[\begin{matrix} \theta; \pi^{q-n} X \\ \sigma \end{matrix} \right]$ as defined by equation (2.2), an infinite sequence of hypergeometric series,

$$(2.3) \quad {}_nF_{q-1} \left[\begin{matrix} \theta^{(\nu)}; \pi^{q-n} X \\ \sigma^{(\nu)} \end{matrix} \right] = \sum_{m=0}^{\infty} A^{(\nu)}(m) X^m.$$

In particular the original series (2.2) corresponds to the case $\nu = 0$. [Note that $A^{(\nu)}(m)$ is defined by equation (2.3) and that this symbol should not be taken to denote the result of applying to $A(m)$ the operation of (2.4).]

Let q' be the number of components, σ_j , of σ such that $\sigma_j \neq 1$. We rearrange the subscripts so that $\sigma_j \neq 1$ for $j \leq q'$. We define the auxiliary functions, g_ν ;

$$(2.4) \quad g_\nu(m) = \frac{A^{(\nu)}(m)}{q' \prod_{j=1}^{q'} (m + \sigma_j^{(\nu)})}.$$

For $x \in \mathfrak{C}$, $a \in [0, p)$ we define

$$\rho(a, x) = \begin{cases} 0 & \text{if } a \leq px' - x, \\ 1 & \text{if } a > px' - x \end{cases}$$

and we put

$$N_\theta(a) = \sum_{i=1}^n \rho(a, \theta_i)$$

and similarly

$$N_\theta(a) = \sum_{j=1}^{q'} \rho(a, \sigma_j),$$

We define $N_{\theta^{(\nu)}}$ and $N_{\sigma^{(\nu)}}$ by similar formulae, using the same value of q' for all $\sigma^{(\nu)}$. For fixed x , the map $a \rightarrow \rho(a, x)$ is discontinuous with a certain jump. We use $\rho(a+, x)$ to denote the limit of $\rho(b, x)$ as b approaches a from the right. A similar meaning for $N_\sigma(a+)$, $N_\theta(a+)$ is to be understood.

LEMMA 2.1. — For $a, \mu \in \mathbb{Z}_+$, $a < p$, we have

$$\begin{aligned} \text{ord} \frac{g_0(a + p\mu)}{g_1(\mu)} &= N_\theta(a) - N_\sigma(a+) + \sum_{i=1}^n \rho(a, \theta_i) \text{ord}(\mu + \theta_i) \\ &\quad + \sum_{j=1}^{q'} (1 - \rho(a+, \sigma_j)) \text{ord}(\mu + \sigma_j) + \beta a. \end{aligned}$$

Proof. — We recall [1, equation 1.3] that for $x \in \mathfrak{C}$,

$$\text{ord} \frac{(x)_{a+p\mu}}{(x')_\mu} = \mu + (1 + \text{ord}(\mu + x')) \rho(a, x)$$

and in particular, for $x = 1$ the right side is just μ . Thus we obtain

$$\begin{aligned} \text{ord} \frac{g_0(a+p\mu)}{g_1(\mu)} &= \sum_{i=1}^n [1 + \text{ord}(\mu + \theta_i)] \rho(a, \theta_i) + \beta a \\ &\quad - \sum_{j=1}^{q'} [1 + \text{ord}(\mu + \sigma_j)] \rho(a, \sigma_j) \\ &\quad + \sum_{j=1}^{q'} (\text{ord}(\mu + \sigma_j) - \text{ord}(a + p\mu + \sigma_j)). \end{aligned}$$

The right-hand side of the last equation is clearly

$$N_0(a) - N_\sigma(a+) + \sum_{i=1}^n \text{ord}(\mu + \theta_i) \rho(a, \theta_i) + \sum_{j=1}^{q'} (Y_j + E_j) + \beta a,$$

where

$$Y_j = (1 - \rho(a+, \sigma_j)) \text{ord}(\mu + \sigma_j),$$

$$E_j = (\rho(a+, \sigma_j) - \rho(a, \sigma_j)) (1 + \text{ord}(\mu + \sigma_j)) - \text{ord}(a + p\mu + \sigma_j).$$

The lemma follows from $E_j = 0$ which in turn follows from the explicit formulae

$$\begin{aligned} \rho(a+, \sigma_j) - \rho(a, \sigma_j) &= \begin{cases} 1 & \text{if } a = p\sigma_j - \sigma_j, \\ 0 & \text{otherwise;} \end{cases} \\ \text{ord}(a + p\mu + \sigma_j) &= \begin{cases} 1 + \text{ord}(\mu + \sigma_j) & \text{if } a = p\sigma_j - \sigma_j, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

LEMMA 2.2. — *If for all $a, \nu \in \mathbb{Z}_+$, $a < p, j = 1, 2, \dots, q'$ we have*

$$(iv) \beta a + N_{0^{(\nu)}}(a) \geq N_{\sigma^{(\nu)}}(a+);$$

$$(v) |\sigma_j^{(\nu)}| = 1$$

then for all ν, g_ν maps \mathbb{Z}_+ into \mathfrak{O} and in particular

$$F \left[\begin{matrix} \theta, \pi^{q-n} X \\ \sigma \end{matrix} \right] \in \mathfrak{O}[[X]].$$

Proof. — Using induction on m , we may use hypothesis (iv) and Lemma 2.1 to show that $g_\nu(m) \in \mathfrak{O}$ for all ν provided this holds for $m = 0$. Since

$$g_\nu(0) = \frac{1}{\prod_{j=1}^{q'} \sigma_j^{(\nu)}},$$

hypothesis (v) shows that $g_\nu(0) \in \mathfrak{O}$. This proves the assertion concerning g_ν , and the lemma then follows from equation (2.4).

3. FORMAL CONGRUENCES FOR HYPERGEOMETRIC SERIES. — We use the same notation as in paragraph 2. Our object is to find sufficient conditions for the applicability of Theorem 1.1 to the series defined by equation (2.2). Throughout this section we shall suppose that θ and σ satisfy the conditions:

$$(v) \quad |\sigma_j^{(\nu)}| = 1 \quad \text{for } j = 1, 2, \dots, q', \nu \in \mathbb{Z}_+ \quad \text{and if } p = 2 \text{ then } q \equiv n \pmod{2};$$

(vi) $\beta a + N_{\theta^{(\nu)}}(a) \geq N_{\sigma^{(\nu)}}(a+) + \mathfrak{S}(N_{\sigma^{(\nu)}}(a+))$, for $a \in [0, p)$, $\nu \in \mathbb{Z}_+$ (where \mathfrak{S} denotes the characteristic function of the set of strictly positive real numbers, viewed as a subset of the reals). Hypothesis (vi) means that for each pair (ν, a) we have either

$$(3.1) \quad N_{\theta^{(\nu)}}(a) = N_{\sigma^{(\nu)}}(a+) = 0$$

or

$$(3.2) \quad \beta a + N_{\theta^{(\nu)}}(a) - N_{\sigma^{(\nu)}}(a+) \geq 1.$$

LEMMA 3.1. — *If $\mu < p^s$ then (for $j = 1, 2, \dots, n'$),*

$$\text{ord}(\sigma_j + \mu) \leq s.$$

Proof. — Suppose otherwise, say $\text{ord}(\sigma_1 + \mu) > s$. Then

$$(3.3) \quad \mu = \sum_{\nu=0}^{s-1} p^\nu (p \sigma_1^{(\nu+1)} - \sigma_1^{(\nu)})$$

since the right side is the minimal (non-negative) integral representative of $-\sigma_1$ modulo p^s . Thus $\mu + \sigma_1 = p^s \sigma_1^{(s)}$. Hence $\sigma_1^{(s)} \equiv 0 \pmod{p}$, which contradicts hypothesis (v).

LEMMA 3.2. — *For $a, \mu, m, s \in \mathbb{Z}_+$, $a < p$, $\mu < p^s$, we have*

$$(3.4) \quad \frac{g_0(a + p\mu + mp^{s+1})}{g_{s+1}(m)} \in \mathfrak{O}.$$

Denoting this ratio by u , we assert that

$$(3.5) \quad \frac{u}{mp^s + \mu + \theta_i} \rho(a, \theta_i) \equiv 0 \pmod{p} \quad \text{for } i = 1, 2, \dots, n,$$

$$(3.6) \quad \frac{u}{\mu + \sigma_j} \rho(a, \sigma_j) \equiv 0 \pmod{p} \quad \text{for } j = 1, 2, \dots, n'.$$

Proof. — The first assertion follows from iteration of Lemma 2.4, using $\mu < p^s$. This assertion is equivalent to

$$(3.7) \quad \frac{g_\nu(N)}{g_{\nu+s}\left(\left[\frac{N}{p^s}\right]\right)} \in \mathfrak{O} \quad (\text{for all } N \in \mathbb{Z}_+)$$

and hence in particular

$$(3.8) \quad \frac{g_1(\mu + mp^s)}{g_{s+1}(m)} \in \mathfrak{O}.$$

Thus in verifying (3.5) we may with no loss in generality replace u by $\frac{g_0(a + p\mu + mp^{s+1})}{g_1(\mu + mp^s)}$. Equation (3.5) now follows from Lemma 2.4.

To verify equation (3.6) we may suppose $j = 1$,

$$a > p\sigma'_1 - \sigma_1$$

so that equation (3.2) holds for $\nu = 0$. It follows from Lemma 2.4 that

$$\text{ord} \frac{g(a + p\mu + mp^{s+1})}{g_1(\mu + mp^s)} \geq 1$$

and hence it is enough to prove

$$(3.9) \quad \frac{g_1(\mu + mp^s)}{g_{s+1}(m)} \in (\mu + \sigma'_1) \mathfrak{O}.$$

Setting $t = \text{ord}(\mu + \sigma'_1)$ the assertion follows from (3.7) unless $t > 0$ as shall now be supposed. However, by the preceding lemma, $t \leq s$.

For $\nu = 0, 1, \dots, s-1$, put

$$T_\nu = \frac{g_{\nu+1}\left(\left[\frac{\mu}{p^\nu}\right] + mp^{s-\nu}\right)}{g_{\nu+2}\left(\left[\frac{\mu}{p^{\nu+1}}\right] + mp^{s-\nu-1}\right)}$$

and we note that the left side of (3.9) is the same as $T_0 T_1 \dots T_{s-1}$. By (3.7) each $T_\nu \in \mathfrak{O}$ and hence we may prove (3.9) if we can show

$$(3.10) \quad T_\nu \in p \mathfrak{O} \quad \text{for } \nu = 0, 1, \dots, t-1.$$

The definition of t shows that

$$\mu = \sum_{\nu=0}^{t-1} p^\nu (p\sigma_1^{(\nu+2)} - \sigma_1^{(\nu+1)}) + p^t M$$

where $M \in \mathbb{Z}_+$.

Hence, for $\nu \leq t-1$, putting $a_\nu = p\sigma_1^{(\nu+2)} - \sigma_1^{(\nu+1)}$, we have

$$\left[\frac{\mu}{p^\nu}\right] \equiv a_\nu \pmod{p}$$

and hence by lemma 2.1, for $\nu \leq t - 1$,

$$\text{ord } T_\nu \geq N_{\theta(\nu+1)}(a_\nu) - N_{\sigma(\nu+1)}(a_\nu +) + \beta a.$$

Now

$$N_{\sigma(\nu+1)}(a_\nu +) \geq \rho(a_\nu +, \sigma_1^{(\nu+1)}) = 1$$

and hence by hypothesis (vi), $\text{ord } T_\nu \geq 1$. This completes the proof of the lemma.

THEOREM 3.1. — *The hypotheses of Theorem 1.1 are satisfied by the functions $A^{(\nu)}$, g_ν defined by equations (2.3), (2.4), it being understood that hypotheses (v), (vi) are valid.*

Proof. — Lemma 2.2 shows that g_ν is a mapping into \mathfrak{O} , hypothesis (ii) of Theorem 1.1 follows from equation (2.4) and hypothesis (i) is trivial. The remainder of the proof concerns the verification of hypothesis (iii).

We may restrict our attention to the case $r = 0$ in that hypothesis and as in paragraph 1 we use A (resp. B) to denote $A^{(0)}$ (resp. $A^{(1)}$).

We must show for $\mu < p^s$,

$$(3.11) \quad \left(A(a + p\mu + mp^{s+1}) - A(a + p\mu) \frac{B(\mu + mp^s)}{B(\mu)} \right) \\ \times \frac{1}{A(a + p\mu)} \in p^{s+1} \frac{g_{s+1}(m)}{g_0(a + p\mu)}.$$

Let

$$(3.12) \quad Y = \frac{\prod_{j=1}^{q'} \left(1 + \frac{mp^s}{\mu + \sigma_j} \right)^{\rho(a, \sigma_j)}}{\prod_{i=1}^n \left(1 + \frac{mp^s}{\mu + \theta_i} \right)^{\rho(a, \theta_i)}}.$$

We assert that (3.11) is implied by

$$(3.13) \quad g_0(a + p\mu + mp^{s+1})(Y' - 1) \in p^{s+1} g_{s+1}(m) \mathfrak{O}$$

for all $Y' \in (1 + p^{s+1})Y$, $\mu < p^s$.

To prove this assertion, we use [1, equations (1.1), (1.2)] which shows that [if $p = 2$, we use here the hypothesis $q \equiv n \pmod{2}$] :

$$A(a + p\mu) \frac{B(\mu + mp^s)}{B(\mu)} = A(a + p\mu + mp^{s+1}) Y'$$

where $Y' \in Y(1 + (p^{s+1}))$ and hence (3.11) is implied by

$$(3.14) \quad (Y' - 1) \frac{A(a + p\mu + mp^{s+1})}{A(a + p\mu)} \in p^{s+1} \frac{g_{s+1}(m)}{g_0(a + p\mu)}$$

It follows from equation (2.4) that

$$\frac{A(a + p\mu + mp^{s+1})}{A(a + p\mu)} = \frac{g_0(a + p\mu + mp^{s+1})}{g_0(a + p\mu)} \prod_{j=1}^{q'} \left(1 + \frac{mp^{s+1}}{\sigma_j + a + p\mu}\right).$$

Since $a + p\mu < p^{s+1}$, it follows from Lemma 3.1 that the factors of the product, $\prod_{j=1}^{q'}$, lie in \mathfrak{O} and hence (3.14) is implied by the same statement with the function A replaced by g_0 . The assertion that equation (3.13) implies (3.11) is now clear. To verify equation (3.13) we consider three cases.

Case I : $s = 0$. — In this case, $\mu = 0$ and hence (3.13) assumes the form

$$(3.15) \quad \frac{g_0(a + pm)}{g_1(m)} (Y' - 1) \in p.$$

There are two possibilities :

(α) Equation (3.1) holds for $\nu = 0$: In this case $Y = 1$; hence $Y' - 1 \in p\mathfrak{O}$ and equation (3.15) then follows from equation (3.7).

(β) Equation (3.2) holds for $\nu = 0$: Let

$$(3.16) \quad T(m) = \sum_{i=1}^n \rho(a, \theta_i) \text{ord}(m + \theta_i).$$

For future use we note

$$(3.16') \quad T(x + mp^s) \geq \text{Min}(s, T(x)).$$

Lemma 2.1 shows that

$$(3.17) \quad \text{ord} \frac{g_0(a + mp)}{g_1(m)} \geq 1 + T(m).$$

On the other hand, equation (3.12), hypothesis (ν) and $\mu = 0$ give

$$\text{ord } Y \geq -T(m).$$

In any case $\text{ord } Y' = \text{ord } Y$ and since $T(m)$ is non-negative we conclude that

$$(3.18) \quad \text{ord}(Y' - 1) \geq -T(m).$$

Equation (3.15) follows from (3.17), (3.18). This completes the proof of (3.13) in Case I.

Case II. — Here we assume $s \geq 1$ and that in the notation of (3.16)

$$(3.19) \quad T(\mu) \geq s.$$

Clearly (3.1) cannot hold (as $s \neq 0$) and hence (3.2) holds for $\nu = 0$. Thus by Lemma 2.1 we have

$$(3.20) \quad \text{ord} \frac{g_0(a + p\mu + mp^{s+1})}{g_1(\mu + mp^s)} \geq 1 + T(\mu + mp^s).$$

From Lemma 3.1 and equation (3.12), we find

$$(3.21) \quad \text{ord } Y' = \text{ord } Y \geq T(\mu) - T(\mu + mp^s).$$

It follows from equations (3.19), (3.16') that

$$1 + T(\mu + mp^s) + \text{Min} \{ 0, T(\mu) - T(\mu + mp^s) \} \geq 1 + s.$$

Hence the sum of $\text{ord}(Y' - 1)$ and the left side of (3.20) is not less than $s + 1$. Equation (3.13) now follows from equation (3.8). This completes the proof in Case II.

Case III. — We assume $x \geq 1$ but that $T(\mu) < s$. Hence in particular, if $a > p\theta'_i - \theta_i$ then

$$(3.22) \quad \text{ord}(\mu + \theta_i) < s.$$

We deduce from this hypothesis, from Lemma 3.1, and from equation (3.12) that

$$(3.23) \quad Y' - 1 \in p^{s+1} \mathfrak{O} + \sum_{i=1}^n \rho(a, \theta_i) \frac{mp^s}{\mu + \theta_i} \mathfrak{O} + \sum_{j=1}^n \rho(a, \sigma_j) \frac{mp^s}{\mu + \sigma_j} \mathfrak{O}.$$

Equation (3.13) follows immediately from this equation and Lemma 3.2. This completes the proof in this case and thus completes the proof of the theorem.

4. RATIO OF SOLUTIONS. — We recall that the differential equation,

$$(4.1) \quad t(1-t) \frac{d^2 u}{dt^2} + (1-2t) \frac{du}{dt} - \frac{1}{4} u = 0,$$

satisfied by $F = F\left(\frac{1}{2}, \frac{1}{2}, 1, t\right)$ has a logarithmic singularity at $t = 0$ and hence there exists a unique (local) solution

$$(4.2) \quad u = \mathfrak{f} + F \log t,$$

specified by the choice of F and the condition that \mathfrak{f} be holomorphic and vanish at $t = 0$. It is well known from the theory of elliptic modular functions that for $p \neq 2$:

$$(4.3) \quad \exp \frac{\mathfrak{f}}{F} \in \mathfrak{O}[[t]].$$

The object of this section is to exhibit this fact as a special case of a property of generalized hypergeometric functions.

We use the notation of paragraph 2. For $x \in \mathbb{C}$, $m \in \mathbb{Z}_+$, let

$$D_x(m) = \begin{cases} 0 & \text{if } m = 0, \\ \sum_{\nu=0}^{m-1} \frac{1}{x + \nu} & \text{if } m > 0. \end{cases}$$

LEMMA 4.1. — For $x \in \mathbb{C}$, $b, t, M \in \mathbb{Z}_+$, $b < p$, we have

$$D_x(bp^t + Mp^{t+1}) - D_x(Mp^{t+1}) \equiv \frac{1}{p^{t+1}} \frac{\rho(b, x^{(t)})}{M + x^{(t+1)}} \pmod{\frac{1}{p^t}}.$$

Proof. — The left side is equal to the sum of the reciprocals of $\nu + x + Mp^{t+1}$ as ν runs through all integers in $[0, bp^t]$. Modulo p^{-t} we need retain only those ν for which $\nu \equiv -x \pmod{p^{t+1}}$. The p -adic and archimedean conditions imply that at most one ν be retained, namely

$$\nu = \sum_{j=0}^t p^j (px^{(j+1)} - x^{(j)}),$$

and this appears only if

$$b > px^{(t+1)} - x^{(t)}.$$

This completes the proof of the Lemma.

We now need a combinatorial result.

LEMMA 4.2. — Let W be any mapping of \mathbb{Z}_+ into (say) Ω . Let \bar{W} denote the “integral”

$$\bar{W}(m, s) = \sum_{j=mp^s}^{(m+1)p^s-1} W(j).$$

Let f be any mapping of \mathbb{Z}_+ into Ω , then for $s \in \mathbb{Z}_+$,

$$\sum_{i=0}^{p^{s+1}-1} f(j) W(j) = f(0) \bar{W}(0, s+1) + \sum_{t=0}^s \sum_{i=0}^{p^{t+s-t}-1} (f(jp^t) - f\left(\left[\frac{j}{p}\right]p^{t+1}\right)) \bar{W}(j, t).$$

Proof. — Let

$$X_t = \sum_{j=0}^{p^{t+s-t}-1} f(jp^t) \bar{W}(j, t)$$

and let Y_t be the corresponding sum with $f(jp^t)$ replaced by $f\left(\left[\frac{j}{p}\right]p^{t+1}\right)$.

Using the definition of \bar{W} ,

$$X_t = \sum_{i=0}^{p^{t+s-t}-1} \sum_{\nu=jp^t}^{(j+1)p^t-1} f(jp^t) W(\nu).$$

But for $\nu \in [jp^t, (j + 1)p^t)$, we have $j = \left[\frac{\nu}{p^t} \right]$. Furthermore we have the disjoint union,

$$[0, p^{s+1}) = \bigcup_{j=0}^{p^{s+1}-1} [jp^t, (j + 1)p^t)$$

which shows that

$$X_t = \sum_{\nu=0}^{p^{s+1}-1} f\left(p^t \left[\frac{\nu}{p^t} \right]\right) W(\nu).$$

Similarly

$$Y_t = \sum_{\nu=0}^{p^{s+1}-1} f\left(p^{t+1} \left[\frac{\nu}{p^{t+1}} \right]\right) W(\nu).$$

We now compute

$$\begin{aligned} \sum_{t=0}^s (X_t - Y_t) &= \sum_{\nu=0}^{p^{s+1}-1} W(\nu) \sum_{t=0}^s \left(f\left(p^t \left[\frac{\nu}{p^t} \right]\right) - f\left(p^{t+1} \left[\frac{\nu}{p^{t+1}} \right]\right) \right) \\ &= \sum_{\nu=0}^{p^{s+1}-1} W(\nu) (f(\nu) - f(0)) \\ &= \sum_{\nu=0}^{p^{s+1}-1} W(\nu) f(\nu) - f(0) \bar{W}(0, s + 1). \end{aligned}$$

This proves the lemma.

Again referring to the notation of paragraph 2 we consider

$$F = {}_nF_{q-1} \left[\begin{matrix} 0, \pi^{q-n} t \\ \sigma \end{matrix} \right]$$

as defined by equation (2.2). It is well known that F is a solution (regular at the origin) of the differential equation

$$(4.4) \quad (\pi^{q-n} t P(\delta) - Q(\delta)) u = 0$$

where $\delta = t \frac{d}{dt}$ and P and Q are the polynomials,

$$\begin{aligned} P(\delta) &= \prod_{i=1}^n (\delta + \theta_i), \\ Q(\delta) &= \delta \prod_{j=1}^{q-1} (\delta + \sigma_j - 1). \end{aligned}$$

By hypothesis, $\sigma_1, \dots, \sigma_{q-1}$ all lies in \mathfrak{C} and hence $1 - \sigma_j$ cannot lie in Z_+ unless $\sigma_j = 1$. Since Q is the indicial polynomial (at $t = 0$) of (4.4), it follows that F is the unique solution holomorphic at the origin.

We now assume that the equation (4.4) has a logarithmic type solution at the origin, i. e. we suppose

$$(4.5) \quad \sigma_{q-1} = 1.$$

Then there exists a power series \mathcal{F} uniquely determined by the conditions that $\mathcal{F}(0) = 0$ and that $\mathcal{F} + F \log t$ is a solution of equation (4.4). The computation of \mathcal{F} is based on the fact that if $R(\delta)$ is a polynomial in δ with constant coefficients and $\log t$ is used to denote the multiplication mapping

$$u \rightarrow u \cdot \log t$$

then

$$R(\delta) \circ \log t = \log t \circ R(\delta) + R'(\delta).$$

The result of the computation is

$$(4.6) \quad \begin{cases} F(t) = \sum A(m) t^m, \\ \mathcal{F}(t) = \sum_{m=0}^{\infty} A(m) D(m) t^m \end{cases}$$

where

$$D(m) = \sum_{i=1}^n D_{\theta_i}(m) - \sum_{j=1}^q D_{\sigma_j}(m)$$

and A is used to denote $A^{(0)}$. We now consider

$$G = {}_nF_{n-1} \left[\begin{matrix} \theta', \pi^{q-n} t \\ \sigma' \end{matrix} \right]$$

and define the power series \mathcal{G} by the condition that $\mathcal{G}(0) = 0$ and that $\mathcal{G} + G \log t$ satisfies the same differential equation as G [which is given by (4.4) with the obvious modifications]. Clearly (using B to denote $A^{(1)}$),

$$(4.6') \quad \begin{cases} G(t) = \sum B(m) t^m, \\ \mathcal{G}(t) = \sum B(m) E(m) t^m \end{cases}$$

where E is given by the same formula as D after replacing each θ_i (resp. σ_j) by its prime.

For $x \in \mathbb{C}$, let us put

$$\begin{aligned} \mathcal{F}_x(t) &= \sum_{m=0}^{\infty} A(m) D_x(m) t^m, \\ \mathcal{G}_x(t) &= \sum_{m=0}^{\infty} B(m) D_x(m) t^m. \end{aligned}$$

THEOREM 4.1. — If θ, σ satisfy conditions (v), (vi) of paragraph 3, then

$$\frac{\mathfrak{G}}{\mathfrak{F}}(t^p) \equiv p \frac{\mathfrak{F}}{\mathfrak{F}}(t) \pmod{p \mathfrak{O}[[t]]}.$$

Remark. — If $G = F$ then equation (4.3) is an immediate consequence.

Proof. — Let $S = \{\theta_1, \dots, \theta_n, \sigma_1, \sigma_2, \dots, \sigma_{q-1}, 1\}$. We will show

$$(4.7) \quad \frac{\mathfrak{G}_{x'}}{\mathfrak{G}}(t^p) \equiv p \frac{\mathfrak{F}_x}{\mathfrak{F}}(t) \pmod{p \mathfrak{O}[[t]]}$$

for each $x \in S$. This will be adequate for the proof of the theorem since \mathfrak{F} is a linear combination (with coefficients ± 1) of the $\{\mathfrak{F}_x\}_{x \in S}$ and \mathfrak{G} is the corresponding linear combination of $\{\mathfrak{G}_{x'}\}_{x \in S}$. It follows from Lemma 2.2 that F and G are units in $\mathfrak{O}[[t]]$ and hence equation (4.7) is equivalent to

$$F(t) \mathfrak{G}_{x'}(t^p) - p \mathfrak{F}_x(t) G(t^p) \in p \mathfrak{O}[[t]]$$

and trivially by computing the coefficient of t^{a+pN} ($a < p$), this is equivalent to the assertion that for all $a, N \in \mathbb{Z}_+, a < p$, we have

$$(4.8) \quad L_x(a + pN) \equiv 0 \pmod{p}$$

where

$$L_x(a + pN) = \sum_{j=0}^N B(N-j) A(a + pj) (D_{x'}(N-j) - p D_x(a + pj)).$$

We compute $D_x(pj) \pmod{\mathfrak{O}}$ by noting that it is the sum of reciprocals of numbers $v + x, v \in [0, pj)$, that the non-units of this type correspond to

$$v = px' - x + p\mu, \quad \mu \in [0, j),$$

and thus

$$(4.9) \quad D_x(pj) \equiv \frac{1}{p} D_{x'}(j) \pmod{\mathfrak{O}}.$$

Furthermore from Lemma 4.1, we obtain

$$D_x(a + pj) - D_x(pj) \equiv \frac{1}{p} \frac{\rho(a, x)}{j + x'} \pmod{\mathfrak{O}}.$$

We assert

$$(4.10) \quad A(a + pj) (p D_x(a + pj) - D_{x'}(j)) \equiv 0 \pmod{p}.$$

Indeed the left side lies in

$$g_0(a + pj) \left(\frac{\rho(a, x)}{j + x'} \mathfrak{O} + p \mathfrak{O} \right)$$

and the assertion (for $x \in S$) follows from Lemma 3.2. (In the statement of that lemma choose s so that $p^s > j$, put $m = 0$, $\mu = j$.) Thus

$$(4.11) \quad L_x(a + pN) \equiv \sum_{j=0}^N A(a + pj) B(N - j) (D_{x'}(N - j) - D_{x'}(j)) \pmod{p}.$$

For N fixed, $j \in \mathbb{Z}_+$, let

$$T(j) = A(a + pj) B(N - j).$$

The right side of equation (4.11) is the same as

$$\sum_{j=0}^N T(j) D_{x'}(N - j) - \sum_{j=0}^N T(j) D_{x'}(j).$$

Replacing j by $N - j$ in the first sum and recalling

$$T(N - j) - T(j) = U_a(j, N),$$

we obtain

$$(4.12) \quad L_x(a + pN) = \sum_{j=0}^N D_{x'}(j) U_a(j, N).$$

We apply Lemma 4.2 by letting W be the mapping

$$j \rightarrow U_a(j, N)$$

and letting f be $D_{x'}$. Choose s so that $p^{s+1} > N$. Thus in the notation of the lemma, $f(0) = 0$, $W(j) = 0$ for $j > N$ and $\overline{W}(m, s) = H_a(m, s, N)$.

We conclude that the right side of equation (4.12) is equal to

$$\sum_{t=0}^s \sum_{j=0}^{p^{t+s-t-1}} Y_{j,t}$$

where

$$Y_{j,t} = \left(D_{x'}(jp^t) - D_{x'}\left(\left[\frac{j}{p}\right] p^{t+1}\right) \right) H_a(j, t, N).$$

Putting $j = b + pM$, $b < p$, we find from equation (4.3) and Lemma 4.4 that

$$Y_{j,t} \in p^{t+1} g_{t+1}(b + pM) \left(p^{-t} \mathfrak{O} + p^{-(t+1)} \frac{\rho(b, x^{(t+1)})}{M + x^{(t+2)}} \mathfrak{O} \right).$$

This ideal lies in $p \mathfrak{O}$ (for each $x \in S$) by an application of Lemma 3.2. This completes the proof of equation (4.8) and hence of the theorem.

COROLLARY. — *Equation (4.7) remains valid if hypothesis (4.5) is dropped. [Indeed that hypothesis is only used in the motivation for the formulation of equation (4.7).]*

5. NUMERICAL EXAMPLES. — For each real number b , let $S(b)$, $S(b+)$ denote the step functions :

$$\begin{aligned} S(b)(x) &= 0 && \text{if } x \leq b, \\ &+ 1 && \text{if } x > b, \\ S(b+)(x) &= 0 && \text{if } x < b, \\ &+ 1 && \text{if } x \geq b. \end{aligned}$$

We apply the notation to the representation of $N_0(a)$ as required for the verification of condition (iv) above.

Example 1 :

$${}_2F_1 \left[\begin{matrix} \frac{1}{3}, \frac{2}{3}, t \\ \frac{1}{2} \end{matrix} \right] \quad (p \neq 2, 3).$$

Case 1 : $p \equiv 1 \pmod{3}$. — Here

$$\theta = \left[\frac{1}{3}, \frac{2}{3} \right] = \theta'$$

and so

$$\begin{aligned} N_0 &= S \left[\frac{p-1}{3} \right] + S \left[2 \frac{p-1}{3} \right], \\ N_\sigma(+) &= S \left[\frac{p-1}{2} + \right] \end{aligned}$$

which shows that

$$N_0 - N_\sigma(+) = S \left[\frac{p-1}{3} \right] - S \left[\frac{p-1}{2} + \right] + S \left[2 \frac{p-1}{3} \right],$$

a representation which clearly exhibits the discontinuities in order of appearance. This shows that condition (iv) is satisfied but that condition (v) is not satisfied since for $\frac{p-1}{2} < a < 2 \frac{p-1}{3}$, the sum in question assumes the value zero instead of 1.

This analysis shows that for $p \equiv 1 \pmod{3}$, the function

$${}_2F_1 \left[\begin{matrix} \frac{1}{3}, \frac{1}{2}, t \\ \frac{2}{3} \end{matrix} \right]$$

would satisfy both (iv) and (vi).

Case 2 : $p \equiv -1 \pmod{3}$. — Here

$$\theta = \left[\frac{1}{3}, \frac{2}{3} \right]$$

is again stable under the mapping $\theta \rightarrow \theta'$ but

$$\left[\frac{1}{3}\right]' = \frac{2}{3}, \quad \left[\frac{2}{3}\right]' = \frac{1}{3}$$

and so

$$N_{\theta} = S\left[\frac{p-2}{3}\right] + S\left[\frac{2p-1}{3}\right].$$

Thus

$$N_0 - N_{\sigma}(+) = S\left[\frac{p-2}{3}\right] - S\left[\frac{p-1}{2} +\right] + S\left[\frac{2p-1}{3}\right].$$

This shows that once again condition (iv) is satisfied but condition (vi) is not satisfied.

Example 2 :

$${}_2F_2 \left[\begin{matrix} 1, 1 \\ \frac{1}{5}, \frac{1}{5} \end{matrix} ; \pi t \right] \quad (p > 5).$$

Case 1 : $p \equiv 1 \pmod{5}$,

$$N_0 - N_{\sigma}(+) = 2S\left[\frac{p-1}{5}\right] - S\left[3\frac{p-1}{5} +\right].$$

[The term βa in conditions (iv), (vi) clearly plays no role since $a \leq p-1$ and $\beta = \frac{1}{p-1}$.] It is clear that conditions (iv) and (vi) are both satisfied.

Case 2 : $p \equiv -1 \pmod{5}$,

$$\theta' = \left[\frac{4}{5}, \frac{4}{5}\right], \quad \sigma' = \left[\frac{2}{5}, 1\right].$$

Thus

$$N_0 - N_{\sigma}(+) = -S\left[\frac{2p-3}{5} +\right] + 2S\left[\frac{4p-1}{5}\right]$$

while

$$N_{\theta'} - N_{\sigma'}(+) = 2S\left[\frac{p-4}{5}\right] - S\left[\frac{3p-2}{5} +\right].$$

Clearly $N_{\theta'} - N_{\sigma'}(+)$ satisfies conditions (iv), (vi) while these conditions do not hold for $N_0 - N_{\sigma}(+)$.

Case 3 : $p \equiv 2 \pmod{5}$,

$$\begin{aligned} \theta' &= \left[\frac{3}{5}, \frac{3}{5}\right], & \sigma' &= \left[\frac{4}{5}, 1\right]; & \theta'' &= \left[\frac{4}{5}, \frac{4}{5}\right], & \sigma'' &= \left[\frac{2}{5}, 1\right]; \\ \theta'' &= \left[\frac{2}{5}, \frac{2}{5}\right], & \sigma' &= \left[\frac{1}{5}, 1\right]. \end{aligned}$$

Thus

$$N_{\theta^{(\nu)}} - N_{\sigma^{(\nu)}} (+) = \begin{cases} 2 S \left[\frac{3p-1}{5} \right] - S \left[\frac{4p-3}{5} + \right] & \text{for } \nu = 0, \\ - S \left[\frac{2p-4}{5} + \right] + 2 S \left[\frac{4p-3}{5} \right] & \text{for } \nu = 1, \\ - S \left[\frac{p-2}{5} + \right] + 2 S \left[\frac{2p-4}{5} \right] & \text{for } \nu = 2, \\ 2 S \left[\frac{p-2}{5} \right] - S \left[\frac{3p-1}{5} \right] & \text{for } \nu = 3. \end{cases}$$

Clearly neither condition (iv) nor (vi) is satisfied.

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