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FREE DIFFERENTIABLE S^1 AND S^3 ACTIONS ON HOMOTOPY SPHERES

By Dan BURGHELEA (*)

0. Introduction. — In this paper we shall study the differentiable free S^1 (resp. S^3)-actions on homotopy spheres Σ^k, and their classifications up to a

(1) differentiable equivalence;

(2) topological equivalence;

(3) differentiable free S^1 (resp. S^3)-cobordism.

Throughout this paper, manifolds, G-principal bundles, cobordisms, etc., are always meant to be oriented, and actions, diffeomorphisms, homeomorphisms, homotopy equivalences, are orientation preserving.

The main results of this paper can be summarised in the following theorems :

Theorem A. — *If* Σ^{2n+1} $(n \geq 3)$ *has a differentiable free* S^1*-action, then there exist infinitely many other differentiable free* S^1*-actions topologically nonequivalent, and non-" rationally free* S^1*-cobordant ".*

Theorem B. — *If* Σ^{4n+3} $(n \geq 4)$ *has a differentiable free* S^3*-action, then it has infinitely many differentiable free* S^3*-actions topologically nonequivalent and non " rationally free* S^3*-cobordant ".*

Notice that if Σ^k has a differentiable or topological free S^1 (*resp.* S^3)-action then $k = 2t + 1$ $(k = 4t + 3)$ for some natural number t.

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THEOREM C. — *Two differentiable free* S^1 (resp. S^3)-*actions on homotopy spheres, which are rational free* S^1 (resp S^3)-*cobordant, are free* S^1 (resp. S^3)-*cobordant.*

THEOREM D. — (*a*). *Two differentiable free* S^3-*actions on a n-homotopy sphere, which are rational free* S^3-*cobordant, are topologically equivalent* (*except for* $n = 7$).

(*b*). *The set of all differentiable free* S^1-*actions on a* $2n+1$-*homotopy sphere* ($n \neq 2$), *rationally free* S^1-*cobordant to a given differentiable free* S^1-*action on* Σ^{2n+1}, *contains only* $2^{d(\Sigma)}$ *topologically nonequivalent* S^1-*actions, where* $d(\Sigma^{2n+1})$ *is a non-negative integer, smaller than or equal to* $\left[\frac{n+1}{2}\right]$ (*for a positive real number* α, $[\alpha]$ *denotes its integral part*).

In some sense *b* is analogous to *a*, with S^1-actions replaced by S^3-actions, but only outside the " world " of the prime number 2.

REMARK E. — The topological equivalence of differentiable free S^1 (resp. S^3)-actions on homotopy spheres, implies their (differentiable) free S^1 (resp. S^3)-cobordism but not their differentiable free S^1 (resp. S^3)-equivalence.

STATEMENT F. — There exist oriented differentiable spin-manifolds M^{4n} with non-trivial topological S^1-action and $\hat{A}(M^{4n}) \neq 0$.

Theorems A and B are derived using the functorialized form of the Browder-Novikov theory (due to Denis Sullivan). This theory has been previously applied to differentiable free S^1 (resp. S^3)-actions by Hsiang [7], who proved the existence of at least one homotopy sphere Σ^{2n+1}, $n \geq 3$ (resp. one homotopy sphere Σ^{4n+3}, $n \geq 2$) with infinitely many differentiable free S^1 (resp. S^3)-actions, topologically nonequivalent.

The main progress represented by theorems A and B lies in their main corollary, namely : S^{2n+1} ($n \geq 3$) (resp. S^{4n+3} ($n \geq 2$)) has infinitely many topologically nonequivalent differentiable free S^1 (resp. S^3)-actions. At the same time they give a partial answer to problem 2 in (G. E. Bredon and C. N. Lee, p. 235 [10]). We should note that theorem A for $n = 3$ has been previously proved by Montgomery and Yang [9] by rather different methods.

Theorem C is derived as a consequence of a nice suggestion of Denis Sullivan to reconsider the old Thom's definition of rational Pontrjagin classes of polyhedral rational homology manifolds (*see* also Milnor [8]). It can be also recovered by using the Atiyah-Singer invariant associated to a differentiable free S^1-action; here, however, we prefer to use the more topological method, apparently simpler.

Statement F is a consequence of the methods and computations used in the previous sections. This statement can be of some interest if we notice that Atiyah and Hirzebruch [1] have shown that any differentiable oriented spin-manifold which admits a non-trivial differentiable S^1-action has $\hat{A}(M^{4n}) = 0$. By statement F, one gets examples of differentiable oriented spin-manifolds showing that :

(1) The Atiyah-Hirzebruch result is not true in the TOP-category, although $\hat{A}$ is still defined.

(2) There exists a compact differentiable manifold whose topological group of all homeomorphisms contains a compact connected Lie group, while that of all diffeomorphisms (with the C^∞-topology) does NOT.

Important : When no confusion could occur we simplify the notation omitting the " indexes "; for instance instead of λ^d_{4k}, λ^t_{4k} we shall write λ_{4k} and instead $K^{M,\partial M}_d$, $K^{M,\partial M}_t$ we shall write K_d or K_t or only K_d or K_t.

1. Review of Sullivan's theory and the subgroup G (M). — In this section we briefly review the Sullivan exact sequences which functorialises the Browder-Novikov theory, and the homotopy type of H/Top. Our contribution consists only in the definition of the subgroup $G(M^{4k}) \subset [M^{4k}; H/O]$ which will be very important for the computations below and whose main property is pointed out by proposition 1.5.

We will follow the standard notation O, Top, H, H/O, H/Top (*see* [13]) recalling that all these space are ∞-loop spaces and all the natural maps $O \to Top \to H \to H/O \to H/Top$ are maps in the category of ∞-loop spaces [16]. We use " d " for differentiable, " t " for topological and when we treat two of them simultaneously, we use " c " for " d " or " t " and C for O or Top. Thus a diffeomorphism (homeomorphism), will be referred to as a d-automorphism (t-automorphism).

Following Sullivan, for any compact c-manifold $(M^n, \partial M^n)$ with possible empty ∂M, one defines $\mathcal{S}_c(M^n)$ (respectively $\mathcal{S}_c(M^n, \partial M^n)$) as the equivalence (concordance) classes of homotopy equivalences $h : (N^n, \partial N^n) \to (M^n, \partial M^n)$ (respectively homotopy equivalences h, which restricts on ∂N to a c-automorphism).

Two homotopy equivalences $h_i : (N^n_i, \partial N^n) \to (M^n, \partial M^n)$, $i = 1, 2$, are concordant, if there exists a c-automorphism $l : (N^n_1, \partial N^n_1) \to (N^n_2, \partial N^n_2)$, such that $h_2 . l$ is homotopic to h_1, (respectively iff there exists a c-automorphism l such the $h_2 . l |_{\partial N_1} = h_1 |_{\partial N_1}$ and $h_2 . l$ and h_1 are homotopic by a homotopy constant on ∂N_1. $\mathcal{S}_c(M^n)$ and $\mathcal{S}_c(M^n, \partial M^n)$ are obviously sets with a natural base point represented by id : $(M^n, \partial M^n) \to (M^n, \partial M^n)$.

One has an obviously defined sequence of based point preserving maps

$$(1)\qquad \begin{cases} \mathcal{S}_c(M^n, \partial M^n) \xrightarrow{J^c} \mathcal{S}_c(M^n) \xrightarrow{\partial} \mathcal{S}_c(\partial M^n) \quad \text{with} \quad \partial . J^c = \star \\ \quad (\star \text{ denotes the base point}) \end{cases}$$

i. e. $\partial . J^c$ sends all elements of $\mathcal{S}_c(M^n, \partial M^n)$ to the based point of $\mathcal{S}_c(\partial M^n)$.

If $n \neq 3, 4$, the " Poincaré conjecture " implies that $\mathcal{S}_t(S^n) = \star$, and $\mathcal{S}_c(S^n) = \theta_n$ (with Milnor-Kervaire notation [17]).

Proposition 1.1. — *If ∂M^n is a homotopy sphere the sequence* (1) *is exact* (*as sequence of base pointed sets*).

(The proof is obvious.)

If $(M^n, \partial M^n)$ is a compact d-manifold, ignoring the differential structure, one gets a compact t-manifold; therefore one has a natural map $u^M : \mathcal{S}_d(M) \to \mathcal{S}_t(M)$ (resp. $u^{M, \partial M} : \mathcal{S}_d(M, \partial M) \to \mathcal{S}_t(M, \partial M)$) such that $J^t . u^{M, \partial M} = u^M . J^d$ and $\partial . u^M = u^{\partial M} . \partial$.

Let $(M^n, \partial M^n)$ be a compact c-manifold with a possibly empty boundary. In [13] D. Sullivan defines the following based point preserving maps :

(i) $K^M_c : \mathcal{S}_c(M^n) \to [M^n; H/C]$ and $K^{M, \partial M}_c : \mathcal{S}_c(M^n, \partial M^n), \to [M^n, \partial M^n; H/C]$

($[\ldots; H/C]$ denotes the abelian group of homotopy classes of continous maps in H/C); if $j^c : [M^n, \partial M^n; H/C] \to [M^n; H/C]$ denotes the group-homomorphism induced by the inclusion $(M^n, \emptyset) \subset (M^n, \partial M^n)$ and $p^{M, \partial M}$ or p^M denotes the group-homomorphism $p^{\cdots} : [\ldots, H/O] \to [\ldots, H/Top]$ induced by $p : H/O \to H/Top$, one has :

$$j^c . K^{M, \partial M}_c = K^M_c . J^c, \qquad p^M . K^M_d = K^M_t . u^M \qquad \text{and} \qquad p^{M, \partial M} . K^{M, \partial M}_d = K^{M, \partial M}_t . u^{M, \partial M}$$

(ii) $$\lambda^c_n : [M^n, M^n; H/C] \to P_n,$$

where $P_n = Z, 0, Z_2, 0$ as $n \equiv 0, 1, 2, 3 \pmod 4$; λ^c_n verifies $\lambda^t_n . p^{M, \partial M} = \lambda^d_n$, and λ^c_n is a group-homomorphism for $n \not\equiv 0 \pmod 4$. We will give below the explicit definition of λ^c_n.

(iii) $$\theta_n(\partial\pi) \xrightarrow{S_n} \mathcal{S}_d(M^n, \partial M^n),$$

where $\theta_n(\partial\pi)$ is the subgroup of θ_n (the group of homotopy spheres) consisting of the elements which bound π-manifolds, hence $\theta_n(\partial\pi) \neq 0$ only for n odd.

In fact S_n is derived from the natural action of θ_n on $\mathcal{S}_d(M, \partial M)$, which is obviously defined using the connected sum (recall $\theta_n = \mathcal{S}_d(S^n)$). S_n is given by the action of $\theta_n(\partial\pi)$, on the based point $\star$ of $\mathcal{S}_d(M, \partial M)$.

Let $P = \theta_n(\partial\pi)$ if $c = d$ and $P = \star$ if $c = t$. One has :

SULLIVAN'S EXACT SEQUENCE THEOREM. — If $(M^n, \partial M^n)$ is a compact c-manifold with $\partial M^n \neq \emptyset$, $n \geq 6$, $\pi_1(N^n) = \pi_1(\partial M^n) = 0$, then in the following commutative diagram the horizontal lines are exact sequences of sets

$$\begin{array}{ccccccc}
P & \xrightarrow{S_n} & \mathcal{S}_c(M^n, \partial M^n) & \xrightarrow{K_c^{M,\partial M}} & [M, \partial M; H/C] & \xrightarrow{\lambda_n^c} & P_n \\
 & & \downarrow & & \downarrow & & \\
 & & \mathcal{S}_c(M^n) & \xrightarrow{K_c^{M}} & [M; H/C] & \longrightarrow & \bigstar .
\end{array}$$

Now, we briefly describe the homotopy type of H/Top, following Sullivan [14], Kirby and Siebenmann [6]. First recall that both BO and H/Top are ∞-loop spaces (H/Top by the work of Broardman-Vogt, and BO by the Bott periodicity). If T is an ∞-loop space, $[\ldots, T]$ is a representable half exact functor in the sense of Dold with value in the category of abelian groups. Let Z, Q be the rings of integers and rational numbers respectively. Put

$$Z_{(2)} = \left\{ \frac{m}{n} \in Q \mid m, n \in Z,\ n \text{ odd} \right\},\ Z_{\text{odd}} = \left\{ \frac{m}{n} \in Q \mid m, n \in Z,\ n = 2^a,\ a \in Z \right\};$$

$Z_{(2)}$ and Z_{odd} are subrings of Q.

It is well known that Q, Z_{odd}, and $Z_{(2)}$ are Z-flat modules, therefore $[\ldots, T] \otimes_Z Q$, $[\ldots; T] \otimes_Z Z_{\text{odd}}$ and $[\ldots; T] \otimes_Z Z_{(2)}$ are also representable-half exact functors and the representation spaces T_{odd}, T_2, T_Q are still ∞-loop spaces.

The natural transformations of half exact functors

$$\begin{array}{ccccc}
 & & [\ldots; H/Top] \otimes_Z Z_{(2)} & & \\
 & \nearrow & & \searrow & \\
[\ldots; H/Top] & & & & [\ldots; H/Top] \otimes_Z Q \\
 & \searrow & & \nearrow & \\
 & & [\ldots; H/Top] \otimes_Z Z_{\text{odd}} & &
\end{array}$$

induce the following diagram in the category of ∞-loop spaces

$$\begin{array}{ccccc}
 & & (H/Top)_{(2)} & & \\
 & \overset{r_2}{\nearrow} & & \overset{i_p}{\searrow} & \\
H/Top & & & & (H/Top)(Q) \\
 & \underset{r_{odd}}{\searrow} & & \underset{r}{\nearrow} & \\
 & & (H/Top)_{(\text{odd})} & &
\end{array}$$

which, viewed in the homotopy category, is cartesian.

Then, by general " categorical " considerations one has

Remark 1.2. — Two continous maps $f, g : M \to H/Top$ are homotopic iff :

(i) $r_2 . f$ and $r_2 . g$ are homotopic;

(ii) $r_{odd} . f$ and $r_{odd} . g$ are homotopic.

Sullivan has shown (combining with Kirby-Siebenmann results [6]) that

$$(H/Top)_{(Q)} = \prod_{i=1}^{\infty} K(Q, 4i), \qquad (H/Top)_{(2)} = \prod_{i=1}^{\infty} K(Z_{(2)}, 4i) \times \prod_{i=0}^{\infty} K(Z_2, 4i+2)$$

and

$$(H/Top)_{(odd)} = BO_{(odd)}.$$

Via these identifications, i_p is given as composition of the projection on the first factor with the natural map induced by $Z_{(2)} \subset Q$, and r viewed as an element of $[BO_{odd};\ H/Top_{(Q)}] = \prod_{i=1}^{\infty} H^{4i}(BO_{odd}, Q)$ is represented by $\mathcal{L} = (l_4, l_8, \dots)$ with $l_{4i} \in H^{4i}(BO, Q)$ the Hirzebruch classes.

The composition of r_2 with the second factor projection of

$$\prod_{i=1}^{\infty} K(Z_{(2)}, 4i) \times \prod_{i=0}^{\infty} K(Z_2, 4i+2)$$

can be viewed as an element of $\prod_{i=0}^{\infty} H^{4i+2}(H/Top, Z_2)$, given by $\mathcal{W} = (\overline{w}_2, \overline{w}_6, \overline{w}_{10}, \dots)$ with $\overline{w}_{4i+2} \in H^{4i+2}(H/Top; Z_2)$.

We define $\lambda^{l}_{4k+2}(f) = \langle W(M) . f^{*}(\mathcal{W}), \mu \rangle$, where μ denotes the fundamental classs of

$$H_n(M, \partial M; Z_2) \quad \text{and} \quad W(M) = 1 + w_1(M) + \dots + w_n(M)$$

the total Stiefel-Whitney class of M; we also define

$$\lambda^{l}_{4k}(f) = \frac{1}{8} \langle L(M) . (r_{odd} . f)^{*}(\mathcal{L}), \mu \rangle \ (^1),$$

where $\mu \in H_{4k}(M^{4k}, \partial M^{4k}; Z)$ denotes the fundamental class M^{4k} given by the orientation, and $L(M) = 1 + l_4(M) + \dots l_{4k}(M)$ the total Hirzebruch class of M.

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(1) One denotes by g^{*} the homomorphism of cohomology groups induced by the continous map $g : X \to Y$.

If $p^{M,\partial M}$ is the group homomorphism induced by $p : H/O \to H/Top$, we put

$$\lambda^d_{4k} = \lambda^t_{4k} . p^{M,\partial M} \quad \text{and} \quad \lambda^d_{4k+2} = \lambda^t_{4k+2} . p^{M,\partial M}$$

respectively (notice that $P_{2k+1} = 0$ hence λ^t_{2k+1} and λ^d_{2k+1} are the trivial maps).

It is not obvious that λ^c_{4k+2} is a group homomorphism, however, special properties of the classes $\overline{w}_{4k+2}$ resulting from the product formula for the Kervaire-Arf invariant [11] imply

$$\lambda^c_{4k+2}(fg) = \lambda^c_{4k+2}(f) . \lambda^c_{4k+2}(g).$$

If we consider $\delta_d : H/O \to BO$, and $\delta_t : H/Top \to BTop$ as natural maps classifying the principal fibrations $O \to H \to H/O$ and $Top \to H \to H/Top$, respectively, then

$$(r_{odd} . f)^* (\mathcal{L}) = (\delta f)^* (l_4) + (\delta f)^* (l_8) + \dots,$$

where $l_{4k} \in H^{4k}(BC; Q)$ represents the universal Hirzebruch classes ($BO \to BTop$ induces an isomorphism of the rational cohomology).

Notice that if t denotes the map $t : M \to BC$ classifying the tangent bundle of M, then $L(M) = t^* (1 + \mathcal{L})$, and consequently

$$\lambda^c_{4k}(fg) = \frac{1}{8} \langle t^* (1 + \mathcal{L}) . (\delta^* (fg)^* \mathcal{L}), \mu \rangle.$$

Since $\delta_c : H/C \to BC$ is a map of ∞-loop spaces we have

$$\begin{aligned} 1 + \delta^* (fg)^* (\mathcal{L}) &= 1 + ((\delta f)\ (\delta g))^* (\mathcal{L}) = (1 + (\delta f)^* \mathcal{L})(1 + (\delta g)^* (\mathcal{L})) \\ &= 1 + (\delta f)^* \mathcal{L} + (\delta g)^* \mathcal{L} + ((\delta f)^* \mathcal{L}) . ((\delta g)^* \mathcal{L}) \end{aligned}$$

and as

$$\lambda_{4k}(f) + \lambda_{4k}(g) = \frac{1}{8} \langle t^* (1 + \mathcal{L}) . ((\delta f)^* \mathcal{L}), \mu \rangle + \frac{1}{8} \langle t^* (1 + \mathcal{L}) . (\delta g^*) (\mathcal{L}), \mu \rangle$$

one gets

$$\lambda_{4k}(fg) - \lambda_{4k}(f) - \lambda_{4k}(g) = \frac{1}{8} \langle t^* (1 + \mathcal{L}) . ((\delta f)^* \mathcal{L}) . ((\delta g)^* \mathcal{L}), \mu \rangle,$$

which shows that λ_{4k} is not in general a group homomorphism.

However we will define a subgroup $G(M^{4k}) \subset [M^{4k}, \partial M^{4k}; H/O]$ such that λ^d_{4k} restricted to $G(M^{4k})$ is always a group homomorphism. For that, let us consider the universal reduced Pontrjagin character viewed as a map $\tilde{P} : BO \to \prod_{i=1}^{\infty} K(Q, 4i)$. If we denote by $p^{\left[\frac{k}{2}\right]}$ the projection of

$\prod_{i=1}^{\infty} K(Q, 4i)$ on $\prod_{i=1}^{[\frac{k}{2}]} K(Q, 4i)$, and by $\tilde{P}^{[\frac{k}{2}]} = p^{[\frac{k}{2}]}.\tilde{P}$, then we define

$$G(M^{4k}) = \mathrm{Ker}\left([M^{4k}, \partial M^{4k}; H/O] \to \prod_{i=1}^{[\frac{k}{2}]} H^{4k}(M^{4k}, Q) \right),$$

where the homomorphism in parentheses is the composition

$$[M^{4k}, \partial M^{4k}; H/O] \to [M^{4k}, \partial M^{4k}; BO] \to [M^{4k}; BO] \xrightarrow{\tilde{P}^{[\frac{k}{2}]}} \prod_{i=1}^{[\frac{k}{2}]} H^{4i}(M^{4k}; Q)$$

(the first homomorphism is induced by δ_d and the second by the inclusion $(M, \emptyset) \subset (M, \partial M)$).

PROPOSITION 1.3. — $\lambda^c_{4k} : G(M^{4k}) \to Z$ *is a group homomorphism.*

Proof. — Using the definitions of $G(M^{4k})$ and $\tilde{P}$, we easily check the equivalence of (i), (ii), (iii) (iv) :

(i) $f \in G(M^{4k})$;

(ii) $(\tilde{P}(\delta f))_i = 0$, $i = 1, 2, \ldots, \left[\frac{k}{2}\right]$, where $(\tilde{P})_i$ denotes the i-th component of the Pontrjagin character;

(iii) $(\delta f)^* p_{4i} = 0$, $i = 1, \ldots, \left[\frac{k}{2}\right]$, p_{4i} being the universal Pontrjagin classes;

(iv) $(\delta f)^* l_{4i} = 0$, $i = 1, \ldots, \left[\frac{k}{2}\right]$.

Then, if $f, g \in G(M^{4k})$

$$(\delta f)^*(\mathcal{L}).(\delta g)^*(\mathcal{L}) = 0$$

hence

$$\lambda_{4k}(fg) = \lambda_{4k}(f) + \lambda_{4k}(g)$$

and similarly

$$\lambda_{4k}(f') = -\lambda_{4k}(f), \ f' \in [M^{4k}, \partial M^{4k}; H/O] \quad \text{and} \quad f'f = 0.$$

PROPOSITION 1.4.

$$\dim_Q (G(M^{4k}) \otimes_Z Q) = \sum_{i=[\frac{k}{2}]+1}^{k} \dim H^{4i}(M; Q)$$

and if we denote $G'(M^{4k}) = \mathrm{Ker}(\lambda_{4k} : G(M^{4k}) \to Z)$, *then*

$$\dim_Q (G'(M^{4k}) \otimes_Z Q) \geqq \sum_{i=\left[\frac{k}{2}\right]+1}^{k} \dim H^{4i}(M^{4k}; Q) - 1.$$

In spite of its lack of geometric meaning the estimation of $\dim_Q(G(M^{4k}) \otimes_Z Q)$ will be very important in the proof of the theorems A and B, namely it will alow us to get estimates of the cardinality of some subsets of $\mathcal{S}_d(M, \partial M)$ and $\mathcal{S}_d(M)$.

2. S^1 AND S^3-ACTIONS AND THE HOMOTOPY THEORETICAL EQUIVALENCE OF VARIOUS CLASSIFICATIONS PROBLEMS. — In this section, the problem of the classification of differentiable (topological) free S^1 (resp. S^3)-actions on a homotopy sphere Σ_0^n will be reduced to a homotopy theoretic problem; also from the existence of a differentiable rational free S^1 (resp. S^3)-cobordism between two differentiable free S^1 (resp. S^3)-actions, we derive the equality of the Pontrjagin numbers of their characteristic maps.

Assume G to be S^1 or S^3, the compact Lie groups of complex numbers z with $|z| = 1$, respectively of quaternionic numbers w with $\|w\| = 1$. Before passing to definitions we invite the reader to keep in mind the conventions from the beginning of the introduction (§ 0).

DEFINITION 2.1. — A differentiable (topological) action of G on the differentiable (topological) manifold M^n, is a differentiable (continuous) map $T : G \times M^n \to M^n$ such that :

(i) $T(g_1 . g_2; x) = T(g_1; T(g_2, x))$;

(ii) $T(e, x) = x$, where "e" is the unit element of G;

(iii) $T(g, \ldots) : M^n \to M^n$ is a diffeomorphism.

If M^n is a manifold with boundary, condition (ii) implies that the boundary is invariant under the action, hence one can consider the restriction of the action to the boundary.

The triple (G, T, M^n) will be called a G-manifold, and the triple $(G, T/G \times \partial M, \partial M)$ will be called the boundary of the G-manifold (G, T, M) (and will be denoted by $\partial(G, T, M^n)$).

DEFINITION 2.2. — *a*. Two differentiable (topological) G-manifolds (G, T_i, M_i^n), $i = 1, 2$, are called differentiably or topologically equivalent, iff there exists an orientation preserving (Lie) group homomorphism $\alpha : G \to G$, and a diffeomorphism, respectively homeomorphism, $h : M_1^n \to M_2^n$, such that $T_2(\alpha(g), h(x)) = h(T_1(g, x))$.

b. Two differentiable G-manifolds (G, T_i, M_i^n), $i = 1, 2$ are called G-cobordant iff there exists a G-manifold with boundary (G, T, W^{n+1}) such that $\partial (G, T, W^{n+1})$ has two connected components differentiably equivalent one to (G, T_1, M_1^n) and the other to $(G, T_2, -M_2^n)$ ($-M_2^n$ denotes the manifold with the same topological underlying space and inverse orientation). Obviously G-cobordism is an equivalence relation. If (G, T, M^n) is a G-manifold and $x \in M^n$ a point, denote by $G_x = \{ g \in G \mid T(g, x) = x \}$ the isotropy group of the action T at the point x.

DEFINITION 2.3. — *a*. The G-manifold (G, T, M^n) is called free (in fact the action T is called free) respectively rationally free if for any $x \in M$, $G_x = 0$, respectively G_x is a finite group. In the topological case the action T is free if $G_x = 0$, $M \to M/G$ is a (locally trivial) principal G-bundle, and M/G is a topological manifold. Both of the last conditions are obviously superfluous if the action is differentiable.

b. If in definition 2.2 *b*. one replaces G-manifold by " free G-manifold " respectively " rationally free G-manifold " we get the corresponding notion of " . . . free G-cobordism " respectively " . . . rationally free G-cobordism ".

Notice that for $G = S^1$ the notion of " rationally free " is equivalent to the notion of " fixed point free ".

Let (G, T, M^n) be a differentiable (topological) free G-manifold; since $M^n \xrightarrow{p} M^n/G$ is a principal G-bundle, it is completely classified by its " characteristic map ", namely a homotopy class $F : M^n/G \to BG$. If (G, T', M'^n) is another differentiable (topological) free G-manifold equivalent to (G, T, M), $f' : M'^n/G \to BG$, its corresponding homotopy class (characteristic map) and $\bar{h} : M^n/G \to M'^n/G$ the homotopy equivalence (considered as an homotopy class) induced by h (*see* definition 2.2), then $f'.\bar{h} = f$, because the automorphism α preserving the orientation (of G) lies in a 1-parameter subgroup, consequently it induces a homotopy equivalence of BG homotopic to the identity. Hence :

Remark 2.4. — The differentiable (topological) free G-manifolds (up to an equivalence) are completely determined by the equivalence classes of pairs $(M/G, f : M/G \to BG)$, f being thought of as a homotopy class. Two such pairs $\left(M_1^n/G \xrightarrow{f_1} BG\right)$, $\left(M_2^n/G \xrightarrow{f_2} BG\right)$ are equivalent iff there exists a diffeomorphism (homeomorphism) $t : M_1^n/G \to M_2^n/G$ such that $f.t \sim f_2$ (2).

(2) $\sim$ means homotopic.

Let us denote the set of all such equivalence classes of pairs by ${}^n\mathrm{Act}_d^G$ respectively ${}^n\mathrm{Act}_t^G$ (the differentiable, respectively topological, case). Notice two differentiable (topological) free G-manifolds (G, T_i, M_i^n), $i = 1, 2$ are differentiably (topologically) free G-cobordant iff the pairs $M_1^n/G \xrightarrow{f_1} BG$ and $M_2^n/G \xrightarrow{f_2} BG$ define the same element in $\Omega^t_{n\text{-}\dim G}(BG)$ $(\Omega^t_{n\text{-}\dim G}(BG))$, the oriented differentiable (topological) bordism of the space BG.

We have the following commutative diagram

$$\begin{array}{ccc} {}^n\mathrm{Act}_d^G & \xrightarrow{\omega^d} & \Omega^d_{n-\dim G}(BG) \\ \downarrow \tilde{u} & & \downarrow \tilde{\tilde{u}} \\ {}^n\mathrm{Act}_t^G & \xrightarrow{\omega^t} & \Omega^t_{n-\dim G}(BG), \end{array}$$

the maps ω^d and ω^t being surjective.

PROPOSITION 2.5. — *If Σ_0^n is a homotopy sphere and* (G, T_0, Σ_0^n) *a differentiable (topological) free* G*-action, then :*

(*a*). *If* $G = S^1$, *then* $n = 2k + 1$.

(*b*). *If* $G = S^3$, *then* $n = 4k + 3$.

(*c*). *The principal* G*-bundle* $\Sigma_0^n \to \Sigma_0^n/G$ *is n-universal* (3).

(*c*) is obvious because of the nullity of the homotopy groups of Σ_0^n up to dimension $n - 1$. *a* and *b* follow from easy spectral sequence arguments (*see* for instance [7]).

Therefore if we choose a fixed differentiable (topological) free G-action on the homotopy sphere Σ_0^n, (G, T_0, Σ_0^n) the characteristic map $f_0 : \Sigma_0^n/G \to BG$ defines a map

$$\sigma_d : \mathcal{S}_d(\Sigma_0^n/G) \to {}_d\mathrm{Act}^n(G), \qquad (\sigma_t : \mathcal{S}_t(\Sigma_0^n/G) \to {}_t\mathrm{Act}^n(G))$$

which by (*c*) is clearly injective, and the diagram

$$\begin{array}{ccc} \mathcal{S}_d(\Sigma_0^n/G) & \to & {}_d\mathrm{Act}^n(G) \\ \downarrow u & & \downarrow \tilde{u} \\ \mathcal{S}_t(\Sigma_0^n/G) & \to & {}_t\mathrm{Act}^n(G) \end{array}$$

is commutative.

Hence we have

PROPOSITION 2.6. — *The equivalence classes of differentiable (topological) free* G*-actions on homotopy spheres of dimension n, can be identified with*

$$\mathcal{S}_d(\Sigma_0^n/G) \ (\mathcal{S}_t(\Sigma_0^n/G)).$$

---

(3) For the definition of n-universal principal G-bundle *see* A. BOREL, *Sur la cohomologie des espaces fibrés principaux* (*Ann. of Math.*, vol. 57, 1953).

It is relatively easy to check that $\mathcal{S}_d(\Sigma_0^n/G)$ ($\mathcal{S}_t(\Sigma_0^n/G)$) have infinite cardinality and that the map $\mathcal{S}_d(\Sigma_0^n/G) \to \mathcal{S}_t(\Sigma_0^n/G)$ (n-dim $G \geqq 6$) has finite fibres by applying the Sullivan exact sequences from paragraph 1.

The first statement is already proved in [7] and the second one is a straightforward consequence of the finiteness of θ_n and of $[\Sigma_0^n/G; \mathrm{Top}/O]$ (wich is finite because Σ_0^n/G is a finite CW complex and Top/O has finite homotopy groups [6], [17]).

In the differentiable case we are not interested in all actions on all homotopy spheres but in that subset of $\mathcal{S}_d(\Sigma_0^n/G)$ which corresponds to the actions on the homotopy sphere Σ_0^n; let us denote this (base pointed) subset by $\mathcal{S}_d^{\Sigma_0}(\Sigma_0^n/G)$.

We will try to characterize the base pointed set $\mathcal{S}_d^{\Sigma_0^n}(\Sigma_0^n/G)$ in such a way as to make possible the estimation of its cardinality.

We start with $B_0 \to \Sigma_0^n/G$, the differentiable disc bundle associated to the fibration $\Sigma_0^n \to \Sigma_0^n/G$ of the differentiable free G-manifold (G, T_0, Σ_0^n). (The dimension of the fiber is 2 for $G = S^1$ and 4 for $G = S^3$.) B_0 is then a differentiable manifold whose boundary is Σ_0^n. Of course, the disc bundle $B_0 \to \Sigma_0^n/G$ defines a base point preserving map $\chi : \mathcal{S}_d(\Sigma_0^n/G) \to \mathcal{S}_d(B_0)$ and one checks using the explicit defination of $K^{\Sigma_0^n/G}$ and K^{B_0} that the following diagram is commutative :

$$\begin{array}{ccc} \mathcal{S}_d(\Sigma_0^n/G) & \xrightarrow{\chi} & \mathcal{S}_d(B_0) \\ \big\downarrow {\scriptstyle K^{\Sigma_0^n/G}} & & \big\downarrow {\scriptstyle K^{B_0}} \\ [\Sigma_0^n/G; H/O] & \xrightarrow{\approx} & [B_0; H/O]. \end{array}$$

Recall that χ is defined as follows $\chi(f : M \to \Sigma_0^n/G) = \chi(f) : \tilde{M} \to B_0$ where $\tilde{M}$ is the total space of the pullback of the differentiable disc bundle $B_0 \to \Sigma_0^n/G$ by a differentiable representative of f, and $\chi(f)$ is then the corresponding covering map which is a homotopy equivalence of pairs $\chi(f) : (\tilde{M}, \partial M) \to (B_0, \partial B_0)$.

Moreover, since $[\Sigma_0^n/G; H/O] \xrightarrow{\approx} [B_0; H/O]$, induced by the homotopy equivalence $B_0 \to \Sigma_0^n/G$, is a group isomorphism and $K^{\Sigma_0^n/G}$ is injective, as follows from Sullivan's exact sequence (§ 1) and Proposition 2.5, *a* and *b*, one derives that $\chi : \mathcal{S}_d(\Sigma_0^n/G) \to \mathcal{S}_d(B_0)$ is also injective.

In paragraph 1 we have defined $\partial : \mathcal{S}_d(B_0) \to \mathcal{S}_d(\partial B_0) = \mathcal{S}_d(\Sigma_0^n)$; applying proposition 1.1 one gets :

PROPOSITION 2.7. — *The equivalence classes of differentiable free* G*-actions on homotopy spheres* Σ_0^n, *i. e. the set* $\mathcal{S}_d^{\Sigma_0^n}(\Sigma_0^n/G)$, *is identified with* Ker $(\partial.\chi) = (\partial.\chi)^{-1}(\star)$, *where* $\star$ *is the base point of* $\mathcal{S}_d(\Sigma_0^n/G)$ *as soon as one knows that* $\mathcal{S}_d(\Sigma_0^n/G) \neq \emptyset$.

Proposition 2.7 is important, since with the aid of this characterisation of $\mathcal{S}_d^{\Sigma_0^n}(\Sigma_0^n/G)$, we will be able to write down a diagram using the Sullivan exact sequences, which makes possible the estimation of the cardinality of $\mathcal{S}_d^{\Sigma_0^n}$, as shown in the next section.

PROPOSITION 2.5. — (*d*) *If* (G, T_0, Σ_0^n) *is a differentiable free* G*-action on the homotopy sphere* Σ_0^n, Σ_0^n/G *has the homotopy type of* $CP_{\frac{n-1}{2}}$ *if* $G = S^1$ *or of* $HP_{\frac{n-3}{4}}$ *if* $G = S^3$.

(*e*) B_0 *is a differentiable manifold with boundary of the same homotopy type manifold with boundary as*

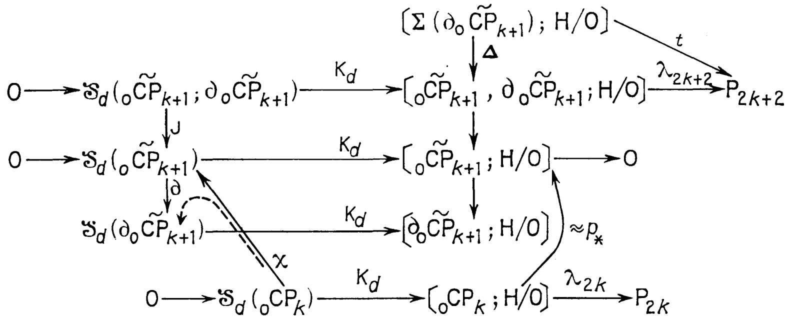
$$\left(CP_{\frac{n+1}{2}} \setminus \mathrm{Int}\, D^{n+1}, \partial D^{n+1}\right) \quad \textit{if } G^1 = S^1$$

and

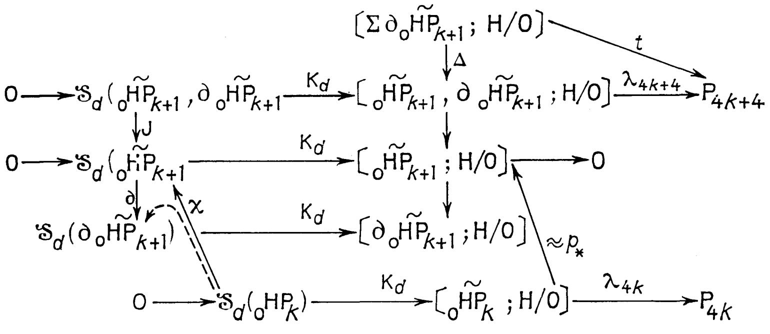
$$\left(HP_{\frac{n+1}{4}} \setminus \mathrm{Int}\, D^{n+1}, \partial D^{n+1}\right) \quad \textit{if } G = S^3.$$

To simplify the notation we will write ${}_0CP_{\frac{n-1}{2}}$ respectively ${}_0HP_{\frac{n-3}{4}}$ for Σ_0^n/G if $G = S^1$ respectively $G = S^3$, and ${}_0\widetilde{CP}_{\frac{n+1}{2}}$ respectively ${}_0\widetilde{HP}_{\frac{n+1}{4}}$ for B_0 if $G = S^1$ respectively S^3. We consider the following diagrams, where the right vertical lines are given by the exact sequence of the pair $({}_0\ldots P\ldots, \partial D\ldots)$ [4].

(1)



(2)



[4] By Σ we denote the suspension.

Almost all of the maps have already been defined in paragraph 1, and the commutativity was also stated by the propositions and remarks of this section and previous ones, except the upper right triangle

$$\begin{array}{ccc} [\Sigma(\partial\, {}_0\widetilde{CP}_{k+1})\,;\, H/O] & \xrightarrow{\quad t \quad} & \\ \downarrow \Delta & & \\ [{}_0\widetilde{CP}_{k+1}, \partial\, {}_0\widetilde{CP}_{k+1}, H/O] & \xrightarrow{\lambda_{2k+2}} & P_{2k+2} \end{array}$$

and the analogous diagram for ${}_0HP$....

Notice that $\Sigma\left(\partial\ {}_0\widetilde{CP}_{k+1}\right)$ is still a topological manifold. Define t exactly as λ, but remark that all the characteristic classes vanish, hence $W\left(\Sigma\left(\partial\,{}_0\widetilde{CP}_{k+1}\right)\right) = 1$ and $L\left(\Sigma\left(\partial\,{}_0\widetilde{CP}_{k+1}\right)\right) = 1$. Therefore if k is even $t(f) = \langle f^*\,\delta^*(\mathcal{W}),\ |\mu|\rangle$ and if k is odd $t(f) = \frac{1}{8}\langle f^*\,\delta^*(\mathcal{L}), \mu\rangle$. With this definition t is always a group homomorphism. Define Δ as the group homomorphism induced by the map of degree 1,

$$\left({}_0\widetilde{CP}_{k+1}/\partial\,{}_0\widetilde{CP}_{k+1}\right) \to \Sigma\left(\partial\,{}_0\widetilde{CP}_{k+1}\right),$$

which occurs in the Puppe-sequence of $\partial\,{}_0\widetilde{CP}_{k+1} \to {}_0\widetilde{CP}_{k+1}$.

One checks easily the commutativity of the triangle using the definitions of t, λ and Δ.

According to Sullivan, all horizontal lines are exact sequences of sets (*see* § 1) and by homotopy arguments the vertical lines are exact sequences, the left ones of pointed sets and the right ones of abelian groups.

The diagrams will allow us to get good estimates of the cardinality of $\mathcal{S}_d^{\Sigma_0^n}(\Sigma_0^n/G)$ as will be explained in the next section.

Next, we would like to make some comments on the case of the differentiable (topological) rationally free G-actions.

We consider then the maps

$$CP_\infty \xrightarrow{h_2} K(Z, 2) \xrightarrow{h} K(Q, 2) \qquad \text{and} \qquad HP_\infty \xrightarrow{h_4} K(Z, 4) \xrightarrow{h} K(Q, 4),$$

where h_k denotes the projections on the k-th Postnikov term in the Postnikov decomposition of CP_∞ [which happens to be $K(Z, 2)$] respectively of HP_∞.

h_2 is a homotopy equivalence, h_4 and h are rational homotopy equivalences (this means they induce an isomorphism for the homotopy groups tensored with Q over Z, and then for rational cohomology).

Assume one has a rationally free differentiable G-manifold (G, T, M^n). We attach a cohomology class $\varepsilon \in H^2(M/G; Q)$ if $G = S^1$, or a cohomology class $\varepsilon \in H^4(M/G; Q)$ if $G = S^3$, which, in the particular cases of free actions, represent the rational Euler class of the oriented fibre bundles $M \to M/G$. The existence of the class ε comes from the Leray spectral sequence for cohomology with rational coefficients associated to the projection $M \to M/G$. As far rational cohomology is concerned $M \to M/G$ ressembles a fibre bundle with spheres as fibres. In fact the map $M \to M/G$ is surjective and the fiber at any point is a rational homology sphere, namely the homogeneous space G/G_x where G_x is a finite subgroup. If $G = S^1$, G/G_x is homeomorphic to S^1 and if $G = S^3$ it is homeomorphic to S^3/G_x, which is always an oriented differentiable manifold with the property that $S^3 \to S^3/G_x$ induces an isomorphism for rational cohomology. Moreover, if we are concerned with cohomology with rational coefficients the Leray spectral sequence of $M \to M/G$ gives a spectral sequence which converges to $\mathcal{G}\, H^*(M; Q)$ whose E^2 is equal to $H^*(M/G; \mathcal{L})$. Here $\mathcal{L}$ denotes the local coefficient system defined by attaching to any point $x \in M/G$ the group $H^*(p^{-1}(x); Q)$ (*see* appendix 1), and because the action is orientation preserving, $\mathcal{L}$ is the trivial local system. Following similar arguments in [5] we can prove the existence of the Euler rational class, which is the standard one, if the action of G is free.

Representing the rational cohomology of M/G as homotopy classes of maps of M/G into K (Q; ...), the Eilenberg-Mac Lane spaces, to any differentiable free G-manifold we attach a pair $M/G \xrightarrow{\hat{f}} K(Q, i)$ ($i = 2$, respectively 4 if $G = S^1$ respectively S^3), where M/G is a polyhedral oriented rational homology manifold. We recall that the triangulability of the space of orbits of a differentiable action of a compact Lie group on a differentiable manifold is proved in [15].

If the action is free, $\hat{f} = h.h_i.f$ ($i = 2$ or 4 as $G = S^1$ or S^3). If (G, T_1, M_1), (G, T_2, M_2) are two differentiable rational free G-manifolds, then $\hat{f}_1 : M_1/G \to K(Q, i)$ and $\hat{f}_2 : M_2/G \to K(Q, i)$, considered as singular " polyhedral oriented rational homology manifolds " are cobordant, hence the Pontrjagin numbers of $\hat{f}_1$ and $\hat{f}_2$, which can be defined since the Pontrjagin classes are defined (for polyhedral oriented rational homology manifolds), are equal. Therefore we have the following :

PROPOSITION 2.8. — *Let* (G, T_k, M^n_k), $k = 1, 2$, *be two differentiable free* G-*manifolds which are differentiably rationally free* G-*cobordant; then* $f_1 : M^n_1/G \to BG$ *and* $f_2 : M^n_2/G \to BG$ *have the same Pontrjagin numbers.*

The proof is immediate if one notices that $\hat{f}_1 = h.h_i.f_1$ and $\hat{f}_2 = h.h_i.f_2$ have the same Pontrjagin numbers because f_k and $\hat{f}_k$, $k = 1, 2$, have the same Pontrjagin numbers ($h.h_i$ is a rational homotopy equivalence).

3. Proof of theorems A and B. — In this section we will prove the theorems A and B, but first we will point out the main steps of these proofs. As we have seen in paragraph 2, proposition 2.6, the equivalence classes of differentiable free G-actions on the homotopy sphere Σ_0^n identifies to $\mathcal{S}_d^{\Sigma_0^n}(\Sigma_0^n/G)$, which by proposition 2.7 identifies to Ker $(\partial.\chi)$. The main diagrams and their exacteness on the horizontal and vertical lines allow us to identify, in diagram (1), Ker $\partial.\chi$ to Ker $\left(\lambda_{\frac{n-1}{2}}\right) \cap \mu_*\left(\text{Ker } \lambda_{\frac{n+1}{2}}\right)$, and in diagram (2) Ker $\partial.\chi$ to Ker $\lambda_{\frac{n-3}{4}} \cap \mu_*\left(\text{Ker } \lambda_{\frac{n+1}{4}}\right)$ using the map K; we have denoted by μ_* the composition

$$\left[{}_0\widetilde{CP}_{k+1}, \partial\, {}_0\widetilde{CP}_{k+1}; H/O\right] \to \left[{}_0\widetilde{CP}_{k+1}; H/O\right] \xrightarrow{p^{*-1}} [{}_0CP_k; H/O]$$

(resp.

$$[{}_0\widetilde{HP}_{k+1}, \partial\, {}_0\widetilde{HP}_{k+1}; H/O] \to [{}_0\widetilde{HP}_{k+1} : H/O] \xrightarrow{p^{*-1}} [{}_0HP_k; H/O]).$$

Therefore, to prove there exist infinitely many nonequivalent differentiable free G-actions on Σ_0^n, it suffices to check that Ker $(\lambda \ldots) \cap \mu_*(\text{Ker } \lambda \ldots)$ is an infinite set. Of course this set lies inside the abelian group $\left[{}_0PC_{\frac{n-1}{2}}; H/O\right]$ $\left(\text{resp. } \left[{}_0HP_{\frac{n-3}{4}}; H/O\right]\right)$ but unfortunately it is not subgroup. However, if $n \geq 11$, $G = S^1$, or $n \geq 19$, $G = S^3$ we can build inside this set a subgroup $\mathcal{M}$ of $[{}_0P \ldots; H/O]$, and prove that card $\mathcal{M} \geq \infty$ by showing that $\dim_Q \mathcal{M} \otimes_Z Q \geq 1$ (theorem 3.1).

If $G = S^1$ and $n = 7$ or $n = 9$ we will check directly that in diagram (1), $\left(\text{Ker } \lambda_{\frac{n-1}{2}}\right) \cap \mu_*\left(\text{Ker } \lambda_{\frac{n+1}{2}}\right)$ is infinite; we will deduce that from the simple remark that the equation $dx^2 + ex + fy = 0$ with d, e, f integers, has infinitely many solutions (x, y), $x, y \in Z$ (theorem 3.2).

In the next step, knowing that $(\text{Ker } \lambda \ldots) \cap \mu_* (\text{Ker } \lambda \ldots)$ is infinite we have to show that one can choose inside this set an infinite number of elements corresponding to actions which are not topologically equivalent, and also an infinite number of elements corresponding to actions which are not free cobordant (they are not even rationally free cobordant by the theorem C which will be proved in paragraph 4). This will be done by theorem 3.3.

Remark. — It seems possible that arguments similar to those used for the case $G = S^1$, $n = 7, 9$, will work for $G = S^3$, $n = 11, 15$, and then, theorem B would be true for $n = 4k + 3$, $k \geq 2$.

Theorem 3.1. — *(a) In the diagram 1, for any $k \geq 5$*

$$\text{card } (\text{Ker } \lambda_{2k} \cap \mu_* \text{ Ker } \lambda_{2k+2}) \geq \infty.$$

(*b*) *In the diagram* 2,

$$\text{card}\,(\text{Ker}\,\lambda_{4k} \cap \mu_* \,\text{Ker}\,\lambda_{4k+4}) \geqq \infty.$$

The proof of (*a*) and (*b*) are based on the following remark : if G is a finitely generated abelian group and G_1 and G_2 subgroups of G, then

$$\dim_Q (G_1 \cap G_2) \otimes_Z Q \geqq \dim_Q (G_1 \otimes_Z Q) + \dim_Q (G_2 \otimes_Z Q) - \dim_Q (G \otimes_Z Q).$$

Proof of (*a*). — Assume $k = 2p$. Denote by

$$\mathcal{M} = G' ({}_0CP_{2p}) \cap \mu_* (\text{Ker}\,\lambda^d_{4p+2})$$

and recall from paragraph 1 that λ^d_{4p+2} is a group homomorphism. Therefore $G'({}_0CP_{2p})$ and $\mu_*(\text{Ker}\,\lambda^d_{4p+2})$ aresub groups of $[{}_0CP_{2p}; H/O]$, finitely generated abelian group, hence

$$\begin{aligned}\dim_Q (\mathcal{M} \otimes_Z Q) \geqq \dim_Q (G' ({}_0CP_{2p}) \otimes_Z Q) + \dim_Q \mu_* \otimes Q\,(\text{Ker}\,\lambda_{4p+2} \otimes Q)\\ - \dim_Q [{}_0CP_{2p}; H/0] \otimes_Z Q = \dim_Q (G' ({}_0CP_{2p}) \otimes_Z Q) \quad (^5),\end{aligned}$$

because $\mu_* \otimes Q$ is an isomorphism and

$$\text{Ker}\,(\lambda_{4p+2} \otimes Q) = [{}_0\widetilde{CP}_{2p+1}, \partial\, {}_0\widetilde{CP}_{2p+1}; H/O] \otimes_Z Q.$$

By proposition 1.4,

$$\dim_Q (G' ({}_0CP_{2p}) \otimes_Z Q) \geqq \sum_{i \geqq \left[\frac{p}{2}\right]+1} \dim_Q H^{4i} ({}_0CP_{2p}; Q) - 1,$$

hence $\geqq 1$ for $p \geqq 3$. Consequently if $p \geqq 3$, card $\mathcal{M} \geqq \infty$, hence card $(\text{Ker}\,\lambda_{4p} \cap \mu_*\,\text{Ker}\,\lambda_{4p+2}) \geqq \infty$ and (*a*) is proved.

Assume $k = 2p + 1$. Denote by $\mathcal{M} = \text{Ker}\,\lambda_{4p+2} \cap \mu_*\, G' ({}_0\widetilde{CP}_{2p+2})$ and remark that $\text{Ker}\,\lambda_{4p+2}$ and $\mu_*\, G'({}_0\widetilde{CP}_{2p+2})$ are subgroups of $[{}_0CP_{2p+1}; H/O]$ which is finitely generated abelian group, hence

$$\begin{aligned}\dim_Q \mathcal{M} \otimes_Z Q \geqq \dim_Q (\text{Ker}\,\lambda_{4p+2} \otimes Q)\\ + \dim_Q \mu_* \otimes Q\,(G' ({}_0\widetilde{CP}_{2p+2}) \otimes_Z Q) - \dim_Q [{}_0CP_{2p+1}; H/O] \otimes_Z Q.\end{aligned}$$

Since in diagram (1) the upper right triangle is commutative, $t \otimes Q$ is an isomorphim and the right vertical line is exact, it follows that $\mu_* \otimes Q$

(5) Often we are working with [X, H/O] as with homotopy classes of base point preserving maps. We have to notice that always X is connected and simply connected (of the homotopy type of CP_k or HP_k) therefore the homotopy classes of continuous maps and homotopy classes of base point preserving continuous maps are the same.

is injective on $G'\left({}_0\widetilde{CP}_{2p+2}\right) \otimes_Z Q$. Because $\lambda_{4p+2} \otimes Q = 0$, the last term of the previous inequality is equal to $\dim_Q\left(G'\left({}_0\widetilde{CP}_{2p+2}\right) \otimes_Z Q\right)$.

By proposition 1.4,

$$\dim_Q G'\left({}_0\widetilde{CP}_{2p+2}\right) \otimes_Z Q \geq \sum_{i \geq \left[\frac{p+1}{2}\right]+1} \dim_Q H^{4i}\left({}_0\widetilde{CP}_{2p+2}, \partial\, {}_0\widetilde{CP}_{2p+2}; Q\right) - 1$$

$$= \sum_{i \geq \left[\frac{p+1}{2}\right]+1} \dim_Q H^{4i}(CP_{2p+2}; Q) - 1 \geq 1$$

as soon as $p \geq 2$, hence card $(\operatorname{Ker} \lambda_{4p+2} \cap \mu_* \operatorname{Ker} \lambda_{4p+4}) \geq$ card $\mathfrak{M} \geq \infty$ for $2p+1 \geq 5$.

Proof of (*b*). — Denote by $\mathfrak{M} = G'({}_0HP_k) \cap \mu_* G'\left({}_0\widetilde{HP}_{k+1}\right)$ where $G'({}_0HP_k)$ and $\mu_* G'\left({}_0\widetilde{HP}_{k+1}\right)$ are subgroups of the finitely generated abelian group $G({}_0HP_k) \subset [{}_0HP_k; H/O]$. [From the definition of G (§ 1), it follows that $\mu_*\left(G\left({}_0\widetilde{HP}_{k+1}\right)\right) \subset (G({}_0HP_k))$; then

$$\dim_Q \mathfrak{M} \otimes_Z Q \geq \dim_Q (G'({}_0HP_k) \otimes_Z Q) + \dim_Q \otimes \mu_* \otimes Q\left(G'\left({}_0\widetilde{HP}_{k+1}\right) \otimes_Z Q\right)$$
$$- \dim_Q G({}_0HP_k) \otimes_Z Q \geq -1 + \dim_Q \mu_* \otimes Q\left(G'\left({}_0\widetilde{HP}_{k+1}\right) \otimes_Z Q.\right]$$

Since in the diagram (2) the upper right triangle is commutative, $t \otimes Q$ is an isomorphism, and the right vertical line is exact, it follows that $\mu_* \otimes Q$ is injective on $G'\left({}_0\widetilde{HP}_{k+1}\right)$. Consequently the last term of the previous inequality is greater than or equal to

$$-2 + \sum_{i \geq \left[\frac{k+1}{2}\right]+1} \dim_Q H^{4i}\left({}_0\widetilde{HP}_{k+1}, \partial\, {}_0\widetilde{HP}_{k+1}; Q\right)$$

$$= -2 + \sum_{i \geq \left[\frac{k+1}{2}\right]+1} \dim_Q H^{4i}(HP_{k+1}; Q)$$

hence

$$\dim_Q \mathfrak{M} \otimes_Z Q \geq 1 \quad \text{for} \quad k \geq 4,$$

hence

$$\text{card}\,(\operatorname{Ker} \lambda_{4k} \cap \mu_* \operatorname{Ker} \lambda_{4k+4}) \geq \infty \quad \text{for} \quad k \geq 4.$$

Q. E. D.

If $\widehat{CP}_{k+1} = {}_0\widetilde{CP}_{k+1}/\partial_0 \widetilde{CP}_{k+1}$ and $K : \left({}_0\widetilde{CP}_{k+1}, \partial_0 \widetilde{CP}_{k+1}\right) \to \left(\widehat{CP}_{k+1}, \star\right)$ the corresponding identification map, one can identify the group $\left[\widehat{CP}_{k+1}; H/O\right]$ with $\left[{}_0\widetilde{CP}_{k+1}, \partial\, {}_0\widetilde{CP}_{k+1}; H/O\right]$ through the isomorphism induced by K.

Some times it will be easier to consider $[\widehat{CP}_{k+1}; H/O]$ instead $[{}_0\widetilde{CP}_{k+1}, \partial_0 \widetilde{CP}_{k+1}; H/O]$ and via the identification induced by K to think at μ_* as induced by the natural inclusion ${}_0CP_k \subset \widehat{CP}_{k+1}$; this is justified because the pairs $(\widehat{CP}_{k+1}, {}_0CP_k)$ and (CP_{k+1}, CP_k) have the same homotopy type.

It is also important to notice that $\widehat{CP}_{k+1}$ is a topological manifold, and up to the top dimension the image of Stiefel-Whitney respectively Pontrjagin classes by the homomorphism K^* induced by $K : {}_0\widetilde{CP}_{k+1} \to CP_{k+1}$ are precisely the Stiefel-Whitney respectively Pontrjagin classes of ${}_0\widetilde{CP}_{k+1}$; also the image by $K : ({}_0\widetilde{CP}_{k+1}, \partial\, {}_0CP_{k+1}) \to (CP_{k+1}, \star)$ of the orientation of ${}_0\widetilde{CP}_{k+1}$ is just the orientation of $\widehat{CP}_{k+1}$.

THEOREM 3.2 .— *For $k = 3, 4$ in the diagram* 1

$$\text{card}\,(\text{Ker}\,\lambda_{2k} \cap \mu_* \,\text{Ker}\,\lambda_{2k+2}) \geqq \infty.$$

Proof. — Recall that $H^*({}_0CP_4; Z) = Z[z]/z^3$, z generator of

$$H^2({}_0CP_4; Z) = Z, \qquad \text{and} \qquad [{}_0CP_4 : BO] = \tilde{K}_0({}_0CP_4) = Z \oplus Z$$

whose generators ω, $\eta\,[\omega = (1, 0), \ \eta = (0, 1)]$ satisfy $\eta = \omega^2$ (with respect to the ring structure of $\tilde{K}_0({}_0CP_4)$). The Pontrjagin classes of ω and η are given by $p_4(\omega) = z^2$, $p_8(\omega) = 0$, $p_4(\eta) = 0$, $p_8(\eta) = 6z^4$. If we consider $\delta^* : [{}_0CP_4; H/O] \to [{}_0CP_4; BO]$ one easily can see that $[{}_0CP_4; H/O]$ contains a subgroup of finite index isomorphic to $Z \oplus Z$ and using the estimation of $\text{Ker}\,([{}_0CP_4; BO] \to [{}_0CP_4; BH])$, ([4], p. 58, manuscrit §8), we can choose as generators of this group the elements ξ_1 and $\xi_2 \in [{}_0CP^4; H/O]$ so that $\delta^*\xi_1 = 24\,\omega + 98\,\eta$, $\delta^*\xi_2 = 240\,\omega$. Then an element $\xi = m\,\xi_1 + n\,\xi_2$, $m, n \in Z$, has

$$p_4(\delta^*\xi) = 24\,mz^2 \qquad \text{and} \qquad p_8(\delta^*\xi) = (am^2 + bm + cn)\,z^4,$$

a, b, c being precised integers.

Let us denote by αz^2 the first Pontrjagin class of ${}_0CP_4$; then $\lambda_8(\xi) = 0$ iff the evaluation

$$\left\langle \left(1 + \frac{\alpha}{3} z^2\right)\left(8\,mz^2 + \frac{7\,am^2 + 7\,bm + 7\,cn - 24^2\,m^2}{45}\,z^4\right), [{}_0CP_4] \right\rangle = 0.$$

As the evaluation equality is equivalent to an equation

$$(\star) \qquad dm^2 + em + fn = 0$$

with d, e, f integers depending on α, the element

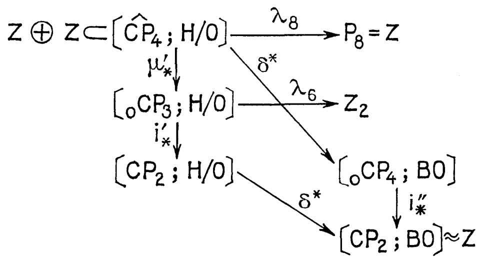
$$\xi = m\,\xi_1 + n\,\xi_2 \in [{}_0CP_4;\ H/O]$$

belongs to $\operatorname{Ker}\lambda_8$ iff m and n satisfying this equation. Because $\pi_9\,(H/O) = Z_2 \oplus Z_2$ we consider the following general solution [of the equation $(\bigstar)$: $m = 8\,ft$, $n = -\,8\,t\,(8\,dft + e)$ and claim that any element $\xi = m\,\xi_1 + n\,\xi_2$ with m and n given by the above formulas belongs to $\operatorname{Ker}\lambda_8 \cap \mu_* \operatorname{Ker}\lambda_{10}$. Such an element belongs to $\operatorname{Ker}\lambda_8$ because it satisfies the equation $(\bigstar)$ and it belongs also to $\operatorname{Im}\mu_*$ because $\xi = 8\,ft\,\xi_1 - 8\,t\,(8\,dft + e)\,\xi_2$ is divisible by 8 and the following sequence is exact $[CP_5;\ H/O] \to [{}_0CP_4;\ H/O] \to Z_2 \oplus Z_2$; moreover

$$\frac{1}{2}(m\,\xi_1 + n\,\xi_2) \in \operatorname{Im}\mu_*,$$

therefore there exists $\gamma \in [CP_5;\ H/O]$ such that $\mu'_*\,(\gamma) = \frac{1}{2}\,(m\,\xi_1 + n\,\xi_2)$. But because $\lambda_{10}\,(2\,\gamma) = 0$ one has $2\,\gamma \in \operatorname{Ker}\lambda_{10}$, hence $\mu_*\,(2\,\gamma) = m\,\xi_1 + n\xi_2$, hence $\xi \in \mu_* \operatorname{Ker}\lambda_{10}$. The set of all $\xi = 8\,ft\,\xi_1 - 8\,t\,(dft + e)\,\xi_2$, $t \in Z$ is an infinite subset of $\operatorname{Ker}\lambda_8 \cap \mu_* \operatorname{Ker}\lambda_{10}$, hence the theorem is proved for $k = 4$.

Assume now $k = 3$. (In this case theorem A has been already proved by Montgomery-Yang by a different method.) Let us consider i : $CP_2 \to {}_0CP_3$ a map so that $i^*\,(z'') = z'$ where z' and z'' are the canonical generators of the cohomology rings $H^*\,(CP_2;\ Z)$ and $H^*\,({}_0CP_3;\ Z)$ (Notice that CP_2 and ${}_0CP_3$ are the base spaces of some precised S^1-principal fibrations; the Euler classes of these fibrations are the canonical generators z' and z''). Such a map exists and it is uniquely defined up to an homotopy. We consider now the following commutative diagram



where μ'_* and i''_* are induced by the inclusion ${}_0CP_3 \subset \widehat{CP}_4$ and the composition $CP_2 \xrightarrow{i} {}_0CP_3 \subset \widehat{CP}_4$, consequently $\mu_*\,(z) = z'$. In what follows we will describe an infinite family of elements in $[{}_0CP_4;\ H/O]$ denoted by **C**, and we will show that (1) $\mu_*\,(\mathbf{C}) \subset \operatorname{Ker}\lambda_6 \cap \mu_* \operatorname{Ker}\lambda_8$ and

(2) card $\mu_* (\mathbf{C}) \geqq \infty$. Clearly this will imply card $(\operatorname{Ker} \lambda_6 \cap \mu_* \operatorname{Ker} \lambda_8) \geqq \infty$. Define $\mathbf{C} = \{ 2\, ft . \xi_1 - 2\, t\, (2\, dft + e)\, \xi_2 \mid t \in \mathrm{Z} \}$ and d, e, f being the coefficients of the equation ($\star$) for α given by $p_4 (\widehat{\mathrm{CP}_4}) = \alpha z^2 \}$. Because $i''_* \delta^* (m\, \xi_1 + n\, \xi_2) = 24\, m\, \overline{\omega}$ where $\overline{\omega}$ is the generator of $[\mathrm{CP}_2 ; \mathrm{BO}] = \mathrm{Z}$, $i''_* (\omega) = \overline{\omega}$ and $i''_* (\eta) = 0$ the set $\mathbf{C}' = i''_* \delta (\mathbf{C}) = \{ 48\, ft\, \overline{\omega} \mid t \in \mathrm{Z} \}$ is an infinite set, hence

$$\operatorname{Card} \mu_* (\mathbf{C}) \geqq \operatorname{card} \delta^* i'_* \mu_* (\mathbf{C}) = \operatorname{Card} i''_* \delta^* (\mathbf{C}) \geqq \infty ;$$

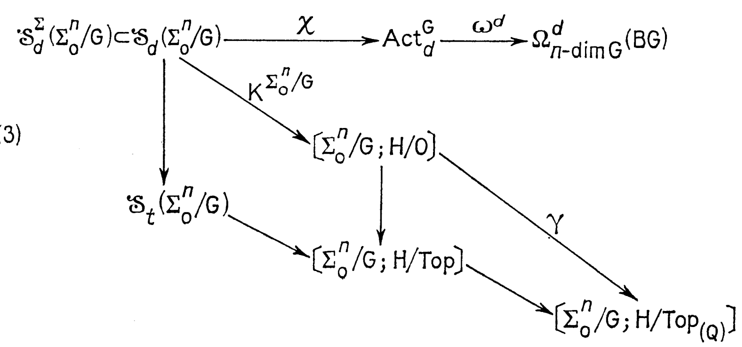
this proves (2). (1) being obvious the theorem is proved.

THEOREM 3.3. — (*a*) *If on the homotopy sphere* Σ_0^{2k+1} *there exist infinitely many differentiable free* S^1*-actions* ($k \geqq 3$) *which are differentiably nonequivalent, then among them there exist infinitely many which are topologically nonequivalent and infinitely many which are not differentiable free* S^1*-cobordant.*

(*b*) *If on the homotopy sphere* Σ_0^{4k+3} *there exist infinitely many differentiable free* S^3*-actions* ($k \geqq 2$) *which are differentiable nonequivalent, then among them there exist infinitely many which are topologically nonequivalent and infinitely many which are not differentiable free* S^3*-cobordant.*

In paragraph 2 we have defined the arrows of the following commutative diagram,

(3)



except γ which is the group homomorphism induced by the composition of the natural maps $\mathrm{H/O} \to \mathrm{H/Top} \to (\mathrm{H/Top})_{(\mathrm{Q})}$; this map is a morphism of ∞-loop spaces and moreover, a rational homotopy equivalence. We denote by $\theta = \omega^d . \chi$ and by $\tau = \gamma . \mathrm{K}$ and remark that τ has finite fibres (K is injective), $[\Sigma_0^n/\mathrm{G}, \mathrm{H/O}]$ is a finitely generated abelian group and $\gamma \otimes_{\mathrm{Z}} \mathrm{Q}$ an isomorphism. The theorem follows immediately from proposition 3.4 (*b*) which states that the fibres of τ and θ coincide.

One gets theorem A by combining theorems 3.1 (*a*), 3.2 and 3.3 (*a*), and theorem B by combining theorems 3.1 (*b*), 3.2 and 3.3 (*b*).

PROPOSITION 3.4. — (*a*). *If* $i : \Sigma_0^n/G \to BG$ *denotes the characteristic map of the principal fibration* $\Sigma_0^n/G \to BG$ *and* $\xi_1, \xi_2 \in \mathcal{S}_d(\Sigma_0^n/G)$ *are represented by* $f_1 : M_1 \to \Sigma_0^n/G$ *respectively* $f_2 : M_2 \to \Sigma_0^n/G$, *then the Stiefel-Whitney numbers of* $i.f_1$ *and* $i.f_2$ *are the same.*

(*b*). *If* $\xi_1, \xi_2 \in \mathcal{S}_d(\Sigma_0^n/G)$, *then* $\theta(\xi_1) = \theta(\xi_2)$ *iff* $\tau(\xi_1) = \tau(\xi_2)$.

Proof. of (*a*). — In proposition 2.5 we have established that Σ_1^n/G and Σ_2^n/G are homotopy equivalent (2.5; *d*), and both $i.f_1$ and $i.f_2$, as characteristic maps of the principal fibrations $\Sigma_1^n \to \Sigma_1^n/G = M_1$ and $\Sigma_2^n \to \Sigma_2^n/G = M_2$, induce isomorphisms of cohomology groups in dimension $\leqq n - \dim G$ (2.5; *c*). As the cohomology ring $H^*(BG; Z_2)$ is a polynomial ring in the generator $z \in H^k(BG; Z_2)$ with $k = 2$ (resp. 4) if $G = S^1$ (resp. S^3) and $(if_j)^*(z)$ represents the generator of $H^*(\Sigma_j^n/G; Z_2)$, $j = 1, 2$, one clearly checks that the Stiefel-Whitney numbers of $i.f_1$ and $i.f_2$ are equal.

Proof of (*b*). — We will show first that $\tau(\xi_1) = \tau(\xi_2)$ implies $\theta(\xi_1) = \theta(\xi_2)$. Notice that $\tau(\xi_1) = \tau(\xi_2)$ implies that $K(\xi_1)(K(\xi_2))^{-1}$ is an element of finite order in $[\Sigma_0^n/G; H/O]$; hence denoting by g_2 a homotopy inverse of f_2 and using the Sullivan's explicit definition of K [13], one concludes that $(g_2 . f_1)^*(p_{4i}(M_2)) = p_{4i}(M_1)$.

As :

(1) $H^*(BG; Q)$ is a polynomial ring in the generator $z \in H^k(BG; Q)$ with $k = 2$ (resp. 4) if $G = S^1$ (resp. S^3);

(2) if_j induces isomorphisms of the cohomology groups in dimension $\leqq n$-dim G [hence $H^*(\Sigma_j^n/G; Q)$ is a truncated polynomial ring in the generator $z_j = (if_j)^*(z)$, $j = 1, 2$] and

(3) $(g_2 f_1)^*(p_{4i}(M_2)) = p_{4i}(M_1)$;

it follows that the Pontrjagin numbers of if_1 and if_2 are equal. Indeed the Pontrjagin numbers of a map $g : M \to BG$ are of the form

$$\mathcal{P}^g_{\alpha, \beta_1, \ldots, \beta_k} \ldots = \langle g^*(z^\alpha) . p_4^{\beta_1}(M_1) \ldots p_{4k}^{\beta_k}(M_1); [M_1] \rangle$$

therefore

$$\begin{aligned} \mathcal{P}^{if_1}_{\alpha, \beta_1, \ldots} &= \langle z_1^\alpha . p_4^{\beta_1}(M_1) \ldots p_{4k}^{\beta_k}(M_1); [M_1] \rangle \\ &= \langle (g_2 f_1)^*(z_2^\alpha) . (g_2 f_1)^*(p_4^{\beta_1}(M_2)) \ldots (g_2 f_1)^*(p_{4k}^{\beta_k}(M_2)); [M_2] \rangle \\ &= \langle z_2^\alpha \, p_4^{\beta_1}(M_2) \ldots p_{4k}^{\beta_k}(M_2); [M_2] \rangle = \mathcal{P}^{if_2}_{\alpha, \beta_1, \ldots, \beta_k}. \end{aligned}$$

As for any k, $H^k(BG; Z)$ is free abelian the equality of the Stiefel-Whitney and Pontrjagin numbers of if_1 and if_2 implies that $\tau(\xi_1) = \tau(\xi_2)$.

Conversely, assuming $\tau(\xi_1) = \tau(\xi_2)$, one has the equality of the Pontrjagin numbers of if_1 and if_2, in particular

$$\mathcal{P}^{if_1}_{\alpha,\beta_1,\beta_2,\dots} = \mathcal{P}^{if_2}_{\alpha,\beta_1,\beta_2,\dots} \quad \text{for} \quad \beta_1 = \dots = \beta_{r-1} = \beta_{r+1} = \dots = 0,$$

$\beta_r = 1$ and $\alpha = \dfrac{n-1-4r}{2}$ $\left(\text{resp. } \dfrac{n-3-4r}{4}\right)$ if $G = S^1$ (resp. S^3); this implies $(g_2 f_1)^* (z_2^\alpha . p_{4r}(M_2)) = z_1^\alpha\, p_{4r}(M_1)$. Because of the (truncated) polynomial structure of $H^*(\Sigma^n/G; Q)$ and because $(g_2 f_1)^* (z_2^\alpha) = z_1^\alpha$, one obtains $(g_2 f_1)^* (p_{4r}(M_2)) = p_{4r}(M_2)$, which clearly implies that $K(\xi_1).(K(\xi_2))^{-1}$ is an element of finite order in the abelian group $[\Sigma_0^n/G : H/O]$, hence $\gamma(K(\xi_1)) = \gamma(K(\xi_2))$ and the proposition is proved.

4. Proof of theorems C and D.

— In this section we will prove Theorem C and D.

Proof of Theorem C. — Let (G, T_1, Σ_1^n) and (G, T_2, Σ_2^n) be two differentiable free G-actions on the homotopy spheres Σ_1^n and Σ_2^n, and let $f_1 : \Sigma_1^n/G \to BG$ and $f_2 : \Sigma_2^n/G \to BG$ be their characteristic maps (*see* § 2).

By proposition 3.4 (*a*) the Stiefel-Whitney numbers of f_1 and f_2 are equal and according to proposition 2.8 the rationally free G-cobordism of these actions implies that the Pontrjagin numbers of f_1 and f_2 are equal.

As $H^k(BG; Z)$ is torsion free and finitely generated for any k, f_1 and f_2 are equal in $\Omega_{n\text{-}\dim G}(BG)$ hence the actions are differentiably free G-cobordant (*see* § 2). Q. E. D.

In order to prove theorem D we need

PROPOSITION 4.1. — (*a*) *The natural group homomorphism*

$$[HP_K; H/Top] \xrightarrow{\eta} [HP_K; Top] \otimes_Z Q$$

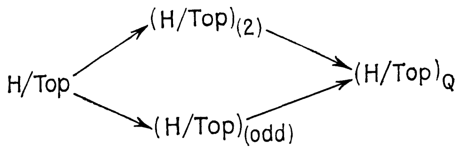
is injective.

(*b*) *The kernel*

$$L = \operatorname{Ker}\{[CP_K; H/Top] \xrightarrow{\eta} [CP_K; (H/Top)_{(Q)}]\}$$

is a Z_2-*vector space with* $\dim_{Z_2} L = \left[\dfrac{k+1}{2}\right]$.

Proof. — Recall from paragraph 1 the cartesian diagram in the homotopy category



which for any CW-complex X induces the following cartesian diagram of abelian groups :

$$(4)\qquad \begin{array}{ccccc} & & [X;(H/Top)_{(2)}] & & \\ & \nearrow & & \searrow & \\ [X;H/Top] & & \overset{\eta}{\dashrightarrow} & & [X,(H/Top)_{(Q)}] \\ & \searrow & & \nearrow & \\ & & [X;(H/Top)_{(odd)}] & & \end{array}$$

We notice that the composite $[X; H/Top] \to [X; (H/Top)_{(2)}] \to [X, (H/Top)_{(Q)}]$ denoted in what follows by η is the same as the natural group homomorphism $[X, H/Top] \to [X, H/Top] \otimes_Z Q$. If $X = HP_k$, then

$$[HP_k; (H/Top)_{(odd)}] = [HP_k; BO] \otimes_Z Z_{odd} = \{\underbrace{Z \oplus \ldots \oplus Z}_{k}\} \otimes_Z Z_{odd}$$

(according to paragraph 1 and [11], theorem 3.1) and

$$[HP_k; (H/Top)_{(2)}] = \underbrace{Z_{(2)} \oplus \ldots \oplus Z_{(2)}}_{k}$$

(according to paragraph 1). As both $[HP_k, (H/Top)_{(odd)}]$ and $[HP_k; (H/Top)_{(2)}]$ have no elements of finite order, the homomorphisms

$$[HP_k; (H/Top)_{(2)}] \to [HP_k; (H/Top)_{(Q)}] \quad \text{and} \quad [HP_k; (H/Top)_{odd}] \to [HP_k; (H/Top)_{(Q)}]$$

are injective; then from the cartesian property of diagram 4 one obtains that $[HP_k; H/Top] \to [HP_k; (H/Top)_{(Q)}]$ is injective.

$[CP_k; (H/Top)_{(odd)}] = [CP_k; BO_{(odd)}] = [CP_k; BO] \otimes_Z Z_{odd}$ (according to paragraph 1) do not contain elements of finite order because the only possible torsion in $[CP_k; BO]$ is that of order 2 (for the computation of $[CP_k; BO]$ we refer to [11], theorem 3.9); hence

$$[CP_k; (H/Top)_{(odd)}] \to [CP_k; (H/Top)_{(Q)}]$$

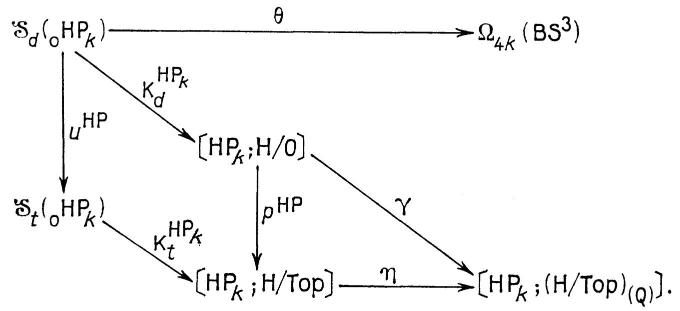
is injective and then the cartesian property of the diagram (4) implies L isomorphic to $\operatorname{Ker}\{[CP_k; (H/Top)_{(2)}] \to [CP_k; (H/Top)_{(Q)}]\}$. From paragraph 1 we know that the homomorphism in parenthesis is just the homomorphism induced by the map (*see* § 1)

$$i_p : \prod_{i=1}^{\infty} K(Z_{(2)}; 4i) \times \prod_{i=0}^{\infty} K(Z_2; 4i+2) \to \prod_{i=1}^{\infty} K(Q, 4i).$$

$$\text{Consequently } L = \left[CP_K; \prod_{i=0}^{\infty} K(Z_2, 4i+2)\right] = \underbrace{Z_2 \oplus \ldots \oplus Z_2}_{\frac{k+1}{2}}.$$

Q. E. D.

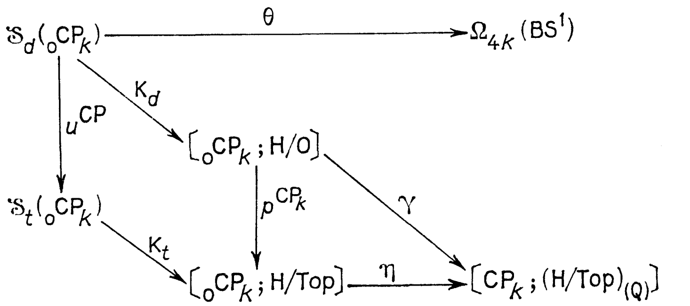
Proof of Theorem D (*a*). — We come back to diagram 3 (§ 3) in the particular case $G = S^3$:



Let $(S^3, T_1, \Sigma_1^{4k+3})$ and $(S^3, T_2, \Sigma_2^{4k+3})$ be two differentiable actions differentiably rationally free S^3-cobordant whose corresponding elements in $\mathcal{S}_d({}_0HP)$ are ξ_1 and ξ_2. By Theorem C, $\theta(\xi_1) = \theta(\xi_2)$ therefore by proposition 3.4, $\tau(\xi_1) = \tau(\xi_2)$.

As by proposition 4.1 (*a*) respectively paragraph 1, η respectively K_t are injective, $\tau(\xi_1) = \tau(\xi_2)$ implies $u(\xi_1) = u(\xi_2)$ which (by proposition 2.6) means that the actions are topologically equivalent.

Proof of Theorem D (*b*). — We consider again diagram 3 (§ 3) in the particular case of $G = S^1$:



By proposition 2.6 and theorem C it suffices to show that

$$\text{Card}\,(u^{\cdots}(\theta^{-1}(\theta(\xi)))) = 2^d \qquad \text{with} \quad d \leq \left[\frac{k+1}{2}\right].$$

By proposition 3.4 we have

$$\text{Card}\,(u^{\cdots}(\theta^{-1}(\theta(\xi)))) = \text{card}\,(u^{\cdots}(\tau^{-1}(\tau(\xi)))) = \text{card}\,(u(\tau^{-1}(x)))$$

with

$$x = \tau(\xi) \in [CP_k; (H/Top)_{(Q)}].$$

Let us denote by $S = \eta^{-1}(x)$, $L = \operatorname{Ker} \eta$, $L_1 = \operatorname{Im} p^{CP_k}$ respectively $L_1 = \operatorname{Im} p^{CP_k} \cap \operatorname{Ker} \lambda^{t}_{2k}$ if k is even respectively odd, and $L^1 = L \cap L_1$. With these notations one can check :

(1) $S = a.L$ (the a-translation of L in $[{}_0CP_k; H/Top]$ with

$$a = p^{CP}.K_d(\xi) \qquad \text{hence} \quad a \in L_1.$$

(2) $S \cap L_1 = a.L^1$.

(3) $(p^{CP}.K_d)^{-1}(S \cap L_1) = (p^{CP}.K_d)^{-1}(S)$.

(4) $\operatorname{Im} u \supset K_t^{-1}(S \cap L_1)$.

(5) $\operatorname{Im} K_t \supset (S \cap L_1)$.

Assuming we have checked (1), (2), (3), (4), (5) applying (3), (4), (5), (2), we get

$$\begin{aligned}\operatorname{Card}(u(u^{-1}(K_t^{-1}(S)))) &= \operatorname{Card}(u(u^{-1}(K_t^{-1}(S \cap L_1))))\\ &= \operatorname{Card}(K_t^{-1}(S \cap L_1)) = \operatorname{Card}(S \cap L_1) = \operatorname{Card} a.L^1.\end{aligned}$$

But $\operatorname{Card} a.L^1 = \operatorname{Card} L^1 = 2^{\dim L^1} = 2^d$ because L^1 is a subgroup of the Z_2-vector space L, hence a Z_2-vector space. At the same time $d = \dim L^1 \leqq \dim L = \left[\frac{k+1}{2}\right]$ and the theorem D (b) is proved. It remains only to check (1), (2), (3), (4) and (5); (1) is obvious and (2) follows immediately from (1) as soon as we remark that L_1 is a subgroup of $[{}_0CP_k; H/Top]$.

Proof of (3). — Take $\xi \in (p^{CP}.K_d)^{-1}(S)$ hence $p^{CP}.K_d(\xi) \in S$. On the other hand $p^{CP}.K_d(\xi)$ belongs to $\operatorname{Im} p^{CP}$ and if k odd (because of Sullivan exact sequence) $p^{CP}.K_d(\xi) \in \operatorname{Ker} \lambda^t_{2k}$, hence $p^{CP}.K_d(\xi) \in S \cap L_1$ and (3) is proved.

Proof of (4). — Assume $\zeta \in K_t^{-1}(S \cap L_1)$, hence $K_t(\zeta) \in S \cap L_1$, hence $K_t(\zeta) \in \operatorname{Im} p^{CP}$ i. e. there exists $b \in [{}_0CP^k; H/O]$ with $p^{CP}(b) = K_t(\zeta)$. On the other hand $\lambda^d_{2k}(b) = \lambda^t_{2k}(K_t(\zeta)) = 0$ [if k is even because $L_1 = \operatorname{Im} p^{CP} \cap \operatorname{Ker} \lambda^t_{2k}$, and if k is odd because $K_t(\zeta)$ is of the form $a.s$ with s an element of finite order and a verifying $\lambda^t_{2k}(a) = 0$; one can check that $\lambda^t_{2k}(a.s) = 0$ applying the explicit description λ^t_{2k}, k even, given in paragraph 1]; therefore, there exists ξ such that $K_d(\xi) = b$; this means that $K_t u(\xi) = K_t(\zeta)$, and from the injectivity of K_t one gets $\zeta = u(\xi)$.

Proof of (5). — If k is odd $L_1 \subset \operatorname{Im} K_t$ (from the Sullivan's exact sequence) and if k is even $S \subset \operatorname{Im} K_t$ because of (1) by the same argument as in the proof of (4). Q. E. D.

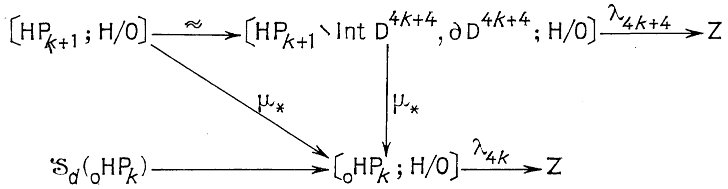
5. Proof of statement F. — In this section we will construct topological S^1-manifolds (S^1, T, M^{4k+4}) such that :

(1) M^{4k+4} is a differentiable manifolds of the homotopy type of HP_{k+1} (therefore a spin manifold).

(2) The fixed point set of T consists of two differentiable submanifolds M_0^{4k} and $M_1^0 =$ point, and the action T is differentiable outside the fixed point M_1^0.

(3) $\hat{A}(M^{4k+4}) \neq 0$ where $\hat{A}$ denotes the $\hat{A}$-genus.

In order to build up these manifolds, as also the action T we need the following part of the diagram (2), paragraph 2



for $(S^3, T, \Sigma_0^{4k+3})$ the standard action of S^3 on S^{4k+3}, in which case ${}_0HP_k = HP_k$, $({}_0\widetilde{HP}_{k+1}, \partial_0\widetilde{HP}_{k+1}) = (HP_{k+1} \setminus \text{Int } D^{4k+4}, \partial D^{4k+4})$, ∂D^{4k+4} and ${}_0\widehat{HP}_{k+1} = HP_{k+1}$.

To understand how one can build up M^{4k+4}, we recall from paragraphs 2 and 3 that $\text{Ker } \lambda_{4k} \cap \mu_* (\text{Ker } \lambda_{4k+4})$ identifies to the equivalence classes of differentiable free S^3-actions on S^{4k+3}, more precisely, the element $\alpha \in \text{Ker } \lambda_{4k} \cap \mu_* (\text{Ker } \lambda_{4k+4})$ can be viewed as a homotopy equivalence $P \xrightarrow{h} HP_k$, and the pull back of the 4-dimensional disc bundle on HP_k gives a differentiable manifold with boundary $(B, \partial B)$ on which S^3 acts, such that the restriction of this action on ∂B is just the free action corresponding to the element f (*see* § 2). Because ∂B is diffeomorphic to S^{4k+3} (*see* § 2) we will construct M^{4k+4} as $B \cup_h D^{4k+4}$ with D^{4k+4} attached to ∂B following a diffeomorphism $h : S^{4k+3} \to \partial B$, and we will extend the action on B radially on D^{4k+4} (via the diffeomorphism h) and get a S^3-action on M. One obtains a differentiable manifold whose differentiable structure depends on h, but not its topological structure, and which clearly satisfies (1).

As S^1 is a subgroup of S^3 we regard M as topological S^1-manifold and notice that (2) is also satisfied but not necessarily (3). However, choosing carefully α, one can hope to get M^{4k+4} so that (3) is also satisfied. In what follows we shall indicate how one can choose α to make sure that (3) is satisfied. In fact instead α we will look for an element

$$\beta \in [HP_{k+1} \setminus \mathrm{Int}\, D^{4k+4}, \partial D^{4k+4}; H/O] = [HP_{k+1}; H/O]$$

such that $\mu_*(\beta) = \alpha$, hence $\beta \in \mathrm{Ker}\, \lambda_{4k+4}$ and $\mu_*(\beta) \in \mathrm{Ker}\, \lambda_{4k}$. It will be convenient to interprete always β as an element of $[HP_{k+1}; H/O]$ instead of $[HP_{k+1} \setminus \mathrm{Int}\, D^{4k+4}, \partial D^{4k+4}; H/O]$ via the natural identification $\xrightarrow{\approx}$.

Inside the abelian group $[HP_{k+1}; H/O]$ we consider the subgroup T' whose elements $f \in T' \subset [HP_{k+1}; H/O]$ verify $(\delta . f)^* (p_{4i}) = 0$ for all $i = 1, \ldots, k-2$ where δ is the natural map $H/O \rightarrow BO$ and p_{4i} the universal rational Pontrjagin classes. In paragraph 1 we denoted this map by δ_d, in order to distinguish betweeen $H/O \rightarrow BO$ and $H/\mathrm{Top} \rightarrow B\,\mathrm{Top}$, but because no confusion could now arise we will omit the index d. Notice that if $k \geqq 4$, then :

(*a*) $\dim T' \otimes_Z Q = 3$.

(*b*) $f, g \in T' \Rightarrow (\delta f + \delta g)^* (p_{4i}) = (\delta f)^* p_{4i} + (\delta g)^* p_{4i}$.

(*c*) Let $\{R_i\}$ be a multiplicative sequence of polynomials with rational coefficients in the sense of Hirtzebruch (*see* [8], § XV) with $r_{4i} \in H^{4i}(B\,\mathrm{Top}; Q)$ being the R-universal characteristic classes defined by it, and $R = 1 + r_4 + r_8 + \ldots$, the total R-class ($r_{4i}$ is a linear combination with rational coefficients of monomials $p_{4i_1} \ldots p_{4i_k}$, $i_1 + \ldots + i_k = i$). If $f, g \in T'$, then $(\delta f + \delta g)^* (R) = (\delta f)^* (R) + (\delta g)^* (R) + 1$ (in particular $\{R_i\}$ can be the multiplicative sequence $\{L_i\}$ or $\{\hat{A}_i\}$).

(*d*) If $f \in T'$ then $(\delta f)^* r_{4i} = c_i\, p_{4i}(\delta f)$ with c_i the coefficient of the monomial p_{4i} in r_{4i}.

We leave it for the reader to check (*a*), (*b*), (*c*), (*d*).

Let us denote by z the canonical generator of $H^*(HP_k; Q)$ (no confusion will occur missing an index « k » for z, because of the naturality of z with respect to the linear imbedding $HP_k \subset HP_{k+1}$), and let us express the total L and $\hat{A}$-classes of HP_{k+1} respectively the total L-class of HP_k by

$$L(\tau(HP_{k+1})) = 1 + m_1 z + \ldots + m_{k+1} z^{k+1}; \qquad \hat{A}(\tau(HP_{k+1})) = 1 + r_1 z + \ldots + r_{k+1} z^{k+1}$$

respectively

$$L(\tau(HP_k)) = 1 + n_1 z + \ldots + n_{k+1} z^{k+1}, \quad m_i, n_i, r_i \in Q$$

with $\tau(HP)$ denoting the tangent bundle of HP.

According to (*a*) and (*b*), it is not difficult to see that one can choose a subgroup T of T' so that T is isomorphic to $Z \oplus Z \oplus Z$ and generated by three elements e_1, e_2, e_3 with the following simple Pontrjagin classes :

$$p_{4i}(\delta . e_1) = 0 \quad \text{for} \quad i \neq k-1$$

and

$$p_{4(k-1)}(\delta . e_1) = d_1 z^{k-1}, \; p_{4i}(\delta . e_2) = 0 \quad \text{for} \quad i \neq k$$

and

$$p_{4k}(\delta . e_2) = d_2 z^k, \quad p_{4i}(\delta . e_3) = 0 \quad \text{for} \quad i \neq k+1$$

and

$$p_{4(k+1)}(\delta e_3) = d_3 z^{k+1},$$

d_1, d_2, d_3 being integer numbers. We will seek β among the elements of T, hence $\beta = A\, e_1 + B\, e_2 + C\, e_3$, $A, B, C \in Z$. (*c*), (*d*) and the explicit definition of λ_{4k} and λ_{4k+4} (§ 1) implies easily that $\beta \in \operatorname{Ker} \lambda_{4k+4} \cap \mu^{*-1} (\operatorname{Ker} \lambda_{4k})$ iff the following equations (i) and (ii) are satisfied :

(i) $a_{k-1}\, d_1\, m_2\, A + a_k\, d_2\, m_1\, B + a_{k+1}\, d_3\, C = 0$;

(ii) $a_{k-1}\, d_1\, n_1\, A + a_k\, d_2\, B = 0$;

the number a_k are the coefficients of p_{4k} in the formal expression $l_k(p_4, \ldots, p_{4k})$.

We are looking now for the condition which has to be added to make sure that M, constructed from $\mu_*(\beta)$, satisfies $\hat{A}(M) \neq 0$.

We notice that the manifold obtained from $\mu^*(\beta)$ as indicated before, after forgetting its differentiable structure can be thought of as the domain of the homotopy equivalence

$$h : M \to HP_{k+1} \quad \text{with} \quad K_t(h) = p^{HP_{k+1}}(\beta), \quad (p^{HP_{k+1}} : [HP_{k+1}; H/O] \to [HP_{k+1}; H/Top]).$$

Its topological stable tangent bundle

$$h_*(\tau(M)) = \tau(HP_{k+1}) \oplus \delta . \beta,$$

hence

$$\hat{A}(\tau(M)) = (1 + r_1 z + r_2 z^2 + \ldots + r_{k+1} z^{k+1}) \times (1 + b_{k-1} d_1 A z^{k-1} + b_k d_2 B z^k + b_{k+1} d_3 (z^{k+1})$$

with b_j the coefficients of p_{4j} in the formal expression of $\hat{A}_j(p_4 \ldots p_{4j})$. Consequently $\hat{A}(M^{4(k+1)}) \neq 0$ is equivalent to

(iii) $r_{k+1} + b_{k-1}\, d_1\, r_2\, A + b_k\, d_2\, r_1\, B + b_{k+1}\, d_3\, C \neq 0$.

Because all the numbers which occur as coefficients of A, B, C are rational numbers it is not difficult to see that one can find integer numbers A, B, C verifying simultaneously (i), (ii), (iii) iff the determinant

$$\Delta_{k+1} = \begin{vmatrix} a_{k-1}\, m_2 & a_k\, m_1 & a_{k+1} \\ b_{k-1}\, r_2 & b_k\, r_1 & b_{k+1} \\ a_{k-1} & a_k & 0 \end{vmatrix} \neq 0,$$

hence we get :

Proposition 5.1. — *If $k \geqq 4$ and $\Delta_{k+1} \neq 0$ then there exists topological S^1-manifolds (S^1, T, M^{4k+4}) so that the* (1), (2), (3) *are verified.*

For $k = 4$ we have $\Delta_5 \neq 0$.

Added in proofs : I am indebted to Don Zagier for showing me how to check $\Delta_{k+1} \neq 0$ for any k.

According to his computations

$$\Delta_{k+1} = -\frac{2^{4k-3}}{45} \frac{B_{k-1}}{(2k-2)!} \frac{B_k}{(2k)!} \frac{B_{k+1}}{(2k+2)!} \Delta'_{k+1}$$

where

$$\Delta'_{k+1} = \begin{vmatrix} 10k^2 - 7k + 42 & 2(2^{2k-1} - 1) & 2^{2k+1} \\ \frac{5}{2}k^2 + \frac{1}{2}k - 3 & -(2^{2k-3} - 1) & 2^{2k-3} - 1 \\ 15k & 2^{2k-1} & 0 \end{vmatrix}$$

and B_k are the Bernouilli numbers. One easily check $\Delta'_{k+1} \neq 0$.

According to Atiyah-Hirzebruch M^{4k+4} constructed before does not admit any differentiable S^1-action, hence we have :

Corollary 5.2. — *For the manifold M^{4k+4} (constructed before) the group of all orientation preserving diffeomorphisms does not contain any compact connected subgroup but the group of all orientation preserving homeomorphisms does contain (compact subgroups).*

In a forthcoming paper we will come back on the problem " which compact connected Lie subgroups of the group of all orientation preserving homeomorphisms are conjugate to compact connected Lie subgroups which come from $\mathrm{Diff}_0(M^k)$ ".

APPENDIX

Let G be a compact connected Lie group acting on the compact differentiable manifold M, possibly with boundary, with finite isotropy groups. Denote by M/G the space of orbits and by $p : M \to M/G$ the continuous map from M to the factor space M/G and consider $\mathcal{L}$, the sheaf associated to the presheaf defined by attaching to any open set $U \subset M/G$ the group $H^*(p^{-1}(U); Q)$ (singular cohomology or Čech cohomology).

THEOREM 1. — *$\mathcal{L}$ is a locally trivial sheaf (hence is a local-coefficient-system).*

Proof. — The proof can be done by showing that for any $x \in M^n/G$ and $W \subset M^n/G$ open neighbourhood, there exists a neighbourhood V, $\overline{V} \subset W$ together with the maps $l : D \to \overline{V}$ and $\chi : D \times G \to p^{-1}(\overline{V})$ such that :

(1) D is a closed disc.

(2) χ and l are inducing isomorphisms of rational cohomology (i. e. the cohomology with coefficients in the field of rational numbers).

(3) The diagram

$$\begin{array}{ccc} p^{-1}(\overline{V}) & \xrightarrow{p} & \overline{V} \\ \uparrow{\scriptstyle\chi} & & \uparrow{\scriptstyle l} \\ D \times G & \xrightarrow{pr_1} & D \end{array}$$

is commutative.

We leave the reader himself to check how (1), (2) and (3) imply that $\mathcal{L}$ is a locally trivial sheaf.

If $x \in M^n/G$ is an orbit corresponding to the isotropy group 0, the existence of D, χ and l follows from the local triviality of $M \setminus P \to (M \setminus P)/G$ where P is the closed subset of all points $y \in M$ with $G_y \neq 0$ (G_y denotes the isotropy group of y).

Let $\tilde{x}$ be a point of M such that $G_{\tilde{x}} \neq 0$ and $x = p(\tilde{x})$. According to ([2], chap. VIII, theorem 3.8), there exists a slice D passing through $\tilde{x}$, namely a closed differentiable imbedded disc D of dimension $k = \dim M - \dim G$ centered in $\tilde{x}$ such that :

(i) D and Int D are $G_{\tilde{x}}$-invariant;

(ii) $g.D \cap D \neq \emptyset$ implies $g \in G_{\tilde{x}}$;

(iii) For any $\omega : T \to G$, T open set in G/G_x and ω local cross-section, $F : (T \times \text{Int } D) \to M$ defined by $F(t, u) = \omega(t).\omega$ is an open imbedding.

We define $\overline{V} = p(D) = p\left(\bigcup_{g \in G} g.D\right)$ which is closed (because D is compact) in M/G, and $V = \text{Int } \overline{V} = p(\text{Int } D) = p\left(\bigcup_{g \in G} g.\text{Int } D\right)$. Put then $l = p \mid D$ and define $\chi : G \times D \to \overline{V}$ by $\chi(g, u) = gu$.

The conditions (1) and (3) are obviously satisfied, so it remains only to check (2). We will get (2) as consequence of the following proposition :

PROPOSITION 2. — *(a) If* M *is a compact manifold possibly with boundary and* H *a finite group acting on* M, *then* $p : M \to M/H$ *induces an injective homomorphism* p^* *of rational cohomology.*

(*b*) *If for any element* $g \in \mathrm{H}$ *the action of* g *is a diffeomorphism homotopic to the identity, then* p *induces an isomorphism of rational cohomology.*

Assuming that proposition 2 is proved, we will go on with the proof of theorem 1. Notice that the finite subgroup $\mathrm{G}_{\tilde{x}}$ act on $\mathrm{G}\times\mathrm{D}$ by $\alpha(g, u) = (g.\alpha^{-1}, \alpha u)$ and the map $\chi : \mathrm{G}\times\mathrm{D} \to \overline{\mathrm{V}}$ factors through $\bar{x} : \mathrm{G}\times\mathrm{D}/\mathrm{G}_{\tilde{x}} \to p^{-1}(\overline{\mathrm{V}})$, which is an homeomorphism [the injectivity follows from (ii)]. By proposition 2 (*b*), $\mathrm{G}\times\mathrm{D} \to \mathrm{G}\times\mathrm{D}/\mathrm{G}_{\tilde{x}}$ induces an isomorphism of rational cohomology, hence χ does.

Because D is compact and invariant by the action of $\mathrm{G}_{\tilde{x}}$, $\mathrm{D} \to \mathrm{D}/\mathrm{G}_{\tilde{x}} \approx \mathrm{V}$ induces also an isomorphism of rational cohomology by proposition 2 (*b*).

Proof of proposition 2. — According to [15] M/H has a well defined triangulation which can be lifted by p to a well defined triangulation compatible with the action of G.

Let us denote by $\mathrm{C}^*(\mathrm{M})$ the simplicial cochain module of M, $\mathrm{C}^*_{\mathrm{H}}(\mathrm{M})$ the submodule of H-invariant cochains (boundary operator transform H-invariant cochains in H-invariant cochains) and by $\mathrm{C}^*(\mathrm{M}/\mathrm{H})$ the module of simplicial cochains of M/H (all are cochains with rational coefficients). Notice that $p : \mathrm{C}^*(\mathrm{M}/\mathrm{H}) \to \mathrm{C}^*(\mathrm{M})$ is the induced cochain map which is injective and $p(\mathrm{C}^*(\mathrm{M}/\mathrm{H})) = \mathrm{C}^*_{\mathrm{H}}(\mathrm{M})$, hence $\mathrm{H}^*(\mathrm{M}/\mathrm{H};\, \mathrm{Q})$ is identified to the cohomology of $\mathrm{C}^*_{\mathrm{H}}(\mathrm{M})$. On the other hand there exists $t : \mathrm{C}^*(\mathrm{M}) \to \mathrm{C}^*_{\mathrm{H}}(\mathrm{M})$ define by $t(\sigma) = \sum_{g\in\mathrm{H}} g.\sigma$ which is also a cochain map, i. e. compatible with the operator δ.

If i denotes the inclusion $i : \mathrm{C}^*_{\mathrm{H}}(\mathrm{M}) \to \mathrm{C}^*(\mathrm{M})$, then $t.\,i : \mathrm{C}^*_{\mathrm{H}}(\mathrm{M}) \to \mathrm{C}^*_{\mathrm{H}}(\mathrm{M})$ is a cochain map with the property that $t.i(\sigma) = \text{card } \mathrm{H}.(\sigma)$; hence $t.i$ induces an isomorphism of rational cohomology and consequently i induces an injective homomorphism i^*; (*a*) is proved.

If σ is a cocycle in $\mathrm{C}^*(\mathrm{M})$, by the hypothesis of (*b*) $g.\sigma$ is cohomologous to σ, hence if we denote by $[\sigma]$ the cohomology class of σ, $[\sigma] = \frac{1}{\text{card } \mathrm{H}}[t\,\sigma]$. Because $[t\,\sigma] \in \text{Imag } i^*$, the previous equality says that i^* is surjective and combining with (*a*) we get i^* is an isomorphism and (*b*) is proved.

Q. E. D.

Both theorem 1 and proposition 2 are very well known but I found it easier to give their proofs than to refer to them in the literature.

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