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BRAUER GROUPS OF ABELIAN SCHEMES

By Raymond T. HOOBLER (*)

Let A be an abelian variety over a field k. Mumford has given a very beautiful construction of the dual abelian variety in the spirit of Grothendieck style algebraic geometry by using the theorem of the square, its corollaries, and cohomology theory. Since the k-points of $\mathbf{Pic}_{A/R}$ is $H^1(A, \mathbf{G}_m)$, it is natural to ask how much of this work carries over to higher cohomology groups where the computations must be made in the étale topology to render them non-trivial. Since $H^2(A, \mathbf{G}_m)$ is essentially a torsion group, the representability of the corresponding functor does not have as much geometric interest as for $H^1(A, \mathbf{G}_m)$. On the other hand, it is closely related to the Brauer group of A and to the arithmetic of A.

The purpose of this paper is to extend the theorem of the cube and several other results about Pic (X) to the corresponding assertions about $H^2(X, \mathbf{G}_m)$, where X is a proper, smooth scheme over S and then to apply this to the question of determining the image of the Brauer group of X in $H^2(X, \mathbf{G}_m)$. Since $H^2(X, \mathbf{G}_m)$ is essentially torsion, many of the results follow from Kunneth type computations if S = Spec k, k an algebraically closed field. The suprising observation is that so many of these results hold for $H^2(X, \mathbf{G}_m)$ if X is defined over a regular or normal base scheme S.

Let $p : X \to S$ be a proper, smooth morphism, l a prime distinct from the residue characteristics of S. The first section shows how various assumptions on the Neron-Severi sheaf of groups of X over S, which is the " non-primitive " piece of $\mathbf{Pic}_{X/S}$, can be used to describe the behaviour of the l-primary component of $R^2 p_* \mathbf{G}_m$. While this section is primarily devoted to preliminary results for the main theorems in the next section, Corollary 1.3 shows that $\mathbf{Pic}^{\tau}_{X/S}$ is a closed subfunctor of $\mathbf{Pic}_{X/S}$ which partially

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answers a question raised by S. Kleiman ([11], Remark 4.8). In the second section, these preliminary results are combined with a duality theorem of Grothendieck's to extend the above mentioned results on Pic to H^2. As an immediate application of the generalized theorem of the cube, we describe the behaviour of the l-primary component of $H^2(A, \mathbf{G}_m)$ under various isogenies where A is an abelian scheme over S. In the third section, the generalized theorem of the cube is combined with some non-abelian sheaf cohomology to show that the l-primary component of $H^2_N(A, \mathbf{G}_m)$ is contained in the Brauer group of A for an abelian scheme A over S. We also show that under strong restrictions on the "non-primitive" elements of Pic (X), the primitive piece of $H^2(X, \mathbf{G}_m)$ is contained in the Brauer group of X. In particular for abelian schemes over a regular base S this establishes (up to p-torsion where $p = \text{char}\,(\Gamma(S, \mathcal{O}_S)))$ M. Artin's conjecture on Brauer groups which was stated in his 1966 Moscow talk.

The questions answered here arose from reading D. Mumford's beautiful book on abelian varieties [13]. The generalization of the Weil-Barsotti formula comes from L. Breen's thesis [2]. In addition I would like to thank L. Breen for several very useful discussions without which I would not have had the tools, e. g., Theorem 2.3, to answer the questions mentioned above.

If $p : X \to S$ is a morphism of schemes, we let $p_T : X_T \to T$ denote the morphism coming from base change by $T \to S$. Moreover p is proper if it is formally proper, formally separated, and *of finite presentation* where formally proper and formally separated means that the functor of points of X over S satisfies the valuative criterion for properness and separation. Finally if H is an abelian group or a sheaf of abelian groups, we let ${}_lH$, $H(l)$, H_{tors}, $l\,H$, and H_l stand for the l-torsion elements of H, the l-primary component of H, the torsion subgroup of H, the image of H under multiplication by l, and the cokernel of multiplication by l on H respectively.

SECTION 1

Let S be an arbitrary scheme. We will be working with the global étale and fppf (fidèlement plat de présentation finie) topologies on S which will be denoted S_{etg} and S_{plg} respectively. The former is the topology [1] on the category of schemes over S generated from the pretopology for which the set of coverings of T consists of families $\{u_i : T_i \to T\}$, where u_i is étale and $T = \bigcup u_i(T_i)$ (set theoretically). The latter is generated from the pretopology for which the set of coverings of T consists of families $\{u_i : T_i \to T\}$, where u_i is flat and locally of finite presentation and

$T = \cup u_i(T_i)$ (set theoretically). We will indicate the various cohomology groups, etc. by the appropriate subscript. $\tilde{S}_*$ and $\hat{S}_*$ will denote the category of sheaves and presheaves on S_* respectively. If there is no subscript, the étale topology is to be understood.

Many of the results we need are stated in terms of the étale or fppf site on S, S_{et} and S_{pl} respectively. These are defined by putting the corresponding induced topologies on the full subcategory of ((Schemes/S)) consisting of those T which are étale or flat and locally of finite presentation over S. The translation of most of these results follows from the observation that $H^n(S, F) \cong H^n(S_{et}, F \mid S_{et})$ for any $F \in \tilde{S}$ and similarly for the fppf topology. Note that there is an obvious morphism $\sigma : S_{pl} \to S_{et}$. Grothendieck has shown ([9], § 11) that if $F \in \tilde{S}_{pl}$ is representable by a smooth group scheme in the étale topology on S, then $R^n \sigma_* F = (0)$ for $n > 0$ and so $H^n(S_{et}, \sigma_* F) \to H^n(S_{pl}, F)$ is an isomorphism. We will usually delete the σ_* for such F since its cohomology is independent of these topologies. In particular these hypotheses are satisfied for $\mathbf{G}_m$, the sheaf of units, and μ_n, the sheaf of n^{th}-roots of unity, if n is relatively prime to all of the residue characteristics of S.

We will want to use the technique of evaluation of fibre functors. If we define a geometric point of S to be a morphism $\xi : \mathrm{Spec}(\bar{k}) \to S$ where k is a separable algebraic closure of $\varkappa(\xi(\mathrm{Spec}(\bar{k})))$, then a geometric point over a given $y \in S$ is determined uniquely up to S-isomorphism and the results of Exposé VIII of [1] carry over to analogous results for $F \in \tilde{S}$. If $y \in S$, we will let $\bar{y}$ denote the corresponding geometric point of S over y and $\tilde{y}$ the corresponding strictly local scheme defined by the strict henselization of $\mathcal{O}_{S,y}$ with respect to $\bar{y}$. Recall that $F \in \hat{S}$ is said to be locally of finite presentation if

$$\varinjlim F(\mathrm{Spec}(A_\alpha)) \to F(\mathrm{Spec}(\varinjlim A_\alpha))$$

is an isomorphism for all filtered inductive systems of affine schemes over S ([5], Chapter IV, 8.8). Thus $F(\tilde{y})$ is the fibre of F over y for any $F \in \tilde{S}$ which is locally of finite presentation.

Given two geometric points $\bar{y}$ and $\bar{y}_1$ of S, recall that a specialization map of $\bar{y}$ to $\bar{y}_1$ is an S-morphism $\tilde{y} \to \tilde{y}_1$ ([1], Exposé VIII). If a specialization map exists, $\bar{y}_1$ is called a specialization of $\bar{y}$. This is equivalent to y_1 being a specialization of y in S since a specialization map is the same as giving an S-morphism $\bar{y} \to \tilde{y}_1$. In particular our procedure of selecting an essentially unique geometric point $\bar{y}$ for each $y \in S$ determines the specialization map uniquely up to an S-automorphism.

DEFINITION. — A set valued presheaf F on S_{etg} is constantly increasing on T if F is locally of finite presentation and for any two geometric points $\bar{y}$ and $\bar{y}_1$ of T with $\bar{y}_1$ a specialization of $\bar{y}$, the specialization map $F(\tilde{y}_1) \to F(\tilde{y})$ is monic. F is constantly increasing if it is constantly increasing on T for all T over S.

Before stating the generalized theorem of the square (which becomes a triviality in this rarefied setting), we need one more definition.

DEFINITION. — If F is presheaf of groups on the category of S-schemes with a section, define $F_N(X) = \operatorname{Ker}(F(q) : F(X) \to F(S))$ where X is a scheme over S with section q.

PROPOSITION 1.1. — *Let* $F \in \tilde{S}$ *be a sheaf of groups which is constantly increasing on the* S-*scheme* T. *If* $x \in H^0(T, F)$, *then* $W = \{t \in T : x|_{\tilde{t}} = 0\}$ *is open and closed in* T *where* $x|_{\tilde{t}}$ *is the image of* x *in* $F(\tilde{t})$. *Moreover if* T *is connected, has a section* q *over* S *and* $F(\tilde{q}(s)) \to F(\overline{q(s)})$ *is monic for some* $s \in S$, *then* $H^0_N(T, F) = (0)$.

Proof. — If t_1 is a specialization of t in T, then $x|_{\tilde{t}} = 0$ if and only if $x|_{\tilde{t}_1} = 0$ which shows that W is open and closed. The rest follows from the observation that $\overline{q(s)} = \bar{s}$.

We will use the base change theorem and the specialization theorem of étale cohomology to produce examples of constantly increasing sheaves, but first to fix notation, let $p : X \to S$ be a proper map. Recall that $R^1 p_* \mathbf{G}_m = \mathbf{Pic}_{X/S}$ (in the étale topology), $\mathbf{Pic}^0_{X/S}$ is the subfunctor whose T-points are elements of $\mathbf{Pic}_{X/S}(T)$ which define invertible sheaves algebraically equivalent to zero on all geometric fibres of $X_T \to T$, and $\mathbf{Pic}^\tau_{X/S}$ is the subfunctor whose T-points are those elements of $\mathbf{Pic}_{X/S}(T)$ which define invertible sheaves τ-equivalent to zero, i. e., some tensor power of the invertible sheaf is algebraically equivalent to zero, on all geometric fibres of $X_T \to T$. Let $\mathbf{NS}_{X/S}$, the étale Néron-Severi sheaf of X/S, be the quotient sheaf $(\mathbf{Pic}_{X/S})/(\mathbf{Pic}^0_{X/S}) \in \tilde{S}$. It is immediate that

$$(\mathbf{NS}_{X/S})_{\text{tors}} = (\mathbf{Pic}^\tau_{X/S})/(\mathbf{Pic}^0_{X/S})$$

and if we let $\mathbf{FNS}_{X/S} = (\mathbf{NS}_{X/S})/(\mathbf{NS}_{X/S})_{\text{tors}} \in \tilde{S}$, then

$$\mathbf{FNS}_{X/S} = (\mathbf{Pic}_{X/S})/(\mathbf{Pic}^\tau_{X/S}).$$

PROPOSITION 1.2. — *Let* $p : X \to S$ *be a proper map with* $p_* \mathcal{O}_X = \mathcal{O}_S$. *universally and let* l *be a prime distinct from the residue characteristics of* S.

(1) *There is an exact sequence in* $\tilde{S}$

$$0 \to \mathbf{FNS}_{X/S} \otimes (\mathbf{Q}_l/\mathbf{Z}_l) \to R^2 p_* \boldsymbol{\mu}_{l^\infty} \to R^2 p_* \mathbf{G}_m(l) \to 0,$$

where $\boldsymbol{\mu}_{l^\infty} = \varinjlim \boldsymbol{\mu}_{l^n}$. *If* S *is strictly local and* $H^0(S, {}_l\mathbf{NS}_{X/S}) = 0$, *then* $H^0(S, \mathbf{Pic}^0_{X/S})$ *is l-divisible.*

(2) *Suppose p is also smooth. Then* $\mathbf{FNS}_{X/S}$, $(\mathbf{FNS}_{X/S})_{l^n}$, *and* $(\mathbf{Pic}^\tau_{X/S})_{l^n}$ *are constantly increasing. Moreover if* T *is an* S*-scheme and* $\mathbf{FNS}_{X/S} \otimes \mathbf{Q}_l/\mathbf{Z}_l$ *is locally constant when restricted to* T_{et}, *then* $R^2 p_* \mathbf{G}_m(l)$ *is also constantly increasing. In particular if* T *is connected and has a section over* S, *then* $H^0_N(T, R^2 p_* \mathbf{G}_m(l)) = (0)$.

Proof. — It is well known that $\mathbf{Pic}_{X/S}$, $\mathbf{Pic}^\tau_{X/S}$, and $\mathbf{Pic}^0_{X/S}$ are locally of finite presentation. Consequently so are all of their quotients formed from them and their tensor products with the constant sheaf $\mathbf{Q}_l/\mathbf{Z}_l$. By standard results on étale cohomology, $R^2 p_* \boldsymbol{\mu}_{l^\infty}$ and $R^2 p_* \mathbf{G}_m(l)$ are also locally of finite presentation.

(1) The Kummer sequence $0 \to \mu_{l^n} \to \mathbf{G}_m \to \mathbf{G}_m \to 0$ is exact on X_{etg}. Take the higher direct images under p_*, break up the resulting long exact sequence into short exact sequences, and then take $\varinjlim$ to get

$$0 \to \varinjlim (\mathbf{Pic}_{X/S})_{l^n} \to R^2 p_* \boldsymbol{\mu}_{l^\infty} \to (R^2 p_* \mathbf{G}_m)(l) \to 0.$$

There is a map $\varinjlim (\mathbf{Pic}_{X/S})_{l^n} \to \mathbf{Pic}_{X/S} \otimes (\mathbf{Q}_l/\mathbf{Z}_l)$, defined in the usual way, which is an isomorphism of sheaves since it is an isomorphism at all stalks. Since the functors involved are all locally of finite presentation, it is enough to verify (1) when the functors are evaluated at S and S is a strictly local scheme with S_0 the closed subscheme corresponding to the separably closed residue field of S and $X_0 = X \times_S S_0$. In this case we have an exact, commutative diagram

$$(1.1)\qquad \begin{array}{ccccccc} 0 \to H^0(S, \mathbf{Pic}_{X/S}) \otimes Q_l/Z_l & \to & H^2(X, \boldsymbol{\mu}_{l^\infty}) & \to & H^2(X, \mathbf{G}_m)(l) & \to 0 \\ \downarrow & & \downarrow \cong & & \downarrow & \\ 0 \to H^0(S_0, \mathbf{Pic}_{X/S}) \otimes Q_l/Z_l & \to & H^2(X_0, \boldsymbol{\mu}_{l^\infty}) & \to & H^2(X_0, \mathbf{G}_m)(l) & \to 0 \end{array}$$

where the middle map is an isomorphism by the étale base change theorem ([1], Exposés XII et XIII). Since $H^0(S_0, \mathbf{Pic}^\tau_{X/S})$ is an extension of an l-divisible group by a finite group, the map between

$$H^0(S_0, \mathbf{Pic}_{X/S}) \otimes Q_l/Z_l \qquad \text{and} \qquad H^0(S_0, \mathbf{FNS}_{X/S}) \otimes Q_l/Z_l$$

si an isomorphism. Since the left hand vertical map of (1.1) is monic, the map between these functors evaluated at S is a monomorphism and so is an isomorphism which proves the first part of (1).

The diagram analogous to (1.1) derived from the Kummer sequence with $n < \infty$ replaces Q_l/Z_l with $Z/l^n Z$. An argument just like the one above then shows that

$$H^0(S, \mathbf{Pic}_{X/S}) \otimes Z/l^n Z \cong H^0(S, \mathbf{NS}_{X/S}) \otimes Z/l^n Z$$

and also that $H^0(S, \mathbf{NS}_{X/S}) \otimes Z/l^n Z$ is contained in $H^0(S_0, \mathbf{NS}_{X/S}) \otimes Z/l^n Z$. Now $\mathbf{Pic}^{\tau}_{X/S}$ is an open subfunctor of $\mathbf{Pic}_{X/S}$ ([11], Theorem 4.7) and so $H^0(S, \mathbf{FNS}_{X/S})$ contained in the finitely generated group $H^0(S_0, \mathbf{FNS}_{X/S})$. Thus it is a finitely generated free abelian group. Consequently if $H^0(S, {}_l\mathbf{NS}_{X/S}) = 0$, then

$$H^0(S, \mathbf{Pic}^0_{X/S}) \otimes Z/l Z \to H^0(S, \mathbf{Pic}_{X/S}) \otimes Z/l Z$$

is monic and hence $H^0(S, \mathbf{Pic}^0_{X/S}) \otimes Z/l Z = 0$ as desired.

(2) Since $\mathbf{Pic}^{\tau}_{X/S} \otimes \mathbf{Q}_l/\mathbf{Z}_l = (0)$, an easy induction argument shows that $[\mathbf{Pic}^{\tau}_{X/S}(l)]_{l^n}$ is isomorphic to $(\mathbf{Pic}^{\tau}_{X/S})_{l^n}$. Since $p_* \mathcal{O}_X = \mathcal{O}_S$ universally, $\mathbf{Pic}^{\tau}_{X/S}(l) \cong R^1 p_* \mu_{l^\infty}$. Now the specialization theorem for étale cohomology ([1], Exposé XVI) says that $R^m p_* \boldsymbol{\mu}_{l^\infty}$ and $R^m p_* \boldsymbol{\mu}_{l^n}$ are locally constant sheaves on S_{et} for all m and n. This and the above remarks show that a specialization map induces an isomorphism of groups for $(\mathbf{Pic}^{\tau}_{X/S})_{l^n}$ and so that $(\mathbf{FNS}_{X/S})_{l^n}$ is constantly increasing as is $\mathbf{FNS}_{X/S}$.

Under the additional assumption that $\mathbf{FNS}_{X/S} \otimes \mathbf{Q}_l/\mathbf{Z}_l$ is locally constant when restricted to T_{et}, the spécialization theorem for $R^2 p_* \boldsymbol{\mu}_{l^\infty}$ and the 5-lemma show that a specialization map induces an isomorphism on $R^2 p_* \mathbf{G}_m(l)$ and so it is constantly increasing. Now if $q : S \to T$ is the section and $t = q(s)$ for some $s \in S$, then $\tilde{t} \to \tilde{s}$ has a section by the universal mapping property of strictly henselian rings. Hence

$$H^0(\tilde{t}, \mathbf{FNS}_{X/S} \otimes \mathbf{Q}_l/\mathbf{Z}_l) \to H^0(\tilde{s}, \mathbf{FNS}_{X/S} \otimes \mathbf{Q}_l/\mathbf{Z}_l)$$

is onto. Since $\varkappa(\tilde{s}) = \varkappa(\tilde{t})$, (1.1) shows that this map is monic as well. Consequently $H^0(\tilde{t}, R^2 p_* \mathbf{G}_m(l)) \to H^0(\tilde{s}, R^2 p_* \mathbf{G}_m(l))$ is an isomorphism by the 5-lemma and so Proposition 1.1 shows that $H^0_N(T, R^2 p_* \mathbf{G}_m(l)) = (0)$.

COROLLARY 1.3. — *Let $p : X \to S$ be a proper, smooth morphism such that $p_* \mathcal{O}_X = \mathcal{O}_S$ universally. Then $\mathbf{Pic}^{\tau}_{X/S}$ is an open and closed subfunctor of $\mathbf{Pic}_{X/S}$.*

Proof. — Let T be any S-scheme, $x \in H^0(T, \mathbf{Pic}^{\tau}_{X/S})$. Since $\mathbf{FNS}_{X/S}$ is constantly increasing, Proposition 1.1 shows that

$$W = \{ t \in T : \bar{x}|^{\bar{t}} = 0 \text{ in } H^0(\bar{t}, \mathbf{FNS}_{X/S}) \}$$

is open and closed in T where $\bar{x}$ is the image of x in $H^0(T, \mathbf{FNS}_{X/S})$. Since the condition for x to belong to $H^0(T, \mathbf{Pic}^{\tau}_{X/S})$ is phrased in terms of geometric points, we see that $\mathbf{Pic}^{\tau}_{X/S}$ is indeed an open and closed subfunctor.

Corollary 1.4. — *Let $p : X \to S$ be a proper, smooth morphism such that $p_* \mathcal{O}_X = \mathcal{O}_S$ universally and let l be a prime distinct from the residue characteristics of* S. *If* T *is an* S*-scheme with section q, then*

$$H^0_N(T, l^n(\mathbf{Pic}^{\tau}_{X/S})) \to H^1_N(T, {}_{l^n}\mathbf{Pic}^{\tau}_{X/S}) \to {}_{l^n}H^1_N(T, \mathbf{Pic}_{X/S}) \to 0$$

is an exact sequence for any n.

Proof. — If $t \in T$, then $H^0(\tilde{t}, \mathbf{FNS}_{X/S}) \to H^0(\bar{t}, \mathbf{FNS}_{X/S})$ is monic by the base change theorem. Moreover

$$H^0(\tilde{t}, (\mathbf{Pic}^{\tau}_{X/S})_{l^n}) \cong H^0(\bar{t}, (\mathbf{Pic}^{\tau}_{X/S})_{l^n}) \qquad \text{since} \quad (\mathbf{Pic}^{\tau}_{X/S})_{l^n} = [\mathbf{Pic}^{\tau}_{X/S}(l)]_{l^n}.$$

Hence the base change theorem for $R^2 p_* \boldsymbol{\mu}_{l^n}$ shows that

$$H^0(\tilde{t}, (\mathbf{FNS}_{X/S})_{l^n}) \to H^0(\bar{t}, (\mathbf{FNS}_{X/S})_{l^n})$$

is monic. Now if $0 \to F' \to F \to F'' \to 0$ is exact in $\tilde{S}$ and $H^0_N(T, F'') = 0$, then $H^1_N(T, F') \cap \operatorname{Ker}[H^1(T, F') \to H^1(T, F)]$ is clearly (0). Since $\mathbf{FNS}_{X/S}$ and $(\mathbf{FNS}_{X/S})_l$ are constantly increasing, Proposition 1.1 and the above remarks show that $H^1_N(T, \mathbf{Pic}^{\tau}_{X/S})(l) \to H^1_N(T, \mathbf{Pic}_{X/S})(l)$ is an isomorphism and so $\mathbf{Pic}_{X/S}$ can be replaced by $\mathbf{Pic}^{\tau}_{X/S}$ in the last cohomology group of the assertion. $(\mathbf{Pic}^{\tau}_{X/S})_{l^n}$ is also constantly increasing and so the same argument shows that $H^1_N(T, l^n(\mathbf{Pic}^{\tau}_{X/S})) \to H^1_N(T, \mathbf{Pic}^{\tau}_{X/S})$ is monic. The desired result now follows by taking the cohomology of the sequence

$$0 \to {}_{l^n}\mathbf{Pic}^{\tau}_{X/S} \to \mathbf{Pic}^{\tau}_{X/S} \to l^n(\mathbf{Pic}^{\tau}_{X/S}) \to 0$$

gotten from factoring multiplication by l^n on $\mathbf{Pic}^{\tau}_{X/S}$.

SECTION 2

In this section we prove a generalized version of thet eorem of the cube and a generalization of the Weil-Barsotti formula. We then extend the result $\mathrm{Pic}^0(X) \cong \mathrm{Pic}^0(A)$ where A is the Albanese variety of a smooth scheme over a field k. One of the key tools in this program is Grothendieck's duality theorem ([4], Exposé XI). Since the only published proof of this theorem ([17], [18]) is not sufficiently general for our purposes, we will outline one here.

Recall that $p : X \to S$ is locally free if p is finite and $p_* \mathcal{O}_X$ is a locally free $\mathcal{O}_S$-module.

Proposition 2.1. — *Let* H *be a locally free commutative group scheme over a connected scheme* S. *Then* $\mathbf{Ext}^1_{\tilde{S}_{pl}}(H, \mathbf{G}_m) = 0$.

Proof. — Since $\mathbf{Ext}^1_{\tilde{S}_{pl}}(H, \mathbf{G}_m)$ is obtained by sheafifying the presheaf

$$U \to \mathrm{Ext}^1_{\tilde{S}_{pl}}(U; H; \mathbf{G}_m)$$

it is enough to show that any element $E' \in \mathrm{Ext}^1_{\tilde{S}_{pl}}(S;\ H, \mathbf{G}_m)$ can be split by an fppf covering of S. Moreover E′ may be interpreted as a short, exact sequence of group schemes over S ([15], § 17, Chapter III). Given such an exact sequence we get a commutative exact diagram of group schemes over S since $n = \mathrm{rank}\,(H)$ annihilates H [16].

$$\begin{array}{rccccccccc}
 & & & 0 & & & & & & \\
 & & & \uparrow & & & & & & \\
E' : & 0 & \longrightarrow & \mathbf{G}_m & \longrightarrow & L & \longrightarrow & H & \longrightarrow & 0 \\
 & & & \uparrow{\scriptstyle n} & & \uparrow{\scriptstyle n} & & \uparrow{\scriptstyle 0} & & \\
 & 0 & \longrightarrow & \mathbf{G}_m & \longrightarrow & L & \longrightarrow & H & \longrightarrow & 0 \\
 & & & \uparrow & & \uparrow & & \uparrow & & \\
E : & 0 & \longrightarrow & \boldsymbol{\mu}_n & \xrightarrow{\alpha} & L_n & \xrightarrow{\beta} & H & \longrightarrow & 0 \\
 & & & \uparrow & & \uparrow & & \uparrow & & \\
 & & & 0 & & 0 & & 0 & &
\end{array}$$

If the sequence E can be split by an fppf covering of S, then so can E′. Let E^D be the Cartier dual of the exact sequence E. This gives another exact sequence of group schemes over S, and it is enough to split this by a covering in S_{pl} since $E^{DD} = E$. But $\boldsymbol{\mu}_n^D = \mathbf{Z}/\mathbf{nZ}_S$ and so it is enough to find an fppf covering $\{ T_i \to S \}$ and $y_i \in ({}_nL)^D(T_i)$ such that $\alpha^D(y_i) = 1$, the generating section of $\mathbf{Z}/\mathbf{nZ}_S$. Since α^D is surjective in $\tilde{S}_{pl}$, this can always be done.

Corollary 2.2. — *Let* H *be a locally free group scheme over a connected scheme* S *with* H^D *an étale group scheme. Then* $\mathbf{Ext}^1_{\tilde{S}}(H, \mathbf{G}_m) = 0$.

Proof. — The morphism $\sigma : S_{pl} \to S_{et}$ gives two spectral sequences converging to the same abutment ([1], Exposé V) :

$$R^p \sigma_* \mathbf{Ext}^q_{\tilde{S}_{pl}}(\sigma^*(H), \mathbf{G}_m) \quad \text{and} \quad \mathbf{Ext}^p_{\tilde{S}}(H, R^q \sigma_* \mathbf{G}_m).$$

Since H is representable $\sigma^* (H) = H \in \tilde{S}_{pl}$, and since $R^q \sigma_* \mathbf{G}_m = 0$ for $q > 0$, the second spectral sequence collapses. The exact sequence of low degree terms then gives an isomorphism

$$R^1 \sigma_* \mathbf{Hom}_{\tilde{S}_{pl}} (H, \mathbf{G}_m) = R^1 \sigma_* H^D \overset{\approx}{\to} \mathbf{Ext}^1_S (H, \mathbf{G}_m).$$

But H^D is étale and so $R^1 \sigma_* H^D = 0$.

Grothendieck's duality theorem now follows from spectral sequence computations. Since we are primarily dealing with the étale topology, we will prove an étale version of it.

THEOREM 2.3. — *Let* S *be a connected scheme,* $p : X \to S$ *a proper map with* $p_* \mathcal{O}_X = \mathcal{O}_S$ *universally,* H *a locally free étale commutative group scheme over* S. *Then for any* T/S, *there is an exact sequence natural in* T *and* H *and also natural in* X/S

$$0 \to \mathrm{Ext}^1_{\tilde{S}} (T; H^D, \mathbf{G}_m) \to H^1 (X_T, H_X) \to \mathbf{Hom}_{\tilde{S}} (H^D, \mathbf{Pic}_{X/S}) (T).$$

If p *has a section, the last map is surjective and* $H^1_N (X_T, H_X)$ *is naturally isomorphic to* $\mathrm{Hom}_{\tilde{T}} (H^D, \mathbf{Pic}_{X/S})$.

On the sheaf level, there is an injection

$$0 \to R^1 p_* (H) \to \mathbf{Hom}_{\tilde{S}} (H^D, \mathbf{Pic}_{X/S})$$

which is an isomorphism if p *has a section locally in* S_{etg}.

Proof. — There are spectral sequences ([1], Exposé V)

$$H^p \left(X_T, \mathbf{Ext}^q_{\tilde{X}} (H^D_X, \mathbf{G}_m)\right) \Rightarrow \mathrm{Ext}^n_{\tilde{X}} (X_T; H^D_X, \mathbf{G}_m)$$

and

$$\mathrm{Ext}^p_{\tilde{S}} (T; H^D, (R^q p_*) \mathbf{G}_m) \Rightarrow \mathrm{Ext}^n_{\tilde{X}} (X_T; H^D_X, \mathbf{G}_m)$$

since $p^* (H) = H_X \in \tilde{X}$. We get an exact diagram (2.1) by piecing together the exact sequences of low degree terms, recalling that $H^D = \mathbf{Hom}_{\tilde{S}} (H, \mathbf{G}_m)$, observing that $p_* \mathbf{G}_{m,X} = \mathbf{G}_{m,S}$, and using Corollary 2.2,

$$(2.1) \qquad \begin{array}{ccc} & & 0 \\ & & \downarrow \\ & & \mathrm{Ext}^1_{\tilde{S}} (T; H^D, \mathbf{G}_m) \\ & & \downarrow \alpha \\ 0 \longrightarrow H^1 (X_T, H_X) & \longrightarrow & \mathrm{Ext}^1_{\tilde{X}} (X_T; H^D_X, \mathbf{G}_m) \longrightarrow 0 \\ & & \downarrow \beta \\ & & \mathbf{Hom}_{\tilde{S}} (H^D, \mathbf{Pic}_{X/S}) (T) \end{array}$$

If p has a section, α has a splitting map, defined from the section, which identifies $\mathrm{Hom}_{\tilde{T}}(H^D, \mathbf{Pic}_{X/S})$ with $H^1_N(X_T, H_X)$ by the naturality of (2.1) and the fact that β is onto since the next map in the exact sequence of low degree terms defining the vertical column also has a splitting map. The statements about sheaves follow from the statements about presheaves by sheafifying and Corollary 2.2. The naturality in X follows from the observation that an S-morphism $\varphi : X' \to X$ induces mappings between the two spectral sequences for X/S and the corresponding ones for X'/S.

We are now ready to prove the theorem of the cube. Suppose we are given a family $\{X_i\}$ of schemes over S indexed by a finite set I together with structure maps $p_i : X_i \to S$ and sections $q_i : S \to X_i$. For $J = \{j_0, \ldots, j_m\} \subseteq I = \{1, \ldots, n\}$, let

$$\mathrm{pr}_{j_0,\ldots,j_m} : X_1 \times \ldots \times X_n \to X_{j_0} \times \ldots \times X_{j_m}$$

denote the projection map and

$$s_{j_0,\ldots,j_m} : X_{j_0} \times \ldots \times X_{j_m} \to X_1 \times \ldots \times X_n$$

the map which inserts q_k into the k^{th} factor for all $k \in I - J$, where the products are always taken over S. Following Mumford a quadratic functor on a category of S-schemes with section to abelian groups is a contravariant functor F such that

$$\text{(2.2)} \qquad F(X_1 \times X_2 \times X_3) \to \Pi_J F(X_i \times X_j)$$

is monic where $J = \{(1, 2), (2, 3), (1, 3)\}$ and the map is the product of the $F(s_{i,j})$, $(i, j) \in J$.

THEOREM 2.4. — *Let $p_i : X_i \to S$ be proper morphisms over a locally noetherian scheme S such that $p_{i*}\mathcal{O}_{X_i} = \mathcal{O}_{S_i}$ universally, $1 \leqq i \leqq 3$, and p_2 and p_3 are smooth. Suppose that l is a prime distinct from the residue characteristics of S and that $\mathbf{FNS}_{X_3/S} \otimes \mathbf{Q}_l/\mathbf{Z}_l$ is locally constant when restricted to $X_1 \times X_{2\,\mathrm{et}}$. Then the natural map*

$$H^2(X_1 \times X_2 \times X_3, \mathbf{G}_m)(l) \to \Pi_J H^2(X_i \times X_j, \mathbf{G}_m)(l)$$

defined as in (2.2) *is monic.*

Proof. — We may assume that S is connected and so $X_1 \times X_2$ *and* X_1 are connected by Zariski's connectedness theorem. The Leray spectral sequence for $\mathrm{pr}_{1,2}$ gives an exact diagram (2.3) where there are zeros at the left and right because $pr_{1,2}$ has a section

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ 0 \to & H^2(X_1 \times X_2, \mathbf{G}_m) & \xrightarrow{pr_{1,2}^*} & F^1 H^2(X_1 \times X_2 \times X_3, \mathbf{G}_m) & \xrightarrow{t} & H^1(X_1 \times X_2, \mathbf{Pic}_{X_3/S}) & \to 0 \\ & & & \downarrow & & & \\ & & & H^2(X_1 \times X_2 \times X_3, \mathbf{G}_m) & & & \\ & & & \downarrow{\scriptstyle \pi_{1,2}} & & & \\ & & & H^0(X_1 \times X_2, R^2 p_{3*} \mathbf{G}_m) & & & \end{array} \tag{2.3}$$

Note that (2.3) is functorial in $X_1 \times X_2$.

Suppose $y \in H^2(X_1 \times X_2 \times X_3, \mathbf{G}_m)(l)$ with $s_{i,j}^*(y) = 0$ for all $(i, j) \in J$. Then $s_1^* \pi_{1,2}(y) = \pi_1 s_{1,3}^*(y) = 0$ in $H^0(X_1, R^2 p_{3*} \mathbf{G}_m(l))$, where π_1 : $H^2(X_1 \times X_3, \mathbf{G}_m) \to H^0(X_1, R^2 p_{3*} \mathbf{G}_m)$ comes from the Leray spectral sequence for pr_1. The assumption on the restriction of $\mathbf{FNS}_{X_3/S} \otimes \mathbf{Q}_l/\mathbf{Z}_l$ to $X_1 \times X_{2\,\mathrm{et}}$ and Proposition 1.2 then show that $\pi_{1,2}(y) = 0$.

Regarding y as an element of $F^1 H^2(X_1 \times X_2 \times X_3, \mathbf{G}_m)(l)$ which is isomorphic to $pr_{1,2}^*(H^2(X_1 \times X_2, \mathbf{G}_m)(l)) \oplus H^1(X_1 \times X_2, \mathbf{Pic}_{X_3/S})(l)$, we see that the component of y in the image of $pr_{1,2}^*$ is o since $s_{1,2}^*(y) = 0$. Thus it is enough to show that $t(y) = 0$ where $s_1^*(t(y)) = s_2^*(t(y)) = 0$ by the functoriality of (2.3) in $X_1 \times X_2$.

By Corollary 1.4, the sequence below is exact and so $t(y) = \beta_{1,2}(z)$ for some $z \in H_N^1(X_1 \times X_2, {}_{l^n}(\mathbf{Pic}_{X_3/S}^\tau))$,

$$H_N^0(X_1 \times X_2, l^n(\mathbf{Pic}_{X_3/S}^\tau)) \xrightarrow{\alpha_{1,2}} H_N^1(X_1 \times X_2, {}_{l^n}(\mathbf{Pic}_{X_3/S}^\tau)) \xrightarrow{\beta_{1,2}} {}_{l^n}H_N^1(X_1 \times X_2, \mathbf{Pic}_{X_3/S}) \to 0.$$

Since $s_1^*(t(y)) = s_2^*(t(y)) = 0$, there are

$$z_i \in H_N^0(X_i, l^n(\mathbf{Pic}_{X_3/S})), \qquad i = 1, 2,$$

with $\alpha_1(z_1) = s_1^*(z)$ and $\alpha_2(z_2) = s_2^*(z)$. Then

$$\beta_{1,2}[z - \alpha_{1,2}(pr_1^*(z_1) + pr_2^*(z_2))] = t(y)$$

and

$$s_i^*[z - \alpha_{1,2}(pr_1^*(z_1) + pr_2^*(z_2))] = 0 \qquad \text{for} \quad i = 1, 2.$$

Thus we may assume that z restricted to $H_N^1(X_i, {}_{l^n}(\mathbf{Pic}_{X_3/S}^\tau))$ along the section s_i is zero.

On the other hand multiplication by l^n is étale on $\mathbf{Pic}_{X_3/S}$ by the base change theorem. Since p_3 is proper and smooth, $\mathbf{Pic}_{X_3/S}$ is formally proper and formally separated over S, and so ${}_{l^n}\mathbf{Pic}_{X_3/S}$ is represented by a finite, étale group scheme over S of order l^n which we will denote by H [14].

By Grothendieck's duality theorem, there is an isomorphism functorial in $X_1 \times X_2$

$$H^1_N(X_1 \times X_2, H) \to \mathbf{Hom}_{\tilde{S}}(H^D, {}_{l^n}\mathbf{Pic}_{X_1 \times X_2/S})(S) = \mathrm{Hom}_{\tilde{S}}(H^D, {}_{l^n}\mathbf{Pic}_{X_1 \times X_2/S}),$$

where we have observed that l^n annihilates H^D. On the other hand

$${}_{l^n}\mathbf{Pic}_{X_i/S} = R^1 p_{i*} \boldsymbol{\mu}_{l^n} \quad \text{and} \quad {}_{l^n}\mathbf{Pic}_{X_1 \times X_2/S} = R^1 (p_1 \mathrm{pr}_1)_* \boldsymbol{\mu}_{l^n}$$

and the maps pr_1^* and pr_2^* induce a homomorphism

$$(2.4) \qquad {}_{l^n}\mathbf{Pic}_{X_1/S} \times {}_{l^n}\mathbf{Pic}_{X_2/S} \to {}_{l^n}\mathbf{Pic}_{X_1 \times X_2/S}.$$

Note that for a proper smooth morphism $p : X \to S$ with $p_* \mathcal{O}_X = \mathcal{O}^S$ universally, the Leray spectral sequence for p and Proposition 1.1 show that the sequence below is exact

$$0 \to H^1(S, \boldsymbol{\mu}_{l^n}) \to H^1(X, \boldsymbol{\mu}_{l^n}) \to H^1(X_{\bar{y}}, \boldsymbol{\mu}_{l^n}),$$

where y is any geometric point of S. This observation shows that (2.4) is an isomorphism if S is strictly local. Hence (2.4) is an isomorphism, and so

$$\mathrm{Hom}_{\tilde{S}}(H^D, {}_{l^n}\mathbf{Pic}_{X_1 \times X_2/S}) \simeq \mathrm{Hom}_{\tilde{S}}(H^D, {}_{l^n}\mathbf{Pic}_{X_1/S}) \times \mathrm{Hom}_{\tilde{S}}(H^D, {}_{l^n}\mathbf{Pic}_{X_2/S}).$$

Thus $\{ z \in H^1_N(X_{1,2}, H) : s_1^*(z) = s_2^*(z) = 0 \} = \{ 0 \}$, and so $t(y) = 0$ as desired.

Corollary 2.5. — *Let* S *be a connected, regular noetherian scheme, l a prime different from* char $(\Gamma(S, \mathcal{O}_S))$. *Then* $H^2(\ \ , \mathbf{G}_m)(l)$ *is a quadratic functor on the category of smooth geometrically connected proper* S*-schemes with a section.*

Proof. — Let $p_i : X_i \to S$, $1 \leq i \leq 3$, be proper, smooth morphisms. First note that $p_{i*} \mathcal{O}_{X_i} = \mathcal{O}_S$ universally since p_i is proper, smooth, and geometrically connected. Let η be the generic point of S. If Y is any smooth scheme over S, then $H^2(Y, \mathbf{G}_m) \to H^2(Y_\eta, \mathbf{G}_m)$ is monic since Y is regular ([8], Corollary 1.8). Hence it is enough to establish the corollary when $S = \mathrm{Spec}\, k$ and k is a field of characteristic unequal to l. But then $\mathbf{FNS}_{X_3/S} \otimes \mathbf{Q}_l/\mathbf{Z}_l$ is locally constant when restricted to $X_1 \times X_{2\,\mathrm{et}}$ since it is constantly increasing and $H^0(\tilde{y}, \mathbf{FNS}_{X_3/S} \otimes \mathbf{Q}_l/\mathbf{Z}_l)$ is contained in $H^0(\bar{y}, \mathbf{FNS}_{X_3/S} \otimes \mathbf{Q}_l/\mathbf{Z}_l)$ which is to equal to $H^0(\bar{s}, \mathbf{FNS}_{X_3/S} \otimes \mathbf{Q}_l/\mathbf{Z}_l)$ for all $y \in X_1 \times X_2$ where s is the unique point of S.

Recall that $p : A \to S$ is an abelian scheme over S if it is a proper, smooth group scheme with geometrically connected fibres. In this case

$\mathbf{Pic}^0_{A/S} = \mathbf{Pic}^{\tau}_{A/S}$ is formally smooth ([12], Proposition 6.7) and if S is noetherian, Grothendieck and Murre have shown that $\mathbf{NS}_{A/S(\mathrm{fpqc})}$ is represented by an unramified, essentially proper group scheme [14] where $\mathbf{NS}_{A/S(\mathrm{fpqc})}$ is the quotient in the faithfully flat, quasi compact topology. (T is essentially proper over a noetherian scheme S if T is locally of finite presentation and satisfies the discrete valuation ring criterion for separation and properness.) Since $\mathbf{Pic}^0_{A/S}$ is locally of finite presentation, $\mathbf{NS}_{A/S(\mathrm{fpqc})}$ is the same as the quotient in the fppf topology. Moreover if $\mathbf{Pic}^0_{A/S}$ is representable, then $R^1\sigma_* \mathbf{Pic}^0_{A/S} = 0$ by Grothendieck's theorem and so $\mathbf{NS}_{A/S(\mathrm{fpqc})}$ is the same as $\mathbf{NS}_{A/S}$. If $\mathbf{Pic}^0_{A/S}$ is representable, it is called the dual abelian scheme and denoted by $\hat{A}$. The Grothendieck-Murre representability theorem shows that A is projective (and so $\hat{A}$ exists) if S is a reduced, connected, geometrically unibranch noetherian scheme. Note finally that $(\mathbf{NS}_{A/S})_{\mathrm{tors}} = 0$ and so $\mathbf{NS}_{A/S} = \mathbf{FNS}_{A/S}$.

Corollary 2.6. — *Let* S *be a connected, reduced, geometrically unibranch noetherian scheme,* $p : A \to S$ *an abelian scheme over* S. *Then for any prime* l, $\mathbf{FNS}_{A/S} \otimes \mathbf{Q}_l/\mathbf{Z}_l$ *is locally constant when restricted to* A^2_{et} *and so if* l *is distinct from the residue characteristics of* S, *then*

$$H^2(A^3, \mathbf{G}_m)(l) \to \Pi_J H^2(A^2, \mathbf{G}_m)(l)$$

is monic where the map is defined as in (2.2) *by using the zero section of* A *over* S.

Proof. — From the above remarks it is enough to show that an essentially proper, unramified group scheme G over a connected, reduced, geometrically unibranch, noetherian scheme T defines a locally constant sheaf on T_{et}. Since T is noetherian, it is enough to show that the specialization map induces an isomorphism $H^0(\tilde{t}_1, G) \to H^0(\tilde{\eta}, G)$, where $\tilde{t}_1$ is a strictly local scheme with generic point η ([1], Exposé IX). Since taking the strict henselization preserves the properties S has, we may assume that T is a strictly local, reduced, geometrically unibranch, noetherian scheme with generic point η. If y is any point of G lying over η, the schematic closure Y of y maps onto T since G is essentially proper. On the other hand Y is quasi-finite over T and so, being irreducible, is finite over T ([5], IV, 18.12.3). But then Y is finite and unramified over T and maps onto the strictly local scheme T and so must be isomorphic to T.

Remark. — If $\mathbf{G}_m$ is replaced by μ_{l^n} in the statement of Theorem 2.4, then the conclusion holds without any assumption on $\mathbf{FNS}_{X_3/S} \otimes \mathbf{Q}_l/\mathbf{Z}_l$.

The argument proceeds just as above but uses the étale base change theorem and the specialization theorem for $R^2 p_{3*} \mu_{l^n}$ instead of the computations in the first section. Moreover, as L. Breen has remarked, if $S = \mathrm{Spec}(k)$, k a separably closed field, then this modification of Theorem 2.5 follows immediately from the étale Kunneth formula.

As an application of the generalized theorem of the cube we can begin to describe the behaviour of $H^2(A, \mathbf{G}_m)(l)$ for an abelian scheme A over S. We will say that A satisfies the generalized theorem of the cube for l if l is a prime and the map

$$H^2(A^3, \mathbf{G}_m)(l) \to \Pi_J H^2(A^2, \mathbf{G}_m)(l)$$

defined in Theorem 2.4 is monic.

Proposition 2.5. — *Let $p : A \to S$ be an abelian scheme satisfying the generalized theorem of the cube for l, $y \in H^2(A, \mathbf{G}_m)(l)$.*

(1) *Given S-morphisms $f, g, h : T \to A$,*

$$(f+g+h)^*(y) - (f+g)^*(y) - (g+h)^*(y) - (f+h)^*(y) + f^*(y) + g^*(y) + h^*(y) = 0,$$

in $H^2(T, \mathbf{G}_m)(l)$.

(2) *Let $n_A : A \to A$ denote the isogeny given by multiplication by n. Then* $n_A^*(y) = n^2 y + \left(\frac{n^2-n}{2}\right)((-1_A)^*(y) - y)$.

(3) *Given sections $f', g' : S \to A$, let $T_{f'}, T_{g'} : A \to A$ be translation by f', g' respectively. Then*

$$T^*_{f'+g'}(y) = T^*_{f'}(y) + T^*_{g'}(y) - y + p^*(z) \quad \textit{for some } z \in H^2(S, \mathbf{G}_m)(l).$$

Proof. — We see that

$$(\mathrm{pr}_1 + \mathrm{pr}_2 + \mathrm{pr}_3)^*(y) - (\mathrm{pr}_1 + \mathrm{pr}_2)^*(y) - (\mathrm{pr}_1 + \mathrm{pr}_3)^*(y) - (\mathrm{pr}_2 + \mathrm{pr}_3)^*(y) + \mathrm{pr}_1^*(y) + \mathrm{pr}_2^*(y) + \mathrm{pr}_3^*(y)$$

is zero when restricted to $S \times A \times A$, $A \times S \times A$, and $A \times A \times S$, and so it is zero in $H^2(A^3, \mathbf{G}_m)(l)$: (1) then follows by the definition of $(f+g+h) : T \to A$, etc; (2) follows by induction on n, and (3) follows from (1) by setting $f = f'p : A \to A$, $g = g'p : A \to A$, and $h = 1_A : A \to A$ since $T_{f'} = 1_A + f'p$, etc.

Corollary 2.8. — *Let A be an abelian scheme over S satisfying the generalized theorem of the cube for l. If $l \neq 2$, then*

$$_{l^n}H^2_N(A, \mathbf{G}_m) \subseteq \mathrm{Ker}\,[(l^n_A)^* : H^2(A, \mathbf{G}_m) \to H^2(A, \mathbf{G}_m)].$$

If $l = 2$, *then*

$$ {}_{2^n}H^2_N(A, \mathbf{G}_m) \subseteq \operatorname{Ker}[(2^{n+1}_A)^* : H^2(A, \mathbf{G}_m) \to H^2(A, \mathbf{G}_m)]. $$

Next we will give an interpretation of the " primitive " elements of $H^2(X, \mathbf{G}_m)$.

DEFINITION. — Let $p : X \to S$ be a morphism with a section, F an étale sheaf on the category of S-schemes. Define

$$ H^i_{N,pr}(X, F) = \operatorname{Ker}[H^i_N(X, F) \to H^0(S, R^i p_*(p^* F))]. $$

This is functorial on the category of S-schemes with a section since the Leray spectral sequence for p is functorial in X. Note that this spectral sequence shows that

$$ H^2_{N,pr}(X, \mathbf{G}_m) \xrightarrow{\approx} H^1(S, \mathbf{Pic}_{X/S}). $$

If A is an abelian scheme over S, there is an alternative way of defining " primitive " elements.

DEFINITION. — Let A be an abelian scheme over S, $F \in \hat{S}$. Define

$$ F_{prim}(A) = \operatorname{Ker}[F(m) - F(pr_1) - F(pr_2) : F(A) \to F(A^2)], $$

where $m : A^2 \to A$ is the multiplication map. For notational purposes we will denote $F(m) - F(pr_1) - F(pr_2)$ by ρ^*.

The work of L. Breen enables us to compare these two definitions of primitive elements as well as to prove an analogue of the Weil-Barsotti formula.

THEOREM 2.9. — *Let* $p : A \to S$ *be an abelian scheme over a quasi-compact, quasi-separated scheme* S *such that* $\hat{A}$ *exists. There is an exact sequence*

$$ 0 \to \mathbf{NS}_{A/S}(S)/\mathbf{Pic}_{A/S}(S) \to \operatorname{Ext}^2(S; A, \mathbf{G}_m) \to H^2_{prim}(A, \mathbf{G}_m) \to H \to 0, $$

where H *is a* 2*-torsion group. For any prime* l *distinct from the residue characteristics of* S,

$$ 2\,H^2_{prim}(A, \mathbf{G}_m)(l) \subseteq H^2_{N,pr}(A, \mathbf{G}_m)(l). $$

If moreover $H^1(S, \hat{A})(l) \to H^1(S, \mathbf{Pic}_{A/S})(l)$ *is surjective, then*

$$ H^2_{N,pr}(A, \mathbf{G}_m)(l) \subseteq H^2_{prim}(A, \mathbf{G}_m)(l). $$

Proof. — L. Breen [2] has constructed two spectral sequences from which the exact sequence (2.5) follows where H is a 2-torsion group,

$$ 0 \to \mathbf{NS}_{A/S}(S)/\mathbf{Pic}_{A/S}(S) \to \operatorname{Ext}^2(S; A, \mathbf{G}_m) \to H^2_{prim}(A, \mathbf{G}_m) \to H \to 0. \tag{2.5} $$

His construction is done in the fppf topology, but it works equally well in the étale topology. He proves the exactness of (2.5) when the base is regular and then concludes that $\mathrm{Ext}^2(\mathrm{S};\ \mathrm{A},\ \mathbf{G}_m)$ is torsion. Since the regularity assumption is only used for this conclusion, (2.5) is exact without this hypothesis. The first inclusion relation between the two definitions of primitive elements follows from the exact sequence

$$0 \to {}_{l^m}\mathrm{A} \to \mathrm{A} \xrightarrow{m_{\mathrm{A}}} \mathrm{A} \to 0$$

and Corollary 2.2 since these imply that $\mathbf{Ext}^2(\mathrm{A},\ \mathbf{G}_m)(l) = (0)$.

For the reverse inclusion let us first show that

$$\mathrm{H}^2_{\mathrm{N,pr}}(\mathrm{A},\ \boldsymbol{\mu}_{l^\infty}) \to \mathrm{H}^2_{\mathrm{N,pr}}(\mathrm{A},\ \mathbf{G}_m)(l)$$

is surjective. The Kummer sequence and the Leray spectral sequence for p give the exact commutative diagram (2.6)

$$(2.6)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}^2(\mathrm{S},\ \boldsymbol{\mu}_{l^n}) & \longrightarrow & \mathrm{F}^1\mathrm{H}^2(\mathrm{A},\ \boldsymbol{\mu}_{l^n}) & \longrightarrow & \mathrm{H}^1(\mathrm{S},\ \mathrm{R}^1 p_* \boldsymbol{\mu}_{l^n}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{H}^2(\mathrm{S},\ \mathbf{G}_m) & \longrightarrow & \mathrm{F}^1\mathrm{H}^2(\mathrm{A},\ \mathbf{G}_m) & \longrightarrow & \mathrm{H}^1(\mathrm{S},\ \mathbf{Pic}_{\mathrm{A/S}}) & \longrightarrow & 0 \end{array}$$

where the right hand terms are the primitive elements in the sense of the first definition. The assumption on l shows that $\mathrm{R}^1 p_* \boldsymbol{\mu}_{l^n} = {}_{l^n}\hat{\mathrm{A}}$, and so the exact cohomology sequence coming from

$$0 \to {}_{l^n}\hat{\mathrm{A}} \to \hat{\mathrm{A}} \xrightarrow{l^n_{\hat{\mathrm{A}}}} \hat{\mathrm{A}} \to 0$$

and the hypothesis on $\mathrm{H}^2(\mathrm{S},\ \mathbf{Pic}_{\mathrm{A/S}})(l)$ give the desired surjectivity. It remains to show that for all n,

$$\mathrm{H}^2_{\mathrm{N,pr}}(\mathrm{A},\ \boldsymbol{\mu}_{l^n}) \subseteq \mathrm{H}^2_{\mathrm{prim}}(\mathrm{A},\ \boldsymbol{\mu}_{l^n}).$$

Let $y \in \mathrm{H}^2_{\mathrm{N,pr}}(\mathrm{A},\ \boldsymbol{\mu}_{l^n})$. Using the notation of (2.3), the Leray spectral sequence for $\mathrm{pr}_1 : \mathrm{A}^2 \to \mathrm{A}$ gives an exact diagram (2.7),

$$(2.7)\qquad \begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ 0 & \to & \mathrm{H}^2(\mathrm{A},\ \boldsymbol{\mu}_{l^n}) & \xrightarrow{\mathrm{pr}_1^*} & \mathrm{F}^1\mathrm{H}^2(\mathrm{A}^2,\ \boldsymbol{\mu}_{l^n}) & \xrightarrow{t} & \mathrm{H}^1(\mathrm{A},\ {}_{l^n}\hat{\mathrm{A}}) & \longrightarrow & 0 \\ & & & & \downarrow & & & & \\ & & & & \mathrm{H}^2(\mathrm{A}^2,\ \boldsymbol{\mu}_{l^n}) & & & & \\ & & & & \downarrow{\scriptstyle \pi_1} & & & & \\ & & & & \mathrm{H}^0(\mathrm{A},\ \mathrm{R}^2 p_* \boldsymbol{\mu}_{l^n}) & & & & \end{array}$$

Since y can be split by an étale covering of the base S, we see that $\pi_1(\rho^*(y)) = 0$. Regarding $\rho^*(y)$ as an element of $F^1 H^2(A^2, \mu_{l^n})$, its component in $H^2(A, \mu_{l^n})$ is $s_1^*(\rho^*(y))$ which is immediately seen to be zero. Thus to finish the proof it is enough to show that $t(\rho^*(y)) = 0$ in $H_N^1(A, {}_{l^n}\hat{A})$. But ${}_{l^n}\hat{A}$ is represented by a finite, étale group scheme over S, and so by Theorem 2.3,

$$H_N^1(A, {}_{l^n}\hat{A}) \xrightarrow{\approx} \mathrm{Hom}_{\tilde{S}}({}_{l^n}\hat{A}^D, \mathbf{Pic}_{A/S}).$$

Since $t(\rho^*(y))$ is split by an étale covering of S, $t(\rho^*(y)) = 0$ as desired.

COROLLARY 2.10. — *Let $p : A \to S$ be an abelian scheme over a regular, noetherian, connected base* S, *l a prime distinct from* char $(\Gamma(S, \mathcal{O}_S))$. *Then*

$$2\, H^2_{\mathrm{prim}}(A, \mathbf{G}_m)(l) \subseteq H^2_{N,\mathrm{pr}}(A, \mathbf{G}_m)(l).$$

If, moreover, $H^1(\eta, \hat{A})(l) \to H^1(\eta, \mathbf{Pic}_{A/S})(l)$ *is surjective where* η *is the generic point of* S, *then* $H^2_{N,\mathrm{pr}}(A, \mathbf{G}_m)(l) \subseteq H^2_{\mathrm{prim}}(A, \mathbf{G}_m)(l)$.

Proof. — Since S is regular, $\hat{A}$ exists. Moreover, the map $A_\eta \to A$ induces an injection on $H^2(A, \mathbf{G}_m)$, $H^2_{\mathrm{prim}}(A, \mathbf{G}_m)$, and $H^2_{N,\mathrm{pr}}(A, \mathbf{G}_m)$. Thus we see that

$$H^2_{\mathrm{prim}}(A_\eta, \mathbf{G}_m) \cap H^2_N(A, \mathbf{G}_m) = H^2_{\mathrm{prim}}(A, \mathbf{G}_m)$$

and

$$H^2_{N,\mathrm{pr}}(A_\eta, \mathbf{G}_m) \cap H^2_N(A, \mathbf{G}_m) = H^2_{N,\mathrm{pr}}(A, \mathbf{G}_m)$$

by Corollary 2.6 and Proposition 1.2. Hence we may replace S by η and the corollary becomes Theorem 2.9.

Remark. — Suppose that S is a regular, noetherian scheme and l is a prime such that either l is distinct from the residue characteristics of S or that S is an excellent, Jacobsen, Dedekind scheme, i. e., noetherian, integral, normal of dimension 1, with 2 distinct from all of the residue characteristics. Given an abelian scheme A over S, let

$$H^2_S(A^2, \mathbf{G}_m) = \{\, y \in H^2(A^2, \mathbf{G}_m) : y = S^*(y) \,\}$$

where S^* is the map on cohomology induced from the map $S : A^2 \to A^2$ which interchanges the two factors. Using Breen's spectral sequences, results from [2], and some of his unpublished work, one can show that there is a subgroup H in $H^2(S, \hat{A})(l)$ and a map

$$H \to [H^2_S(A^2, \mathbf{G}_m)/\rho^*(H^2(A, \mathbf{G}_m))](l)$$

whose kernel and cokernel are 2-torsion.

Finally we want to investigate the behaviour of primitive elements under the Albanese map. Let $p : X \to S$ be a proper morphism with a section such that $p_* \mathcal{O}_X = \mathcal{O}_S$ universally. Following Grothendieck we say that A (X), the Albanese scheme of X/S, exists if $\mathbf{Pic}_{X/S}$ exists and there is a (necessarily unique) abelian subscheme $\mathbf{Pic}^{00}_{X/S}$ which as a point set is $\mathbf{Pic}^{0}_{X/S}$ whose dual abelian scheme and its dual abelian scheme exist. The universal mapping property of $\mathbf{Pic}_{X/S}$ shows that $(\hat{\mathbf{Pic}}^{00}_{X/S})$ is universal for maps of X into abelian schemes over S where $X \to (\hat{\mathbf{Pic}}^{00}_{X/S})$ is defined from the universal sheaf on $X \times_S \mathbf{Pic}_{X/S}$ pulled back to $X \times_S \mathbf{Pic}^{00}_{X/S}$. Since $(\hat{\mathbf{Pic}}^{00}_{X/S})$ satisfies the universal mapping property for Albanese schemes, we will denote it by A (X). Conditions for its existence are investigated in paragraph 3, Exposé 236 of [6].

PROPOSITION 2.11. — *Let $p : X \to S$ be a proper morphism with a section over a noetherian scheme S such that $p_* \mathcal{O}_X = \mathcal{O}_S$ universally. Suppose that A (X) exists and l is a prime such that either l is distinct from the residue characteristics of S or l is an arbitrary prime and $\mathbf{Pic}^0_{X/S}$ is represented by an abelian scheme over S. If*

$$H^1(S, \mathbf{Pic}^0_{X/S})(l) \to H^1(S, \mathbf{Pic}_{X/S})(l)$$

is onto, then $H^2_{N,\mathrm{pr}}(A(X), \mathbf{G}_m)(l) \to H^2_{N,\mathrm{pr}}(X, \mathbf{G}_m)(l)$ *is onto. If moreover*

$$H^1(S, \mathbf{Pic}^0_{A(X)/S})(l) \to H^1(S, \mathbf{Pic}_{A(X)/S})(l)$$

is onto, then $H_X(l)/H_{A(X)}(l)$ *is the kernel of the map on primitive elements where* $H_Y = H^0(S, \mathbf{NS}_{Y/S})/H^0(S, \mathbf{Pic}_{Y/S})$ *for any Y over S.*

Proof. — Let $f : X \to A(X)$ be the Albanese map. f defines a spectral sequence homomorphism from the Leray spectral sequence for $A(X) \to S$ to the Leray spectral sequence for $X \to S$. The identification of $H^2_{N,\mathrm{pr}}(X, \mathbf{G}_m)$ with $H^1(S, \mathbf{Pic}_{X/S})$ and similarly for A (X) shows that the map on primitive elements induced by f

$$H^1(S, \mathbf{Pic}_{A(X)/S}) \to H^1(S, \mathbf{Pic}_{X/S})$$

comes from $\mathbf{Pic}_{A(X)/S} \to \mathbf{Pic}_{X/S}$. The duality theorem for abelian schemes and the definition of A (X) shows that $\mathbf{Pic}^{00}_{X/S}$ may be canonically identified with $\mathbf{Pic}^0_{A(X)/S}$. The exact cohomology sequences coming from the short exact sequences defining $\mathbf{NS}_{X/S}$ and $\mathbf{NS}_{A(X)/S}$ give a commutative exact diagram (2.8),

$$\begin{array}{ccccccc} 0 \longrightarrow & H_{A(X)}(l) & \longrightarrow & H^1(S, \mathbf{Pic}^{00}_{X/S})(l) & \xrightarrow{\delta} & H^1(S, \mathbf{Pic}_{A(X)/S})(l) \\ & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ 0 \longrightarrow & H_X(l) & \longrightarrow & H^1(S, \mathbf{Pic}^0_{X/S})(l) & \longrightarrow & H^1(S, \mathbf{Pic}_{X/S})(l) \end{array} \tag{2.8}$$

where β comes from the inclusion $j : \mathbf{Pic}^{00}_{X/S} \to \mathbf{Pic}^0_{X/S}$.

In the first case if Q is the cokernel of j in $\tilde{S}$, then multiplication by l on Q is an isomorphism since it is surjective on $\mathbf{Pic}^{00}_{X/S}$ and the étale group schemes ${}_l\mathbf{Pic}^{00}_{X/S}$ and ${}_l\mathbf{Pic}^{0}_{X/S}$ are equal. In the second case j is the identity map. Thus β is an isomorphism in either case and the 5-lemma gives the desired result.

Finally the next proposition gives a set of conditions which will insure that the hypothesis " $H^1(S, \mathbf{Pic}^0_{X/S})(l) \to H^1(S, \mathbf{Pic}_{X/S})(l)$ is onto " is satisfied.

PROPOSITION 2.12. — *Let $p : X \to S$ be a proper morphism over a noetherian scheme* S. *If* ${}_l\mathbf{NS}_{X/S} = 0$, $H^0(S, \mathbf{Pic}_{X/S}) \to H^0(S, \mathbf{NS}_{X/S})$ *has torsion cokernel, and*

$$H^0(S, \mathbf{FNS}_{X/S})_l \to H^0(S, (\mathbf{FNS}_{X/S})_l)$$

is an isomorphism, then $H^1(S, \mathbf{Pic}^0_{X/S})(l) \to H^1(S, \mathbf{Pic}_{X/S})(l)$ *is surjective.*

If A *is an abelian scheme over* S *such that* $\hat{A}$ *exists, then the kernel of* $H^1(S, \hat{A}) \to H^1(S, \mathbf{Pic}_{A/S})$ *is* 2-*torsion. The map is surjective on l-primary components if and only if* $H^0(S, \mathbf{NS}_{A/S})_l \to H^0(S, (\mathbf{NS}_{A/S})_l)$ *is an isomorphism.*

Proof. — The sequence $0 \to \mathbf{FNS}_{X/S} \to \mathbf{FNS}_{X/S} \to (\mathbf{FNS}_{X/S})_l \to 0$ gives the exact sequence

$$0 \to H^0(S, \mathbf{FNS}_{X/S})_l \to H^0(S, (\mathbf{FNS}_{X/S})_l) \to {}_lH^1(S, \mathbf{FNS}_{X/S}) \to 0.$$

Thus the exact sequence defining $\mathbf{NS}_{X/S}$ and the isomorphism l : $(\mathbf{NS}_{X/S})_{\text{tors}} \to (\mathbf{NS}_{X/S})_{\text{tors}}$ defined by multiplication by l give the first assertion.

The assertion for an abelian scheme A over S such that $\hat{A}$ exists follows immediately from the fact that $H^0(S, \mathbf{NS}_{A/S})/H^0(S, \mathbf{Pic}_{A/S})$ is a 2-torsion group which we will now prove. Given an invertible sheaf L on A, $m^*(L) - \mathrm{pr}_1^*(L) - \mathrm{pr}_2^*(L)$ defines an element of $H^0(A, \mathbf{Pic}^0_{A/S})$ and so a homomorphism $\Lambda(L) : A \to \hat{A}$, where $m : A^2 \to A$ is the multiplication map. Moreover $\Lambda(L) = 0$ if and only if $L \in H^0(S, \mathbf{Pic}^0_{A/S})$. If $y \in H^0(S, \mathbf{NS}_{A/S})$, there is an étale covering $T \to S$ and a representative $L_T \in H^0(T, \mathbf{Pic}_{A/S})$ for y. By descent $\Lambda(L_T) : A_T \to \hat{A}_T$ defines a homomorphism $\Lambda(y)$: $A \to \hat{A}$. In this manner we may regard $H^0(S, \mathbf{NS}_{A/S}) \subseteq \mathrm{Hom}_S(A, \hat{A})$. But then if P is the universal sheaf on $A \times_S \hat{A}$, we have

$$2\Lambda(y) = \Lambda(2y) = \Lambda(M), \qquad \text{where} \quad M = (1_A, \Lambda(y))^*(P), \quad (1_A, \Lambda(y)) : A \to A \times_S \hat{A},$$

by the rigidity lemma ([12], Proposition 6.1) and the observation that this is true over the geometric points of S [13].

SECTION 3

In this section we will apply the generalized version of the theorem of the cube to obtain a strengthening of the Grothendieck-Auslander-Goldman theorem [8] that $H^2(X, \mathbf{G}_m) = Br(X)$, where $Br(X)$ denotes the Brauer group of X [7], when X is a regular, noetherian scheme of dimension ≤ 2.

We begin by introducing a new topology. Let S_f, the finite topology, be the topology on the category of locally free schemes over S generated from the pretopology for which the set of coverings of T is the set of single morphisms $u : T' \to T$ such that u is locally free and $T = u(T')$ (set theoretically), i. e., u is affine and $u_* \mathcal{O}_{T'}$ is a coherent, faithfully flat $\mathcal{O}_T$-module. Note that there is an obvious morphism of sites $\tau : S_{pl} \to S_f$ and so a Leray spectral sequence for any $F \in \tilde{S}_{pl}$. If $F \in \hat{S}$, let

$$F(T)_f = \{ y \in F(T) : \text{there is a covering } u : T' \to T \text{ in } S_f \text{ with } F(u)(y) = 0 \}.$$

Thus $F(T)_f$ consists of those elements which can be split by a locally free, faithfully flat covering.

Before stating the key proposition of this section, let us recall the relationship between $Br(S)$ and $H^2(S, \mathbf{G}_m)$. Let S be a quasi-compact, quasi-separated scheme, and let $\mathbf{GL}_n$ be the sheaf of automorphisms of $\bigoplus_1^n \mathcal{O}_S$ on S_{pl}. The exact sequence (3.1) of sheaves of groups on S_{pl}

$$0 \to \mathbf{G}_m \to \mathbf{GL}_n \to \mathbf{PGL}_{n-1} \to 1 \tag{3.1}$$

is a central extension of sheaves of groups, and so there is a boundary map ([3]; [10] if we work in $\tilde{S}$)

$$d^1 : \quad H^1(S_{pl}, \mathbf{PGL}_{n-1}) \to H^2(S_{pl}, \mathbf{G}_m).$$

By fppf descent, $H^1(S_{pl}, \mathbf{GL}_n)$ classifies locally free coherent $\mathcal{O}_S$-modules of rank n and $H^1(S_{pl}, \mathbf{PGL}_{n-1})$ classifies sheaves of Azymaya algebras of rank n^2 [7]. Thus by the exact sequence of cohomology sets and taking $\varinjlim$ over n, we get a natural injection

$$Br(S) \to H^2(S_{pl}, \mathbf{G}_m),$$

where $Br(S)$ is the Brauer group of the scheme S.

PROPOSITION 3.1. — *Let* S *be a quasi-compact, quasi-separated scheme. Then* $\check{H}^2(S_{pl}, \mathbf{G}_m)_f \subseteq Br(S)$, *where* $\check{H}^2(S_{pl}, F)$ *is the second Čech cohomology group of the sheaf* $F \in \tilde{S}_{pl}$.

Proof. — Let $f: T \to S$ be a locally free, faithfully flat morphism. Since any invertible module over a semi-local ring is free, $R^1 f_* \mathbf{G}_{m,T} = 0$ in $\tilde{S}_{pl}$. Thus the Leray spectral sequence for f shows that

$$H^2(S_{pl}, f_* \mathbf{G}_{m,T}) \to H^2(T_{pl}, \mathbf{G}_{m,T})$$

is monic. The spectral sequence comparing Čech cohomology with sheaf cohomology shows that

$$\check{H}^2(S_{pl}, F) \to H^2(S_{pl}, F)$$

is monic for all S and all $F \in \tilde{S}_{pl}$, and so

$$\check{H}^2(S_{pl}, f_*, \mathbf{G}_{m,T}) \to \check{H}^2(T_{pl}, \mathbf{G}_{m,T})$$

is monic.

Now if $y \in \check{H}^2(S_{pl}, \mathbf{G}_{m,S})$ is split by a locally free, faithfully flat morphism $f: T \to S$, then $y \in \mathrm{Ker}\left[\check{H}^2(S_{pl}, \mathbf{G}_{m,S}) \to \check{H}^2(S_{pl}, f_* \mathbf{G}_{m,T})\right]$. Let $i: \mathbf{G}_{m,S} \to f_* \mathbf{G}_{m,T}$ be the inclusion. Then we can choose a quasi-compact covering map $\varphi: X \to S$ in S_{pl} and a Čech 2-cocycle $u \in H^0(X^3, \mathbf{G}_{m,S})$ representing y such that there is a Čech 1-cochain

$$v \in H^0(X^2, f_*, \mathbf{G}_{m,T}) = H^0(X_T^2, \mathbf{G}_{m,T}), \qquad \text{with} \quad \partial^1(v) = i(u) \in H^0(X^3, f_* \mathbf{G}_{m,T}),$$

where

$$\partial^1: \quad \check{C}^1(X \to S, f_* \mathbf{G}_{m,T}) \to \check{C}^2(X \to S, f_* \mathbf{G}_{m,T})$$

is the Čech coboundary map on Čech cochains defined, of course, by the usual formula (3.2)

$$(3.2) \qquad \partial^1(v) = \mathrm{pr}_{2,3}^*(v) - \mathrm{pr}_{1,3}^*(v) + \mathrm{pr}_{1,2}^*(v).$$

Let $\mathcal{F}$ be the faithfully flat, locally free, coherent $\mathcal{O}_S$-module $f_*\mathcal{O}_T$ and denote $\Psi^* \mathcal{F}$ by $\mathcal{F}_{S'}$ where $\Psi: S' \to S$. Note that $\mathcal{F}_{S'}$. is naturally isomorphic to $(f_{S'})_* (\mathcal{O}_{T \times_S S'})$ since f is a finite, faithfully flat morphism in the cartesian diagram (3.3)

$$(3.3) \qquad \begin{array}{ccc} T \times_S S' & \longrightarrow & T \\ \downarrow{\scriptstyle f_{S'}} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{\ \Psi\ } & S \end{array}$$

Consequently we will identify $F_{S'}$ with $(f_{S'})_* (\mathcal{O}_{T \times_S S'})$.

We have an exact sequence of *presheaves* of groups on S_{pl}

$$(3.4) \qquad 0 \to \mathbf{G}_{m,S} \xrightarrow{j} \mathbf{Aut}_{\mathcal{O}_S}(\mathcal{F}) \xrightarrow{c} \mathrm{Aut}\left(\mathbf{End}_{\mathcal{O}_S}(\mathcal{F})\right) \to 1.$$

Where the first two presheaves are sheaves and Aut $\left(\mathbf{End}_{\mathcal{O}_S}(\mathbf{F})\right)$ is the presheaf whose values at T′ are those algebra automorphisms of $\mathbf{End}_{\mathcal{O}_{T'}}(\mathcal{F}_{T'})$ defined by conjugating any $\mathcal{O}_{T'}$-endomorphism of $\mathcal{F}_{T'}$ by an $\mathcal{O}_{T'}$-automorphism of $\mathcal{F}_{T'}$. j is the map sending a unit to the automorphism defined by multiplication by the unit. Now there is an injection

$$k : \ f_* \mathbf{G}_{m,T} \rightarrow \mathbf{Aut}_{\mathcal{O}_S}(\mathcal{F})$$

since $H^0(S', f_* \mathbf{G}_{m,T}) = H^0(S' \times_S T, \mathbf{G}_{m,T})$ and multiplication by any element of this group defines an $\mathcal{O}_{S'}$-automorphism of $\mathcal{F}_{S'}$. Identifying $f_* \mathbf{G}_{m,T}$ with a subsheaf of $\mathbf{Aut}_{\mathcal{O}_S}(\mathcal{F})$, the Čech 1-cochain v becomes an automorphism $k(v) : \mathcal{F}_{X^2} \rightarrow \mathcal{F}_{X^2}$ and so $k(v)$ is in $\check{C}^1(X \rightarrow S, \mathbf{Aut}_{\mathcal{O}_S}(\mathcal{F}))$. Then

$$\mathrm{pr}^*_{2,3}(k(v)) \circ \mathrm{pr}^*_{1,2}(k(v)) \circ \mathrm{pr}^*_{1,3}(k(v))^{-1} : \ \mathcal{F}_{X^3} \rightarrow \mathcal{F}_{X^3}$$

is precisely $j(i(u))$ by (3.2) and the assumption on v and u. Thus using non-abelian Čech cohomology [10],

$$c(k(v)) \in \check{Z}^1\left(X \rightarrow S, \mathrm{Aut}\left(\mathbf{End}_{\mathcal{O}_S}(\mathcal{F})\right)\right),$$

the Čech 1-cocycles of Aut $\left(\mathbf{End}_{\mathcal{O}_S}(\mathcal{F})\right)$ for the covering $X \rightarrow S$. Let y_1 be the corresponding cohomology class in $\check{H}^1\left(S_{pl}, \mathrm{Aut}\left(\mathbf{End}_{\mathcal{O}_S}(\mathcal{F})\right)\right)$. Since (3.4) is a central extension of presheaves of groups on S_{pl}, we have a boundary map

$$\check{d}^1 : \ \check{H}^1\left(S_{pl}, \mathrm{Aut}\left(\mathbf{End}_{\mathcal{O}_S}(\mathcal{F})\right)\right) \rightarrow \check{H}^2(S_{pl}, \mathbf{G}_m)$$

defined in the obvious manner [10], and the calculation above shows that $\check{d}^1(y_1) = y$.

On the other hand sheafifying the sequence (3.4), we get a central extension of sheaves of groups on X_{pl} [compare with (3.1)]

$$0 \rightarrow \mathbf{G}_{m,S} \xrightarrow{j} \mathbf{Aut}_{\mathcal{O}_S}(\mathcal{F}) \rightarrow \mathbf{Aut}\left(\mathbf{End}_{\mathcal{O}_S}(\mathcal{F})\right) \rightarrow 1,$$

where the cokernel of j is now the usual sheaf of $\mathcal{O}_S$-algebra automorphisms of $\mathbf{End}_{\mathcal{O}_S}(\mathcal{F})$ in S_{pl} ([7], Theorem 5.10). If y'_1 is the image of y_1 under

$$\check{H}^1\left(S_{pl}, \mathrm{Aut}\left(\mathbf{End}_{\mathcal{O}_S}(\mathcal{F})\right)\right) \rightarrow H^1\left(S_{pl}, \mathbf{Aut}\left(\mathbf{End}_{\mathcal{O}_S}(\mathcal{F})\right)\right),$$

then Giraud's construction [3] of the boundary map

$$d^1 : \ H^1\left(S_{pl}, \mathbf{Aut}\left(\mathbf{End}_{\mathcal{O}_S}(\mathcal{F})\right)\right) \rightarrow H^2(S_{pl}, \mathbf{G}_m)$$

shows that $d^1(y'_1) = y$, where $\check{H}^2(S_{pl}, \mathbf{G}_m) \subseteq H^2(S_{pl}, \mathbf{G}_m)$. The final observation needed to finish the proof is that $H^1(S_{pl}, \mathbf{Aut}(\mathbf{End}_{\mathcal{O}_S}(\mathcal{F})))$ classifies Azumaya algebras locally isomorphic to $\mathbf{End}_{\mathcal{O}_S}(\mathcal{F})$ in S_{pl} and the above map is compatible with the construction sketched after (3.1) which gave the inclusion $\mathrm{Br}(S) \subseteq H^2(S_{pl}, \mathbf{G}_m) = H^2(S, \mathbf{G}_m)$.

The next lemma enables us to describe some of the elements in $\check{H}^2(S_{pl}, \mathbf{G}_m)_f$ in a more reasonable way.

LEMMA 3.2. — *Let* S *be a scheme,* $\tau : S_{pl} \to S_f$ *the corresponding morphism of sites. Then* $\tau^* : H^2(S_f, \mathbf{G}_m) \to H^2(S_{pl}, \mathbf{G}_m)$ *factors through the inclusion* $\check{H}^2(S_{pl}, \mathbf{G}_m) \to H^2(S_{pl}, \mathbf{G}_m)$.

Proof. — The morphism of sites τ gives a commutative diagram (3.5), up to natural transformations, of categories and functors where the maps in the right hand square are $\check{H}^0_{pl}$ and $\check{H}^0_f$ respectively and Ab is the category of abelian groups,

(3.5)
$$\begin{array}{ccccc} \tilde{S}_{pl} & \xrightarrow{i} & \hat{S}_{pl} & \longrightarrow & \mathrm{Ab} \\ {\scriptstyle \tau_*}\downarrow & & {\scriptstyle \tau_\cdot}\downarrow & & \| \\ \tilde{S}_f & \xrightarrow{j} & \hat{S}_f & \longrightarrow & \mathrm{Ab} \end{array}$$

$\tau_\cdot$ is the direct image morphism on the category of presheaves of abelian groups on S_{pl} and i, j are the respective inclusion maps of sheaves into presheaves. The upper and lower lines define the spectral sequence comparing Čech and sheaf cohomology on S_{pl} and S_f respectively, and so we have a morphism

$$[\check{H}^p(S_f, \mathcal{H}^q_f(\mathbf{G}_m)) \Rightarrow H^n(S_f, \mathbf{G}_m)] \to [\check{H}^p(S_{pl}, \mathcal{H}^q_{pl}(\mathbf{G}_m)) \Rightarrow H^n(S_{pl}, \mathbf{G}_m)],$$

where $\mathcal{H}^q_*(\mathbf{G}_m)(T) = H^q(T_*, \mathbf{G}_m)$. This gives a commutative, exact diagram (3.6) by noting that $\mathcal{H}^1_{pl}(\mathbf{G}_m) = \mathrm{Pic}$,

(3.6)
$$\begin{array}{ccccccc} 0 \longrightarrow & \check{H}^2(S_f, \mathbf{G}_m) & \longrightarrow & H^2(S_f, \mathbf{G}_m) & \longrightarrow & \check{H}^1(S_f, \mathcal{H}^1_f(\mathbf{G}_m)) \\ & \downarrow & & \downarrow & & \downarrow{\scriptstyle \lambda} \\ 0 \longrightarrow & \check{H}^2(S_{pl}, \mathbf{G}_m) & \longrightarrow & H^2(S_{pl}, \mathbf{G}_m) & \longrightarrow & \check{H}^1(S_{pl}, \mathrm{Pic}) \end{array}$$

Now λ factors through $\check{H}^1(S_f, \mathrm{Pic}) \to \check{H}^1(S_{pl}, \mathrm{Pic})$. But given any locally free, faithfully flat morphism $T \to S$ and an invertible sheaf L on $T \times_S T$, i. e., any element of $\check{C}^1(T \to S, \mathrm{Pic})$, there is a Zariski covering $\{U_i\}$ of S such that $L|_{p^{-1}(U_i)}$ is trivial for all i, where $p : T \times_S T \to S$ is the structure morphism since any invertible module over a semi-local ring is free. Thus given any 1-cochain $L \in \check{C}^1(T \to S, \mathrm{Pic})$, there is a refinement of $T \to S$ in S_{pl} which trivializes L, and so λ factors through the zero map which gives the desired result.

THEOREM 3.3. — *If $p : A \to S$ is an abelian scheme over a noetherian scheme S satisfying the generalized theorem of the cube for l, then* $H^2_N(A, \mathbf{G}_m)(l) \subseteq \mathrm{Br}(A)$.

Proof. — We will show that $\tau^*(H^2(A_f, \mathbf{G}_m)) \supseteq H^2_N(A, \mathbf{G}_m)(l)$. Since $R^1\tau_*\boldsymbol{\mu}_{l^n}$ is the sheaf on A_f associated to the presheaf $Y \to H^1(Y_{pl}, \boldsymbol{\mu}_{l^n})$ and this latter group classifies principal homogeneous spaces in $\tilde{Y}_{pl}$ for the finite, locally free group scheme $\boldsymbol{\mu}_{l^n}$, we see that $R^1\tau_*\boldsymbol{\mu}_{l^n} = (0)$. Hence the Leray spectral sequence for τ shows that $H^2(A_f, \boldsymbol{\mu}_{l^n})$ maps onto $H^2(A_{pl}, \boldsymbol{\mu}_{l^n})_f$ and so it is enough to demonstrate that $H^2_N(A_{pl}, \boldsymbol{\mu}_{l^\infty})_f$ maps onto $H^2_N(A, \mathbf{G}_m)(l)$.

Suppose $y \in H^2_N(A_{pl}, \mathbf{G}_m)$ with $l^n y = 0$. The filtration on the Leray spectral sequence for τ and the Kummer sequence gives an exact commutative diagram (3.7),

$$
\begin{array}{ccccccc}
 & & & & \mathrm{Pic}(A) & \xrightarrow{\pi'} & H^0(A_f, (R^1\tau_*\mathbf{G}_m)_{l^n}) \\
 & & & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle \delta} \\
(3.7)\quad 0 & \longrightarrow & H^2(A_{pl}, \boldsymbol{\mu}_{l^n})_f & \longrightarrow & H^2(A_{pl}, \boldsymbol{\mu}_{l^n}) & \xrightarrow{\pi} & H^0(A_f, R^2\tau_*\boldsymbol{\mu}_{l^n}) \\
 & & \downarrow & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle j_*} \\
0 & \longrightarrow & H^2(A_{pl}, \mathbf{G}_m)_f & \longrightarrow & H^2(A_{pl}, \mathbf{G}_m) & \longrightarrow & H^0(A_f, R^2\tau_*\mathbf{G}_m)
\end{array}
$$

Choose $y_1 \in H^2_N(A_{pl}, \boldsymbol{\mu}_{l^n})$ with $i_*(y_1) = y$. The isogeny $l^{2n}_A : A \to A$ splits y by Corollary 2.8, and there is $z \in \mathrm{Pic}_N(A) = \mathbf{Pic}_{A/S}(S)$ with $\delta\pi'(z) = \pi((l^{2n}_A)^*(y_1)) \in H^0(A_f, R^2\tau_*\boldsymbol{\mu}_{l^n})$. But one of the corollaries of the theorem of cube for $\mathbf{Pic}_{A/S}$ is the formula of Proposition 2.7, (2). Thus $(l^{2n}_A)^*(z) = l^n z_1$ for some $z_1 \in \mathbf{Pic}_{A/S}(S)$. Hence $\pi'(z) = 0$ and so $\pi((l^{2n}_A)^*(y_1)) = (l^{2n}_A)^*(\pi(y_1)) = 0$. Thus $\pi(y_1) = 0$ as desired.

COROLLARY 3.4. — *Let $p : X \to S$ be a proper morphism with a section over a noetherian scheme S such that $p_*\mathcal{O}_X = \mathcal{O}_S$ universally. Suppose that $H^1(S, \mathbf{Pic}^0_{X/S})(l) \to H^1(S, \mathbf{Pic}_{X/S})(l)$ is onto and either $A(X)$ exists and satisfies the generalized theorem of the cube for a prime l distinct from all the residue characteristics of S or that $\mathbf{Pic}^0_{X/S}$ is represented by an abelian scheme over S satisfying the generalized theorem of the cube for l. Then* $H^2_{N,pr}(X, \mathbf{G}_m)(l) \subseteq \mathrm{Br}(X)$.

Proof. — In either case $H^2_{N,pr}(A(X), \mathbf{G}_m)(l) \to H^2_{N,pr}(X, \mathbf{G}_m)(l)$ is onto and the former group is contained in $\mathrm{Br}(A(X))$.

Remark. — (1) By using the results of this section, the corollary can be proven for X proper over a noetherian scheme S under the assumptions that the presheaf $T \to \mathbf{Pic}_{X/S}(T)/\mathbf{Pic}^0_{X/S}(T)$ is constant when restricted to S_f and l is a prime distinct from the residue characteristics of S such that $H^1(S, \mathbf{Pic}^0_{X/S})(l) \to H^1(S, \mathbf{Pic}_{X/S})(l)$ is onto.

(2) Suppose $p : X \to S$ is a proper, smooth morphism with $p_* \mathcal{O}_X = \mathcal{O}_S$, where $S = \mathrm{Spec}(\mathbf{Z})$ or a curve over a finite field, such that $\mathbf{NS}_{X/S}$ is constant and $\mathbf{Pic}_{X/S}(S) \to \mathbf{NS}_{X/S}(S)$ is surjective. If p has a section and $\mathbf{Pic}^0_{X/S}$ is represented by an abelian scheme over S, then $H^2_{N,pr}(X, \mathbf{G}_m)$ is the " Tate-Šafarevič " group of $\mathbf{Pic}^0_{X/S}$ restricted to the generic point η of S; that is, $H^2_{N,pr}(X, \mathbf{G}_m)$ classifies principal homogeneous spaces for $\mathbf{Pic}^0_{X_\eta/\eta}$ which have a point in all completions of primes of S. Thus the second part of Corollary 3.5 may be viewed as a generalization of M. Artin's result ([9], § 3) which interpretes the Tate-Šafarevič group of a curve X over S as the Brauer group of X.

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