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EXPANSIONAL IN BANACH ALGEBRAS

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ABSTRACT. — Properties of expansionals are discussed in connection with an evolution equation and a cocycle equation in Banach space.

1. Introduction

The covariant perturbation theory for quantized field, which has first been developed by Tomonaga [9], Schwinger [8] and Feynman [4], in independent and different styles and has been united by Dyson [3], contains a powerful computational tool which does not seem to be widely known among mathematicians. We shall briefly discuss one aspect of this theory in a context of Banach algebra.

The relevant computational algorithm has been suggested by Feynman [5] and later formulated clearly by Fujiwara [6] as an Expansional : Let A and B be two elements in a Banach algebra with a unit. Define

$$\text{(1.1)} \qquad E_r(A; B) = \sum_{n=0}^{\infty} \int_0^1 dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n \, A(t_n) \ldots A(t_1),$$

$$\text{(1.2)} \qquad E_l(A; B) = \sum_{n=0}^{\infty} \int_0^1 dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n \, A(t_1) \ldots A(t_n),$$

where

$$\text{(1.3)} \qquad A(t) = e^{tB} A e^{-tB}.$$

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If A and B commute, then $A(t) = A$ and $E_r(A; B) = E_l(A; B) = \exp A$. In the general case, we have

$$\| A(t) \| \leq \| A \| \exp 2t \| B \|,$$

$$\int_0^1 dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n = (n!)^{-1}.$$

Hence (1.1) and (1.2) are absolutely convergent.

Important formulas are

$$E_r(A; B) = e^{A+B} e^{-B}, \tag{1.4}$$

$$E_l(A; -B) = e^{-B} e^{A+B}. \tag{1.5}$$

(*See* Theorem 3.) By rewriting $A + B$ and $-B$ as A and B in (1.4) and as B and A in (1.5), we obtain

$$e^A e^B = E_r(A + B; -B) = E_l(A + B; A). \tag{1.6}$$

It is important that the expansions in (1.1) and (1.2) converge for arbitrary A and B, which is in a sharp contrast with the Baker-Campbell-Hausdorff formula

$$e^A e^B = e^C, \qquad C = A + B + \frac{[A, B]}{2} + \ldots \tag{1.7}$$

whose convergence is guaranteed in a general context only if both $\| A \|$ and $\| B \|$ are sufficiently small.

This comparison seems to provide a sufficient motivation for an investigation of various mathematical properties of expansionals. An application has been given in [1]. Another application will be given in a forthcoming paper [2].

2. Definition and Simple Properties

Let M be a Banach algebra with a unit, and $A(t)$ be a M-valued continuous function of positive reals $t \geq 0$ satisfying

$$\sup_{0 \leq t \leq T} \| A(t) \| \equiv r_T < \infty \qquad (T \geq 0). \tag{2.1}$$

The continuity can be either relative to the norm in M or, if M is a strongly closed subalgebra of linear operators on a Banach space H, then relative to strong operator topology. In either case,

$$A(t_1), \quad \ldots, \quad A(t_n)$$

is continuous in $(t_1, \ldots, t_n)$ relative to the relevant topology.

Define

$$(2.2)\qquad \operatorname{Exp}_r\left(\int_0^t ; \mathrm{A}(s)\, ds\right) = \sum_{n=0}^{\infty} \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n\, \mathrm{A}(t_n) \ldots \mathrm{A}(t_1),$$

$$(2.3)\qquad \operatorname{Exp}_l\left(\int_0^t ; \mathrm{A}(s)\, ds\right) = \sum_{n=0}^{\infty} \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n\, \mathrm{A}(t_1) \ldots \mathrm{A}(t_n).$$

By the continuity assumption, the integrals exist in M and by (2.1) and (1.3) the series converge absolutely.

Remark. — Let

$$(2.4)\qquad \mathrm{T}(\mathrm{A}(t_1) \ldots \mathrm{A}(t_n)) = \mathrm{A}(t_{p_1}) \ldots \mathrm{A}(t_{p_n}),$$

$$(2.5)\qquad \overline{\mathrm{T}}(\mathrm{A}(t_1) \ldots \mathrm{A}(t_n)) = \mathrm{A}(t_{p_n}) \ldots \mathrm{A}(t_{p_1}),$$

if $(p_1, \ldots, p_n)$ is a permutation of $(1, \ldots, n)$ and

$$t_{p_1} > t_{p_2} > \ldots > t_{p_n}.$$

Then they are defined up to a Null set and satisfy

$$(2.6)\qquad \operatorname{Exp}_r\left(\int_0^t ; \mathrm{A}(s)\, ds\right) = \sum_{n=0}^{\infty} \int_0^t dt_1 \ldots \int_0^t dt_n \frac{\overline{\mathrm{T}}(\mathrm{A}(t_1), \ldots, \mathrm{A}(t_n))}{n!},$$

$$(2.7)\qquad \operatorname{Exp}_l\left(\int_0^t ; \mathrm{A}(s)\, ds\right) = \sum_{n=0}^{\infty} \int_0^t dt_1 \ldots \int_0^t dt_n \frac{\mathrm{T}(\mathrm{A}(t_1), \ldots, \mathrm{A}(t_n))}{n!}.$$

For this reason, the following notation is used in physics :

$$\operatorname{Exp}_r\left(\int_0^t ; \mathrm{A}(s)\, ds\right) = \overline{\mathrm{T}} \exp \int_0^t \mathrm{A}(s)\, ds,$$

$$\operatorname{Exp}_l\left(\int_0^t ; \mathrm{A}(s)\, ds\right) = \mathrm{T} \exp \int_0^t \mathrm{A}(s)\, ds.$$

$\mathrm{T}(\mathrm{A}(t_1), \ldots, \mathrm{A}(t_n))$ is called the time-ordered product (chronological product, T-product) and plays an important role in quantum field theory. It was denoted as $\mathrm{P}(\mathrm{A}(t_1), \ldots, \mathrm{A}(t_n))$ when introduced by Dyson [3].

Proposition 1. — *Let* $\lambda \geqq 0$:

$$(2.8)\qquad \operatorname{Exp}_r\left(\int_0^t ; \lambda\, \mathrm{A}(\lambda s)\, ds\right) = \operatorname{Exp}_r\left(\int_0^{\lambda t} ; \mathrm{A}(s)\, ds\right),$$

$$(2.9)\qquad \operatorname{Exp}_l\left(\int_0^t ; \lambda\, \mathrm{A}(\lambda s)\, ds\right) = \operatorname{Exp}_l\left(\int_0^{\lambda t} ; \mathrm{A}(s)\, ds\right),$$

$$\text{Exp}_r\left(\int_0^t ; A(t-s)\,ds\right) = \text{Exp}_l\left(\int_0^t ; A(s)\,ds\right), \tag{2.10}$$

$$\text{Exp}_l\left(\int_0^t ; A(t-s)\,ds\right) = \text{Exp}_r\left(\int_0^t ; A(s)\,ds\right). \tag{2.11}$$

Proof. — The change of integration variables $t'_k = \lambda\, t_k$ yields (2.8) and (2.9), $t'_k = t - t_k$ yields (2.10) and (2.11). Q. E. D.

Remark. — It would be more economical to write everything in terms of

$$\text{Exp}_r\, A(.) \equiv \text{Exp}_r\left(\int_0^1 ; A(s)\,ds\right) \quad \text{and} \quad \text{Exp}_l\, A(.) \equiv \text{Exp}_l\left(\int_0^1 ; A(s)\,ds\right).$$

However some formulas such as above are easier to remember when integration symbol is written. For later formulas such as (3.5), (3.6), (3.17) and (3.18), it is more natural to introduce

$$\text{Exp}_r\left(\int_{t'}^t ; A(s)\,ds\right) \equiv \text{Exp}_r\left(\int_0^{t-t'} ; A(t'+s)\,ds\right),$$

$$\text{Exp}_l\left(\int_{t'}^t ; A(s)\,ds\right) \equiv \text{Exp}_l\left(\int_0^{t-t'} ; A(t'+s)\,ds\right).$$

However, we shall need formulas for

$$\text{Exp}_r\left(\int_0^t ; A(s)\,ds\right) \quad \text{and} \quad \text{Exp}_l\left(\int_0^t ; A(s)\,ds\right)$$

in our application in [2] and hence we shall write out all formulas in terms of them.

PROPOSITION 2 :

$$\frac{d}{dt}\text{Exp}_r\left(\int_0^t ; A(s)\,ds\right) = \text{Exp}_r\left(\int_0^t ; A(s)\,ds\right) A(t), \tag{2.12}$$

$$\frac{d}{dt}\text{Exp}_l\left(\int_0^t ; A(s)\,ds\right) = A(t)\,\text{Exp}_l\left(\int_0^t ; A(s)\,ds\right). \tag{2.13}$$

Proof. — Immediate from definitions (2.2) and (2.3). Q. E. D.

Remark. — The subscripts r and l indicate the direction (right and left) in which A (t) comes out in Proposition 2.

PROPOSITION 3 :

$$\text{Exp}_r\left(\int_0^t ; A(s)\,ds\right)\text{Exp}_l\left(\int_0^t ; -A(s)\,ds\right) = 1. \tag{2.14}$$

Proof. — Consider

$$f(t) = \mathrm{Exp}_r\left(\int_0^t ; \mathrm{A}(s)\, ds\right) \mathrm{Exp}_l\left(\int_0^t ; -\mathrm{A}(s)\, ds\right) \qquad (t \geqq 0).$$

By (2.12) and (2.13), we have

$$f'(t) = \mathrm{Exp}_r\left(\int_0^t ; \mathrm{A}(s)\, ds\right) \{ \mathrm{A}(t) - \mathrm{A}(t) \} \mathrm{Exp}_l\left(\int_0^t ; -\mathrm{A}(s)\, ds\right) = 0$$

Since $f(0) = 1$,

$$f(t) = 1 + \int_0^t f'(s)\, ds = 1.$$

Q. E. D.

PROPOSITION 4 :

$$\mathrm{Exp}_l\left(\int_0^t ; -\mathrm{A}(s)\, ds\right) \mathrm{Exp}_r\left(\int_0^t ; \mathrm{A}(s)\, ds\right) = 1. \tag{2.15}$$

Proof. — Let n th term of (2.2) and (2.3) be $\mathrm{E}_r^n(t; \mathrm{A})$ and $\mathrm{E}_l^n(t; \mathrm{A})$. Due to absolute convergence, it is enough to prove

$$\sum_{k=0}^{n} \mathrm{E}_l^k(t; -\mathrm{A})\, \mathrm{E}_r^{n-k}(t; \mathrm{A}) = 0 \qquad (n \geqq 1). \tag{2.16}$$

We have

$$(-1)^k \mathrm{E}_l^k(t; -\mathrm{A})\, \mathrm{E}_r^{n-k}(t; \mathrm{A}) = \left(\int_{\mathrm{I}_k} + \int_{\mathrm{I}_{k+1}}\right) dt_1 \ldots dt_n\, \mathrm{A}(t_1) \ldots \mathrm{A}(t_n)$$

for $0 \leqq k \leqq n$, $n \geqq 1$, where I_k, $1 \leqq k \leqq n$, denote

$$\mathrm{I}_k = \{ (t_1, \ldots, t_n);\, t \geqq t_1 \geqq \ldots \geqq t_k \geqq 0,\ t \geqq t_n \geqq \ldots \geqq t_k \geqq 0 \},$$

while I_{n+1} and I_0 are empty set. (2.16) is now immediate due to cancellations.

Q. E. D.

Remark. — A similar proof holds also for Proposition 3. Propositions 3 and 4 imply that both $\mathrm{Exp}_r\left(\int_0^t ; \mathrm{A}(s)\, ds\right)$ and $\mathrm{Exp}_l\left(\int_0^t ; \mathrm{A}(s)\, ds\right)$ are invertible.

If M is a $\star$-Banach algebra, then

$$\mathrm{Exp}_r\left(\int_0^t ; \mathrm{A}(s)\, ds\right)^* = \mathrm{Exp}_l\left(\int_0^t ; \mathrm{A}(s)^*\, ds\right), \tag{2.17}$$

$$\mathrm{Exp}_l\left(\int_0^t ; \mathrm{A}(s)\, ds\right)^* = \mathrm{Exp}_r\left(\int_0^t ; \mathrm{A}(s)^*\, ds\right). \tag{2.18}$$

In particular, if $A(t)$ is skew-hermitien $[A(t)^* = -A(t)]$, then both

$$\operatorname{Exp}_r\left(\int_0^t ; A(s)\,ds\right) \quad \text{and} \quad \operatorname{Exp}_l\left(\int_0^t ; (s)\,ds\right)$$

are unitary.

3. Differential equation view point

THEOREM 1. — *The differential equation*

$$f'(t) = f(t)\,A(t), \qquad f(t) \in M, \tag{3.1}$$

for a given initial value $f(0)$ *has a unique solution*

$$f(t) = f(0)\operatorname{Exp}_r\left(\int_0^t ; A(s)\,ds\right). \tag{3.2}$$

The differential equation

$$g'(t) = A(t)\,g(t), \qquad g(t) \in M, \tag{3.3}$$

for a given initial value $g(0)$ *has a unique solution*

$$g(t) = \operatorname{Exp}_l\left(\int_0^t ; A(s)\,ds\right) g(0). \tag{3.4}$$

Proof. — Consider

$$F(t) = f(t)\operatorname{Exp}_l\left(\int_0^t ; -A(s)\,ds\right).$$

By (2.13) and (3.1), $F'(t) = 0$. Since $F(0) = f(0)$, we have $F(t) = f(0)$. By (2.15), we have (3.2). Conversely (3.2) satisfies (3.1) due to (2.12). Similar proof holds for $g(t)$. Q. E. D.

Remark. — Theorem 1 and its proof hold also for the case where $g(t)$ is an element of a Banach space H and $A(t)$ is a bounded linear operator on H. The expression (2.3) for the solution is the Neumann-Liouville series (the iterative solution) of the Volterra equation

$$g(t) = g(0) + \int^t A(t)\,g(t).$$

Being solutions of evolution equations,

$$\operatorname{Exp}_r\left(\int_0^t ; A(s)\,ds\right) \quad \text{and} \quad \operatorname{Exp}_l\left(\int_0^t ; A(s)\,ds\right)$$

satisfy the following chain rule :

PROPOSITION 5 :

$$(3.5)\quad \mathrm{Exp}_r\left(\int_0^{t} ; A(s)\,ds\right)\mathrm{Exp}_r\left(\int_0^{t'} ; A(s+t)\,ds\right) = \mathrm{Exp}_r\left(\int_0^{t+t'} ; A(s)\,ds\right),$$

$$(3.6)\quad \mathrm{Exp}_l\left(\int_0^{t'} ; A(s+t)\,ds\right)\mathrm{Exp}_l\left(\int_0^{t} ; A(s)\,ds\right) = \mathrm{Exp}_l\left(\int_0^{t+t'} ; A(s)\,ds\right),$$

Proof. — $\mathrm{Exp}_r\left(\int_0^{t+t'} ; A(s)\,ds\right)$ is the unique solution of (3.1) at $t+t'$ for the initial value $f(0)=1$. It is then the unique solution of

$$f'(s) = f(s)\,A(s+t), \qquad f(0) = \mathrm{Exp}_r\left(\int_0^{t} ; A(\sigma)\,d\sigma\right)$$

at $s = t'$ and hence is given by the left hand side of (3.5) due to (3.2).

A similar proof holds for (3.6). Q. E. D.

For $B(s) \in M$, we define

$$(3.7)\qquad (B \star A)(t) = \mathrm{Exp}_r\left(\int_0^{t} ; B(s)\,ds\right) A(t)\,\mathrm{Exp}_l\left(\int_0^{t} ; -B(s)\,ds\right),$$

$$(3.8)\qquad (B \mathbin{\hat{\star}} A)(t) = \mathrm{Exp}_l\left(\int_0^{t} ; -B(s)\,ds\right) A(t)\,\mathrm{Exp}_r\left(\int_0^{t} ; B(s)\,ds\right).$$

Obviously

$$(3.9)\qquad (B \mathbin{\hat{\star}} (B \star A))(t) = A(t) = (B \star (B \mathbin{\hat{\star}} A))(t).$$

PROPOSITION 6 :

$$(3.10)\qquad \mathrm{Exp}_r\left(\int_0^{t} ; (B \star A)(s)\,ds\right)\mathrm{Exp}_r\left(\int_0^{t} ; B(s)\,ds\right) = \mathrm{Exp}_r\left(\int_0^{t} ; (A(s)+B(s))\,ds\right),$$

$$(3.11)\qquad \mathrm{Exp}_l\left(\int_0^{t} ; B(s)\,ds\right)\mathrm{Exp}_l\left(\int_0^{t} ; \{(-B) \star A\}(s)\,ds\right) = \mathrm{Exp}_l\left(\int_0^{t} ; (A(s)+B(s))\,ds\right).$$

Proof. — Consider

$$f(t) = \mathrm{Exp}_r\left(\int_0^{t} ; (B \star A)(s)\,ds\right)\mathrm{Exp}_r\left(\int_0^{t} ; B(s)\,ds\right).$$

By (2.12), we have

$$f'(t) = \mathrm{Exp}_r\left(\int_0^t ; (\mathrm{B} \star \mathrm{A})(s)\, ds\right)(\mathrm{B} \star \mathrm{A})(t)\, \mathrm{Exp}_r\left(\int_0^t ; \mathrm{B}(s)\, ds\right) + f(t)\, \mathrm{B}(t).$$

By (2.15), we have

$$(\mathrm{B} \star \mathrm{A})(t)\, \mathrm{Exp}_r\left(\int_0^t ; \mathrm{B}(s)\, ds\right) = \mathrm{Exp}_r\left(\int_0^t ; \mathrm{B}(s)\, ds\right)\mathrm{A}(t).$$

Hence

$$f'(t) = f(t)(\mathrm{A}(t) + \mathrm{B}(t)).$$

We also have $f(0) = 1$. Therefore we have (3.10) by Theorem 1.

A similar proof holds for (3.11). Q. E. D.

We now introduce a further notation

$$\mathrm{A}_t(s) = \mathrm{A}(t - s). \tag{3.12}$$

then

$$(\mathrm{A}_t \star \mathrm{B}_t)_t(s) = \mathrm{Exp}_r\left(\int_0^{t-s} ; \mathrm{A}(t-\sigma)\, d\sigma\right)\mathrm{B}(s)\, \mathrm{Exp}_l\left(\int_0^{t-s} ; -\mathrm{A}(t-\sigma)\, d\sigma\right)$$
$$= \mathrm{Exp}_l\left(\int_0^{t-s} ; \mathrm{A}(\sigma + s)\, d\sigma\right)\mathrm{B}(s)\, \mathrm{Exp}_r\left(\int_0^{t-s} ; -\mathrm{A}(\sigma + s)\, d\sigma\right), \tag{3.13}$$

$$(\mathrm{A}_t \,\hat{\star}\, \mathrm{B}_t)_t(s) = \mathrm{Exp}_l\left(\int_0^{t-s} ; -\mathrm{A}(t-\sigma)\, d\sigma\right)\mathrm{B}(s)\, \mathrm{Exp}_r\left(\int_0^{t-s} ; \mathrm{A}(t-\sigma)\, d\sigma\right)$$
$$= \mathrm{Exp}_r\left(\int_0^{t-s} ; -\mathrm{A}(\sigma + s)\, d\sigma\right)\mathrm{B}(s)\, \mathrm{Exp}_l\left(\int_0^{t-s} ; \mathrm{A}(\sigma + s)\, d\sigma\right), \tag{3.14}$$

where (2.10) and (2.11) have been used.

[We may also write $\mathrm{Exp}_l\left(\int_s^t ; \mathrm{A}(\sigma)\, d\sigma\right)\mathrm{B}(s)\, \mathrm{Exp}_r\left(\int_s^t ; -\mathrm{A}(\sigma)\, d\sigma\right)$ for the last expression in (3.13) and

$$\mathrm{Exp}_r\left(\int_s^t ; -\mathrm{A}(\sigma)\, d\sigma\right)\mathrm{B}(s)\, \mathrm{Exp}_l\left(\int_s^t ; \mathrm{A}(\sigma)\, d\sigma\right)$$

for the last expression of (3.14).]

Proposition 7 :

$$\mathrm{Exp}_r\left(\int_0^t ; \mathrm{A}(s)\, ds\right)\mathrm{Exp}_r\left(\int_0^t ; \{(-\mathrm{A})_t \star \mathrm{B}_t\}_t(s)\, ds\right)$$
$$= \mathrm{Exp}_r\left(\int_0^t ; (\mathrm{A}(s) + \mathrm{B}(s))\, ds\right), \tag{3.15}$$

$$\text{(3.16)} \qquad \operatorname{Exp}_l\left(\int_0^t ; (A_t \star B_t)_t (s)\, ds\right) \operatorname{Exp}_l\left(\int_0^t ; A(s)\, ds\right) = \operatorname{Exp}_l\left(\int_0^t ; (A(s) + B(s))\, ds\right).$$

Proof. — By using (2.10) in (3.11), we obtain

$$\operatorname{Exp}_r\left(\int_0^t ; B(t-s)\, ds\right) \operatorname{Exp}_r\left(\int_0^t ; \{(-B) \star A\}_t (s)\, ds\right) = \operatorname{Exp}_r\left(\int_0^t ; (A(t-s) + B(t-s))\, ds\right).$$

Substituting A_t and B_t into B and A of this equation, we obtain (3.15).

A similar proof holds for (3.16). Q. E. D.

PROPOSITION 8 :

$$\text{(3.17)} \qquad \operatorname{Exp}_r\left(\int_0^t ; A(s)\, ds\right) \operatorname{Exp}_r\left(\int_0^t ; B(s)\, ds\right) = \operatorname{Exp}_r\left(\int_0^t ; \{(B \hat{\star} A)(s) + B(s)\}\, ds\right) = \operatorname{Exp}_r\left(\int_0^t ; \{A(s) + \{(-A)_t \hat{\star} B_t\}_t (s)\}\, ds\right),$$

$$\text{(3.18)} \qquad \operatorname{Exp}_l\left(\int_0^t ; A(s)\, ds\right) \operatorname{Exp}_l\left(\int_0^t ; B(s)\, ds\right) = \operatorname{Exp}_l\left(\int_0^t ; \{(B_t \hat{\star} A_t)_t (s) + B(s)\}\, ds\right) = \operatorname{Exp}_r\left(\int_0^t ; \{A(s) + \{(-A) \hat{\star} B\}(s)\}\, ds\right).$$

Proof. — Substituting $B \hat{\star} A$ into A of (3.10) and $\{(-A)_t \star B_t\}_t$ into B of (3.15), we obtain (3.17).

A similar proof holds for (3.18). Q. E. D.

Remark. — If $[A(t), B(s)] = 0$ for all t and s, then $\operatorname{Exp}_r\left(\int_0^t ; B(s)\, ds\right)$ commutes with $A(t)$ and hence $B \star A = A$ by (2.14).

Hence we have

$$\text{(3.19)} \quad \operatorname{Exp}_r\left(\int_0^t ; A(s)\, ds\right) \operatorname{Exp}_r\left(\int_0^t ; B(s)\, ds\right) = \operatorname{Exp}_r\left(\int_0^t ; \{A(s) + B(s)\}\, ds\right).$$

Similarly

$$(3.20)\quad \mathrm{Exp}_l\left(\int_0^t ; A(s)\,ds\right)\mathrm{Exp}_l\left(\int_0^t ; B(s)\,ds\right) = \mathrm{Exp}_l\left(\int_0^t ; \{A(s)+B(s)\}\,ds\right).$$

PROPOSITION 9 :

$$(3.21)\qquad (B \star C) \star (B \star A) = (C + B) \star A,$$

$$(3.22)\qquad (B \star A)(t) = e^{tB} A(t) e^{-tB} \quad \text{if} \quad B(t) = B.$$

Proof. — By definition, we have

$$\{(B \star C) \star (B \star A)\}(t) = \mathrm{Exp}_r\left(\int_0^t ; (B \star C)(s)\,ds\right)\mathrm{Exp}_r\left(\int_0^t ; B(s)\,ds\right) A(t)$$
$$\times \mathrm{Exp}_l\left(\int_0^t ; -B(s)\,ds\right)\mathrm{Exp}_l\left(\int_0^t ; -(B \star C)(s)\,ds\right).$$

By (3.10) and (3.11), we obtain (3.21), where we use (3.11) with A and B replaced by $-C$ and $-B$. If $B(t) = B$, then we have

$$\mathrm{Exp}_r\left(\int_0^t ; B(s)\,ds\right) = e^{tB} \quad \text{and} \quad \mathrm{Exp}_l\left(\int_0^t ; -B(s)\,ds\right) = e^{-tB}.$$

Hence we have (3.22). Q. E. D.

PROPOSITION 10 :

$$(3.23)\qquad \mathrm{Exp}_r\left(\int_0^t ; A(s)\,ds\right) = \mathrm{Exp}_l\left(\int_0^t ; (A \star A)(s)\,ds\right),$$

$$(3.24)\qquad \mathrm{Exp}_l\left(\int_0^t ; A(s)\,ds\right) = \mathrm{Exp}_r\left(\int_0^t ; \{(-A) \star A\}(s)\,ds\right).$$

Proof. — We set $B = -A$ in (3.11) and multiply $\mathrm{Exp}_r\left(\int_0^t ; A(s)\,ds\right)$ from the left. By (2.14), we obtain (3.23).

A similar proof holds for (3.24). Q. E. D.

Remark. — If $A(t)$ is defined for $t \in (-\infty, \infty)$, $\sup_{-T \leq t \leq T} \|A(t)\| < \infty$ and $A(t)$ is continuous, then Theorem 1 and Propositions 1-10 hold for $\lambda \neq 0$, $t \in (-\infty, \infty)$ and $t' \in (-\infty, \infty)$.

4. Cocycle View Point

We consider a continuous one-parameter semigroup $\alpha(t)$, $t \geq 0$, of continuous endomorphisms of M and

$$(4.1)\qquad \alpha(t) A = A(t), \qquad A \in M.$$

THEOREM 2. — *Given* $\alpha(t)$, $t \geqq 0$ *and* $A \in M$. *A unique differentiable solution of the cocycle equation*

$$f(s)\{\alpha(s) f(t)\} = f(s+t), \tag{4.2}$$

with the boundary condition

$$f(0) = 1, \qquad f'(+0) = A, \tag{4.3}$$

is given by

$$f(s) = \operatorname{Exp}_r\left(\int_0^s ; \alpha(s) A\, ds\right). \tag{4.4}$$

A unique differentiable solution of

$$\{\alpha(t) g(s)\} g(t) = g(s+t), \tag{4.5}$$

$$g(0) = 1, \qquad g'(+0) = A, \tag{4.6}$$

is given by

$$g(s) = \operatorname{Exp}_l\left(\int_0^s ; \alpha(s) A\, ds\right). \tag{4.7}$$

Proof. — Due to (3.5) and

$$\operatorname{Exp}_r\left(\int_0^t ; \alpha(\sigma + s) A\, d\sigma\right) = \alpha(s) \operatorname{Exp}_r\left(\int_0^t ; \alpha(\sigma) A\, d\sigma\right),$$

$f(s)$ given by (4.4) is a solution of (4.2). It also satisfies (4.3).

By differentiating (4.2) by t and setting $t = 0$, we obtain

$$f(s)\{\alpha(s) A\} = f'(s).$$

By Theorem 1, (4.4) is the unique solution.

A similar proof holds for $g(t)$. Q. E. D.

Remark. — If $\alpha(t)$, $-\infty < t < \infty$, is a continuous one-parameter group of automorphisms of M, then Theorem 2 holds with $-\infty < s < \infty$ and $-\infty < t < \infty$.

PROPOSITION 11. — *If* $\alpha(t)$, $-\infty < t < \infty$, *is a continuous one-parameter group of continuous automorphisms of* M, *then*

$$\operatorname{Exp}_r\left(\int_0^t ; \alpha(s) A\, ds\right) = \alpha(t) \operatorname{Exp}_l\left(\int_0^{-t} ; -\alpha(s) A\, ds\right), \tag{4.8}$$

$$\operatorname{Exp}_l\left(\int_0^t ; \alpha(s) A\, ds\right) = \alpha(t) \operatorname{Exp}_r\left(\int_0^{-t} ; -\alpha(s) A\, ds\right), \tag{4.9}$$

Proof. — In the cocycle equation (4.5), we set $s = -t$, substitute $g(t) = \mathrm{Exp}_l\left(\int_0^t ; -\alpha(s)\,\mathrm{A}\,ds\right)$ [namely $g(t)$ with $g'(+0) = -\mathrm{A}$] and multiply $\mathrm{Exp}_r\left(\int_0^t ; \alpha(s)\,\mathrm{A}\,ds\right)$ form the left. We then obtain (4.8) due to (2.15).

A similar proof holds for (4.9). Q. E. D.

Remark. — We use the definitions (2.2) and (2.3) also for negative t. The right hand side of (2.6) and (2.7) have to be interchanged for $t < 0$ according to this definition. Proposition 11 can also be proved from (2.10) and (2.11).

PROPOSITION 12. — *Let* $\alpha(t)$, $t \geqq 0$, *be a continuous one-parameter semi-group of continuous endomorphisms of* M *and* $\mathrm{B} \in \mathrm{M}$. *Let*

$$(4.10)\quad \beta(t)\,\mathrm{A} = \{(\alpha(.)\,\mathrm{B}) \star (\alpha(.)\,\mathrm{A})\}(t) = \mathrm{Exp}_r\left(\int_0^t ; \alpha(s)\,\mathrm{B}\,ds\right)\{\alpha(t)\,\mathrm{A}\}\,\mathrm{Exp}_l\left(\int_0^t ; -\alpha(s)\,\mathrm{B}\,ds\right), \qquad \mathrm{A} \in \mathrm{M}.$$

Then $\beta(t)$ *is a continuous one-parameter semigroup of continuous endomorphisms of* M *and*

$$(4.11)\qquad \frac{d}{dt}(\beta(t) - \alpha(t))\Big|_{t=0} = \delta_{\mathrm{B}}$$

where δ_{B} *denotes the inner derivation*

$$\delta_{\mathrm{B}}\,\mathrm{A} = \mathrm{BA} - \mathrm{AB}.$$

Proof. — By Propositions 3 and 4, $\mathrm{Exp}_r\left(\int_0^t ; \alpha(s)\,\mathrm{B}\,ds\right)$ is an invertible element of M and its inverse is $\mathrm{Exp}_l\left(\int_0^t ; -\alpha(s)\,\mathrm{B}\,ds\right)$. Hence $\beta(t)$ is a continuous endomorphism of M.

By cocycle equations (4.2) and (4.5), we have

$$\beta(t_1 + t_2) = \beta(t_1)\,\beta(t_2).$$

Since $\mathrm{Exp}_r\left(\int_0^t ; \alpha(s)\,\mathrm{B}\,ds\right)$ and $\mathrm{Exp}_l\left(\int_0^t ; -\alpha(s)\,\mathrm{B}\,ds\right)$ are differentiable in t, $\beta(t)\,\mathrm{A}$ is continuous in t. We also have $\beta(0) = \alpha(0) = 1$. Hence $\beta(t)$ is a continuous one-parameter group of endomrophisms of M.

Due to (2.12) and (2.13),

$$\lim_{t\to+0} t^{-1}\left(\mathrm{Exp}_r\left(\int_0^t ; \alpha(s)\,\mathrm{B}\,ds\right) - 1\right) = \mathrm{B},$$

$$\lim_{t\to+0} t^{-1}\left(\mathrm{Exp}_l\left(\int_0^t ; -\alpha(s)\,\mathrm{B}\,ds\right) - 1\right) = -\mathrm{B}.$$

Since $\mathrm{Exp}_r\left(\int_0^t ; \alpha(s)\,\mathrm{B}\,ds\right)$ is uniformly bounded by $\exp\int_0^{\mathrm{T}} \|\alpha(s)\,\mathrm{B}\|\,ds$, for $0 \leq t \leq \mathrm{T}$, we have (4.11). Q. E. D.

Remark 1. — If we assume a condition, which guarantees the uniqueness of a semigroup of operators with a given generator, such as the norm continuity of $\alpha(t)\,\mathrm{A}$ in t for each $\mathrm{A} \in \mathrm{M}$, then (4.10) is a unique continuous semigroup of continuous endomorphisms satisfying (4.11).

Remark 2. — (4.10) can be written as

$$\mathrm{Exp}_r\left(\int_0^t ; \alpha(s)\,\mathrm{B}\,ds\right)\alpha(t)\,\mathrm{A} = \{\beta(t)\,\mathrm{A}\}\,\mathrm{Exp}_r\left(\int_0^t ; \alpha(s)\,\mathrm{B}\,ds\right), \tag{4.12}$$

$$\{\alpha(t)\,\mathrm{A}\}\,\mathrm{Exp}_l\left(\int_0^t ; -\alpha(s)\,\mathrm{B}\,ds\right) = \mathrm{Exp}_l\left(\int_0^t ; -\alpha(s)\,\mathrm{B}\,ds\right)\beta(t)\,\mathrm{A}. \tag{4.13}$$

Namely, $\mathrm{Exp}_r\left(\int_0^t ; \alpha(s)\,\mathrm{B}\,ds\right)$ and $\mathrm{Exp}_l\left(\int_0^t ; -\alpha(s)\,\mathrm{B}\,ds\right)$ are invertible intertwining operators of two endomorphisms $\alpha(t)$ and $\beta(t)$.

Remark 3. — If $\alpha(t)$, $-\infty < t < \infty$, is a one-parameter group of continuous automorphisms of M, then $\beta(t)$ is a one-parameter group of continuous automorphisms of M satisfying (4.11).

Remark 4. — Define

$$\rho(t)\,\mathrm{A} = \mathrm{Exp}_r\left(\int_0^t ; \alpha(s)\,\mathrm{B}\,ds\right)\alpha(t)\,\mathrm{A}. \tag{4.14}$$

Then

$$\rho(t_1 + t_2) = \rho(t_1)\,\rho(t_2). \tag{4.15}$$

5. Perturbation View Point

Proposition 13. — *Let* $\mathrm{A}(t), \mathrm{B} \in \mathrm{M}$ *and*

$$\alpha^{\mathrm{B}}\,\mathrm{A}(t) = e^{\mathrm{B}}\,\mathrm{A}(t)\,e^{-\mathrm{B}}. \tag{5.1}$$

Then

$$\mathrm{Exp}_r\left(\int_0^t ; (\mathrm{A}(s) + \mathrm{B})\,ds\right) = \mathrm{Exp}_r\left(\int_0^t ; \alpha^{s\mathrm{B}}\,\mathrm{A}(s)\,ds\right)e^{t\mathrm{B}}, \tag{5.2}$$

$$\text{(5.3)} \qquad \mathrm{Exp}_l\left(\int_0^t ; (\mathrm{A}(s)+\mathrm{B})\, ds\right) = e^{t\mathrm{B}}\, \mathrm{Exp}_l\left(\int_0^t ; \alpha^{-s\mathrm{B}} \mathrm{A}(s)\, ds\right).$$

Proof. — In (3.10) and (3.11), we set B (t) = B and use (3.22) as well as

$$\mathrm{Exp}_r\left(\int_0^t ; \mathrm{B}\, ds\right) = \mathrm{Exp}_l\left(\int_0^t ; \mathrm{B}\, ds\right) = e^{t\mathrm{B}}.$$

We then obtain (5.2) and (5.3). Q. E. D.

THEOREM 3 :

$$\text{(5.4)} \qquad e^{\mathrm{A}+\mathrm{B}}\, e^{-\mathrm{B}} = \mathrm{Exp}_r\left(\int_0^1 ; \alpha^{s\mathrm{B}} \mathrm{A}\, ds\right),$$

$$\text{(5.5)} \qquad e^{-\mathrm{B}}\, e^{\mathrm{A}+\mathrm{B}} = \mathrm{Exp}_l\left(\int_0^1 ; \alpha^{-s\mathrm{B}} \mathrm{A}\, ds\right).$$

Proof. — We obtain (5.4) and (5.5) by setting A (t) = A in (5.2) and (5.3). Q. E. D.

PROPOSITION 14 :

$$\text{(5.6)} \qquad e^{\mathrm{A}}\, e^{\mathrm{B}}\, e^{-(\mathrm{A}+\mathrm{B})} = \mathrm{Exp}_r\left(\int_0^t ; \mathrm{C}_1(s)\, ds\right)$$

$$\text{(5.7)} \qquad e^{-(\mathrm{A}+\mathrm{B})}\, e^{\mathrm{A}}\, e^{\mathrm{B}} = \mathrm{Exp}_l\left(\int_0^1 ; \mathrm{C}_2(s)\, ds\right),$$

where

$$\text{(5.8)} \quad \mathrm{C}_1(t) = e^{(\mathrm{A}+\mathrm{B})t}\, e^{-\mathrm{B}t}(\mathrm{A}+\mathrm{B})\, e^{\mathrm{B}t}\, e^{-(\mathrm{A}+\mathrm{B})t} - (\mathrm{A}+\mathrm{B}) = (\alpha^{(\mathrm{A}+\mathrm{B})t}\, \alpha^{-\mathrm{B}t} - 1)(\mathrm{A}+\mathrm{B}),$$

$$\text{(5.9)} \quad \mathrm{C}_2(t) = e^{-(\mathrm{A}+\mathrm{B})t}\, e^{\mathrm{A}t}(\mathrm{A}+\mathrm{B})\, e^{-\mathrm{A}t}\, e^{(\mathrm{A}+\mathrm{B})t} - (\mathrm{A}+\mathrm{B}) = (\alpha^{-(\mathrm{A}+\mathrm{B})t}\, \alpha^{\mathrm{A}t} - 1)(\mathrm{A}+\mathrm{B}).$$

Proof. — By (5.4) and (5.5), we have

$$\text{(5.10)} \quad e^{\mathrm{A}}\, e^{\mathrm{B}} = \mathrm{Exp}_r\left(\int_0^1 ; \alpha^{-s\mathrm{B}}(\mathrm{A}+\mathrm{B})\, ds\right) = \mathrm{Exp}_l\left(\int_0^1 ; \alpha^{s\mathrm{A}}(\mathrm{A}+\mathrm{B})\, ds\right).$$

By substituting $\alpha^{-\mathrm{B}t}(\mathrm{A}+\mathrm{B}) - (\mathrm{A}+\mathrm{B})$ and $\mathrm{A}+\mathrm{B}$ into A (t) and B of (5.2), we obtain

$$e^{\mathrm{A}}\, e^{\mathrm{B}} = \mathrm{Exp}_r\left(\int_0^1 \mathrm{C}_1(s)\, ds\right) e^{\mathrm{A}+\mathrm{B}}$$

which proves (5.6). A similar proof using (5.3) yields (5.7). Q. E. D.

Remark. — Due to

$$\text{(5.11)} \qquad \alpha^{t\mathrm{B}}\, \mathrm{A} = e^{t\mathrm{B}}\, \mathrm{A}\, e^{-t\mathrm{B}} = (\exp t\, \delta_{\mathrm{B}})\, \mathrm{A},$$

both $\mathrm{C}_1(t)$ and $\mathrm{C}_2(t)$ are sums of multiple commutators of A and B (i. e., they are in closed Lie algebra generated by A and B).

PROPOSITION 15 :

$$(5.12)\qquad e^{A_1}\dots e^{A_n} = \operatorname{Exp}_r\left(\int_0^1 ;\ \sum_{m=1}^{n} \alpha^{-A_n s}\dots\alpha^{-A_{m+1} s} A_m\, ds\right)$$

$$(5.13)\qquad = \operatorname{Exp}_l\left(\int_0^1 ;\ \sum_{m=1}^{n} \alpha^{A_1 s}\dots\alpha^{A_{m-1} s} A_m\, ds\right)$$

$$(5.14)\qquad = \operatorname{Exp}_r\left(\int_0^1 ;\ \sum_{m=1}^{n} \alpha^{A_1 (1-s)}\dots\alpha^{A_{m-1}(1-s)} A_m\, ds\right)$$

$$(5.15)\qquad = \operatorname{Exp}_l\left(\int_0^1 ;\ \sum_{m=1}^{n} \alpha^{-A_n (1-s)}\dots\alpha^{-A_{m+1}(1-s)} A_m\, ds\right).$$

Proof. — By (5.2) and (5.3), we have

$$\operatorname{Exp}_r\left(\int_0^1 ;\ C_{m-1}(s)\, ds\right) e^{A_m} = \operatorname{Exp}_r\left(\int_0^1 ;\ C_m(s)\, ds\right),$$
$$C_m(s) = \alpha^{-A_m s}\, C_{m-1}(s) + A_m,$$

$$e^{A_m} \operatorname{Exp}_l\left(\int_0^1 ;\ D_{m+1}(s)\, ds\right) = \operatorname{Exp}_l\left(\int_0^1 ;\ D_m(s)\, ds\right),$$
$$D_m(s) = \alpha^{A_m s}\, D_{m+1}(s) + A_m.$$

Starting from $C_1(s) = A_1$ and $D_n(s) = A_n$, we obtain recursively

$$C_m(s) = \sum_{k=1}^{m} \alpha^{-A_m s}\dots\alpha^{-A_{k+1} s} A_k,$$

$$D_m(s) = \sum_{k=m}^{n} \alpha^{A_m s}\dots\alpha^{A_{k-1} s} A_k.$$

Thus we obtain (5.12) and (5.13) from

$$e^{A_1}\dots e^{A_n} = \operatorname{Exp}_r\left(\int_0^1 ;\ C_n(s)\, ds\right) = \operatorname{Exp}_l\left(\int_0^1 ;\ D_1(s)\, ds\right).$$

By using (2.10) and (2.11) in (5.12) and (5.13), we obtain (5.14) and (5.15). Q. E. D.

Remark. — Proposition 15 can also be proved from Proposition 8.

PROPOSITION 16. — *Let* H_0 *be a selfadjoint operator and* V *be a bounded linear operator on a Hilbert space* H. *Let* $D(H_0)$ *denote the domain of* H_0.

Let $\Psi(t) \in \mathrm{H}$, $-\infty < t < \infty$, *satisfy the following conditions.*

(1) $\Psi(t)$ *is bounded over a compact set of* t.

(2) $(\Phi, \Psi(t))$ *is differentiable in* t *for every* $\Phi \in \mathrm{D}(\mathrm{H}_0)$.

(3) $\frac{d}{dt}(\Phi, \Psi(t)) = (-i(\mathrm{H}_0 + \mathrm{V}^*)\Phi, \Psi(t))$ *for every* $\Phi \in \mathrm{D}(\mathrm{H}_0)$.

Such $\Psi(t)$ *exists and is unique for a given initial value* $\Psi(0)$. *It is given by*

$$(5.16)\quad \Psi(t) = \mathrm{Exp}_r\left(\int_0^t ; i\,\alpha^{i\mathrm{H}_0 s}\,\mathrm{V}\,ds\right) e^{i\mathrm{H}_0 t}\,\Psi(0) = e^{i\mathrm{H}_0 t}\,\mathrm{Exp}_l\left(\int_0^t ; i\,\alpha^{-i\mathrm{H}_0 s}\,\mathrm{V}\,ds\right)\Psi(0),$$

where

$$(5.17)\quad \alpha^{i\mathrm{H}_0 s}\,\mathrm{V} = e^{i\mathrm{H}_0 s}\,\mathrm{V}\,e^{-i\mathrm{H}_0 s}.$$

Proof. — By taking $\alpha(t) = \alpha^{i\mathrm{H}_0 t}$ and $\mathrm{A} = i\,\mathrm{V}^*$ in (4.8), we obtain

$$(5.18)\quad e^{i\mathrm{H}_0 t}\,\mathrm{Exp}_l\left(\int_0^{-t} ; -i\,\alpha^{i\mathrm{H}_0 s}\,\mathrm{V}^*\,ds\right) = \mathrm{Exp}_r\left(\int_0^t ; i\,\alpha^{i\mathrm{H}_0 s}\,\mathrm{V}^*\,ds\right) e^{i\mathrm{H}_0 t}.$$

By (2.12), we have

$$\frac{d}{dt}\left\{\mathrm{Exp}_r\left(\int_0^t ; i\,\alpha^{i\mathrm{H}_0 s}\,\mathrm{V}^*\,ds\right) e^{i\mathrm{H}_0 t}\right\}\Phi = \left\{\mathrm{Exp}_r\left(\int_0^t ; i\,\alpha^{i\mathrm{H}_0 s}\,\mathrm{V}^*\,ds\right) e^{i\mathrm{H}_0 t}\right\}\{i(\mathrm{H}_0 + \mathrm{V}^*)\}\Phi$$

if $\Phi \in \mathrm{D}(\mathrm{H}_0)$. We also have

$$\frac{d}{dt}\mathrm{Exp}_l\left(\int_0^{-t} ; -i\,\alpha^{i\mathrm{H}_0 s}\,\mathrm{V}^*\,ds\right)\Phi = i\{\alpha^{-i\mathrm{H}_0 t}\,\mathrm{V}^*\}\,\mathrm{Exp}_l\left(\int_0^{-t} ; -i\,\alpha^{i\mathrm{H}_0 s}\,\mathrm{V}^*\,ds\right)\Phi$$

by (2.13). As we have seen

$$(5.19)\quad \Phi_1(t) \equiv e^{i\mathrm{H}_0 t}\,\mathrm{Exp}_l\left(\int_0^{-t} ; -i\,\alpha^{i\mathrm{H}_0 s}\,\mathrm{V}^*\,ds\right)\Phi$$

is strongly differentiable in t and hence $e^{i\mathrm{H}_0 s}\,\Phi_1(t)$ is strongly differentiable in s at $s = 0$ if $\Phi \in \mathrm{D}(\mathrm{H}_0)$. Therefore $\Phi_1(t)$ is in the domain of H_0 if $\Phi \in \mathrm{D}(\mathrm{H}_0)$ and

$$(5.20)\quad \frac{d}{dt}\Phi_1(t) = i(\mathrm{H}_0 + \mathrm{V}^*)\,\Phi_1(t)$$

in the strong topology.

Let $\Phi \in \mathrm{D}(\mathrm{H}_0)$ and

$$\Psi_1(t) = e^{-i\mathrm{H}_0 t}\,\mathrm{Exp}_l\left(\int_0^t ; -i\,\alpha^{i\mathrm{H}_0 s}\,\mathrm{V}\,ds\right)\Psi(t).$$

By (2.14), we have

$$(5.21)\quad \Psi(t) = \mathrm{Exp}_r\left(\int_0^t ; i\,\alpha^{i\mathrm{H}_0 s}\,\mathrm{V}\,ds\right) e^{i\mathrm{H}_0 t}\,\Psi_1(t).$$

By (2.18), we have

$$(\Phi, \Psi_1(t)) = \left(\operatorname{Exp}_r\left(\int_0^t ; i\,\alpha^{i H_0 s}\, V^*\, ds\right) e^{i H_0 t}\, \Phi, \Psi(t)\right) = (\Phi_1(t), \Psi(t))$$

by (5.18) and (5.19). Since $\Phi_1(t) \in D(H_0)$, we have

$$\lim_{s \to t} \{(\Phi_1(t), \Psi(s) - \Psi(t))/(s-t)\} = (-i(H_0 + V^*)\,\Phi_1(t), \Psi(t))$$

by assumption (2).

By assumption (1), $\Psi(t)$ is locally bounded and by assumption (2), $(\Phi, \Psi(t))$ is continuous in t for Φ in a dense set $D(H_0)$. Hence $\Psi(t)$ is weakly continuous in t and

$$\mathrm{w}-\lim_{s \to t} \Psi(s) = \Psi(t).$$

Together with (5.20), we have

$$\lim_{s \to t} \{(\Phi_1(s) - \Phi_1(t), \Psi(s))/(s-t)\} = (i(H_0 + V^*)\,\Phi_1(t), \Psi(t)).$$

Hence

$$\frac{d}{dt}(\Phi, \Psi_1(t)) = 0$$

for all $\Phi \in D(H_0)$. This implies

$$(\Phi, \Psi_1(t) - \Psi_1(0)) = 0$$

for Φ in a dense set $D(H_0)$ and hence

$$\Psi_1(t) = \Psi_1(0) = \Psi(0).$$

By (5.21), we have the first equality of (5.16).

By (5.18) where V^* is replaced by V and by (2.9) with $\lambda = -1$, we obtain

$$\operatorname{Exp}_r\left(\int_0^t i\,\alpha^{i H_0 s}\, V\, ds\right) e^{i H_0 t} = e^{i H_0 t} \operatorname{Exp}_l\left(\int_0^t ; i\,\alpha^{-t H_0 s}\, V\, ds\right),$$

the use of negative λ in (2.9) is allowed for a group of automorphisms $\alpha^{i H_0 s}$. Hence we have the second equality in (5.16).

Conversely, the last expression in (5.16) implies

$$\frac{d}{dt}(\Phi, \Psi(t)) = (-i(H_0 + V^*)\,\Phi, \Psi(t))$$

for $\Phi \in D(H_0)$ and hence satisfies the assumptions (2) and (3). $\Psi(t)$ is obviously bounded for any bounded t and satisfies assumption (1).

Q. E. D.

Remark. — If $V^* = V$, then $H_0 + V$ is selfadjoint and

$$\exp i (H_0 + V) t = \mathrm{Exp}_r \left(\int_0^t ; i \alpha^{i H_0 s} V \, ds \right) e^{i H_0 t} = e^{i H_0 t} \mathrm{Exp}_l \left(\int_0^t ; i \alpha^{-i H_0 s} V \, ds \right),$$

where the right hand sides can be directly proved to be a continuous one-parameter group of unitaries by cocycle equations, the domain of the generator is exactly $D(H_0)$ and the generator is $i(H_0 + V)$ on $D(H_0)$. The selfadjointness of such $H_0 + V$ is a very special case of the Kato perturbation theorem.

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