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# Prolongations of linear partial differential equations. I. A conjecture of Élie Cartan

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# PROLONGATIONS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS : I. A CONJECTURE OF ÉLIE CARTAN

BY HUBERT GOLDSCHMIDT (*).

This paper is motivated by the work of Élie Cartan on exterior differential systems which culminated in the Cartan-Kähler theorem for *involutive* systems. In his book [3], Élie Cartan attacks the problem of finding solutions of systems of partial differential equations which are not involutive and asks the following question : " étant donnée une solution particulière d'un système différentiel donné, peut-elle être obtenue comme solution non singulière d'un système en involution susceptible d'être déduit du système donné par un procédé régulier ?... Le procédé régulier auquel il est fait allusion repose sur la notion de *prolongement* d'un système différentiel ". This is a fundamental problem : determine conditions under which a system can be " prolonged " to a compatible system which admits the same solutions as the given one and under which such a system can be deduced from the original one in a *finite* number of steps.

Cartan distinguishes two cases. First, if the system is compatible, Cartan affirms that, by prolonging a system a sufficient number of times one obtains an involutive system whose solutions are the solutions of the original system " sous certaines conditions qu'il n'est du reste pas facile de préciser ". In 1957, Kuranishi [8] established this result, which is known today as the Cartan-Kuranishi prolongation theorem. In the case of an incompatible system, Cartan says that one must add to the given system equations expressing the compatibility conditions of the system and its prolongations.

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In this paper, we deal only with linear systems of partial differential equations and show that Cartan's conjecture holds under certain regularity conditions. Our regularity assumption is satisfied in particular by constant coefficient equations.

If a system $R_k$ of linear partial differential equations of order $k$ satisfies our regularity condition, we show that, by prolonging the system $m_0 - k$ times and by adding to this prolonged system of order $m_0$ the finite number of equations expressing the obstructions to extending a solution of order $m_0$ to a solution of order $m_0 + l_0$, we obtain a system $R_{m_0}^{(l_0)}$ of order $m_0$ which is compatible or formally integrable and which has the same solutions and formal solutions as the original system $R_k$ (Theorem 1). This new system is obtained from the original system $R_k$ by adding finitely many equations to the system $R_k$. By the Cartan-Kuranishi prolongation theorem, one can choose $m_0$ such that the system $R_{m_0}^{(l_0)}$ is involutive.

In paragraphs 1 and 2, we define the symbol cohomology of a partial differential equation introduced by Spencer [15]. The vanishing of these cohomology groups was shown by Serre to be equivalent to Cartan's notion of involutiveness. For a regular equation $R_k$, we also introduce the cosymbol cohomology, which was already considered by Quillen [14] under more restrictive hypotheses on $R_k$. Most of the results of this paper including the prolongation theorem (Theorem 1) follow from the $\delta$-Poincaré lemma for the cosymbol cohomology (Lemma 3). Our proof of this lemma is based on the work of Grothendieck [7] on the Hilbert scheme in algebraic geometry.

The remainder of this paper is devoted to other consequences of lemma 3 and of our prolongation theorem, most of which are extensions of certain results of Quillen [14]. In paragraph 4, we define the *naive Spencer sequence* of a regular partial differential equation $R_k$; our construction is slightly different from Bott's (*see* R. Bott [1], D. G. Quillen [14], D. C. Spencer [15] and S. Sternberg [16]). We prove that under certain regularity conditions the cohomology of the naive Spencer sequences stabilizes (Theorem 2); we are thus able to define the *Spencer cohomology* of equations which are not necessarily formally integrable as the cohomology of one of the stable naive Spencer sequences. Furthermore, the stable naive sequences are formally exact (Corollary 2) and by our prolongation theorem, under the hypotheses of Theorem 1, the Spencer cohomology of $R_k$ depends only on the formal solutions of $R_k$ (Corollary 1). Finally, we show that the cohomology of the *sophisticated Spencer sequence* of a formally integrable equation is isomorphic to the Spencer cohomology of $R_k$, a result due to Quillen [14].

In paragraph 5, using Spencer's estimate and our prolongation theorem, we prove that the analytic stable naive sequences of an analytic partial differential equation, satisfying the regularity assumption of Theorem 1, are exact.

Throughout this paper, we use the notation of [5]. The author wishes to express his gratitude to Professors D. Mumford and S. Sternberg for several helpful conversations concerning this paper.

1. Differential operators. — Let X be a differentiable manifold of class $C^\infty$ of dimension $n$. We shall denote by T the tangent bundle of X and by $T^\star$ the cotangent bundle of X. If E is a vector bundle over X, we denote by $E_x$ the fiber of E at $x \in X$, by $\mathcal{E}$ the sheaf of germs of sections of E and by $J_k(E)$ the bundle of $k$-jets of E; we set $J_k(E) = 0$, if $k < 0$. We shall always assume that the fibers of a vector bundle have the same dimension. We have a natural sheaf morphism $j_k : \mathcal{E} \to \mathcal{J}_k(\mathcal{E})$, a morphism $p_l(id_k) : J_{k+l}(E) \to J_l(J_k(E))$ of vector bundles and an exact sequence

$$0 \longrightarrow S^k T^\star \otimes E \xrightarrow{\varepsilon} J_k(E) \xrightarrow{\pi_{k-1}} J_{k-1}(E) \longrightarrow 0$$

of vector bundles over X (*see* [5]).

Throughout this paper, E, F will denote vector bundles over X. Let $\varphi : J_k(E) \to F$ be a morphism of vector bundles; such a morphism $\varphi$ is a *differential operator* of order $k$ from E to F. This morphism induces sheaf maps $\varphi : \mathcal{J}_k(\mathcal{E}) \to \mathcal{F}$ and $\varphi \circ j_k : \mathcal{E} \to \mathcal{F}$; the latter map is also called a differential operator of order $k$ from E to F. A solution of $\varphi$ is a germ $s \in \mathcal{E}$ belonging to the kernel of $\varphi \circ j_k$; we denote by $\mathcal{S}$ the sheaf of germs of solutions of $\varphi$. The map $\varphi$ also induces a morphism $J_l(\varphi) : J_l(J_k(E)) \to J_l(F)$ of vector bundles. The $l$-th prolongation $p_l(\varphi) : J_{k+l}(E) \to J_l(F)$ is the composition $J_l(\varphi) \circ p_l(id_k)$; this map induces a morphism $\sigma_l(\varphi) : S^{k+l} T^\star \otimes E \to S^l T^\star \otimes F$; the morphism $\sigma(\varphi) = \sigma_0(\varphi)$ is called the *symbol* of $\varphi$. In particular, if $\varphi$ is the identity map $id_k$ of $J_k(E)$, the map $\sigma_1(id_k)$ induces a morphism $\delta : S^{k+1} T^\star \otimes E \to T^\star \otimes S^k T^\star \otimes E$ (*see* [5]). We set

$$R_{k+l} = \ker p_l(\varphi), \qquad Q_l = \operatorname{coker} p_l(\varphi), \qquad g_{k+l} = \ker \sigma_l(\varphi),$$
$$p_l = \operatorname{coker} \sigma_l(\varphi) \qquad \text{for} \quad l \geq 0,$$

and

$$R_{k+l} = J_{k+l}(E), \qquad Q_l = 0, \qquad g_{k+l} = S^{k+l} T^\star \otimes E, \qquad p_l = 0 \qquad \text{for} \quad l < 0.$$

Let $h_{k+l}$ be the cokernel of the map $\pi_{k+l} : R_{k+l+1} \to R_{k+l}$ induced by the map $\pi_{k+l} : J_{k+l+1}(E) \to J_{k+l}(E)$.

Definition 1. — A partial differential equation of order $k$ on E is the kernel $R_k$ of a differential operator $\varphi : J_k(E) \to F$ of order $k$.

In particular, any sub-bundle of $J_k(E)$ is such an equation.

The following diagram is commutative and exact :

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & g_{k+l} & \longrightarrow & S^{k+l}T^*\otimes E & \xrightarrow{\sigma_l(\varphi)} & S^lT^*\otimes F & \longrightarrow & p_l & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\varepsilon} & & \\
0 & \longrightarrow & R_{k+l} & \longrightarrow & J_{k+l}(E) & \xrightarrow{p_l(\varphi)} & J_l(F) & \longrightarrow & Q_l & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\pi_{k+l-1}} & & \downarrow{\scriptstyle\pi_{k+l-1}} & & \downarrow{\scriptstyle\pi_{l-1}} & & \downarrow{\scriptstyle\pi_{l-1}} & & \\
0 & \longrightarrow & R_{k+l-1} & \longrightarrow & J_{k+l-1}(E) & \xrightarrow{p_{l-1}(\varphi)} & J_{l-1}(F) & \longrightarrow & Q_{l-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & h_{k+l-1} & & 0 & & 0 & & & & \\
 & & \downarrow & & & & & & & & \\
 & & 0 & & & & & & & &
\end{array}$$

In fact, this diagram induces maps $\varepsilon : p_l \to Q_l$, $\pi_{l-1} : Q_l \to Q_{l-1}$ such that the last column is exact. Moreover, the diagram induces a monomorphism $\iota : h_{k+l-1} \to p_l$ such that, if $q_l$ is the kernel of $\pi_{l-1} : Q_l \to Q_{l-1}$, the sequence

$$0 \to h_{k+l-1} \xrightarrow{\iota} p_l \xrightarrow{\varepsilon} q_l \to 0$$

is exact.

Definition 2. — We say that a differential operator $\varphi : J_k(E) \to F$ is *regular* if, for each $l \geqq 0$, the morphism $p_l(\varphi) : J_{k+l}(E) \to J_l(F)$ has constant rank.

We shall henceforth assume that the morphism $\varphi : J_k(E) \to F$ is a regular differential operator.

The following diagram is commutative by Proposition 4.3 of [6] :

$$(1)\qquad \begin{array}{ccccccc}
J_{k+l+m}(E) & \xrightarrow{p_{l+m}(\varphi)} & J_{l+m}(F) & \longrightarrow & Q_{l+m} & \longrightarrow & 0 \\
\downarrow{\scriptstyle p_m(id_{k+l})} & & \downarrow{\scriptstyle p_m(id_l)} & & \downarrow{\scriptstyle p_m(id_l)} & & \\
J_m(J_{k+l}(E)) & \xrightarrow{J_m(p_l(\varphi))} & J_m(J_l(F)) & \longrightarrow & J_m(Q_l) & \longrightarrow & 0
\end{array}$$

Because $\varphi$ is regular, $Q_l$ is a vector bundle and so $J_m(Q_l)$ is well-defined and the bottom row is exact. Therefore the diagram induces a map $p_m(id_l) : Q_{l+m} \to J_m(Q_l)$. It is easily seen that the diagrams

$$(2)\qquad \begin{array}{ccc}
Q_{l+1} & \xrightarrow{p_1(id_l)} & J_1(Q_l) \\
\downarrow{\scriptstyle\pi_l} & & \downarrow{\scriptstyle\pi_0} \\
Q_l & \longrightarrow & Q_l
\end{array}$$

$$(3)\qquad \begin{array}{ccc}
Q_{l+1} & \xrightarrow{p_1(id_l)} & J_1(Q_l) \\
\downarrow{\scriptstyle\pi_l} & & \downarrow{\scriptstyle J_1(\pi_{l-1})} \\
Q_l & \xrightarrow{p_1(id_{l-1})} & J_1(Q_{l-1})
\end{array}$$

commute, using Proposition 4.3 of [6]. Hence there is a map $\delta : q_{l+1} \to T^* \otimes q_l$ such that the diagram

(4)
$$\begin{tikzcd}
q_{l+1} \arrow[r, "\delta"] \arrow[d] & T^* \otimes q_l \arrow[r] & T^* \otimes Q_l \arrow[d, "\varepsilon"] \\
Q_{l+1} \arrow[rr, "p_1(id_l)"] & & J_1(Q_l)
\end{tikzcd}$$

commutes. The diagram

(5)
$$\begin{tikzcd}
0 \arrow[r] & g_{k+l+1} \arrow[r] \arrow[d, dashed, "\delta"] & S^{k+l+1} T^* \otimes E \arrow[r, "\sigma_{l+1}(\varphi)"] \arrow[d, "\delta"] & S^{l+1} T^* \otimes F \arrow[r] \arrow[d, "\delta"] & p_{l+1} \arrow[r] \arrow[d, dashed, "\delta"] & 0 \\
0 \arrow[r] & T^* \otimes g_{k+l} \arrow[r] & T^* \otimes S^{k+l} T^* \otimes E \arrow[r, "\sigma_l(\varphi)"] & T^* \otimes S^l T^* \otimes F \arrow[r] & T^* \otimes p_l \arrow[r] & 0
\end{tikzcd}$$

commutes and so induces morphisms

$$\delta : \quad g_{k+l+1} \to T^* \otimes g_{k+l}, \qquad \delta : \quad p_{l+1} \to T^* \otimes p_l.$$

It is then easily verified that the exact diagram

(6)
$$\begin{tikzcd}
0 \arrow[r] & h_{k+l} \arrow[r, "\iota"] \arrow[d, dashed, "\delta"] & p_{l+1} \arrow[r, "\varepsilon"] \arrow[d, "\delta"] & q_{l+1} \arrow[r] \arrow[d, "\delta"] & 0 \\
0 \arrow[r] & T^* \otimes h_{k+l-1} \arrow[r, "\iota"] & T^* \otimes p_l \arrow[r, "\varepsilon"] & T^* \otimes q_l \arrow[r] & 0
\end{tikzcd}$$

is also commutative; hence this diagram induces a map $\delta : h_{k+l} \to T^* \otimes h_{k+l-1}$.

If $\pi_{k+l-1} : R_{k+l} \to R_{k+l-1}$ has constant rank, then $g_{k+l}$, $h_{k+l-1}$, $p_l$ are vector bundles and the sequence

$$J_1(R_{k+l}) \xrightarrow{J_1(\pi_{k+l-1})} J_1(R_{k+l-1}) \longrightarrow J_1(h_{k+l-1}) \longrightarrow 0$$

is exact. Under our hypotheses on $\varphi$, we have

$$R_{m+1} = J_1(R_m) \cap J_{m+1}(E) \qquad \text{for} \quad m \geqq k \quad (\textit{see } [5]).$$

The exact diagram

(7)
$$\begin{tikzcd}
R_{k+l+1} \arrow[r, "\pi_{k+l}"] \arrow[d, "p_1(id_{k+l})"] & R_{k+l} \arrow[r] \arrow[d, "p_1(id_{k+l-1})"] & h_{k+l} \arrow[r] \arrow[d, dashed, "\delta"] & 0 \\
J_1(R_{k+l}) \arrow[r, "J_1(\pi_{k+l-1})"] & J_1(R_{k+l-1}) \arrow[r] & J_1(h_{k+l-1}) \arrow[r] & 0
\end{tikzcd}$$

is clearly also commutative and so induces a map $\delta : h_{k+l} \to J_1(h_{k+l-1})$. We claim that this map $\delta$ is the composition of the map $\delta : h_{k+l} \to T^* \otimes h_{k+l-1}$ defined above and of the monomorphism $\varepsilon : T^* \otimes h_{k+l-1} \to J_1(h_{k+l-1})$. Indeed, the three-dimensional diagram (8) is easily seen to be exact and commutative. In diagram (8), the map $\delta : S^{k+l+1} T^* \otimes E \to J_1(S^{k+l} T^* \otimes E)$

[resp. $\delta : S^{l+1}T^*\otimes F \to J_1(S^l T^*\otimes F)$, $\delta : p_{l+1} \to J_1(p_l)$] is the composition of the maps

$$\delta : \quad S^{k+l+1}T^*\otimes E \to T^*\otimes S^{k+l}T^*\otimes E \qquad \text{and} \qquad \varepsilon : \quad T^*\otimes S^{k+l}T^*\otimes E \to J_1(S^{k+l}T^*\otimes E)$$

$$[\text{resp. } \delta : S^{l+1}T^*\otimes F \to T^*\otimes S^l T^*\otimes F \quad \text{and} \quad \varepsilon : T^*\otimes S^l T^*\otimes F \to J_1(S^l T^*\otimes F),$$
$$\delta : p_{l+1} \to T^*\otimes p_l \quad \text{and} \quad \varepsilon : T^*\otimes p_l \to J_1(p_l)].$$

If follows that the diagram

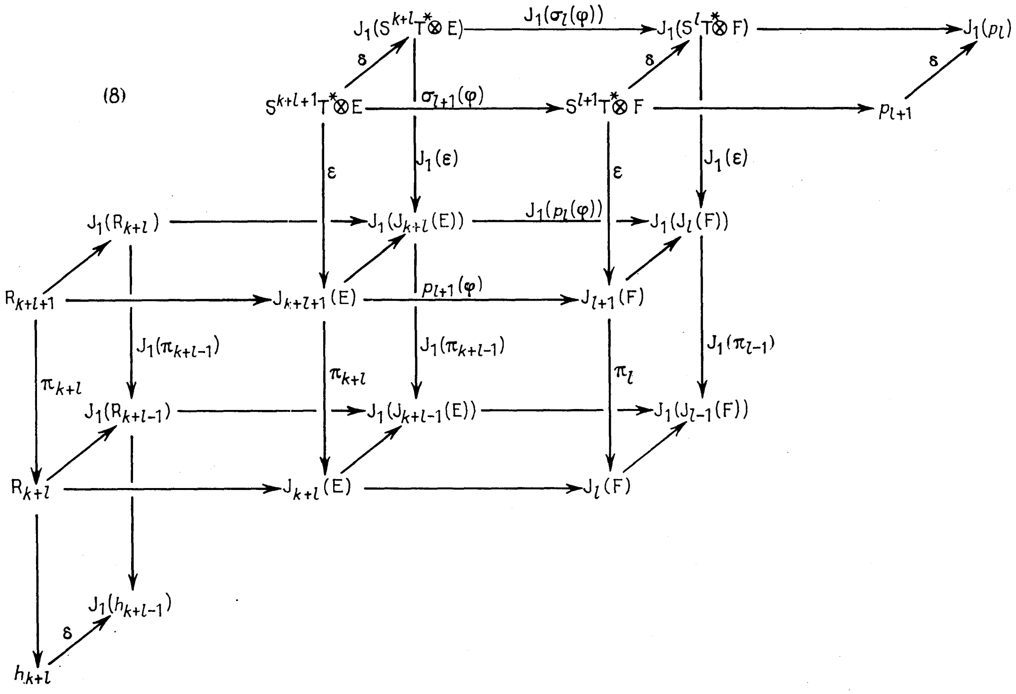
$$\begin{array}{ccc} h_{k+l} & \xrightarrow{\delta} & J_1(h_{k+l-1}) \\ \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle J_1(\iota)} \\ p_{l+1} & \xrightarrow{\delta} & J_1(p_l) \end{array}$$

commutes. Since $\delta : p_{l+1} \to T^*\otimes p_l$ is the composition of $\delta : p_{l+1} \to T^*\otimes p_l$ and of the monomorphism $\varepsilon : T^*\otimes p_l \to J_1(p_l)$, and since the diagram

$$\begin{array}{ccc} T^*\otimes h_{k+l-1} & \xrightarrow{\varepsilon} & J_1(h_{k+l-1}) \\ \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle J_1(\iota)} \\ T^*\otimes p_{l-1} & \xrightarrow{\varepsilon} & J_1(p_{l-1}) \end{array}$$

commutes, it follows that $\delta(h_{k+l}) \subset \varepsilon(T^*\otimes h_{k+l-1})$ and hence that $\delta : h_{k+l} \to J_1(h_{k+l-1})$ is the composition of the map $\delta : h_{k+l} \to T^*\otimes h_{k+l-1}$ defined above and of the monomorphism $\varepsilon : T^*\otimes h_{k+l-1} \to J_1(h_{k+l-1})$.

(8)




*Remark.* — Let $\tilde{F} = J_k(E)/R_k$ and $\tilde{\varphi} : J_k(E) \to \tilde{F}$ be the canonical projection. Then clearly $\tilde{F}$ is a sub-bundle of F and $R_{k+l} = \ker p_l(\tilde{\varphi})$. Therefore, if $\varphi$ is regular, the families of vector spaces $R_{k+l}$, $g_{k+l}$, $h_{k+l}$ and the maps $\partial : g_{k+l} \to T^* \otimes g_{k+l-1}$ and $\partial : h_{k+l} \to T^* \otimes h_{k+l-1}$ depend only on the sub-bundle $R_k$ of $J_k(E)$.

2. SYMBOL AND COSYMBOL COHOMOLOGIES. — We consider a family of graded vector spaces $r = \bigoplus_{l \in \mathbf{Z}} r_l$ over X, where each $r_l$ is a family of finite dimensional vector spaces, and a linear map $\partial : r \to T^* \otimes r$ of degree $-1$. We extend $\partial$ to a linear map

$$\partial : \quad \Lambda^j T^* \otimes r \to \Lambda^{j+1} T^* \otimes r$$

of degree $-1$ by setting

$$\partial(\omega \otimes u) = (-1)^j \omega \wedge \partial u \qquad \text{if} \quad \omega \in \Lambda^j T^*, \quad u \in r.$$

We let $M_l$ denote the family of vector spaces dual to the family $r_l$ and we write $M = \bigoplus_{l \in \mathbf{Z}} M_l$. We obtain a dual map

$$\partial^* : \quad T \otimes M \to M$$

of degree $1$. We write $\partial^*(t \otimes m) = t.m$, if $t \in T$, $m \in M$. The following lemma is an easy consequence of the definitions :

LEMMA 1 (*see* D. G. Quillen [14]). — *The dual of the map*

$$\partial : \quad \Lambda^j T^* \otimes r \to \Lambda^{j+1} T^* \otimes r$$

*is the map*

$$\partial^* : \quad \Lambda^{j+1} T \otimes M \to \Lambda^j T \otimes M$$

*defined by*

$$\partial^*((t_1 \wedge \ldots \wedge t_{j+1}) \otimes m) = \sum_{i=1}^{j+1} (-1)^{i+1} (t_1 \wedge \ldots \wedge \hat{t}_i \wedge \ldots \wedge t_{j+1}) \otimes t_i.m,$$

*if* $t_1, \ldots, t_{j+1} \in T$, $m \in M$. *Moreover, the sequence*

$$0 \to r \xrightarrow{\partial} T^* \otimes r \xrightarrow{\partial} \Lambda^2 T^* \otimes r \xrightarrow{\partial} \ldots \to \Lambda^n T^* \otimes r \to 0$$

*is a complex if and only if* $\partial^* : T \otimes M \to M$ *induces on* M *the structure of an* ST-*module. If one of these last two conditions holds, then*

$$0 \to \Lambda^n T \otimes M \to \ldots \xrightarrow{\partial^*} \Lambda^2 T \otimes M \xrightarrow{\partial^*} T \otimes M \xrightarrow{\partial^*} M \to 0 \tag{9}$$

*is the Koszul complex of the* ST-*module* M; *we denote by* $H_j(M)$ *the homology of the sequence* (9) *at* $\Lambda^j T^* \otimes M$. *The complex* (9) *is the direct sum of the complexes*

$$0 \to \Lambda^n T^* \otimes M_{l-n} \to \ldots \xrightarrow{\partial^*} \Lambda^2 T^* \otimes M_{l-2} \xrightarrow{\partial^*} T^* \otimes M_{l-1} \to M_l \to 0$$

*whose homology at* $\Lambda^j T^* \otimes M_{l-j}$ *we denote by* $H_j(M)_{l-j}$. *Then* $H_j(M) = \bigoplus_{l \in \mathbf{Z}} H_j(M)_l$.

Let $r_l = S^l T^* \otimes E$ and let $\delta$ be induced by the unique derivation $\delta : ST^* \to T^* \otimes ST^*$ of degree $-1$ extending the identity map of $T^*$, that is

$$\delta(\xi_1 \dots \xi_l) = \sum_{i=1}^{l} \xi_i \otimes (\xi_1 \dots \hat{\xi}_i \dots \xi_l),$$

if $\xi_i \in T^*$, $1 \leq i \leq l$. One easily verifies that the sequences (9) are complexes and that $\delta : S^l T^* \otimes E \to T^* \otimes S^{l-1} T^* \otimes E$ is the map defined in paragraph 1. In fact, the ST-module structure of M given by Lemma 1 is precisely the same as the ST-module structure of $ST \otimes E^*$ under the identification of $(ST^*)^*$ with ST described in [5]. Moreover, the sequence

$$(10) \qquad 0 \to S^m T^* \otimes E \xrightarrow{\delta} T^* \otimes S^{m-1} T^* \otimes E \xrightarrow{\delta} \dots \to \Lambda^n T^* \otimes S^{m-n} T^* \otimes E \to 0$$

is exact for $m \geq 1$.

Let $\varphi : J_k(E) \to F$ be a regular differential operator from E to F. Let $r_l$ be one of the following families of vector spaces $g_l$, $p_l$, $q_l$, $h_{k+l-1}$ over X, and let $\delta : r_l \to T^* \otimes r_{l-1}$ be the corresponding map defined in paragraph 1. We obtain the following sequences :

$$(11) \qquad 0 \to g_{k+l} \xrightarrow{\delta} T^* \otimes g_{k+l-1} \xrightarrow{\delta} \Lambda^2 T^* \otimes g_{k+l-2} \xrightarrow{\delta} \dots \to \Lambda^n T^* \otimes g_{k+l-n} \to 0;$$

$$(12) \qquad 0 \longrightarrow p_l \xrightarrow{\delta} T^* \otimes p_{l-1} \xrightarrow{\delta} \Lambda^2 T^* \otimes p_{l-2} \xrightarrow{\delta} \dots \longrightarrow \Lambda^n T^* \otimes p_{l-n} \longrightarrow 0;$$

$$(13) \qquad 0 \longrightarrow q_l \xrightarrow{\delta} T^* \otimes q_{l-1} \xrightarrow{\delta} \Lambda^2 T^* \otimes q_{l-2} \xrightarrow{\delta} \dots \longrightarrow \Lambda^n T^* \otimes q_{l-n} \longrightarrow 0;$$

$$(14) \qquad 0 \to h_{k+l} \xrightarrow{\delta} T^* \otimes h_{k+l-1} \xrightarrow{\delta} \Lambda^2 T^* \otimes h_{k+l-2} \xrightarrow{\delta} \dots \to \Lambda^n T^* \otimes h_{k+l-n} \to 0.$$

Since $\delta : g_{k+l} \to T^* \otimes g_{k+l-1}$, $\delta : p_l \to T^* \otimes p_{l-1}$ are induced by the maps $\delta : S^{k+l} T^* \otimes E \to T^* \otimes S^{k+l-1} T^* \otimes E$, $\delta : S^l T^* \otimes F \to T^* \otimes S^{l-1} T^* \otimes F$ respectively, it is clear that (11) and (12) are complexes. The commutativity of diagram (6) implies that (13) and (14) are also complexes.

Definition 3. — The symbol [resp. cosymbol] cohomology of $\varphi$ is the cohomology of the sequences (11) [resp. (14)]. We denote by $H^{k+l-j,j}(g_k)$ [resp. $H^{k+l-j,j}(h)$] the cohomology of the sequence (11) [resp. (14)] at $\Lambda^j T^* \otimes g_{k+l-j}$ [resp. $\Lambda^j T^* \otimes h_{k+l-j}$]. We say that $g_m$ is involutive, with $m \geq k$, if $H^{m+l,j}(g_k) = 0$ for $l \geq 0$, $j \geq 0$.

We recall that $g_{k+l}$ depends only on the family of subspaces $g_k$ of $S^k T^* \otimes E$ and that the sequences

$$0 \to g_{k+l+1} \xrightarrow{\delta} T^* \otimes g_{k+l} \xrightarrow{\delta} \Lambda^2 T^* \otimes g_{k+l-1}$$

are exact for $l \geq 0$ (*see* [5], [6]).

We now state the $\delta$-Poincaré lemmas :

Lemma 2. — *There exists an integer $k_0 \geq k$ depending only on $n$, $k$ and* $\dim E$ *such that $H^{k_0+m,j}(g_k) = 0$, for all $m \geq 0$, $j \geq 0$.*

LEMMA 3. — *There exists an integer* $k_1 \geqq k$ *depending only on* $n$, $k$, $\dim E$ *and* $\dim R_{k+l}$ $(l \geqq 0)$, *such that* $H^{k_1+m, j}(h) = 0$, *for all* $m \geqq 0$, $j \geqq 0$.

The remainder of this section is devoted to the proof of these lemmas.

Let $A = \bigoplus_{k \in \mathbf{Z}} A_k$ be the graded ring $\mathbf{R}[X_1, \ldots, X_n]$, where $A_k$ is the subspace of A consisting of all homogeneous polynomials of degree $k$; we write $A_k = 0$ for $k < 0$. We denote by $A(p)$ the graded A-module

$$A(p) = \bigoplus_{k \in \mathbf{Z}} A(p)_k,$$

where $A(p)_k = A_{p+k}$.

If $M = \bigoplus_{l \in \mathbf{Z}} M_l$ is a graded A-module of finite type, we denote by $H_j(M) = \bigoplus_{l \in \mathbf{Z}} H_j(M)_l$ the $j$-th Koszul homology group of M.

Following Grothendieck [7], we make the next :

DEFINITION 4. — A family of graded A-modules $M_\alpha$, $\alpha \in I$, of finite type is said to be bounded if :

(i) there exist integers $p$, $q$ such that the graded A-module

$$M'_\alpha = \bigoplus_{l \geqq 0} (M'_\alpha)_l, \qquad \text{where} \quad (M'_\alpha)_l = (M_\alpha)_{p+l},$$

is a quotient of $A^q$, for all $\alpha \in I$;

(ii) a finite number of polynomials occur as Hilbert polynomials of the A-modules $M_\alpha$.

PROPOSITION 1 (*see* D. Mumford [13], Lecture 14). — *Let* $M_\alpha$, $\alpha \in I$, *be a bounded family of graded* A-*modules of finite type satisfying condition* (i) *of Definition* 4. *Then there exists an integer* $n_0$, *depending only on* $p$, $q$ *and the Hilbert polynomials of the* A-*modules* $M_\alpha$, $\alpha \in I$, *such that* $H_j(M_\alpha)_l = 0$ *for all* $l \geqq n_0$, $j \geqq 0$.

PROPOSITION 2 (*see* A. Grothendieck [7]). — *Let* $k$, $p$, $q \geqq 0$ *be given integers. The family of all kernels and cokernels of all homomorphisms from* $A^q$ *to* $A(k)^p$ *of graded* A-*modules of degree* 0 *is bounded.*

Let $M = \bigoplus_{m \in \mathbf{Z}} g^*_m$, $P = \bigoplus_{l \in \mathbf{Z}} p^*_l$, $N = \bigoplus_{l \in \mathbf{Z}} h^*_{k+l-1}$; by Lemma 1, these are ST-modules. By the commutativity of diagram (5), we have the exact sequence of graded ST-modules

$$0 \to P \to ST \otimes F^* \xrightarrow{\sigma^*(\varphi)} ST \otimes E^* \to M \to 0,$$

where $\sigma^*(\varphi)$ is the direct sum of the maps $\sigma_l(\varphi)^*$ and is an ST-homomorphism of degree $k$. The commutativity of diagram (6) implies that there is an epimorphism of graded ST-modules from P to N of degree 0.

*Proof of Lemma* 2. — Let $p = \dim E$, $q = \dim F$; then by Proposition 2, $\{M_x\}_{x \in X}$, $\{P_x\}_{x \in X}$ are bounded families of graded A-modules. We deduce, from Proposition 1, that exists an integer $k_0 \geqq k$ depending only on $n$, $k$, $\dim E$ and $\dim F$ such that $H_j(M_x)_{k_0+l} = 0$, for all $x \in X$, $l \geqq 0$, $j \geqq 0$. Since M does not depend on F and since $(H_j(M)_{k_0+l})^*$ is isomorphic to $H^{k_0+l, j}(g_k)$, we obtain the desired result.

*Proof of Lemma* 3. — By Proposition 2, a finite number of polynomials $\Phi_1, \ldots, \Phi_s$ occur as Hilbert polynomials of the graded A-modules $M_x$, $x \in X$, and moreover these polynomials depend only on $n$, $k$ and $\dim E$. Hence if $x \in X$, there exists an integer $i$, with $1 \leqq i \leqq s$, such that $\dim (g_{k+l})_x = \Phi_i(k+l)$, for all sufficiently large $l$. Since $R_{k+l}$ is a vector bundle, for $l \geqq 0$, only a finite number of polynomials can occur as Hilbert polynomials of the graded A-modules $N_x$, $x \in X$; moreover these polynomials depend only on $n$, $k$, $\dim E$, and $\dim R_{k+l}$ $(l \geqq 0)$. Now $\{P_x\}_{x \in X}$ is a bounded family of graded A-modules and there exist integers $p$, $q$ depending only on $n$, $k$, $\dim E$ and $\dim F$ such that condition (i) of Definition 4 holds for the family $\{P_x\}_{x \in X}$. Because $N_x$ is a quotient of $P_x$, it is finitely generated and condition (i) of Definition 4 holds for the family $\{N_x\}_{x \in X}$ with the same integers $p$, $q$. Hence $\{N_x\}_{x \in X}$ is a bounded family of graded A-modules. By Proposition 1, there exists an integer $k_1 \geqq k$, depending only on $n$, $k$, $\dim E$, $\dim F$ and $\dim R_{k+l}$ $(l \geqq 0)$, such that $H_j(N_x)_{k_1+l} = 0$ for all $x \in X$, $l \geqq 0$, $j \geqq 0$. Since $N_x$ is independent of F, we obtain the desired result.

*Remark*. — In fact $H^{k_0+l, 0}(h) = 0$ for all $l \geqq 1$, where $k_0$ is the integer given by Lemma 2 depending only on $n$, $k$ and $\dim E$. By the commutativity and exactness of diagram (6), the map $\delta : h_{k+l} \to T^* \otimes h_{k+l-1}$ is injective if $\delta : p_{l+1} \to T^* \otimes p_l$ is injective. From the exactness of (10) and the commutativity and exactness of diagram (5), we deduce that $\delta : p_{l+1} \to T^* \otimes p_l$ is injective if and only if $H^{k+l-1, 2}(g_k) = 0$, for $l \geqq 1$. Hence $H^{k_0+l, 0}(h) = 0$ if $H^{k_0+l-1, 2}(g_k) = 0$. The proof of the prolongation theorem (Theorem 1) uses only this result and not the full statement of Lemma 3.

3. The prolongation theorem. — We define the set of formal sections of the vector bundle E to be

$$J_\infty(E) = \varprojlim J_k(E).$$

Let $\varphi : J_k(E) \to F$ be a differential operator of order $k$ from E to F. We let $p_\infty(\varphi) : J_\infty(E) \to J_\infty(F)$ be the map

$$p_\infty(\varphi) = \varprojlim p_{k+l}(\varphi).$$

A formal solution of $\varphi$ is an element $u \in J_\infty(E)$ satisfying $p_\infty(\varphi)(u) = 0$. The set of all formal solutions of $\varphi$ is $R_\infty = \varprojlim R_{k+l}$. We denote by $\overline{R}_m$ the projection $\pi_m(R_\infty)$ of $R_\infty$ in $R_m$.

We denote by $R_m^{(l)}$ the family of subspaces $\pi_m R_{m+l}$ of $J_m(E)$. If $R_m^{(l)}$ is a sub-bundle of $J_m(E)$, then it has the same solutions as $\varphi$. We have a descending chain of families of subspaces of $J_m(E)$

$$\ldots \subset R_m^{(l+1)} \subset R_m^{(l)} \subset \ldots \subset R_m^{(0)} \subset J_m(E), \tag{15}$$

where $R_m^{(0)} = R_m$. We set $\tilde{R}_m = \bigcap_{l \geq 0} R_m^{(l)}$; it is clear that $\overline{R}_m \subset \tilde{R}_m$. Since (15) is a chain of families of finite dimensional vector spaces, for each point $x \in X$, and each $m$, there exists an integer $l$, depending on $x$ and $m$ such that

$$\tilde{R}_{m,x} = R_{m,x}^{(l)}.$$

Hence we can find an integer $p$ such that

$$\tilde{R}_{m,x} = \pi_m(R_{m+p})_x, \qquad \tilde{R}_{m+1,x} = \pi_{m+1}(R_{m+p})_x.$$

It follows that the map $\pi_m : J_{m+1}(E) \to J_m(E)$ induces a surjective map $\pi_m : \tilde{R}_{m+1} \to \tilde{R}_m$. Hence $\tilde{R}_m \subset \overline{R}_m$, which implies that

$$\overline{R}_m = \bigcap_{l \geq 0} R_m^{(l)}$$

and $R_\infty = \varprojlim \overline{R}_m$.

Definition 5. — A differential operator $\varphi : J_k(E) \to F$ is said to be *formally integrable* if $\varphi$ is regular and $R_m = \overline{R}_m$, for $m \geq k$.

The second condition is equivalent to the fact that $\pi_{k+l} : R_{k+l+1} \to R_{k+l}$ is an epimorphism, for $l \geq 0$.

Following Cartan, we make the next :

Definition 6. — A differential operator $\varphi : J_k(E) \to F$ is said to be *involutive* if $R_{k+1}$ is a vector bundle, if the map $\pi_k : R_{k+1} \to R_k$ is surjective, and if $g_k$ is involutive.

The Cartan-Kähler theorem implies that every involutive differential operator is formally integrable (*see* [5], [6]).

If $R_k$ is a sub-bundle of $J_k(E)$ and if $\varphi : J_k(E) \to F$ is any morphism such that $\ker \varphi = R_k$, we recall that $R_{k+l} = \ker p_l(\varphi)$, for $l \geq 0$, is independent of $\varphi$ and is equal to the $l$-th prolongation $J_l(R_k) \cap J_{k+l}(E)$ of $R_k$. Moreover, if the $l$-th prolongation $R_{k+l}$ of $R_k$ is a sub-bundle of $J_{k+l}(E)$, the $m$-th prolongation $R_{(k+l)+m}$ of $R_{k+l}$ is the same as the $(l+m)$-th prolongation $R_{k+(l+m)}$ of $R_k$ (*see* [5]). We say that the equation $R_k$ is regular [resp. formally integrable, involutive] is any such morphism $\varphi$ is regular [resp. formally integrable, involutive]. We recall that, if $\varphi$ is regular, the families of vector spaces $g_{k+l}$, $h_{k+l}$, the maps $\delta : g_{k+l} \to T^* \otimes g_{k+l-1}$, $\delta : h_{k+l} \to T^* \otimes h_{k+l-1}$ are independent of the choice of $\varphi$.

The remainder of this section is devoted to the proof of the following :

THEOREM 1 (PROLONGATION THEOREM). — *Let* $R_k \subset J_k(E)$ *be a regular partial differential equation of order* $k$ *on* E. *Assume that the maps* $\pi_m : R_{m+r} \to R_m$ *have constant rank, for all* $m \geqq k$, $r \geqq 0$. *Then there exist integers* $l_0 \geqq 0$, $m_0 \geqq k$ *such that the equation* $R_{m_0}^{(l_0)}$ *of order* $m_0$ *on* E *is a formally integrable involutive equation, which has the same formal solutions as* $R_k$, *and whose r-th prolongation is* $R_{m_0+r}^{(l_0)}$.

*Proof.* — The hypotheses imply that $R_m^{(l)}$ is a sub-bundle of $J_m(E)$, for all $m \geqq k$, $l \geqq 0$. For each $m \geqq k$, (15) is a descending chain of sub-bundles of $J_m(E)$. This chain must obviously stabilize; hence for each $m \geqq k$, there exists an integer $r_m$, depending only on $m$, such that

$$R_m^{(r_m)} = \bigcap_{l \geqq 0} R_m^{(l)} = \bar{R}_m.$$

We denote by $(R_m^{(l)})_{+r}$ the $r$-th prolongation of the equation $R_m^{(l)} \subset J_m(E)$ and first prove the following :

LEMMA 4. — $R_{m+r}^{(l)} \subset (R_m^{(l)})_{+r}$ *for all* $l$, $r \geqq 0$, $m \geqq k$.

*Proof.* — We have

$$\begin{aligned} R_{m+r}^{(l)} &= \pi_{m+r} R_{m+l+r} = \pi_{m+r} R_{(m+l)+r} \\ &= \pi_{m+r}(J_r(R_{m+l}) \cap J_{m+l+r}(E)) \subset J_r(\pi_m)\, J_r(R_{m+l}) \cap J_{m+r}(E) \end{aligned}$$

since the diagram

$$\begin{array}{ccc} J_{m+l+r}(E) & \xrightarrow{p_r(id_{m+l})} & J_r(J_{m+l}(E)) \\ \big\downarrow{\scriptstyle \pi_{m+r}} & & \big\downarrow{\scriptstyle J_r(\pi_m)} \\ J_{m+r}(E) & \xrightarrow{p_r(id_m)} & J_r(J_m(E)) \end{array}$$

commutes. Because the map $\pi_m : R_{m+l} \to R_m$ has constant rank,

$$J_r(\pi_m)\, J_r(R_{m+l}) = J_r(\pi_m R_{m+l}) = J_r(R_m^{(l)});$$

we therefore obtain the desired inclusion

$$R_{m+r}^{(l)} \subset J_r(R_m^{(l)}) \cap J_{m+r}(E) = (R_m^{(l)})_{+r}.$$

To prove the theorem, it suffices to show the following :

(I) There exists an integer $l_0$, independent of $m$, such that

$$\bar{R}_m = R_m^{(l)} \quad \text{for all } m \geqq k,\ l \geqq l_0.$$

(II) For each $l \geq 0$, there exists an integer $p_l$ such that

$$R^{(l)}_{p_l+r} = (R^{(l)}_{p_l})_{+r} \quad \text{for all } r \geq 0.$$

Indeed, let $m_0 \geq p_{l_0}$, where $l_0$ is the integer given by (I). Then, for $r \geq 0$,

$$(16) \qquad (R^{(l_0)}_{m_0})_{+r} = R^{(l_0)}_{m_0+r} = \bar{R}_{m_0+r}.$$

Since $R^{(l_0)}_{m_0+r}$ is a sub-bundle of $J_{m_0+r}(E)$, for $r \geq 0$, it follows that $R^{(l_0)}_{m_0}$ is formally integrable and has the same formal solutions as $R_k$. Because of (16), Lemma 2 shows that we can choose $m_0 \geq p_{l_0}$ such that $g^{(l_0)}_{m_0}$ is involutive and such that $R^{(l_0)}_{m_0}$ satisfies the desired properties.

For $m \geq k$, the map $\pi_m : J_{m+1}(E) \to J_m(E)$ induces a map $\pi_m : R^{(l)}_{m+1} \to R^{(l)}_m$, whose kernel and cokernel we denote by $g^{(l)}_{m+1}$, $h^{(l)}_m$ respectively. Then $g^{(0)}_m = g_m$, $h^{(0)}_m = h_m$. We have exact sequences

$$(17) \qquad 0 \to g^{(l)}_{m+1} \xrightarrow{\varepsilon} R^{(l)}_{m+1} \xrightarrow{\pi_m} R^{(l)}_m \to h^{(l)}_m \to 0;$$

$$(18) \qquad 0 \to R^{(l+1)}_m \to R^{(l)}_m \to h^{(l)}_m \to 0$$

since $\pi_m(R^{(l)}_{m+1}) = R^{(l+1)}_m$. It follows that $h^{(l)}_m$ and $g^{(l)}_{m+1}$ are both vector bundles, for $m \geq k$. We therefore have a descending chain

$$(19) \qquad \ldots \subset g^{(l+1)}_m \subset g^{(l)}_m \subset \ldots \subset g^{(0)}_m \subset S^m T^* \otimes E$$

of sub-bundles of $g^{(-1)}_m = S^m T^* \otimes E$, for $m \geq k+1$. Set $g^{(l)}_m = S^m T^* \otimes E$, for $m < k$.

*Proof of* (I). — The image of $g^{(l)}_{m+1}$ under the map

$$\delta : \quad S^{m+1} T^* \otimes E \to T^* \otimes S^m T^* \otimes E$$

is contained in $T^* \otimes g^{(l)}_m$ and the diagram

$$(20) \qquad \begin{tikzcd} g^{(l)}_{m+1} \arrow[r, "\delta"] \arrow[d] & T^* \otimes g^{(l)}_m \arrow[d] \\ g^{(l-1)}_{m+1} \arrow[r, "\delta"] & T^* \otimes g^{(l-1)}_m \end{tikzcd}$$

commutes. Indeed, the map $\delta$ is induced by $p_1(id_m) : J_{m+1}(E) \to J_1(J_m(E))$ and, for $m \geq k$, $p_1(id_m)$ maps $R^{(l)}_{m+1}$ into $J_1(R^{(l)}_m)$ by Lemma 4. Hence

$$\delta(g^{(l)}_{m+1}) \subset (T^* \otimes R^{(l)}_m) \cap (T^* \otimes S^m T^* \otimes E) = T^* \otimes g^{(l)}_m,$$

and it is clear that (20) is commutative, for all $m \in \mathbf{Z}$.

Let $M^{(l)} = \bigoplus_{m \in \mathbf{Z}} g^{(l)*}_m$; according to Lemma 1, $M^{(l)}$ is an ST-module and $M^{(l+1)}$ is a quotient of $M^{(l)}$ as graded ST-modules. Let $K^{(l)}$ be the kernel of the natural projection of $M^{(-1)} = ST \otimes E^*$ onto $M^{(l)}$. We obtain an ascending chain

$$0 \subset K^{(0)} \subset \ldots \subset K^{(l)} \subset K^{(l+1)} \subset \ldots \subset M^{(-1)}$$

of ST-submodules of $M^{(-1)}$. Choose an arbitrary point $x \in X$; then $ST_x \otimes E_x^*$ is a noetherian module and therefore there exists an integer $l_0(x)$ such that

$$K_x^{(l)} = K_x^{(l_0(x))} \qquad \text{for} \quad l \geqq l_0(x).$$

This implies that $M_x^{(l)} = M_x^{(l_0(x))}$, for $l \geqq l_0(x)$ and hence that $g_{m,x}^{(l)} = g_{m,x}^{(l_0(x))}$ for all $m \geqq 0$, $l \geqq l_0(x)$. Since $g_m^{(l)}$ is a vector bundle for $m \geqq k+1$, the ehain (19) stabilizes and

$$g_m^{(l)} = g_m^{(l_0(x))} \quad \text{for all } m \geqq k+1, \quad l \geqq l_0(x).$$

Let $r$ be an integer such that

$$\overline{R}_k = R_k^{(r)}$$

and let $l_0 = \max(r, l_0(x))$. We claim that

$$\text{(21)} \qquad R_m^{(l)} = R_m^{(l_0)} \qquad \text{for} \quad l \geqq l_0, \quad m \geqq k.$$

This statement clearly implies (I). We prove (21) by induction on $m$. The integer $l_0$ was chosen so that (21) holds for $m = k$. Assume (21) is true for $m$, with $m \geqq k$. For $l \geqq l_0$, the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & g_{m+1}^{(l)} \arrow[r] \arrow[d, "\varepsilon"] & g_{m+1}^{(l_0)} \arrow[r] \arrow[d, "\varepsilon"] & 0 \\
0 \arrow[r] & R_{m+1}^{(l)} \arrow[r] \arrow[d, "\pi_m"] & R_{m+1}^{(l_0)} \arrow[d, "\pi_m"] & \\
0 \arrow[r] & R_m^{(l)} \arrow[r] \arrow[d] & R_m^{(l_0)} \arrow[r] & 0 \\
 & 0 & &
\end{tikzcd}$$

is clearly commutative and exact, since $\pi_m(R_{m+1}^{(l)}) = R_m^{(l+1)} = R_m^{(l_0)}$, by our induction hypothesis. This implies that $\pi_m : R_{m+1}^{(l_0)} \to R_m^{(l_0)}$ is surjective. The diagram shows that $R_{m+1}^{(l)} = R_{m+1}^{(l_0)}$ for $l \geqq l_0$.

*Proof of* (II). — To prove (II), it is enough to show that for each $l \geqq 0$, there exists an integer $p_l$ such that

$$\text{(22)} \qquad R_{m+1}^{(l)} = (R_m^{(l)})_{+1} \quad \text{for all } m \geqq p_l.$$

In fact, this condition implies (II) by induction on $r$. Clearly, $R_{p_l+1}^{(l)} = (R_{p_l}^{(l)})_{+1}$. Assume that $R_{p_l+r}^{(l)} = (R_{p_l}^{(l)})_{+r}$; then since $R_{p_l}^{(l)}$ is a vector bundle

$$R_{p_l+(r+1)}^{(l)} = R_{(p_l+r)+1}^{(l)} = (R_{p_l+r}^{(l)})_{+1} = ((R_{p_l}^{(l)})_{+r})_{+1} = (R_{p_l}^{(l)})_{+(r+1)}.$$

We shall show the existence of $p_l$, by induction on $l$.

By Lemma 4, $p_1(id_m) : J_{m+1}(E) \to J_1(J_m(E))$ induces maps

$$p_1(id_m) : \quad R_{m+1}^{(l+1)} \to J_1(R_m^{(l+1)}), \qquad p_1(id_m) : \quad R_{m+1}^{(l)} \to J_1(R_m^{(l)}).$$

The diagram

$$(23) \qquad \begin{tikzcd} 0 \arrow[r] & R_{m+1}^{(l+1)} \arrow[r] \arrow[d, "p_1(id_m)"] & R_{m+1}^{(l)} \arrow[r] \arrow[d, "p_1(id_m)"] & h_{m+1}^{(l)} \arrow[r] \arrow[d, dashed, "\delta"] & 0 \\ 0 \arrow[r] & J_1(R_m^{(l+1)}) \arrow[r] & J_1(R_m^{(l)}) \arrow[r] & J_1(h_m^{(l)}) \arrow[r] & 0 \end{tikzcd}$$

is clearly exact and commutative, for $m \geq k$, and so induces a map $\delta : h_{m+1}^{(l)} \to J_1(h_m^{(l)})$.

LEMMA 5. — *If* $R_{m+1}^{(l)} = (R_m^{(l)})_{+1}$, *for some* $m \geq k$, *then the following statements are equivalent :*

(i) $R_{m+1}^{(l+1)} = (R_m^{(l+1)})_{+1}$;

(ii) $\delta : h_{m+1}^{(l)} \to J_1(h_m^{(l)})$ *is injective.*

*Proof.* — We have

$$\begin{aligned}(R_m^{(l+1)})_{+1} &= J_1(R_m^{(l+1)}) \cap J_{m+1}(E) = J_1(R_m^{(l+1)}) \cap J_1(R_m^{(l)}) \cap J_{m+1}(E) \\ &= J_1(R_m^{(l+1)}) \cap (R_m^{(l)})_{+1} \quad = J_1(R_m^{(l+1)}) \cap R_{m+1}^{(l)}.\end{aligned}$$

From diagram (23) and this equality, we deduce the lemma.

We apply Lemma 5, with $l = 0$. It is clear that the map $\delta : h_{m+1} \to J_1(h_m)$ is precisely the map $\delta$ considered in paragraph 1. Hence Lemma 3 implies the existence of an integer $p_0$ such that the map $\delta : h_{m+1} \to J_1(h_m)$ is injective for all $m \geq p_0$, since this map is the composition of $\delta : h_{m+1} \to T^* \otimes h_m$ and the monomorphism $\varepsilon : T^* \otimes h_m \to J_1(h_m)$. The hypothesis of Lemma 5 is clearly satisfied for all $m \geq k$, and so Lemma 5 shows that $R_{m+1}^{(1)} = (R_m^{(1)})_{+1}$, for all $m \geq p_0$.

Assume that we have shown the existence of an integer $p_l$ such that (22) holds for all $m \geq p_l$. Consider the equation $R_{p_l}^{(l)} \subset J_{p_l}(E)$. Our induction hypothesis implies that $(R_{p_l}^{(l)})_{+r} = R_{p_l+r}^{(l)}$; since (17) is exact and $h_m^{(l)}$ is a vector bundle, for $m \geq k$, the above argument together with Lemma 5 shows the existence of an integer $p_{l+1}$ such that

$$(R_m^{(l+1)})_{+1} = R_{m+1}^{(l+1)} \quad \text{for all } m \geq p_{l+1},$$

completing the proof of (II).

*Remark.* — In our proof of Theorem 1, we have preferred to prove (II) rather than to show that, if $l_0$ denotes the integer given by (I), there exists an integer $m'_0$ such that

$$R_{m+1}^{(l_0)} = (R_m^{(l_0)})_{+1} \quad \text{for all } m \geq m'_0.$$

This last fact together with (I) implies Theorem 1. We shall use (II) in certain applications of Theorem 1 rather this weaker statement which can be proved using an argument due to Kuranishi [8] as follows. By Lemma 4, we have

$$R^{(l_0)}_{m+1} \subset (R^{(l_0)}_m)_{+1} \quad \text{for all } m \geqq k.$$

By (I), the map $\pi_m : R^{(l_0)}_{m+1} \to R^{(l_0)}_m$ is surjective, for all $m \geqq k$. Let $(g^{(l_0)}_m)_{+1}$ denote the kernel of $\pi_m : (R^{(l_0)}_m)_{+1} \to R^{(l_0)}_m$. Since $g^{(l_0)}_m$ is a vector bundle for all $m \geqq k+1$ and since $M^{(l_0)}$ is a quotient of $ST \otimes E^*$, we see that $\{M^{(l_0)}_x\}_{x \in X}$ is a bounded family of graded A-modules; by Proposition 1 there exists an integer $m'_0 \geqq k$ such that the sequence

$$0 \to g^{(l_0)}_{m+1} \xrightarrow{\delta} T^* \otimes g^{(l_0)}_m \xrightarrow{\delta} \Lambda^2 T^* \otimes g^{(l_0)}_{m-1}$$

is exact, for all $m \geqq m'_0$. This implies that

$$g^{(l_0)}_{m+1} = (g^{(l_0)}_m)_{+1} \quad \text{for all } m \geqq m'_0 \quad (\textit{see } [5], [6]).$$

We recall that the map $\pi_m : R^{(l_0)}_{m+1} \to R^{(l_0)}_m$ is surjective, for $m \geqq k$. The exact and commutative diagram, for $m \geqq m'_0$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & g^{(l_0)}_{m+1} \arrow[r] \arrow[d, "\varepsilon"] & g^{(l_0)}_{m+1} \arrow[r] \arrow[d, "\varepsilon"] & 0 \\
0 \arrow[r] & R^{(l_0)}_{m+1} \arrow[r] \arrow[d, "\pi_m"] & (R^{(l_0)}_m)_{+1} \arrow[d, "\pi_m"] & \\
0 \arrow[r] & R^{(l_0)}_m \arrow[r] \arrow[d] & R^{(l_0)}_m \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 &
\end{tikzcd}$$

shows that the desired result holds. Matsuda [12], using Kuranishi's argument, noted that the first prolongation of $\overline{R}_m$ is $\overline{R}_{m+1}$ for all sufficiently large $m$.

## 4. The Spencer cohomology of a differential equation.

— Let

$$C^j_k(E) = \Lambda^j T^* \otimes J_k(E)/\delta(\Lambda^{j-1} T^* \otimes S^{k+1} T^* \otimes E) \qquad \text{for } j \geqq 1,$$

and

$$C^0_k(E) = J_k(E).$$

Applying Proposition 5.1 of [5] to the equation $R_k = J_k(E)$ on E, we obtain :

Proposition 3. — *There exists a unique differential operator*

$$\rho : \quad J_1(J_k(E)) \to C^1_k(E)$$

*of order* 1, *whose symbol is the natural projection* $\tau$ *of* $T^{*}\otimes J_k(E)$ *onto* $C_k^1(E)$ *such that the sequence*

$$0 \longrightarrow J_{k+1}(E) \xrightarrow{p_1(id_k)} J_1(J_k(E)) \xrightarrow{\rho} C_k^1(E) \longrightarrow 0$$

*is exact.*

Proposition 4. — *There is a unique differential operator*

$$\lambda : \quad J_1(J_k(E)) \to T^{*}\otimes J_{k-1}(E)$$

*such that :*

(i) $J_{k+1}(E) \subset \ker \lambda$;

(ii) *The symbol of* $\lambda$ *is the projection* $\pi_{k-1}$ *of* $T^{*}\otimes J_k(E)$ *onto* $T^{*}\otimes J_{k-1}(E)$.

*Proof.* — From Proposition 3, it follows that any such morphism $\lambda$ satisfying (i) is of the form $\bar{\lambda}\circ\rho$, where $\bar{\lambda}$ is a morphism of vector bundles from $C_k^1(E)$ to $T^{*}\otimes J_{k-1}(E)$ and its symbol is the composition $\bar{\lambda}\circ\tau$. The unique map $\bar{\lambda}$ satisfying $\bar{\lambda}\circ\tau = \pi_{k-1}$ is the natural projection of $C_k^1(E)$ onto $T^{*}\otimes J_{k-1}(E)$ induced by the projection $\pi_{k-1}$ of $J_k(E)$ onto $J_{k-1}(E)$. Clearly $\lambda = \bar{\lambda}\circ\rho$ has the desired properties.

Proposition 5. — *The morphism* $\lambda$ *of Proposition* 4 *is determined by*

$$\varepsilon\lambda = J_1(\pi_{k-1}) - p_1(id_{k-1})\circ\pi_0;$$

*moreover, if* $D = \lambda\circ j_1 : \mathcal{J}_k(E) \to \mathcal{T}^{*}\otimes\mathcal{J}_{k-1}(\mathcal{E})$, *the sequence*

$$(24) \qquad 0 \to \mathcal{E} \xrightarrow{j_k} \mathcal{J}_k(\mathcal{E}) \xrightarrow{D} \mathcal{T}^{*}\otimes\mathcal{J}_{k-1}(\mathcal{E})$$

*is exact.*

*Proof.* — It is easily seen that

$$\pi_0 . J_1(\pi_{k-1}) = \pi_0 . p_1(id_{k-1}) . \pi_0$$

as maps from $J_1(J_k(E))$ to $J_{k-1}(E)$ and that the diagram

$$\begin{array}{ccc} J_{k+1}(E) & \xrightarrow{p_1(id_k)} & J_1(J_k(E)) \\ \downarrow{\scriptstyle \pi_k} & & \downarrow{\scriptstyle J_1(\pi_{k-1})} \\ J_k(E) & \xrightarrow{p_1(id_{k-1})} & J_1(J_{k-1}(E)) \end{array}$$

commutes by Proposition 4.3 of [6]. Hence $\varepsilon^{-1}(J_1(\pi_{k-1}) - p_1(id_{k-1}).\pi_0)$ is a well-defined morphism from $J_1(J_k(E))$ to $T^{*}\otimes J_{k-1}(E)$ satisfying condition (i) of Proposition 4. Moreover, the symbol of $\varepsilon^{-1}(J_1(\pi_{k-1}) - p_1(id_{k-1}).\pi_0)$ is $\pi_{k-1}$ since it is determined by the symbol of $J_1(\pi_{k-1})$ which is precisely $\varepsilon\circ\pi_{k-1}$. The exactness of the sequence (24) follows from Lemmas 5.2 or 5.3 of [5].

Note that condition (ii) of Proposition 4 is equivalent to

$$D(fu) = df \otimes \pi_{k-1}(u) + f\,Du \quad \text{for all } f \in \Lambda^0 \mathcal{T}^*, \quad u \in \mathcal{J}_k(\mathcal{E}). \tag{25}$$

The following proposition is easily verified.

PROPOSITION 6. — *If* $\varphi : J_k(E) \to F$ *is a differential operator, the diagram*

$$\begin{tikzcd} J_1(J_{k+l}(E)) \arrow[r, "\lambda"] \arrow[d, "J_1(p_l(\varphi))"] & T^* \otimes J_{k+l-1}(E) \arrow[d, "p_{l-1}(\varphi)"] \\ J_1(J_l(F)) \arrow[r, "\lambda"] & T^* \otimes J_{l-1}(F) \end{tikzcd}$$

*commutes.*

Let us now compute the map

$$\lambda . J_1(\varepsilon) : \quad J_1(S^k T^* \otimes E) \to T^* \otimes J_{k-1}(E).$$

Since $\pi_{k-1} . \varepsilon = 0$,

$$\begin{aligned} \varepsilon . \lambda . J_1(\varepsilon) &= (J_1(\pi_{k-1}) - p_1(id_{k-1}) . \pi_0) . J_1(\varepsilon) \\ &= - p_1(id_{k-1}) . \pi_0 . J_1(\varepsilon) \\ &= - p_1(id_{k-1}) . \varepsilon . \pi_0. \end{aligned}$$

Hence the diagram

$$\begin{tikzcd} J_1(S^k T^* \otimes E) \arrow[r, "\pi_0"] \arrow[d, "J_1(\varepsilon)"] & S^k T^* \otimes E \arrow[r, "\delta"] & T^* \otimes S^{k-1} T^* \otimes E \arrow[d, "\varepsilon"] \\ J_1(J_k(E)) \arrow[rr, "\lambda"] & & T^* \otimes J_{k-1}(E) \end{tikzcd}$$

commutes; therefore so does the diagram

$$\begin{tikzcd} \mathcal{S}^k \mathcal{T}^* \otimes \mathcal{E} \arrow[r, "-\delta"] \arrow[d, "\varepsilon"] & \mathcal{T}^* \otimes \mathcal{S}^{k-1} \mathcal{T}^* \otimes \mathcal{E} \arrow[d, "\varepsilon"] \\ \mathcal{J}_k(\mathcal{E}) \arrow[r, "D"] & \mathcal{T}^* \otimes \mathcal{J}_{k-1}(\mathcal{E}) \end{tikzcd} \tag{26}$$

We extend D to a differential operator

$$\mathbb{D} : \quad \Lambda^j \mathcal{T}^* \otimes \mathcal{J}_k(\mathcal{E}) \to \Lambda^{j+1} \mathcal{T}^* \otimes \mathcal{J}_{k-1}(\mathcal{E})$$

by setting

$$D(\omega \otimes u) = d\omega \otimes \pi_{k-1}(u) + (-1)^j \omega \wedge Du, \tag{27}$$

if $\omega \in \Lambda^j \mathcal{T}^*$, $u \in \mathcal{J}_k(\mathcal{E})$. It is easily seen from (25) that D is well-defined; furthermore (27) and the commutativity of diagram (26) imply that the diagram

$$\begin{tikzcd} \Lambda^j \mathcal{T}^* \otimes \mathcal{S}^k \mathcal{T}^* \otimes \mathcal{E} \arrow[r, "-\delta"] \arrow[d, "\varepsilon"] & \Lambda^{j+1} \mathcal{T}^* \otimes \mathcal{S}^{k-1} \mathcal{T}^* \otimes \mathcal{E} \arrow[d, "\varepsilon"] \\ \Lambda^j \mathcal{T}^* \otimes \mathcal{J}_k(\mathcal{E}) \arrow[r, "D"] & \Lambda^{j+1} \mathcal{T}^* \otimes \mathcal{J}_{k-1}(\mathcal{E}) \end{tikzcd} \tag{28}$$

commutes.

We obtain the naive Spencer sequence for E

$$0 \to \mathcal{E} \xrightarrow{j_k} \mathcal{J}_k(\mathcal{E}) \xrightarrow{\mathrm{D}} \mathfrak{T}^* \otimes \mathcal{J}_{k-1}(\mathcal{E}) \xrightarrow{\mathrm{D}} \Lambda^2 \mathfrak{T}^* \otimes \mathcal{J}_{k-2}(\mathcal{E}) \xrightarrow{\mathrm{D}} \ldots \to \Lambda^n \mathfrak{T}^* \otimes \mathcal{J}_{k-n}(\mathcal{E}) \to 0$$

which is a complex (*see* R. Bott [1], D. G. Quillen [14], D. C. Spencer [15] or S. Sternberg [16]).

If $\varphi : \mathrm{J}_k(\mathrm{E}) \to \mathrm{F}$, $\psi : \mathrm{J}_l(\mathrm{F}) \to \mathrm{G}$ are differential operators, we say that the sequence

$$\mathcal{E} \xrightarrow{\mathrm{D}_0} \mathcal{F} \xrightarrow{\mathrm{D}_1} \mathcal{G}, \tag{29}$$

where $\mathrm{D}_0 = \varphi \circ j_k$, $\mathrm{D}_1 = \psi \circ j_l$, is formally exact if the sequence

$$\mathrm{J}_\infty(\mathrm{E}) \xrightarrow{p_\infty(\varphi)} \mathrm{J}_\infty(\mathrm{F}) \xrightarrow{p_\infty(\psi)} \mathrm{J}_\infty(\mathrm{G}) \tag{30}$$

is exact. We note that if $k = l = 0$, then if the sequence (29) is exact, so is the sequence (30) by Lemma 3.3 of [5].

Lemma 6. — *If* $\varphi : \mathrm{J}_k(\mathrm{E}) \to \mathrm{F}$, $\psi : \mathrm{J}_l(\mathrm{F}) \to \mathrm{G}$ *are differential operators, the sequence* (29) *is formally exact if the sequences of vector bundles*

$$\mathrm{J}_{k+l+m}(\mathrm{E}) \xrightarrow{p_{l+m}(\varphi)} \mathrm{J}_{l+m}(\mathrm{F}) \xrightarrow{p_m(\psi)} \mathrm{J}_m(\mathrm{G})$$

*are exact for* $m \geq 0$.

*Proof.* — Since finite dimensional vector spaces are artinian, the lemma is a direct consequence of Corollary 2, § 3, n° 5 of Bourbaki [2].

Proposition 7. — *The naive Spencer sequence for* E *is exact and formally exact, for* $k \geq 0$.

*Proof.* — For $k = 0$, the statement is trivial. We deduce the proposition from the exactness of (10) and from the commutative diagram (31).

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & \mathcal{S}^k \mathfrak{T}^* \otimes \mathcal{E} \arrow[r, "-\delta"] \arrow[d, "\varepsilon"] & \mathfrak{T}^* \otimes \mathcal{S}^{k-1} \mathfrak{T}^* \otimes \mathcal{E} \arrow[r, "-\delta"] \arrow[d, "\varepsilon"] & \ldots \arrow[r] & \Lambda^n \mathfrak{T}^* \otimes \mathcal{S}^{k-n} \mathfrak{T}^* \otimes \mathcal{E} \arrow[r] \arrow[d, "\varepsilon"] & 0 \\
0 \arrow[r] & \mathcal{E} \arrow[r, "j_k"] \arrow[d] & \mathcal{J}_k(\mathcal{E}) \arrow[r, "\mathrm{D}"] \arrow[d, "\pi_{k-1}"] & \mathfrak{T}^* \otimes \mathcal{J}_{k-1}(\mathcal{E}) \arrow[r, "\mathrm{D}"] \arrow[d, "\pi_{k-2}"] & \ldots \arrow[r] & \Lambda^n \mathfrak{T}^* \otimes \mathcal{J}_{k-n}(\mathcal{E}) \arrow[r] \arrow[d, "\pi_{k-n-1}"] & 0 \\
0 \arrow[r] & \mathcal{E} \arrow[r, "j_{k-1}"] \arrow[d] & \mathcal{J}_{k-1}(\mathcal{E}) \arrow[r, "\mathrm{D}"] \arrow[d] & \mathfrak{T}^* \otimes \mathcal{J}_{k-2}(\mathcal{E}) \arrow[r, "\mathrm{D}"] \arrow[d] & \ldots \arrow[r] & \Lambda^n \mathfrak{T}^* \otimes \mathcal{J}_{k-n-1}(\mathcal{E}) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & & 0 &
\end{tikzcd} \tag{31}$$

Now let $\mathrm{R}_k \subset \mathrm{J}_k(\mathrm{E})$ be a regular partial differential equation of order $k$ on E. By Proposition 6, the operator

$$\mathrm{D} : \quad \Lambda^j \mathfrak{T}^* \otimes \mathcal{J}_m(\mathcal{E}) \to \Lambda^{j+1} \mathfrak{T}^* \otimes \mathcal{J}_{m-1}(\mathcal{E})$$

induces a first order differential operator

$$D: \Lambda^j \mathcal{T}^* \otimes \mathcal{R}_m \to \Lambda^{j+1} \mathcal{T}^* \otimes \mathcal{R}_{m-1}.$$

We obtain a complex

$$(32) \qquad 0 \to \mathcal{S} \xrightarrow{j_m} \mathcal{R}_m \xrightarrow{D} \mathcal{T}^* \otimes \mathcal{R}_{m-1} \xrightarrow{D} \Lambda^2 \mathcal{T}^* \otimes \mathcal{R}_{m-2} \xrightarrow{D} \cdots \to \Lambda^n \mathcal{T}^* \otimes \mathcal{R}_{m-n} \to 0$$

which is always exact at $\mathcal{S}$ and at $\mathcal{R}_m$, which we call the $m$-th naive Spencer sequence of the equation $R_k$.

THEOREM 2. — *Let* $R_k \subset J_k(E)$ *be a regular partial differential equation of order* $k$ *on* E. *Assume that the maps* $\pi_m : R_{m+1} \to R_m$ *have constant rank, for all* $m \geqq k$. *Then there exists an integer* $m_1 \geqq k$ *such that the cohomology of the sequence* (32) *is independent of* $m$, *for* $m \geqq m_1$.

*We call a sequence* (32), *with* $m \geqq m_1$, *a stable naive Spencer sequence of* $R_k$ *and call its cohomology the* Spencer cohomology *of* $R_k$. *We say that* $R_k$ *has stable naive Spencer sequences.*

*Proof.* — Under our assumption, $h_m$ is a vector bundle for $m \geqq k$. We first show that the diagram

$$\begin{array}{ccc} J_1(R_m) & \xrightarrow{\lambda} & T^* \otimes R_{m-1} \\ \downarrow{\scriptstyle J_1(\varkappa)} & & \downarrow{\scriptstyle \varkappa} \\ J_1(h_m) & \xrightarrow{-\delta \circ \pi_0} & T^* \otimes h_{m-1} \end{array}$$

commutes, where $\varkappa : R_m \to h_m$ is the natural projection for $m \geqq k$. It suffices to verify that

$$\varepsilon \varkappa \lambda = J_1(\varkappa) . \varepsilon . \lambda = - \delta . \pi_0 . J_1(\varkappa)$$

as maps from $J_1(R_m)$ to $J_1(h_{m-1})$. We have

$$\begin{aligned} J_1(\varkappa) . \varepsilon . \lambda &= J_1(\varkappa) \, (J_1(\pi_{m-1}) - p_1(id_{m-1}) \pi_0) \\ &= J_1(\varkappa . \pi_{m-1}) - J_1(\varkappa) \, p_1(id_{m-1}) \pi_0 \\ &= - J_1(\varkappa) \, p_1(id_{m-1}) \pi_0 = - \delta . \pi_0 . J_1(\varkappa) \end{aligned}$$

by the commutativity of diagram (7). It follows from (27) that the diagram

$$\begin{array}{ccc} \Lambda^j \mathcal{T}^* \otimes \mathcal{R}_m & \xrightarrow{D} & \Lambda^{j+1} \mathcal{T}^* \otimes \mathcal{R}_{m-1} \\ \downarrow & & \downarrow \\ \Lambda^j \mathcal{T}^* \otimes h_m & \xrightarrow{-\delta} & \Lambda^{j+1} \mathcal{T}^* \otimes h_{m-1} \end{array}$$

commutes. Hence, by the commutativity of (28), we see that the diagram (33) is commutative.

$$
(33)\qquad
\begin{array}{ccccccccccccc}
 & & 0 & & 0 & & 0 & & & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & 0 \longrightarrow & g_{m+1} & \xrightarrow{-\delta} & \mathcal{T}^{\star}\otimes g_m & \xrightarrow{-\delta} & \Lambda^2\mathcal{T}^{\star}\otimes g_{m-1} & \xrightarrow{-\delta} & \ldots & \longrightarrow & \Lambda^n\mathcal{T}^{\star}\otimes g_{m-n+1} & \longrightarrow & 0\\
 & \downarrow & \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\varepsilon} & & & & \downarrow{\scriptstyle\varepsilon} & & \\
0\longrightarrow \mathcal{S} & \xrightarrow{j_{m+1}} & \mathcal{R}_{m+1} & \xrightarrow{D} & \mathcal{T}^{\star}\otimes \mathcal{R}_m & \xrightarrow{D} & \Lambda^2\mathcal{T}^{\star}\otimes \mathcal{R}_{m-1} & \xrightarrow{D} & \ldots & \longrightarrow & \Lambda^n\mathcal{T}^{\star}\otimes \mathcal{R}_{m-n+1} & \longrightarrow & 0\\
\downarrow & & \downarrow{\scriptstyle\pi_m} & & \downarrow{\scriptstyle\pi_{m-1}} & & \downarrow{\scriptstyle\pi_{m-2}} & & & & \downarrow{\scriptstyle\pi_{m-n}} & & \\
0\longrightarrow \mathcal{S} & \xrightarrow{j_m} & \mathcal{R}_m & \xrightarrow{D} & \mathcal{T}^{\star}\otimes \mathcal{R}_{m-1} & \xrightarrow{D} & \Lambda^2\mathcal{T}^{\star}\otimes \mathcal{R}_{m-2} & \xrightarrow{D} & \ldots & \longrightarrow & \Lambda^n\mathcal{T}^{\star}\otimes \mathcal{R}_{m-n} & \longrightarrow & 0\\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & 0 \longrightarrow & h_m & \xrightarrow{-\delta} & \mathcal{T}^{\star}\otimes h_{m-1} & \xrightarrow{-\delta} & \Lambda^2\mathcal{T}^{\star}\otimes h_{m-2} & \xrightarrow{-\delta} & \ldots & \longrightarrow & \Lambda^n\mathcal{T}^{\star}\otimes h_{m-n} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & & & 0 & &
\end{array}
$$

We set $m_1 = \max(k_0 + n, k_1 + n)$, where $k_0$, $k_1$ are the integers given by Lemmas 2 and 3 respectively, and obtain the desired conclusion.

Proposition 8. — *Let* $R_k \subset J_k(E)$ *be a regular partial differential equation of order* $k$ *on* E. *If the maps* $\pi_m : R_{m+r} \to R_m$ *have constant rank, for all* $m \geq k$, $r \geq 0$, *then, for all* $l \geq 0$, *there exists an integer* $p_l \geq k$ *such that the equation* $R^{(l)}_{p_l} \subset J_{p_l}(E)$ *has stable naive Spencer sequences and its Spencer cohomology is isomorphic to the Spencer cohomology of* $R_k$.

*Proof.* — Let $p_l \geq k$ be the integer given in the proof of Theorem 1, such that $R^{(l)}_{p_l+r} = (R^{(l)}_{p_l})_{+r}$, for all $r \geq 0$. Since $h^{(l)}_m$ is a vector bundle, the exactness of (17) implies that $\pi_m : R^{(l)}_{m+1} \to R^{(l)}_m$ has constant rank, for $m \geq k$. Hence, by Theorem 2, $R^{(l)}_{p_l}$ has stable naive Spencer sequences. It suffices to show that the Spencer cohomology of $R^{(l)}_{p_l}$ is isomorphic to the Spencer cohomology of $R^{(l+1)}_{p_{l+1}}$, for $l \geq 0$. By the exactness of (17) and (18), diagram (34) is commutative and its columns are exact, for $m \geq \max(p_l + n,\ p_{l+1} + n)$.

$$
(34)\qquad
\begin{array}{ccccccccccccc}
 & & 0 & & 0 & & 0 & & & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
0\longrightarrow \mathcal{S} & \xrightarrow{j_m} & \mathcal{R}^{(l+1)}_m & \xrightarrow{D} & \mathcal{T}^{\star}\otimes \mathcal{R}^{(l+1)}_{m-1} & \xrightarrow{D} & \Lambda^2\mathcal{T}^{\star}\otimes \mathcal{R}^{(l+1)}_{m-2} & \xrightarrow{D} & \ldots & \longrightarrow & \Lambda^n\mathcal{T}^{\star}\otimes \mathcal{R}^{(l+1)}_{m-n} & \longrightarrow & 0\\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
0\longrightarrow \mathcal{S} & \xrightarrow{j_m} & \mathcal{R}^{(l)}_m & \xrightarrow{D} & \mathcal{T}^{\star}\otimes \mathcal{R}^{(l)}_{m-1} & \longrightarrow & \Lambda^2\mathcal{T}^{\star}\otimes \mathcal{R}^{(l)}_{m-2} & \xrightarrow{D} & \ldots & \longrightarrow & \Lambda^n\mathcal{T}^{\star}\otimes \mathcal{R}^{(l)}_{m-n} & \longrightarrow & 0\\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & 0 \longrightarrow & h^{(l)}_m & \xrightarrow{-\delta} & \mathcal{T}^{\star}\otimes h^{(l)}_{m-1} & \xrightarrow{-\delta} & \Lambda^2\mathcal{T}^{\star}\otimes h^{(l)}_{m-2} & \xrightarrow{-\delta} & \ldots & \longrightarrow & \Lambda^n\mathcal{T}^{\star}\otimes h^{(l)}_{m-n} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & & & 0 & &
\end{array}
$$

Applying Lemma 3 to the equation $R^{(l)}_{p_l} \subset J_{p_l}(E)$, the bottom row of diagram (34) is exact for all $m$ sufficiently large. This clearly implies that the cohomology of the top row is isomorphic to the cohomology of the middle row, for all $m$ sufficiently large, proving the desired result.

COROLLARY 1. — *Let $R_k \subset J_k(E)$ be a regular partial differential equation of order $k$ on $E$. Assume that the maps $\pi_m : R_{m+r} \to R_m$ have constant rank for all $m \geq k$, $r \geq 0$. Then the Spencer cohomology of $R_k$ depends only on the formal solutions $R_\infty$ of $R_k$.*

*Proof.* — Let $m_0$, $l_0$ be the integers given by Theorem 1. By Proposition 8, since $m_0 \geq p_{l_0}$, the Spencer cohomology of $R^{(l_0)}_{m_0}$ is isomorphic to the Spencer cohomology of $R_k$. Because $R^{(l_0)}_m = \overline{R}_m = \pi_m(R_\infty)$ for $m \geq m_0$, the corollary follows.

The following theorem establishes the existence of resolutions for regular differential operators. In [5], we proved the first part of this theorem for formally integrable operators (*see* also M. Kuranishi [9]). The proof given here is based in part on an argument of Quillen [14] which he used to prove a weaker version of this theorem.

THEOREM 3. — *Let $\varphi : J_k(E) \to F$ be a regular differential operator of order $k$ from $E$ to $F$; let $D_0 = \varphi \circ j_k$. Then there exists a formally exact complex*

$$(35) \qquad 0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{E} \xrightarrow{D_0} \mathcal{G}_0 \xrightarrow{D_1} \mathcal{G}_1 \xrightarrow{D_2} \mathcal{G}_2 \xrightarrow{D_3} \cdots \longrightarrow \mathcal{G}_{r-1} \xrightarrow{D_r} \mathcal{G}_r \xrightarrow{D_{r+1}} \cdots,$$

*where $G_r$ is a vector bundle and $G_0 = F$, and $D_r = \psi_r \circ j_{l_r} : \mathcal{G}_{r-1} \to \mathcal{G}_r$ is a differential operator of order $l_r$; moreover the sequences*

$$(36) \qquad 0 \longrightarrow R_{k+m} \longrightarrow J_{k+m}(E) \xrightarrow{p_m(\varphi)} J_m(G_0) \xrightarrow{p_{m-l_1}(\psi_1)} J_{m-l_1}(G_1) \longrightarrow \cdots$$
$$\longrightarrow J_{m-l_1-\ldots-l_r}(G_r) \longrightarrow \cdots$$

*are exact at $R_{k+m}$ and $J_{k+m}(E)$ for $m \geq 0$, at $J_m(G_0)$ for $m \geq l_1$, and at $J_{m-l_1-\ldots-l_r}(G_r)$ for $m \geq l_1 + \ldots + l_{r+1}$, $r \geq 1$.*

*Furthermore, if the maps $\pi_m : R_{m+1} \to R_m$ have constant rank, for all $m \geq k$, the cohomology of* (35) *is isomorphic to the Spencer cohomology of $R_k$.*

*Proof.* — Set $l_1 = \max(k_0, k_1) - k + 1$, where $k_0$, $k_1$ are the integers given by Lemmas 2 and 3 respectively. Let $G_1 = Q_{l_1}$ and let $\psi_1 : J_{l_1}(F) \to Q_{l_1}$ be the natural projection. By the commutativity of diagram (1), to show that the sequences

$$(37) \qquad J_{k+l_1+m}(E) \xrightarrow{p_{l_1+m}(\varphi)} J_{l_1+m}(F) \xrightarrow{p_m(\psi_1)} J_m(G_1)$$

are exact for $m \geq 0$, it is sufficient to prove that the map $p_m(id_{l_1}) : Q_{l_1+m} \to J_m(Q_{l_1})$ is injective for all $m \geq 0$. We shall show in fact that $p_m(id_l) : Q_{l+m} \to J_m(Q_l)$ is injective for all $l \geqq l_1$, $m \geqq 0$. It suffices to prove this for $m = 1$, since the diagram

$$\begin{tikzcd}
Q_{l+m+1} \arrow[r, "p_{m+1}(id_l)"] \arrow[d, "p_1(id_{l+m})"] & J_{m+1}(Q_l) \arrow[d, "p_1(id_m)"] \\
J_1(Q_{l+m}) \arrow[r, "J_1(p_m(id_l))"] & J_1(J_m(Q_l))
\end{tikzcd}$$

is commutative, where $p_1(id_m)$ is a monomorphism. By the commutativity of diagrams (2) and (4), it is clear that the kernel of $p_1(id_l) : Q_{l+1} \to J_1(Q_l)$ is contained in the kernel of $\delta : q_{l+1} \to T^* \otimes q_l$. Hence it is enough to show that $\delta : q_{l+1} \to T^* \otimes q_l$ is injective for $l \geqq l_1$. From diagram (6), we deduce that the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & h_{k+l} \arrow[r, "\iota"] \arrow[d, "\delta"] & p_{l+1} \arrow[r, "\varepsilon"] \arrow[d, "\delta"] & q_{l+1} \arrow[r] \arrow[d, "\delta"] & 0 \\
0 \arrow[r] & T^* \otimes h_{k+l-1} \arrow[r, "\iota"] \arrow[d, "\delta"] & T^* \otimes p_l \arrow[r, "\varepsilon"] \arrow[d, "\delta"] & T^* \otimes q_l \arrow[r] & 0 \\
0 \arrow[r] & \Lambda^2 T^* \otimes h_{k+l-2} \arrow[r, "\iota"] & \Lambda^2 T^* \otimes p_{l-1} & &
\end{tikzcd}$$

is commutative and its rows are exact. Hence $\delta : q_{l+1} \to T^* \otimes q_l$ is injective if the sequences

(38) $$h_{k+l} \xrightarrow{\delta} T^* \otimes h_{k+l-1} \xrightarrow{\delta} \Lambda^2 T^* \otimes h_{k+l-2}$$

and

(39) $$0 \to p_{l+1} \xrightarrow{\delta} T^* \otimes p_l$$

are exact. Now (38) is exact for $l \geqq k_1 - k + 1$ and (39) is exact for $l \geqq k_0 - k + 1$ by the commutativity and exactness of diagram (5) and the exactness of (10).

The differential operator $\psi_1 : J_{l_1}(F) \to G_1$ is formally integrable by the exactness of (37). Therefore we can apply the above result or Corollary 4.2 of [5] to $\psi_1$ to obtain the complex (35) and the exact sequences (36). By Lemma 6, it follows that (35) is formally exact. The construction of $G_r$, $D_r$, with $r > 1$, given in [5] shows without appealing to Lemma 6 that the sequence

$$\mathcal{G}_0 \xrightarrow{D_1} \mathcal{G}_1 \xrightarrow{D_2} \dots \longrightarrow \mathcal{G}_{r-1} \xrightarrow{D_r} \mathcal{G}_r \xrightarrow{D_{r+1}} \dots$$

is formally exact.

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{S} \arrow[r] \arrow[d, "j_{k+m}"] & \mathcal{E} \arrow[r, "D_0"] \arrow[d, "j_{k+m}"] & \mathcal{G}_0 \arrow[r, "D_1"] \arrow[d, "j_m"] & \cdots \arrow[r] & \mathcal{G}_r \arrow[r] \arrow[d, "j_{m-l_1-\cdots-l_r}"] & \cdots \\
0 \arrow[r] & \mathcal{R}_{k+m} \arrow[r] \arrow[d, "D"] & \mathcal{J}_{k+m}(\mathcal{E}) \arrow[r, "p_m(\varphi)"] \arrow[d, "D"] & \mathcal{J}_m(\mathcal{G}_0) \arrow[r, "p_{m-l_1}(\psi_1)"] \arrow[d, "D"] & \cdots \arrow[r] & \mathcal{J}_{m-l_1-\ldots-l_r}(\mathcal{G}_r) \arrow[r] \arrow[d, "D"] & \cdots \\
0 \arrow[r] & \mathcal{T}^* \otimes \mathcal{R}_{k+m-1} \arrow[r] \arrow[d, "D"] & \mathcal{T}^* \otimes \mathcal{J}_{k+m-1}(\mathcal{E}) \arrow[r, "p_{m-1}(\varphi)"] \arrow[d, "D"] & \mathcal{T}^* \otimes \mathcal{J}_{m-1}(\mathcal{G}_0) \arrow[r, "p_{m-l_1-1}(\psi_1)"] \arrow[d, "D"] & \cdots \arrow[r] & \mathcal{T}^* \otimes \mathcal{J}_{m-l_1-\ldots-l_r-1}(\mathcal{G}_r) \arrow[r] \arrow[d, "D"] & \cdots \\
0 \arrow[r] & \Lambda^2 \mathcal{T}^* \otimes \mathcal{R}_{k+m-2} \arrow[r] \arrow[d, "D"] & \Lambda^2 \mathcal{T}^* \otimes \mathcal{J}_{k+m-2}(\mathcal{E}) \arrow[r, "p_{m-2}(\varphi)"] \arrow[d, "D"] & \Lambda^2 \mathcal{T}^* \otimes \mathcal{J}_{m-2}(\mathcal{G}_0) \arrow[r, "p_{m-l_1-2}(\psi_1)"] \arrow[d, "D"] & \cdots \arrow[r] & \Lambda^2 \mathcal{T}^* \otimes \mathcal{J}_{m-l_1-\ldots-l_r-2}(\mathcal{G}_r) \arrow[r] \arrow[d, "D"] & \cdots \\
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & \\
0 \arrow[r] & \Lambda^n \mathcal{T}^* \otimes \mathcal{R}_{k+m-n} \arrow[r] \arrow[d] & \Lambda^n \mathcal{T}^* \otimes \mathcal{J}_{k+m-n}(\mathcal{E}) \arrow[r, "p_{m-n}(\varphi)"] \arrow[d] & \Lambda^n \mathcal{T}^* \otimes \mathcal{J}_{m-n}(\mathcal{G}_0) \arrow[r, "p_{m-l_1-n}(\psi_1)"] \arrow[d] & \cdots \arrow[r] & \Lambda^n \mathcal{T}^* \otimes \mathcal{J}_{m-l_1-\ldots-l_r-n}(\mathcal{G}_r) \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 & & 0 &
\end{tikzcd}
\tag{40}
$$

We now assume that the maps $\pi_m : R_{m+1} \to R_m$ have constant rank for $m \geqq k$, so that $R_k$ has stable naive Spencer sequences by Theorem 2. The commutative diagram (40) has exact and formally exact columns, except for the first one, by Proposition 7. The exactness of the sequence (36) of vector bundles implies that the Spencer cohomology of $R_k$ is isomorphic to the cohomology of (35), completing the proof of Theorem 3.

Moreover, since the sequence (35) is formally exact, we deduce that the stable naive Spencer sequences of $R_k$ are formally exact.

COROLLARY 2. — *Let* $R_k \subset J_k(E)$ *be a regular partial differential equation of order* $k$ *on* E. *Assume that the maps* $\pi_m : R_{m+1} \to R_m$ *have constant rank for all* $m \geqq k$. *Then the stable naive Spencer sequences of* $R_k$ *are formally exact.*

*Remark.* — Let $\varphi : J_k(E) \to F$ be an arbitrary differential operator of order $k$ from E to F. Assume that there exists a complex

$$\mathcal{E} \xrightarrow{D_0} \mathcal{F} \xrightarrow{D_1} \mathcal{G},$$

where G is a vector bundle and $D_1 = \psi \circ j_l : \mathcal{F} \to \mathcal{G}$ is a differential operator of order $l$, such that the sequences

$$0 \longrightarrow R_{k+l+m} \longrightarrow J_{k+l+m}(E) \xrightarrow{p_{l+m}(\varphi)} J_{l+m}(F) \xrightarrow{p_m(\psi)} J_m(G)$$

are exact for $m \geqq 0$. By Lemma 3.3 of [5], for $m \geqq 0$, $R_{k+l+m}$ is a vector bundle over each connected component of X. Hence the condition that $\varphi$ be regular is essentially necessary and sufficient for the existence of the complex (35) of Theorem 3.

Assume that $R_k \subset J_k(E)$ is a formally integrable involutive equation of order $k$ on E. Following Quillen [14], we apply Theorem 3 to the differential operator

$$\rho : \quad J_1(C^0) \to C^1$$

defined in paragraph 5 of [5] and to the sophisticated Spencer sequence of $R_k$

$$0 \to \mathcal{S} \xrightarrow{j_k} \mathcal{C}^0 \xrightarrow{D^0} \mathcal{C}^1 \xrightarrow{D^1} \mathcal{C}^2 \xrightarrow{D^2} \ldots \to \mathcal{C}^n \to 0$$

constructed in [5], which is formally exact, and we obtain :

COROLLARY 3 (*see* D. G. Quillen [14]). — *If* $R_k \subset J_k(E)$ *is a formally integrable involutive equation of order* $k$ *on* E, *then the cohomology of the sophisticated Spencer sequence of* $R_k$ *is isomorphic to the Spencer cohomology of* $R_k$.

5. The Spencer cohomology of an analytic differential equation. — Now assume that X is a real analytic manifold and that the vector bundle E is analytic. For any such analytic vector bundle E, we denote by $\mathcal{E}_\omega$ the subsheaf of $\mathcal{E}$ of analytic germs. If $R_k$ is an analytic subbundle of $J_k(E)$, we say that $R_k$ is an analytic equation. We let $\mathcal{S}_\omega$ denote the subsheaf of $\mathcal{S}$ of analytic germs of solutions of $R_k$.

Theorem 4. — *Let $R_k \subset J_k(E)$ be a regular analytic partial differential equation of order $k$ on E. Assume that the maps $\pi_m : R_{m+1} \to R_m$ have constant rank for all $m \geqq k$. Then the analytic stable naive Spencer sequences*

$$(41)\qquad 0 \to \mathcal{S}_\omega \xrightarrow{j_m} (\mathcal{R}_m)_\omega \xrightarrow{D} (\mathcal{T}^* \otimes \mathcal{R}_{m-1})_\omega \xrightarrow{D} (\Lambda^2 \mathcal{T}^* \otimes \mathcal{R}_{m-2})_\omega \xrightarrow{D} \ldots \to (\Lambda^n \mathcal{T}^* \otimes \mathcal{R}_{m-n})_\omega \to 0$$

*are exact except possibly at $(\mathcal{T}^* \otimes \mathcal{R}_{m-1})_\omega$.*

*If moreover, the maps $\pi_m : R_{m+r} \to R_m$ have constant rank for all $m \geqq k$, $r \geqq 0$, then the analytic stable naive Spencer sequences are exact.*

*Proof.* — Let F be an analytic vector bundle and let $\varphi : J_k(E) \to F$ be an analytic differential operator such that $\ker \varphi = R_k$. Set $G_0 = F$ and let

$$(42)\qquad 0 \longrightarrow \mathcal{S}_\omega \longrightarrow \mathcal{E}_\omega \xrightarrow{D_0} \mathcal{F}_\omega \xrightarrow{D_1} (\mathcal{G}_1)_\omega \xrightarrow{D_2} \ldots \longrightarrow (\mathcal{G}_{r-1})_\omega \xrightarrow{D_r} (\mathcal{G}_r)_\omega \xrightarrow{D_{r+1}} \ldots$$

be a complex, where $G_r$ is an analytic vector bundle, $D_r = \psi_r \circ j_{l_r} : \mathcal{G}_{r-1} \to \mathcal{G}_r$ is an analytic differential operator of order $l_r$ for $r \geqq 1$. Assume that $G_1$, $D_1$ are constructed by Theorem 3 and that $G_r$, $D_r$ are constructed by Corollary 4.2 of [5] such that the sequences (36) are exact. Using Spencer's estimate (*see* L. Ehrenpreis, V. W. Guillemin, and S. Sternberg [4] and W. J. Sweeney [17]), we showed in [5] that the sequence (42) is exact at $(G_r)_\omega$, for $r \geqq 1$. By Theorems 2 and 3, the cohomology of the sequence (41) is independent of $m$ and isomorphic to the cohomology of (42) for all $m$ sufficiently large, proving the first part of the theorem. Note moreover, that if $R_k$ is formally integrable, we can construct $G_r$, $D_r$, for $r \geqq 1$, by Corollary 4.2 of [5] such that the sequence (36) is exact; in this case the sequence (42) is exact and the analytic stable naive Spencer sequences are exact.

Now assume moreover that the maps $\pi_m : R_{m+r} \to R_m$ have constant rank, for all $m \geqq k$, $r \geqq 0$. Let $l_0$, $m_0$ be the integers given by Theorem 1. Then by Proposition 8, the cohomology of (41) is isomorphic to the cohomology of the sequence

$$(43)\qquad 0 \to \mathcal{S}_\omega \xrightarrow{j_m} (\mathcal{R}_m^{(l_0)})_\omega \xrightarrow{D} (\mathcal{T}^* \otimes \mathcal{R}_{m-1}^{(l_0)})_\omega \xrightarrow{D} (\Lambda^2 \mathcal{T}^* \otimes \mathcal{R}_{m-2}^{(l_0)})_\omega \to \ldots \to (\Lambda^n \mathcal{T}^* \otimes \mathcal{R}_{m-n}^{(l_0)})_\omega \to 0$$

if $m$ is sufficiently large. Since $R^{(l_0)}_{m_0}$ is formally integrable and $R^{(l_0)}_{m_0+r} = (R^{(l_0)}_{m_0})_{+r}$, for $m \geqq 0$, the sequence (43) is exact for all sufficiently large $m$ by the above argument and so the cohomology of (41) vanishes for all sufficiently large $m$.

From Theorems 3 and 4, we deduce :

COROLLARY 4. — *Let* $\varphi : J_k(E) \to F$ *be a regular analytic differential operator of order* $k$ *from* E *to* F. *Assume that the maps* $\pi_m : R_{m+r} \to R_m$ *have constant rank for all* $m \geqq k$, $r \geqq 0$. *If* G *is any analytic vector bundle and* $\psi : J_l(F) \to G$ *is any analytic differential operator of order* $l$ *from* F *to* G *such that the sequences*

$$J_{k+l+m}(E) \xrightarrow{p_{l+m}(\varphi)} J_{l+m}(F) \xrightarrow{p_m(\psi)} J_m(G)$$

*are exact for* $m \geqq 0$, *then the sequence*

$$\mathcal{E}_\omega \xrightarrow{D} \mathcal{F}_\omega \xrightarrow{D'} \mathcal{G}_\omega$$

*where* $D = \varphi \circ j_k$, $D' = \psi \circ j_l$, *is exact.*

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