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Annales scientifiques de l'É.N.S. 4[e] série, tome 1, nº 1 (1968), p. 149-159
<http://www.numdam.org/item?id=ASENS_1968_4_1_1_149_0>

Ann. scient. Éc. Norm. Sup.,
4e série, t. 1, 1968, p. 149 à 159.

RESIDUES OF DIFFERENTIALS ON CURVES [1]

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This lecture contains a definition of the residues of differentials on curves, in terms of traces of certain linear operators on infinite dimensional vector spaces. All the standard theorems on residues follow easily from this definition, by proofs which are natural and independent of the characteristic of the ground field. In particular, the fact that "the sum of the residues is zero on a complete curve X" results directly, without computation, from the finiteness of the cohomology groups $H^i(X, \mathcal{O}_X)$, for $i = 0$ and 1, almost as though one had an abstract Stoke's Theorem available. I arrived at this treatment of residues by considering the special features of the one-dimensional case, after discussing with Mumford an approach of Cartier to Grothendieck's higher dimensional residue symbol (*see* HARTSHORNE, *Residues and Duality*, Springer lecture notes in Mathematics, vol. 20, 1966, p. 195).

For a good general account of the subject of residues and duality on curves, with references to other approaches, *see* SERRE, *Groupes algébriques et corps de classes*, Hermann, Paris, 1959, p. 24-35 and also p. 76-81.

1. TRACES. — Let k be a fixed ground field and V a vector space over k. We say that an endomorphism θ of V is *finite potent if* $\theta^n V$ is finite dimensional for some n. For such θ, a trace $\mathrm{Tr}_V(\theta) \in k$ may be defined, having the properties :

$(T)_1$ *If* V *is finite dimensional, then* $\mathrm{Tr}_V(\theta)$ *is the ordinary trace;*

[1] This paper is a slight revision and expansion of a lecture given at the Advanced Science Seminar in Algebraic Geometry, sponsored by the National Science Foundation, held at Bowdoin College, Brunswick, Maine, in the Summer of 1967. I wish to thank James Milne who wrote up the Notes of that lecture, which served as a first draft of this paper.

(T_2) *If* W *is a subspace of* V, *and* $\theta W \subset W$, *then*

$$\mathrm{Tr}_V(\theta) = \mathrm{Tr}_W(\theta) + \mathrm{Tr}_{V/W}(\theta);$$

(T_3) *If* θ *is nilpotent, then* $\mathrm{Tr}_V(\theta) = 0$.

Note that T_1, T_2, T_3 characterize traces; if W is a finite dimensional subspace of V such that $\theta W \subset W$ and $\theta^n V \subset W$, for some n, then $\mathrm{Tr}_V(\theta) = \mathrm{Tr}_W(\theta)$. Such W exist, for we may take $W = \theta^n V$ for some large n. (In fact, there is a unique minimal such W, which equals $\theta^n V$ for all sufficiently large n.)

(T_4) *If* F *is a finite potent subspace of* End(V) (*i. e., if there exists an* n *such that for any family of* n *elements* $\theta_1, \ldots, \theta_n \in F$, *the space* $\theta_1 \ldots \theta_n V$ *is finite dimensional*) *then* $\mathrm{Tr}_V : F \to k$ *is* k*-linear.*

Proof. — We may take F to be finite dimensional and compute the traces of all elements of F on the finite dimensional subspace $W = F^n V$.

Property (T_4) seems the natural linearity property for Tr_V, and is sufficient for our applications. I doubt whether the rule

$$\mathrm{Tr}_V \theta_1 + \mathrm{Tr}_V \theta_2 = \mathrm{Tr}_V(\theta_1 + \theta_2)$$

holds in general, i. e., whenever all three endomorphisms θ_1, θ_2 and $\theta_1 + \theta_2$ are finitepotent, although I do not know a counter example. (If a counter example exists at all, then there will be one with θ_1 and θ_2 nilpotent, because every finitepotent endomorphism is the sum of a nilpotent one and one with finite range.)

(T_5) *If* $\varphi : V' \to V$ *and* $\psi : V \to V'$ *are* k*-linear and* $\varphi\psi$ *is finite potent, then* $\psi\varphi$ *is finite potent, and*

$$\mathrm{Tr}_V(\varphi\psi) = \mathrm{Tr}_{V'}(\psi\varphi).$$

Indeed, for large n the maps φ and ψ induce mutually inverse isomorphisms between the subspaces $W' = (\psi\varphi)^n V'$ and $W = (\varphi\psi)^n V$, under which the endomorphisms $\psi\varphi \mid W'$ and $\varphi\psi \mid W$ correspond.

Fix V. A subspace A of V is " not much bigger " than a subspace B (notation $A < B$) if $(A + B)/B$ is finite dimensional, or equivalently, if $A \subset (B + W)$ for some finite dimensional W; and A is " about the same size " as B (notation $A \sim B$) if $A < B$ and $B < A$. The following rules are easy to check :

$A < B$ and $B < C \Rightarrow A < C$;

$A < B \Rightarrow \varphi(A) < \varphi(B)$, for any k-linear map φ, and

$$\sum_{i=1}^{m} A_i < \bigcap_{j=1}^{n} B_j \quad \Leftrightarrow \quad A_i < B_j \quad \text{all } i \text{ and } j.$$

Fix a subspace A of V; then define subspaces E, E_0, E_1, E_2 of End(V) by

$$\theta \in E \iff \theta A < A, \quad \theta \in E_1 \iff \theta V < A, \quad \theta \in E_2 \iff \theta A < (o),$$
$$\theta \in E_0 \iff \theta V < A \quad \text{and} \quad \theta A < (o).$$

Proposition 1. — *E is a k-subalgebra of* End (V); *the E_i are two-sided ideals in* E; *the* E's *depend only on the $\sim$-equivalence class of* A; *we have* $E_1 \cap E_2 = E_0$ *and* $E_1 + E_2 = E$; *and* E_0 *is finite potent.*

Proof. — Let $\pi : V \to A$ be a linear projection. Then $1 - \pi \in E_2$, $\pi \in E_1$ and $\pi + (1 - \pi) = 1$, so $E_1 + E_2 = E$. The other statements are obvious.

Thus there is a k-linear map $\mathrm{Tr}_V : E_0 \to k$.

Proposition 2. — *Suppose either $\varphi \in E_0$ and $\psi \in E$, or $\varphi \in E_1$ and $\psi \in E_2$. Then the commutator $[\varphi, \psi] = \varphi\psi - \psi\varphi$ is in E_0 and has zero trace.*

Proof. — Trivial from the definition of the E_i and (T_5).

2. Abstract Residues. — Let K be a commutative k-algebra (with 1), V a K-module, and A a k-subspace of V such that $fA < A$ for all $f \in K$. With notations E and E_i (relative to V and A) as above, this last condition means that K operates on V through $E \subset \mathrm{End}_k(V)$, and we shall in what follows habitually use the same letter f to denote an element of K and its image in E.

Theorem 1 (Definition of residue). — *In the situation just described there exists a unique k-linear " residue map "*

$$\mathrm{res}_A^V : \quad \Omega^1_{K/k} \to k$$

such that for each pair of elements f and g in K *we have*

$$\mathrm{res}_A^V(f\,dg) = \mathrm{Tr}_V([f_1, g_1])$$

for every pair of endomorphisms f_1 and g_1 in E *satisfying the following conditions :*

(*a*) *Both* $f \equiv f_1 \pmod{E_2}$ *and* $g_1 \equiv g \pmod{E_2}$;

(*b*) *Either* $f_1 \in E_1$ *or* $g_1 \in E_1$.

Given f and g in K it is always possible to find f_1 and g_1 satisfying (*a*) and (*b*) because $E = E_1 + E_2$. Then $[f_1, g_1] \in E_1$ by (*b*) and $[f_1, g_1] \equiv [f, g] = o \pmod{E_2}$ by (*a*). Hence $[f_1, g_1] \in E_1 \cap E_2 = E_0$ and $\mathrm{Tr}_V([f_1, g_1])$ is defined. By proposition 2 this quantity is unaltered if f_1 or g_1 is changed by an element of E_2 provided that the other is in E_1, and by (T_4) it is a k-bilinear function of f and g. Thus there is a linear map $r : K \otimes_k K \to k$, such that $r(f \otimes g) = \mathrm{Tr}_V([f_1, g_1])$. Recall that by the very definition of Ω^1 there is a k-linear map

$$c : \quad K \otimes_k K \to \Omega^1_{K/k}$$

such that $c(f \otimes g) = f\,dg$ and such that :

(i) c is surjective;

(ii) $\ker(c)$ is generated over k by elements of the form

$$f \otimes gh - fg \otimes h - fh \otimes g.$$

By (i), res_A^V (if it exists) can only be the unique map r' such that $r'c = r$ in the diagram

$$\begin{tikzcd} K \otimes_k K \arrow[r, "r"] \arrow[d, "c"'] & k \\ \Omega^1_{K/k} \arrow[ur, dashed, "r' = \operatorname{res}_A^V"'] & \end{tikzcd}$$

Such a map r' exists if and only if r vanishes on the kernel of c. To see that it does, let f, g, and $h \in K$, choose suitable f_1, g_1, and h_1 in E_1, and then use $(fg)_1 = f_1 g_1$, etc., and the identity

$$[f_1, g_1 h_1] - [f_1 g_1, h_1] - [f_1 h_1, g_1] = 0.$$

Thus res_A^V exists and is unique.

Remark. — For given f and g in K, the computation of $\operatorname{Res}(f\,dg)$ can be effected in finite terms as follows. Let

$$B = A + gA,$$
$$C = B \cap f^{-1}(A) \cap (fg)^{-1}(A) = \{ v \in B \mid fv \in A \text{ and } fgv \in A \}.$$

Let π be a k-linear projection of $(A + fA + fgA)$ onto A. Then $\dim(B/C)$ is finite and

$$(\star) \qquad \operatorname{res}_A^V(f\,dg) = \operatorname{Tr}_{B/C}([\pi f, g]).$$

Indeed, if we extend π to a projection of all of V onto A, then $\pi f \in E_1$ and $\pi f \equiv f \pmod{E_2}$, so $\operatorname{res}_A^V(f\,dg) = \operatorname{Tr}_V([\pi f, g])$. On the other hand $[\pi f, g] = \pi f g - g \pi f$ maps V into B, and C into 0 (because $fg = gf$). Hence $(\star)$ holds by property (T_2) of Tr_V.

Properties of res_A^V :

(R_1) *If* $V \supset V' \supset A$ *and* $KV' = V'$, *then* $\operatorname{res}_A^V = \operatorname{res}_A^{V'}$. *Moveover* $\operatorname{res}_A^V = \operatorname{res}_{A'}^V$ *if* $A \sim A'$.

These statements are obvious from the above remark and from the definition of res. In view of the first statement we can usually omit the superscript V on res_A^V from now on.

(R_2) (Continuity in f and g) *If* $fA + fgA + fg^2A \subset A$ *then* $\operatorname{res}_A(f\,dg) = 0$. *In particular, this is so if* $fA \subset A$ *and* $gA \subset A$. *The function* Res_A *is idenically ot if* A *is a* K-*submodule of* V.

Proof. — The first condition on f and g implies $B = C$ in the above remark, and the later statements follow from the first.

(R_3) *Let* $g \in K$. *Then* $\mathrm{res}_A(g^n\,dg) = 0$ *for all integers* $n \geq 0$, *and moreover the same holds for all* $n \leq -2$ *if* g *is invertible in* K. *In particular,* $\mathrm{res}_A(dg) = 0$ *for all* $g \in K$.

Proof. — Choose $g_1 \in E_1$, such that $g_1 \equiv g \pmod{E_2}$. Then if $n \geq 0$ we have $\mathrm{res}_A(g^n\,dg) = \mathrm{Tr}_V([g_1^n, g_1]) = 0$ because g_1^n and g_1 commute. If g is invertible then $g^{-2-n}\,dg = -(g^{-1})^n\,d(g^{-1})$, which has zero residue by the preceding statement, if $n \geq 0$.

(R_4) *If* g *is invertible in* K, *and* $h \in K$ *is such that* $hA \subset A$, *then*

$$\mathrm{res}_A(hg^{-1}\,dg) = \mathrm{Tr}_{A/(A\cap gA)}(h) - \mathrm{Tr}_{gA/(A\cap gA)}(h).$$

In particular, if g *is invertible and* $gA \subset A$, *then*

$$\mathrm{res}_A(g^{-1}\,dg) = \dim_k(A/gA).$$

Proof. — Take $f = hg^{-1}$ and apply the above remark. We have $[\pi f, g] = \pi h - \pi_1 h$, where $\pi_1 = g\pi g^{-1}$ is a projection of V onto gA. Since both A and gA are stable under h we have

$$\mathrm{res}_A(f\,dg) = \mathrm{Tr}_{(A+gA)/(A\cap gA)}(\pi h) - \mathrm{Tr}_{(A+gA)/(A\cap gA)}(g\pi g^{-1}h)$$

and the result follows.

(R_5) *Suppose that* B *is another* k*-subspace of* V *such that* $fB < B$ *for all* $f \in K$, *then*

$$f(A+B) < A+B \quad \textit{and} \quad f(A\cap B) < A\cap B, \quad \textit{for all } f \in K,$$

and we have

$$\mathrm{res}_A + \mathrm{res}_B = \mathrm{res}_{A+B} + \mathrm{res}_{A\cap B}.$$

Proof. — It is easy to see that we can choose projections π_A, π_B, π_{A+B}, $\pi_{A\cap B}$ of V onto A, B, $A+B$, $A\cap B$ respectively, such that

$$\pi_A + \pi_B = \pi_{A+B} + \pi_{A\cap B}.$$

Notice that if we knew that Tr_V was linear, we would be finished. Nevertheless, both $[\pi_A f, g]$ and $[\pi_{A+B} f, g]$ carry V into $A+B$, and $A+B$ into A, and A into (0) (mod finite dimensional subspaces), so they belong to a finite potent subspace of End V. Hence,

$$\begin{aligned}\mathrm{res}_A f\,dg - \mathrm{res}_{A+B} f\,dg &= \mathrm{Tr}_V([\pi_A f, g]) - \mathrm{Tr}_V([\pi_{A+B} f, g]) \\ &= \mathrm{Tr}_V([(\pi_A - \pi_{A+B})f, g)] \\ &= \mathrm{Tr}_V([(\pi_{A\cap B} - \pi_B)f, g)]\end{aligned}$$

which, by a similar argument, may be shown to equal $\mathrm{res}_{A\cap B} f\,dg - \mathrm{res}_B f\,dg$.

(R_6) *Let* K' *be a commutative* K *algebra which is a free* K-*module of finite rank. Let* $V' = K' \otimes_k V$ *and let* $A' = \sum_i x_i \otimes A \subset V'$, *where* (x_i) *is a* K-*base for* K'. *Then* $f'A' < A'$ *for all* $f' \in K'$, *the* $\sim$-*equivalence class of* A' *depends only on that of* A, *not on the choice of basis* (x_i), *and we have*

$$\mathrm{Res}_{A'}(f'\,dg) = \mathrm{Res}_A((\mathrm{Tr}_{K'/K} f)\,dg) \qquad \textit{for} \quad f' \in K' \quad \textit{and} \quad g \in K.$$

Proof. — A k-endomorphism φ of V' can be expressed as an $n \times n$ matrix (φ_{ij}) of endomorphisms of V by the rule

$$\varphi\left(\sum_j x_j \otimes v_j\right) = \sum_{ij} x_i \otimes \varphi_{ij} v_j$$

for $v_j \in V$. If F is a finite potent subspace of $\mathrm{End}_k V$, then the φ's such that $\varphi_{ij} \in F$ for all i, j form a finite potent subspace F' of $\mathrm{End}_k V'$, and we have $\mathrm{Tr}_{V'}\varphi = \sum_i \mathrm{Tr}_V(\varphi_{ii})$ for all $\varphi \in F'$, as one sees by decomposing the matrix (φ_{ij}) into the sum of a diagonal matrix and two nilpotent triangular matrices, one of the latter having zeros on and below the diagonal, the other having zeros on and above the diagonal. Now write $f'x_j = \sum x_i f_{ij}$ with $f_{ij} \in K$. Let π be a k-linear projection of V on A and put $\pi'\left(\sum x_i \otimes v_i\right) = \sum x_i \otimes \pi v_i$. Then π' is a projection of V' onto A', and

$$[f'\pi', g]_{ij} = [f_{ij}\pi, g].$$

The desired result follows now because $\mathrm{Tr}_{K'/K} f = \sum_i f_{ii}$.

3. Algebraic Curves. — Let X be a connected, regular scheme of dimension 1, proper over a ground field k, and let $K = k(X)$ be its function field. Then X is determined up to a k-isomorphism by K and K may be any function field in one variable over k.

The closed points p of X correspond to the discrete valuation rings O_p with field of fractions K which contain k. Write $A_p = \hat{O}_p$, the completion of O_p, and write K_p for the field of fractions of A_p (so K_p is the completion of K with respect to the valuation defined by O_p).

Definition. — $\mathrm{res}_p : \Omega^1_{K/k} \to k$ *is the* k-*linear map such that*

$$\mathrm{res}_p f\,dg = \mathrm{res}_{A_p}^{K_p}(f\,dg).$$

This definition makes sense, for if t_p is a prime element in A_p, then the residue field $k(p) = A_p/t_pA_p$ has finite dimension (equal to the degree of p relative to k) and so, by induction, $A_p \sim t_p^n A_p$ for all integers n.

Now, for any non-zero f in K (or K_p) we have $fA_p = t_p^n A_p$ for some n, hence in particular, $fA_p < A_p$ for all $f \in K_p$.

THEOREM 2. — *Let p be a k-rational point of* X, *so* $k[[t]] \approx A_p$ *and* $k((t)) \approx K_p$. *If* $f = \sum_{\nu \gg -\infty} a_\nu t^\nu$ *and* $g = \sum_{\mu \gg -\infty} b_\mu t^\mu$ *are two elements of* K (*or* K_p), *then*

$$\operatorname{res}_p f\,dg = \textit{coefficient of } t^{-1} \textit{ in } f(t)\,g'(t) = \sum_{\nu+\mu=0} \mu a_\nu b_\mu.$$

Proof. — By (R_2), we may assume that only finitely many of the a_ν and b_μ are non-zero. Then $f\,dg = f(t)\,g'(t)\,dt$, and by (R_3) only the term in t^{-1} can give non-zero residue. By (R_4) we have

$$\operatorname{res}_A(t^{-1}\,dt) = \dim_k k(p) = 1.$$

Remark. — One often defines $\operatorname{res}_p f\,dg$ by the above expression, but in characteristic $\neq 0$, it is not immediately obvious that the coefficient in question is independent of the choice of the uniformizing parameter t.

THEOREM 3. — *Let* S *be any set of closed points* p. *Put* $O(S) = \bigcap_{p \in S} O_p \subset K$ *Then for* $\omega \in \Omega^1_{K/k}$ *we have*

$$\sum_{p \in S} \operatorname{res}_p(\omega) = \operatorname{res}^K_{O(S)}(\omega).$$

almost all terms of the sum being zero.

COROLLARY. — *We have* $\sum \operatorname{res}_p(\omega) = 0$ *if the sum is taken over all closed points p of the complete curve* X.

The corollary follows from the theorem because, X being proper over k, the space $O(X) = H^0(X, \mathcal{O}_X)$ is finite dimensional. Hence $\operatorname{res}_{O(X)} = 0$, by (R_1), because $O(X) \sim (0)$.

To prove the theorem, let

$$A_S = \prod_{p \in S} A_p,$$

$$V_S = \prod_{p \in S}{}' K_p = \{ f = (f_p) \mid f_p \in K_p \text{ for all } p \quad \text{and} \quad f_p \in A_p \text{ for almost all } p \}.$$

We may assume S non-empty. Then K may be regarded as a subspace of V_S by means of the diagonal embedding $f \mapsto (f_p)$, where $f_p = f$ for all

$p \in S$. Considering the lattice of subspaces of V_S pictured at the right, and using (R_5) we conclude that

$$\mathrm{res}_{A_S} + \mathrm{res}_K = \mathrm{res}_{O(S)} + \mathrm{res}_{(K+A_S)}.$$

$$\begin{tikzcd}[column sep=small]
 & V_S \arrow[d, dash] & \\
 & K + A_S \arrow[dl, dash] \arrow[dr, dash] & \\
K \arrow[dr, dash] & & A_S \arrow[dl, dash] \\
 & K \cap A_S = O(S) \arrow[d, dash] & \\
 & (0) &
\end{tikzcd}$$

But $\mathrm{res}_K = 0$ because K is a K-module, and $\mathrm{res}_{K+A_S} = 0$ because $V_S/(K + A_S)$ is finite dimensional (*see* below). Thus we have only to prove $\mathrm{res}_{A_S}(\omega) = \sum_{p \in S} \mathrm{res}_p(\omega)$.

Let $\omega = f\,dg$, let S' be a *finite* subset of S which contains all poles of f or g, and let $T = S - S'$. Then

$$V_S = V_T \times \prod_{p \in S'} K_p \qquad \text{and} \qquad A_S = A_T \times \prod_{p \in S'} A_p.$$

From (R_5) it follows that

$$\mathrm{res}_{A_S}(f\,dg) = \mathrm{res}_{A_T}(f\,dg) + \sum_{p \in S'} \mathrm{res}_p(f\,dg).$$

But by our choice of S', we have $\mathrm{res}_{A_T}(f\,dg) = 0$ and $\mathrm{res}_p(f\,dg) = 0$ for $p \in T$.

To prove that $V_S/(K + A_S)$ is finite dimensional it suffices to treat the case $S = X$ because the projection $V_X \to V_S$ is surjective. For $S = X$ we have $V_X/(K + A_X) \simeq H^1(X, \mathcal{O}_X)$ which is finite dimensional because X is proper over k. This last well-known isomorphism follows from the exact sequence

$$0 \to \mathcal{O}_X \to K^0 \xrightarrow{\delta} K^1 \to 0$$

of abelian sheaves on X, in which for any open $U \subset X$ we let $K^0(U) =$ Image of K in V_U, and $K^1(U) = V_U/A_U = \bigoplus_{p \in U} K_p/A_p$, the map $\delta(U)$ being induced by $V_U \to V_U/A_U$. The restriction maps for $U' \subset U$ are surjective; hence the sheaves K^i have trivial cohomology in dimensions greater than zero.

The homomorphism δ is surjective because $K + A_p = K_p$ for each p, and $(\operatorname{Ker}\delta) = \underline{o}_X$ because

$$(\operatorname{Ker}\delta)(U) = \operatorname{Ker}(\delta(U)) = O(U) \quad \text{for each } U.$$

Thus the sequence is exact, and

$$H^1(X, \mathcal{O}_X) = \operatorname{Coker}(\delta(X)) = V_X/(K + A_X).$$

THEOREM 4. — *Let* $X' \to X$ *be a surjective morphism of curves of the type we are considering, corresponding to the inclusion of function fields* $K \subset K'$. *Then for* $f' \in K'$, $g \in K$, *and* $p \in X$

$$\sum_{p' \mapsto p} \operatorname{res}_{p'}(f'\, dg) = \operatorname{res}_p((\operatorname{Tr}_{K'/K} f')\, dg).$$

Similarly, if $p' \in X'$ *has image* p *in* X, $f' \in K'_{p'}$, *and* $g \in K_p$, *then*

$$\operatorname{res}_{p'}(f'\, dg) = \operatorname{res}_p((\operatorname{Tr}_{K'_p/K_p} f')\, dg).$$

These formulas (each of which implies the other in virtue of the fact that " the global trace is the sum of the local traces ") both follow immediately from (R_6), because the integral closure of O_p (resp. A_p) in K' (resp. $K'_{p'}$) is a finite $O_{p'}$ (resp. $A_{p'}$) module.

Remark. — The standard proof that the sum of the residues is zero is to use Theorem 4 to reduce to the case X is the projective line, and then to verify that the sum is zero by direct computation in that special case.

4. DUALITY. — For completeness we finish with a rough sketch of the " duality theorem ". The idea is that for an arbitrary regular curve X proper over k there is a " dualizing sheaf " $J_{X/k}$, and if X is smooth over k, then $J_{X/k}$ can be identified with $\Omega^1_{X/k}$ via the theorems on residues.

Let X, K, etc., be as in the preceding section, and let $V = V_X$ and $A = A_X$. For each divisor D on X, let

$$V(D) = \{f = (f_p) \in V \mid \operatorname{ord}_p f_p \geqq - \operatorname{ord}_p D \text{ for all } p \in X\}.$$

Thus for example $V(o) = A$. By the same method as in the paragraph before theorem 4 above, one shows that for each D

$$H^1(X, \mathcal{O}_X(D)) \approx V/(K + V(D)).$$

The " dualizing sheaf " on X/k is the invertible sheaf $J_{X/k}$ whose stalk at the generic point is

$$J_{K/k} = \{\lambda \in \mathrm{Hom}_k(V, k) \mid \lambda(K + V(D)) = 0 \text{ for some divisor } D\}$$
$$\approx \varinjlim_{D} (H^1(X, \mathcal{O}_X(D))^*, \qquad \text{where } ^* \text{ denotes } k\text{-linear dual},$$

and whose stalk at each special point p is

$$J_p = \{\lambda \in J_{K/k} \mid \lambda(A_p) = 0\},$$

so that for each open $U \subset X$ we have $J_{X/k}(U) = \bigcap_{p \in U} J_p \subset J_{K/k}$. For the proof that this $J_{X/k}$ is an invertible sheaf on X, *see* Serre, *loc. cit.*, or Chevalley, *Introduction to the theory of algebraic functions of one variable*, *Math. Surveys*, VI, New York, 1951. The sheaf $J_{X/k}$ is just constructed in such a way that the duality theorem

$$H^0(J_{X/k}(-D)) = \mathrm{Hom}_k(V/(K + V(D)), k) \simeq H^1(X, \mathcal{O}_X(D))^*$$

is a tautology and more generally it is easy to show that

$$H^0(J_{X/k} \otimes \mathcal{O}_X L^{-1}) \approx H^1(X, L)^*$$

for every locally free sheaf L of finite rank over $\mathcal{O}_X$.

There is a canonical homomorphism

$$\Omega^1_{X/k} \xrightarrow{c} J_{X/k}$$

which is characterized by the following action at the generic stalk

$$(c\omega)(f) = \langle f, \omega \rangle = \sum_{p \in X} \mathrm{res}_p(f_p \omega)$$

for $\omega \in \Omega^1_{K/k}$ and $f = (f_p) \in V$. [Note that $(c\omega)(K) = 0$ by the corollary to theorem 3, and $(c\omega)(V(D)) = 0$ for some D by (R_2). Property (R_2) also shows that the given c at the generic stalk extends uniquely to a sheaf homomorphism, since $J_{X/k}$ is torsion free.]

Chevalley (*loc. cit.*) *defined* the differentials of K/k to be elements $\lambda \in J_{K/k}$, but then had to go to some length to explain his " differential " dx, and to even greater length to prove $d(x+y) = dx + dy$! The key fact is

Theorem 5. — *The homomorphism* $c : \Omega^1_{X/k} \to J_{X/k}$ *is an isomorphism at all points* p *where* X *is smooth over* k.

(In particular the map $\Omega^1_{X/k} \to J_{K/k}$ at the generic stalk is an isomorphism, i. e., non-zero, if K/k is separably generated.)

Corollary. — *If* X/k *is smooth (and in particular if* k *is perfect), then for every invertible sheaf* L *on* X *we have*

$$H^1(X, L) \approx H^0(\Omega^1_{X/k} \otimes L^{-1})^*.$$

Proof of Theorem 5. — Suppose X/k is smooth at p. Then Ω^1_p and J_p are free O_p-modules of rank 1, and J_p is generated by any element $\lambda \in J_p$ such that $\lambda \notin t_p J_p$, i. e., such that $\lambda(t_p^{-1} A_p) \neq 0$, where t_p is a prime element of O_p. The element $\lambda = c(dt_p)$ has this property if the residue field $k(p)$ is separable over k, by (R_4). The general case can be reduced to this one by a ground field extension $k \to k'$, or can be treated directly by a projection of X onto the projective line which is étale at p.

(Manuscrit reçu le 27 novembre 1967.)