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SYSTOLIC FREEDOM OF ORIENTABLE MANIFOLDS*

BY IVAN BABENKO AND MIKHAIL KATZ

ABSTRACT. – In 1972, Marcel Berger defined a metric invariant that captures the 'size' of k-dimensional homology of a Riemannian manifold. This invariant came to be called the k-dimensional *systole.* He asked if the systoles can be constrained by the volume, in the spirit of the 1949 theorem of C. Loewner. We construct metrics, inspired by M. Gromov's 1993 example, which give a negative answer for large classes of manifolds, for the product of systoles in a pair of complementary dimensions $(k, n-k)$. An obstruction (restriction on k modulo 4) to constructing further examples by our methods seems to reside in the free part of real Bott periodicity. The construction takes place in a split neighbourhood of a suitable k-dimensional submanifold whose connected components (rationally) generate the k-dimensional homology group of the manifold. Bounded geometry (combined with the coarea inequality) implies a lower bound for the k-systole, while calibration with support in this neighbourhood provides a lower bound for the systole of the complementary dimension. In dimension 4, everything reduces to the case of $S^2 \times S^2$. © Elsevier, Paris

RÉSUMÉ. – En 1972, Marcel Berger a défini un invariant métrique qui exprime la « taille » d'une classe d'homologie k-ième d'une variété riemannienne. Aujourd'hui cet invariant s'appelle la *systole* de dimension k. Il a demandé si les systoles peuvent être contraintes par le volume, dans l'esprit du théorème de C. Loewner de 1949. Nous construisons des métriques, inspirées par l'exemple de M. Gromov de 1993, qui donnent une réponse négative pour une classe de variétés assez large, pour le produit des systoles d'une paire de dimensions complémentaires $(k, n-k)$. Une obstruction (restriction sur k modulo 4) à la construction d'exemples encore plus généraux par notre méthode semble résider dans la partie libre de la périodicité réelle de Bott. La construction a lieu dans un voisinage scindé d'une sous-variété convenable à k dimensions, dont les composantes connexes engendrent (rationnellement) le k-ième groupe d'homologie de la variété. La géométrie bornée et l'inégalité de la coaire entrainent une minoration de la k-systole, tandis que la calibration à support dans ce voisinage fournit une minoration de la systole de la dimension complémentaire. En dimension 4, tout se réduit au cas de $S^2 \times S^2$. © Elsevier, Paris

0. Introduction

In 1972, Marcel Berger defined metric invariants that capture the 'size' of k-dimensional homology of a closed Riemannian manifold X. Given a nonzero integer homology class $\alpha \in H_k(X, \mathbf{Z})$, let

$$(0.1) \qquad \|\alpha\| = \inf_{M \in \alpha} (\operatorname{vol}_k(M)),$$

where the infimum is taken over all cycles represented by maps of k-dimensional manifolds to X (this choice of cycles is explained in Lemma 4.1). Moreover given a nontorsion

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class α, let

$$\|\alpha\|_s = \lim_{q\to\infty} \frac{1}{q}\|q\alpha\| \tag{0.2}$$

be the stable norm. We define the *k-dimensional systole* $\operatorname{sys}_k(X)$ of M. Berger by

$$\operatorname{sys}_k(X) = \inf_{\alpha\neq 0} \|\alpha\| \tag{0.3}$$

as well as the *stable systole*, or *mass*, by

$$\operatorname{mass}_k(X) = \inf \|\alpha\|_s, \tag{0.4}$$

where the infimum is taken over nontorsion classes. M. Berger asked if these invariants can be constrained by the volume (see section 1 for a history of the problem). From our vantage point, what could be the reasons for expecting such constraints? There are three reasons: (1) Loewner's theorem, (2) stable systolic inequalities of Gromov and Hebda, and (3) Gromov's theorem on the 1-systole. Let us describe these briefly.

1. By Loewner's theorem (*cf.* [34], p. 295-296 and [43], [6], [7]), every metric 2-torus T^2 satisfies

$$\frac{\operatorname{length}^2(C)}{\operatorname{area}(\mathrm{T}^2)} \leq \frac{2}{\sqrt{3}}, \tag{0.5}$$

where C is its shortest noncontractible loop, so that $\operatorname{length}(\mathrm{C}) = \operatorname{sys}_1(\mathrm{T}^2)$.

2. Every metric g on the complex projective space $\mathbf{C}\mathrm{P}^n$ satisfies

$$\operatorname{mass}_2(g)^n \leq n! \operatorname{vol}(g). \tag{0.6}$$

Similar results hold for products of spheres $(\mathrm{S}^k \times \mathrm{S}^{n-k}, g)$:

$$\operatorname{mass}_k(g) \operatorname{mass}_{n-k}(g) \leq C_n \operatorname{vol}(g), \tag{0.7}$$

with a constant C_n depending only on the dimension (*cf.* [30], p. 60; or [35], sections 4.36-38). Further generalisations are due to J. Hebda [36].

3. Let $\pi\operatorname{sys}_1(g)$ be the length of the shortest noncontractible loop for the metric g on X. M. Gromov [31] studied the following inequality in 1983:

$$(\pi\operatorname{sys}_1(g))^n \leq Const_n \ \operatorname{vol}(g). \tag{0.8}$$

He explained how one can characterize topologically the class of manifolds for which such an inequality holds for a positive $Const_n > 0$ independent of the metric. For such manifolds, moreover, the constant depends only on $n = \dim(X)$. This class, called *essential*, includes aspherical manifolds as well as real projective spaces. An oriented manifold X is *inessential* if the inclusion of X in the classifying space $B\pi_1 = K(\pi_1(X), 1)$ admits a retraction (fixing the 1-skeleton) to the $(n-1)$-skeleton of $B\pi_1$. Then the 1-systole is not constrained by the volume, as shown by the first author in [1], p. 34. We will apply a similar technique which involves maps to complexes obtained from X by attaching cells of dimension at most $n-1$ (*cf.* section 6). That one cannot expect the inequality (0.8) to be true for all manifolds is obvious from considering a simple expansion-contraction on a product of a circle and an $(n-1)$-sphere: either stretch one factor or shrink the other. Thinking about this counter-example might lead one to conjecture that it is the *product* of the 1-systole and the $(n-1)$-systole which is bounded by the volume. M. Gromov showed that this is also false (see section 1.3).

In the context of the three results cited above, one may find surprising that the systoles actually possess a lot of freedom relative to the volume of the manifold. Let us cite two typical results in the following geometric form (more general statements are given in section 1.6).

THEOREM A. – *Let X be a manifold of even dimension $n = 2m$, and assume that its middle dimensional homology group is free abelian. Then X admits metrics of arbitrarily small volume such that every middle dimensional submanifold with m-volume less than unity necessarily bounds, provided that $m \geq 3$.*

THEOREM B. – *Every compact orientable manifold X of dimension $n \geq 3$ admits metrics g of arbitrarily small volume satisfying the inequality*

$$\operatorname{vol}_{n-1}(M) \ \operatorname{length}(\mathrm{C}) > 1 \tag{0.9}$$

for every noncontractible closed curve $C \subset X$ and every orientable non-separating hypersurface $M \subset X$.

The result of Theorem B is meaningful whenever the first Betti number of X is nonzero. Theorem A is proved in a separate paper [2]. The present article will develop results along the lines of Theorem B, and will present some results in the direction of the case of dimension $n = 4$ missing from Theorem A (see section 6). Can Theorem B be generalized to an arbitrary pair of complementary dimensions? Such a generalization seems to meet a certain topological obstruction. Namely, an obstruction to constructing further examples by our methods seems to reside in the free part of real Bott periodicity. It translates into the modulo 4 condition on k of Theorem 3 in section 1.6 below (see Lemma 5.5). What are the other interesting special cases? We discuss the projective spaces (Example 5.3 and Remark 5.6), the 3-torus (Example 3.5), and products of spheres (Proposition 4.2). What is the starting point for these constructions? It is an example described in 1993 by M. Gromov (see section 1.3). The inspiration for generalizing this example derives from the following theorem of R. Thom which in our notation reads as follows.

THEOREM (R. Thom, [46], p. 32, Theorem II.4). – *Let $\alpha \in \mathrm{H}_k(X^n, \mathbf{Z})$ be an integer homology class of an orientable manifold X^n, where k is odd or else $2k < n$. Then there exists a nonzero integer N depending only on n such that the multiple class $N.\alpha$ can be represented by a submanifold whose bundle of normal vectors is trivial.*

From the point of view of Thom's theorem, the construction can be described as follows. Following Gromov ([34], section $4.A_5$), we use Thom's theorem to choose a submanifold below the middle dimension satisfying the following two properties:

(a) its normal bundle is trivial;

(b) its connected components generate $\mathrm{H}_k(X, \mathbf{Q})$.

The construction takes place within a fixed trivialized tubular neighbourhood of the submanifold. Gromov described an expansion-contraction procedure (*cf.* [33], section 2) with the desired effect on the systole of the complementary dimension. Now, our idea is to combine the expansion-contraction with a volume-saving twist. In more detail, Thom's theorem allows us to carry out the expansion-contraction procedure in a neighbourhood of a suitable k-dimensional submanifold described above. The resulting metrics k-dimensional submanifold. The resulting metrics have large $(n-k)$-dimensional systole (compared to the volume). Here the $(k, n-k)$ systolic inequality is not violated, as the volume is too large. To decrease the volume, we introduce a nondiagonal coefficient in the matrix of the Riemannian metric. This requires a splitting of both the k-dimensional class and its tubular neighbourhood. It is at this point that the modulo 4 condition on k is necessary (see Theorem 1.3). The construction works whenever a suitable multiple of each k-dimensional homology class contains a representative A with the following two properties:

(a) the normal bundle of A is trivial;

(b) A splits off a circle factor in a Cartesian product.

It is sufficient to find a representative which is a sphere with trivial normal bundle. This is done using Thom's theorem (section 5.2) and Bott periodicity (see Lemma 5.5). See section 1.7 for more details on the construction.

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1. Earlier work, statement of results

1.1 The origin of the problem. Following the pioneering work of C. Loewner and P. Pu [43] (and, later, R. Accola, C. Blatter), M. Berger [6] defined the systolic invariants in 1972 in the pages of the Annales Scientifiques de l'École Normale Supérieure, and asked if they can be constrained by the volume (see the excellent survey [8], as well as [10], section TOP.1.E, page 123).

1.2 Is volume a constraint? As late as 1992, M. Gromov was optimistic about the existence of inequalities for systoles similar to the stable systole case (see equation (0.7)). Thus at the end of section $4.\mathrm{A}_3$ of [32] one finds a discussion of the systolic $(k, n-k)$ inequality, and a program of study in terms of conditions on geometry and curvature. Section $4.A_5$ of [32] contains the following statement: "In general, we conjecture that all non-trivial intersystolic inequalities for simply connected manifolds are associated to multiplicative relations in the cohomology in the corresponding dimensions. This conjecture applies to the mass as well as to the volume and moreover it should be refined in the case of mass [...]" (the text goes on to propose such a refinement). The version of [32] published in 1996 modifies this sentence (see [34], p. 354) in the light of Gromov's subsequent developments.

1.3 The first counter-examples. It was given by M. Gromov in 1993 on $X = \mathrm{S}^3 \times \mathrm{S}^1$ and described by M. Berger in [8], p. 301 (see also [35], section 4.45 and [38], section 1). The same paper [8] states Gromov's results concerning the existence of further examples on products of spheres. The second author wrote to M. Berger in February, 1994, asking about the details of Gromov's construction. M. Berger immediately replied [9] that he was awaiting a "new manuscript from Gromov, or now from you!" It is the warm encouragement of M. Berger that initially stimulated the second author's interest in the problem. With the emergence of Gromov's 1993 example, the situation changed drastically. Most inequalities involving the higher systoles are now conjectured to fail (but see [49], Remark 1.3 regarding coefficients modulo 2). The inequalities tend to be violated even among homogeneous metrics, as the 1993 example shows. It would be interesting to explore the Lie-theoretic origin of the $S^3 \times S^1$ example and its generalizations. A homogeneous example in middle dimension on a product of spheres appears in [5] and [48], Theorem D.3.

1.4 'Massive' metrics and instability. To present a different point of view on Gromov's construction, it is convenient to introduce the following terminology. A sequence g_j of metrics is called *massive* if each g_j satisfies equality in the mass inequality (0.7) (for a fixed constant C_n independent of j); it is called *k-unstable* if

$$\lim_{j\to\infty} \frac{\mathrm{sys}_k(g_j)}{\mathrm{mass}_k(g_j)} = \infty \,. \tag{1.1}$$

In this language, the 1993 metric is massive and 1-unstable (*cf.* Remark 3.2). Note that $\mathrm{mass}_{n-1} = sys_{n-1}$ by [23], statement 5.10, p.394 (*cf.* [30], p. 59). Its 1-instability is due to the fact that at the level of the universal cover, the step of the covering transformation T of $S^3 \times \mathbf{R}$ is far larger than the displacement of the hypersurface S^3 (*i.e.* the distance between S^3 and $T(S^3)$), due to a rapid Hopf rotation. Gromov privately speculated that the same could be achieved for $X = S^2 \times S^1$ despite the absence of a free circle action, by stretching in the neighbourhood of the fixed points. Such stretching was formalized by L. Bérard Bergery and the second author in [4] in terms of the nilmanifold N of the Heisenberg group.

1.5 Semidirect products and Sol Geometry. Note also that the fundamental group of N is a semidirect product. C. Pittet [42] clarified the role of semidirect products in the construction of the example on $S^2 \times S^1$, reformulating it in terms of Sol geometry (see Remark 2.2). He generalized this example to manifolds of the form $M \times S^1$ satisfying $H_1(M) = 0$ by localizing the construction of [4].

1.6 Statement of results. There are two rather different systolic problems: in middle dimension, on the one hand, and in a pair of complementary dimensions, on the other. The case of middle-dimensional freedom is treated in [2]. In the present paper we are concerned with the case of a pair of distinct complementary dimensions, and also with the special case $n = 4$ (*see* Theorem 4 below and Remark 3.6).

DEFINITION. – *We say that X is systolically '$(k, n-k)$-free' if*

$$\inf_g \frac{\mathrm{vol}\,(g)}{\mathrm{sys}_{n-k}(g)\,\mathrm{sys}_k(g)} = 0, \tag{1.2}$$

where the infimum is taken over all Riemannian metrics on X. If $n = 2k$, we say that X is k-free.

We now state our main results. Most of the results in this area are due to M. Gromov. As described above, the first results on freedom published with detailed proofs are by L. Bérard Bergery and M. Katz [4], M. Katz [38], and C. Pittet [42].

THEOREM 1. – *Let X be a compact orientable n-manifold, $n \geq 3$. Then X is systolically $(1, n-1)$-free.*

THEOREM 2. – *Let X be a compact n-manifold, $n \geq 5$, whose fundamental group is free abelian. Then X is systolically $(2, n-2)$-free.*

THEOREM 3. – *Let X be a compact orientable $(k-1)$-connected n-manifold, where $k < \frac{n}{2}$. Then X is systolically $(k, n-k)$-free if k is not divisible by 4. If $k = 4$, then X is $(4, n-4)$ free provided that $p_1(X) = 0$.*

The 2-freedom of a simply connected 4-manifold X can be characterized as follows: X admits metrics of arbitrarily small volume such that every noncontractible surface inside it has at least unit area.

THEOREM 4. – *The following three assertions are equivalent:* (i) *the complex projective plane* CP^2 *is* 2*-free;* (ii) $\mathrm{S}^2 \times \mathrm{S}^2$ *is* 2*-free;* (iii) *every simply connected* 4*-manifold is* 2*-free**.

1.7 The construction. To explain the idea with greater care, let $A \subset X$ be a k-dimensional submanifold whose connected components (rationally) generate $\mathrm{H}_k(X)$. Assume that the normal bundle of A is trivial, so that its tubular neighbourhood in X is diffeomorphic to $A \times B^{n-k}$ (*cf.* [14]). Let $R \subset B^{n-k}$ be a codimension 2 submanifold with trivial normal bundle (for example, an $(n-k-2)$-sphere). Then the boundary of a tubular neighbourhood of $R \subset B^{n-k}$ is diffeomorphic to $R \times \mathrm{T}^1$. Assume furthermore that A splits off a circle, *i.e.* $A = \mathrm{C} \times S$ where C is the circle. Let $\Sigma \subset X$ be the hypersurface

$$\Sigma = A \times R \times \mathrm{T}^1 = \mathrm{T}^2 \times L$$

where $\mathrm{T}^2 = \mathrm{C} \times \mathrm{T}^1$ is the 2-torus and $L = S \times R$. A tubular neighbourhood of $\Sigma \subset X$ is a cylinder $\Sigma \times I = Y \times L$ where $Y = \mathrm{T}^2 \times I$ is a cylinder on the 2-torus. We construct direct sum metrics on $\Sigma = Y \times L$ which are fixed on L. Meanwhile, we will describe a sequence of special metrics on Y in Lemma 2.1, which give rise to the free metrics on X. Theorem 3 may be viewed as the most general result one can obtain by generalizing the construction of free metrics on products of spheres (see Proposition 4.2). One may ask if its modulo 4 hypothesis can be removed. While this may turn out to be possible, the free metrics on products of spheres would probably still be the starting point. Therefore, this paper may be viewed as an effort to understand the domain of applicability of the construction described in the previous paragraph, starting with a codimension 2 submanifold in the fiber of the (trivial) normal bundle of a split submanifold in X whose connected components (rationally) generate $\mathrm{H}_k(X)$. Similar results for the spectrum of the Laplacian may be found in [20], [17].

1.8 The geometry of the free metrics. We construct a sequence of metrics g_j for which the quotient (1.2) becomes smaller and smaller. The metric g_j contains two regions: one where the geometry is fixed (*i.e.* independent of j), and another (locally isometric to the left-invariant metric on the Heisenberg group) where the geometry is 'periodic' with j periods. The bounded geometry implies a uniform lower bound for the k-systole. Our key technical tools here are the coarea inequality and the isoperimetric inequality of Federer and Fleming. A suitable calibrating form with support in the 'periodic' region shows that the $(n-k)$-systole grows faster than the volume.

In section 2, we describe a local version of the construction of free metrics used in [4]. We exploit it in section 3 to prove Theorem 1. We generalize the construction to higher k-systoles in section 4 for products of spheres and in section 5 in the general case. Theorem 4 is proved in section 6, using Whitehead products and pullback arguments (*cf.* [1]) for metrics.

* The 2-freedom of orientable 4-manifolds is established in [49].

2. Calibration, bounded geometry, and the Heisenberg group

We present a local version of the construction of [4] in Lemma 2.1 below. The significance of the lemma is that the product of the 2-torus and the interval admits metrics with enough room for cylinders of unexpectedly large area (but see (2.9)). To distinguish two circles which play different roles in the construction, we denote them, respectively, T^1 and C. Consider the cylinder $M = \mathrm{T}^1 \times I$, circle C, and the manifold

$$Y = \mathrm{C} \times M = \mathrm{T}^2 \times I.$$

We view M as a relative cycle in $\mathrm{H}_2(Y, \partial Y)$. The 2-mass of the class $[M]$ is the infimum of areas of *rational* cycles representing it, where $\mathrm{area}\,(\Sigma_i(r_i\sigma_i)) = \Sigma_i |r_i|\,\mathrm{area}\,(\sigma_i)$, $r_i \in \mathbf{Q}$.

LEMMA 2.1. – *The manifold* $Y = \mathrm{T}^2 \times I$ *satisfies*

$$\inf_g \frac{\mathrm{vol}\,(g)}{\mathrm{sys}_1(g)\,\mathrm{mass}_2([M])} = 0,$$

where the infimum is taken over all metrics whose restriction to each component of the boundary $\partial Y = \mathrm{T}^2 \times \partial I$ *is the standard 'unit square' torus satisfying* $\mathrm{length}\,(\mathrm{C}) = \mathrm{length}\,(\mathrm{T}^1) = 1$.

Proof. – More precisely, we will show that there exists a sequence of metrics $Y_j = (Y, g_j)$ satisfying the following four conditions:

(i) the restriction of g_j to the boundary $\mathrm{T}^2 \times \partial I$ at each endpoint is the standard unit square metric for which T^1 and C have unit length; (ii) the 1-systole of g_j is uniformly bounded from below; (iii) the volume of g_j grows at most linearly in j; (iv) the 2-mass of $[M] \in \mathrm{H}_2(Y, \partial Y)$ grows at least quadratically in j.

We present a shortcut to the explicit formula for the solution. A reader interested in the method of arriving at such a formula geometrically can consult [4], p. 630. Consider the following metric $h(x)$ in the y,z-plane depending on a parameter $x \in \mathbf{R}$:

$$h(x)(y,z) = (x\mathrm{d}y - \mathrm{d}z)^2 + \mathrm{d}y^2 \tag{2.0}$$

Let $I = [0, 2j]$. For $x \in I$, set $\widehat{x} = j - |x - j| = \min(x, 2j - x)$. Consider the fundamental domain $D = \{0 \le x \le 2j,\ 0 \le y \le 1,\ 0 \le z \le 1\}$. The metric

$$g_j = h(\widehat{x})(y,z) + \mathrm{d}x^2, \text{ where } (x,y,z) \in D \tag{2.1}$$

gives rise to a metric g_j on $\mathrm{T}^2 \times I$ once we identify the opposite sides of the unit square in the yz-plane. The circles $\mathrm{T}^1, \mathrm{C} \subset \mathrm{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$ are parametrized, respectively, by the y- and z-axis. The circles T^1 and C have lengths, respectively, $\sqrt{\widehat{x}^2 + 1}$ and 1 with respect to the metric $h(\widehat{x}) = (1 + \widehat{x}^2)\mathrm{d}y^2 + \mathrm{d}z^2 - 2\widehat{x}\mathrm{d}y\,\mathrm{d}z$ on $\mathrm{T}^2 \times \{x\}$. The length of T^1 thus stretches from 1 to $\sqrt{j^2 + 1}$ and then shrinks back to 1, so as to satisfy (i). Our choice of the metric such that the length of $\mathrm{T}^1 \subset \mathrm{T}^2 \times \{x\}$ is $\sqrt{1 + \widehat{x}^2}$ rather than simply $\widehat{x}$ results in local homogeneity (see below). The metric g_j is symmetric with respect to the midpoint $x = j$. The map

$$\psi : (x,y,z) \mapsto (x+1, y+z, z) \tag{2.2}$$

defines an isometry from $\mathrm{T}^2 \times [i-1, i]$ to $\mathrm{T}^2 \times [i, i+1]$ for $i = 1, \ldots, j-1$. Thus the metric g_j on $\mathrm{T}^2 \times [0, j]$ is 1-periodic (*cf.* (2.6)), proving (ii). The x-axis projection $Y \to I$ is a Riemannian submersion over an interval of length $2j$ with fibers of unit area, hence the volume estimate (iii). We obtain the lower bound (iv) for the 2-mass of $[M]$ as in [4], pp. 625-626 by a calibration argument. First we calculate the area of the cylinder M. Recall that the universal cover of M is a subset of the xy-plane in our coordinates. Since $\mathrm{T}^1 \times \{x\}$ has length $\sqrt{x^2+1}$, we have

$$\text{(2.3)} \qquad \operatorname{area}(M) = \int_0^{2j} \operatorname{length}(\mathrm{T}^1 \times \{x\}) \mathrm{d}x = 2 \int_0^j \sqrt{x^2+1}\, \mathrm{d}x \sim j^2.$$

The calibrating form is the pullback of the area form of M by the nearest-point projection. The projection, while not distance-decreasing, is area-decreasing. Consider the 2-form

$$\text{(2.4)} \qquad \alpha = \sqrt{1+x^2} \mathrm{d}x \wedge \mathrm{d}\lambda \text{ where } \lambda = y - \frac{x}{1+x^2} z.$$

Note that $\sqrt{1+x^2}\alpha = *(\mathrm{d}z)$ where $*$ is the Hodge star of the left-invariant metric. This formula is easily verified with respect to the orthonormal basis of left-invariant forms $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z - x\mathrm{d}y$ (we owe this remark to T. Hangan). (Note that in Gromov's example, the calibrating form is in fact the Hodge star of the projection to the circle fibre, suitably normalized). The restriction of α to M coincides with the area form of M for $x \leq j$, and α is of unit norm (see [4], pp. 627-628 for details). Let $\phi_j(x)$ be a partition of unity type function with support in $]0, j[$ and such that $\phi_j(x) = 1$ for $x \in [1, j-1]$. The form $\phi_j \alpha$ is closed. Let M' be any rational cycle representing the class $\epsilon[M] \in \mathrm{H}_2(Y, \partial Y)$, where $\epsilon = \pm 1$. As in [4], p. 626, we have from (2.3):

$$\text{(2.5)} \qquad \operatorname{mass}(\epsilon[M]) = \inf_{M'} \operatorname{area}(M') \geq \int_{M'} \epsilon \phi_j \alpha = \int_M \phi_j \alpha \geq \int_1^{j-1} x \mathrm{d}x \sim j^2,$$

proving Lemma 2.1.

Remark 2.2. – The isometry $\psi : (x, y, z) \mapsto (x+1, y+z, z)$ acts in the yz-plane by the unipotent matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Pittet [42] replaced this action by a hyperbolic one, producing relative 2-cycles with exponential growth of the 2-mass, rather than quadratic as in (iv) above.

Remark 2.3. – The universal cover of the 'half' $\mathrm{T}^2 \times [0, j]$ is isometric to the subset defined by the condition $0 \leq x \leq j$ of the Heisenberg group of unipotent matrices

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}; \; x, y, z \in \mathbf{R}$$

with the standard left-invariant metric $\mathrm{d}x^2 + \mathrm{d}y^2 + (\mathrm{d}z - x\mathrm{d}y)^2$ (*cf.* [18], p. 67; [26], p. 227). The isometry ψ is left multiplication by the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $N = G/\Gamma$ be the standard nilmanifold of the Heisenberg group G, where Γ consists of matrices with integer entries. Factoring by the iterates of ψ, we obtain a projection $f : \mathrm{T}^2 \times [0, j] \to N$ which is a local isometry. Now take the interval $I = [0, 2j]$ and fold it in two at $x = j$, *i.e.* send x to $\min(x, 2j - x)$. Let $g : Y_j \to \mathrm{T}^2 \times [0, j]$ be the resulting folding map on Y_j. The *distance decreasing* projection $f \circ g : Y_j \to N$ induces a monomorphism at the level of the fundamental groups. Therefore

$$\operatorname{sys}_1(Y_j) \geq \pi \operatorname{sys}_1(N), \tag{2.6}$$

where $\pi \operatorname{sys}_1$ is the length of the shortest noncontractible loop. Meanwhile, the induced homomorphism in 1-dimensional homology sends the class $[\mathrm{C}]$ to 0, as the z-axis in G projects to a loop in the center of the fundamental group Γ of N, which is also its commutator subgroup.

Remark 2.4. – The group Γ is the Heisenberg group over $\mathbf{Z}$. It is presented by generators x, y, z and relations $[x, y] = z$, $[x, z] = 1$, $[y, z] = 1$, where $[a, b] = aba^{-1}b^{-1}$. Let $a^{(b)} = bab^{-1}$. Note that for every positive integer j we have the following relation in Γ:

$$z^j = y^{(x^j)} y^{-1}, \tag{2.7}$$

which is the combinatorial antecedent of (2.8) below. Let

$$v = \begin{bmatrix} 1 \\ j \end{bmatrix}, \quad A = \begin{bmatrix} 1 + j^2 & -j \\ -j & 1 \end{bmatrix},$$

so that ${}^t v A v = 1$. It follows that the shortest loop in the class $[\mathrm{T}^1 + j\mathrm{C}] \in \mathrm{H}_1(\mathrm{T}^2 \times \{j\})$ has unit length, since the torus $\mathrm{T}^2 \times \{j\}$ is equipped with the metric $\mathrm{d}y^2 + (\mathrm{d}z - j\,\mathrm{d}y)^2$. Thus by sliding the curve $\mathrm{C} \subset \mathrm{T}^2$ to the value $x = j$, we obtain

$$\operatorname{mass}_1[\mathrm{T}^1 + j\mathrm{C}] \leq 1. \tag{2.8}$$

Remark 2.5. – The metric h_x, defined by the matrix $\begin{bmatrix} 1 + \widehat{x}^2 & -\widehat{x} \\ -\widehat{x} & 1 \end{bmatrix}$, where $\widehat{x} = \min(x, 2j - x)$, is flat of unit area. The flat tori of unit area and unit 1-systole used in our construction all lie in a compact part of the moduli space of tori, namely the interval $s + \sqrt{-1}$ for $s \in [-\frac{1}{2}, \frac{1}{2}]$ in the standard fundamental domain in $\mathbf{C}$. The diameter of each of these tori is less than 1, so that our manifold Y_j is rather narrow:

$$\operatorname{diam}_1(Y_j) < 1, \tag{2.9}$$

where the 1-diameter of a Riemannian manifold X, denoted $\operatorname{diam}_1 X$, is the infimum of numbers $\varepsilon > 0$ such that there exists a continuous map from X to a graph with the property that the inverse image of every point has diameter $\leq \varepsilon$. More precisely, given a map $f : X \to \gamma$, from a Riemannian manifold X to a graph γ, define the size $s(f)$ by $s(f) = \sup_{x \in \gamma} \operatorname{diam}(f^{-1}(x))$. Then the 1-diameter of X is the least size of a map from X to a graph: $\operatorname{diam}_1(X) = \inf_{\gamma, f} s(f)$, where the infimum is taken over all graphs γ and all continuous maps $f : X \to \gamma$. Setting $\gamma = [0, 2j]$ and $f =$ the x-coordinate, we obtain (2.9).

Remark 2.6. – The construction of the manifold Y of Lemma 2.1 can be summarized as follows. One takes a torus bundle, say N, over a circle, whose glueing automorphism A

is not an isometry. Let v be a lattice vector whose images increase indefinitely in length under the iterates of A^{-1}. One chooses a fixed metric on N and pulls it back to the infinite 3-dimensional cylinder $Y_\infty = \mathrm{T}^2 \times \mathbf{R}$, where $\mathbf{R}$ is the universal cover of the circle. This produces a periodic metric on Y_∞. Consider the 2-dimensional cylinder $M \subset Y_\infty$ which is the circle subbundle spanned by the direction v. The key point now is that the area of M grows faster than the volume of Y_∞ due to the choice of the direction v. One then takes a long piece of Y_∞ and doubles it as described above to obtain the desired metric.

Remark 2.7. – A. Besicowitch [11] in 1952 exhibited a different type of (1,2)-freedom on a 3-dimensional manifold with boundary, namely the cylinder $D^2 \times [0,1]$ (*cf.* [13], p. 296), disproving a conjecture of Loewner.

3. Construction in dimension/codimension one

PROPOSITION 3.1. – *Every orientable n-manifold with $b_1(X) = 1$ is systolically $(1, n-1)$-free if $n \geq 3$.*

Proof. – Such metrics on X can be obtained by pulling back free metrics on $\mathrm{S}^1 \times \mathrm{S}^{n-1}$ of [42] by a simplicial map of nonzero degree. We describe a construction which lends itself to a generalisation to the case $b_1 > 1$. The construction of free metrics is local in a neighbourhood of a loop C which generates $\mathrm{H}_1(X, \mathbf{Z})$ modulo torsion. Since X is orientable, the normal bundle of C is trivial, and its tubular neighbourhood in X is diffeomorphic to $\mathrm{C} \times B^{n-1}$. Let $L \subset B^{n-1}$ be a codimension 2 submanifold which then has trivial normal bundle (for example, an $(n-3)$-sphere). The boundary of a tubular neighbourhood of $L \subset B^{n-1}$ is diffeomorphic to $L \times \mathrm{T}^1$. Let $\Sigma \subset X$ be the hypersurface

$$\Sigma = \mathrm{C} \times L \times \mathrm{T}^1 = \mathrm{T}^2 \times L \tag{3.1}$$

where $\mathrm{T}^2 = \mathrm{C} \times \mathrm{T}^1$ is the 2-torus. A tubular neighbourhood of $\Sigma \subset X$ is a cylinder $\Sigma \times I = Y \times L$ where $Y = \mathrm{T}^2 \times I$ is a cylinder on the 2-torus. We construct direct sum metrics on $Y \times L$ which are fixed on L. The special metrics on Y of Lemma 2.1 give rise to the free metrics on X. A similar technique was used by C. Pittet [42]. His idea was to use the torus $\Sigma = \mathrm{T}^{n-1}$ (where we use $\mathrm{T}^2 \times L$), but only 2 circles are actually needed. What happens metrically can be described as follows. We choose a metric on X which is a direct sum in

$$\Sigma \times I = \mathrm{T}^2 \times L \times I, \tag{3.2}$$

where T^2 is the standard unit square torus (with C and T^1 both of unit length) and L has unit volume. We now modify the metric in $\Sigma \times I = Y \times L$ by means of the metric $Y = (Y, g_j)$ of Lemma 2.1, while L keeps the same fixed metric of unit volume, and the metric on $Y_j \times L$ is a direct sum. Condition (i) of Lemma 2.1 ensures that the metric varies continuously across ∂I. Denote the resulting Riemannian manifold X_j. The metric of X_j stays the same on the complement of $\Sigma \times I$, while on the region $Y_j \times L$ it is periodic in the sense given in the proof of Lemma 2.1. A loop of length less than 1 is contained either in the cylinder, or in the 1-neighbourhood of its complement. In the latter case, it can be viewed as a loop in X_j for $j = 1$. Thus

$$\operatorname{sys}_1(X_j) \geq \min(\operatorname{sys}_1(X_1), \pi \operatorname{sys}_1(N)) \tag{3.3}$$

by (2.6). Hence the 1-systole (as well as the homotopy 1-systole, justifying the inequality (0.9) of Theorem B) is uniformly bounded from below as j increases. The calibrating $(n-1)$-form β is supported on $Y \times L$. It is obtained from the form (2.4) of Lemma 2.1 by exterior product with the volume form vol_L of L:

$$\beta = \phi_j \alpha \wedge \mathrm{vol}_L . \tag{3.4}$$

Choose a hypersurface $\tilde{M} \subset X$ 'dual' to C, which has standard intersection B^{n-1} with a tubular neighbourhood of C. Then $\tilde{M} \cap Y_j = M$, the 2-dimensional cylinder of Lemma 2.1. Hence

$$\mathrm{sys}_{n-1}(X_j) \geq \mathrm{mass}_{n-1}([\tilde{M}]) \geq \int_{\tilde{M}} \beta = \mathrm{vol}\,(L) \int_M \phi_j \alpha \sim j^2, \tag{3.5}$$

proving Proposition 3.1.

Remark 3.2. – This is a convenient time to explain why these metrics on X are 1-unstable, *i.e.* the 1-mass tends to 0. Indeed, by construction the curve T^1 is contractible in X. Hence we have from (2.8),

$$\mathrm{mass}_1[\mathrm{C}] = \frac{1}{j}\,\mathrm{mass}_1[j\mathrm{C}] = \frac{1}{j}\,\mathrm{mass}_1[\mathrm{T}^1 + j\mathrm{C}] \leq \frac{1}{j} \to 0. \tag{3.6}$$

The map $f \circ g$ of section 2.3 extends to a distance-decreasing map h to a fixed CW complex $X \cup (N \times L)$,

$$h : X_j \to X \cup (N \times L), \tag{3.7}$$

where manifolds X and $N \times L$ are glued along the common hypersurface $\mathrm{T}^2 \times L$. The induced homomorphism at the level of fundamental groups has nontrivial kernel generated by the curve C (the commutator of T^1 and the other generator of $\pi_1(N)$). Note that a contrary, and false, claim was made in [4], p. 626 concerning the homomorphism p_5, in order to prove a uniform lower bound for the 1-systole (see (4.9)). However, such a bound follows immediately from the bounded geometry resulting from the construction, without using the fixed CW complex.

Remark 3.3. – To a myopic observer, X_j looks like an interval of length $2j$. Indeed, let γ be the graph defined by the connected components of the level sets of the distance function from the subset $X_+ \subset X_j$. By (2.9),

$$\mathrm{diam}_1(X_j) \leq \max(1, \mathrm{diam}\,(X_+)). \tag{3.8}$$

Remark 3.4. – We explain the choice of C and M above in terms of Morse theory. Let a be a generator of the cohomology group $\mathrm{H}^1(X, \mathbf{Z}) = \mathbf{Z}$. Then a defines a map $X \to \mathrm{S}^1$ which can be made into a Morse map. The inverse image of a regular point is a smooth submanifold M, which may be assumed to be connected by the Rearrangement Theorem (see J. Milnor, *Lectures on h-cobordism Theorem*). We cut X open along M (*i.e.* cut S^1 at the regular point) to obtain a cobordism with top and bottom given by M, while the map to S^1 becomes an ordinary Morse function on this cobordism. Next, take the 'same' point at top and bottom (*i.e.* the same as a point of M) and join them in the cobordism by a path with the following two properties: (i) it avoids the critical points of the Morse function; (ii) the Morse function is strictly monotone along the path. Now if we glue top and bottom to form X again, this path becomes a loop C which meets M transversely in exactly one point $p \in X$.

Proof of Theorem 1. – In the general case $b = b_1 \geq 1$, we will first construct curves $C_1, \ldots, C_b$ and hypersurfaces $M_1, \ldots, M_b$ such that the intersection $M_i \cap C_k$ contains exactly δ_{ik} points, as follows. We choose a $\mathbf{Z}$-basis $a_1, \ldots, a_b$ for the 1-dimensional cohomology $\mathrm{H}^1(X, \mathbf{Z})$. We define M_i to be the inverse image of a regular point of a map $X \to \mathrm{S}^1$ defined by a_i. The b maps define the period map $X \to \mathrm{T}^b$ which is surjective at the level of the fundamental groups: $\pi_1(X) \to \pi_1(\mathrm{T}^b) = \mathbf{Z}^b$ (Poincaré duality). We now choose curves $C_1, \ldots, C_b \subset X$ whose images represent the standard generators of $\pi_1(\mathrm{T}^b) = \pi_1(\mathrm{S}^1) + \ldots + \pi_1(\mathrm{S}^1)$. Then $a_i(C_k) = \delta_{ik}$ and so the algebraic number of points in the intersection $M_i \cap C_k$ is exactly δ_{ik}. To eliminate points of intersection with negative intersection index, we choose two *adjacent* points on C_k with opposite intersection indices. We now perform a surgery on M_i by removing a little disk around each of the two points and attaching a thin tube to M_i along the piece of C_k joining the two points. In this way, we remove all negative intersections. In particular, we may assume that the intersection $M_i \cap C_k = \emptyset$ is empty if $i \neq k$. We choose a small $\epsilon > 0$ so that $M_i \cap (\cup_\epsilon C_k) = \emptyset$ for all $i \neq k$. We then insert b copies of $Y_j \times L$ inside $\cup_\epsilon C_k$ as in the argument following formula (3.2) for $k = 1, \ldots, b$, to obtain a new Riemannian manifold X_j diffeomorphic to X. The calibration argument is generalized as follows. Let $\beta_k = \phi_j \alpha \wedge \mathrm{vol}_L$ be the closed $(n-1)$-form supported in $\cup_\epsilon C_k$. Then

$$\int_{M_i} \beta_k = 0 \text{ if } i \neq k. \tag{3.9}$$

Take any nonzero integer class $m = \sum_i \epsilon_i d_i [M_i] \in \mathrm{H}_{n-1}(X)$ where $\epsilon_i = \pm 1$ and $d_i \geq 0$, $i = 1, \ldots, b$. We use the signs ϵ_i to specify a calibration form $\beta = \sum_k \epsilon_k \beta_k$. Since the supports of the β_k are disjoint, the form β has norm 1. Let $M' \in m$ be any rational cycle. Then

$$\mathrm{vol}_{n-1}(M') \geq \int_{M'} \beta = \int_{M'} \sum \epsilon_k \beta_k = \sum_{i,k} \epsilon_i d_i \int_{M_i} \epsilon_k \beta_k. \tag{3.10}$$

In view of (3.9) we have

$$\mathrm{vol}_{n-1}(M') \geq \sum_i d_i \int_{M_1} \beta_1 \geq \int_{M_1} \beta_1 \sim j^2. \tag{3.11}$$

This completes the proof of Theorem 1.

3.5. Example of the 3-torus. Let $X = \mathrm{T}^3 = \mathbf{R}^3/\mathbf{Z}^3$. Define three curves $C_1, C_2, C_3 \subset \mathrm{T}^3$ respectively as the projections of the lines $\{(t, \frac{1}{3}, \frac{1}{3})\}$, $\{(\frac{1}{3}, t, \frac{2}{3})\}$, and $\{(\frac{2}{3}, \frac{2}{3}, t)\}$, where $t \in \mathbf{R}$. Let $M_i \subset \mathrm{T}^3$ be the 2-torus which is the projection of the coordinate plane perpendicular to C_i, *e.g.* M_1 is the projection of $\{(0, s, t); s, t \in \mathbf{R}\}$. Then $M_i \cap C_k$ consists of δ_{ik} points. The boundary of the $\frac{1}{9}$-neighbourhood of C_i is a 2-torus

$$\mathrm{T}_i^2 = \partial(\cup_{\frac{1}{9}} C_i) = C_i \times \mathrm{T}^1. \tag{3.12}$$

We modify the flat metric in a $\frac{1}{27}$-neighbourhood of each T_i^2 to make it into the direct sum of circles C_i and T^1 of unit length. We cut X open along each T_i^2 and insert three copies of Y_j of Lemma 2.1 along the cuts. The resulting manifold (T^3, g_j) has 1-systole uniformly bounded below as j increases, 2-systole growing as j^2, and volume as j. This metric bears a formal resemblance to the 'Hedlund' metrics described by V. Bangert [3], p. 278, though it is not obtained by a conformal coordinate change from the flat metric.

Remark 3.6. – Taking a product of (T^3, g_j) with a circle of length j^2, we obtain metrics on the 4-torus with 2-systole growing faster than the square root of the volume. In other words, T^4 admits metrics of arbitrarily small volume such that every surface inside it representing a nonzero class in $\mathrm{H}_2(\mathrm{T}^4)$ has at least unit area.

4. Product of spheres, coarea inequality, and intersection number

The product of the $(1, n-1)$-free metrics of section 3 on an n-manifold X with the sphere S^{k-1} yields $(k, n-1)$-free metrics on $X \times \mathrm{S}^{k-1}$. As we will see, it turns out that it is sufficient to have a product structure at the level of a k-dimensional class containing a representative A with trivial normal bundle. If such an A splits off a circle C:

$$A = B \times \mathrm{C} \tag{4.1}$$

for some $(k-1)$-dimensional B, we can start pasting in the special metrics of Lemma 2.1. The idea of the construction is to keep the factor B of A as a direct summand, while inserting, as in the case $k = 1$, a cylinder obtained by doubling a piece of the Heisenberg group of length $j \to \infty$. Here the circle C plays the same role as in the local construction of Lemma 2.1 on $Y_j = \mathrm{T}^2 \times I$. We will illustrate the construction for the product of two spheres (Proposition 4.2), and treat the general case in the next section. All of our lower bounds for the k-systole for $k \geq 2$ rely upon a technique which combines the coarea inequality and the existence of an intersection number of cycles (dual to the cup product of the dual classes). We now present the relevant lemma. All manifolds are assumed orientable. Recall that the intersection pairing in an n-manifold X with boundary,

$$\mathrm{H}_p(X) \otimes \mathrm{H}_q(X, \partial X) \to \mathrm{H}_{p+q-n}(X),$$

is the homological operation Poincaré-Lefschetz dual to the cup product in cohomology

$$\mathrm{H}^{n-p}(X, \partial X) \otimes \mathrm{H}^{n-q}(X) \to \mathrm{H}^{2n-p-q}(X, \partial X).$$

Recall also that the intersection in question needs to be transverse. Since a treatment of transverse intersection of arbitrary cycles does not seem to be readily available in the literature, we will state precisely the result we need and prove it using only the intersection number of cycles of complementary dimension, treated in [39].

LEMMA 4.1. – *Let D, E, and G be submanifolds of a manifold X meeting transversely in a single point, and consider the transverse intersection*

$$F = D \cap E. \tag{4.2}$$

Let $d \in [D]$ be a cycle defined by the map of a manifold into X. Then for a dense open set of such maps, the intersection $f = d \cap E$ is a cycle but not a boundary.

Proof. – If D and E have complementary dimensions, the lemma is immediate from the existence of the intersection number of two cycles, Lemma 10 of J. Schwartz [44] p. 31 (*cf.* [19], [27], [39], [22], vol. 2, 201O, p. 771). In the general case, the fact that the class of the intersection of the representative cycles is independent of the representatives seems to be difficult to find in the literature. Note that, by the Jiggling Lemma ([44], p. 24), the intersection $d \cap E$ may as well be assumed to be the image of a manifold. In particular, $d \cap E$ is a cycle. By the associativity of set-theoretic intersection, we have

$$(d \cap E) \cap G = d \cap (E \cap G). \tag{4.3}$$

Applying the above lemma on the intersection number twice, we obtain

$$[f] \cdot [G] = [f \cap G] = [d \cap (E \cap G)] = [d] \cdot [E \cap G] \tag{4.4}$$

and therefore

$$[f] \cdot [G] = [D] \cdot [E \cap G] = [D \cap E \cap G] = 1 \in \mathrm{H}_0(X). \tag{4.5}$$

Thus $[f]$ is nonzero. Note that we need to work with maps from manifolds rather than arbitrary cycles to be able to apply the Jiggling lemma. Note also that we do not prove that $[f] = [F]$, only that the class $[f]$ is nonzero.

PROPOSITION 4.2. – *The manifold $X = \mathrm{S}^k \times \mathrm{S}^{n-k}$ is $(k, n-k)$-free for all $n \geq 3$ and all k, except possibly $\mathrm{S}^2 \times \mathrm{S}^2$.*

Proof. – For $k = 1$, the proposition is a special case of Theorem 1. The k-freedom of $\mathrm{S}^k \times \mathrm{S}^k$ for $k \geq 3$ was established in [38]. We may thus assume that $2 \leq k < \frac{n}{2}$. Let $A \subset X$ be a copy of the sphere S^k. We represent the class $[A] \in \mathrm{H}_k(X)$ by an imbedded product $\mathrm{S}^{k-1} \times \mathrm{C} \subset X$, where C is a circle (*cf.* [28], p. 33). This can be done inside an imbedding with trivial normal bundle of $A \times I$ in X, where I is an interval. Join two points of $A \subset A \times I$ by a path disjoint from A. Remove little ϵ-disks around its endpoints in A. Attach to A the boundary of the tubular ϵ-neighbourhood of the path. The resulting hypersurface $\mathrm{S}^{k-1} \times \mathrm{C} \subset A \times I$ has trivial normal bundle in $A \times I$ and hence also in X. Hence its tubular ϵ'-neighbourhood $\cup_{\epsilon'}(\mathrm{S}^{k-1} \times \mathrm{C})$ is diffeomorphic to $\mathrm{S}^{k-1} \times \mathrm{C} \times B^{n-k}$. Let $M \subset X$ be a copy of the sphere S^{n-k} which meets $\mathrm{S}^{k-1} \times \mathrm{C}$ in a single point. We can assume that the intersections are standard: $M \cap (\cup_{\epsilon'}(\mathrm{S}^{k-1} \times \mathrm{C})) = B^{n-k}$. Let $\mathrm{T}^1 \subset B^{n-k}$ be an imbedded circle. For sufficiently small ϵ'', the boundary of the tubular ϵ''-neighbourhood of T^1 is diffeomorphic to $\mathrm{T}^1 \times \mathrm{S}^{n-k-2} \subset B^{n-k} \subset M$. The hypersurface

$$\Sigma = \mathrm{S}^{k-1} \times \mathrm{C} \times \mathrm{T}^1 \times \mathrm{S}^{n-k-2} \subset \cup_{\epsilon'}(\mathrm{S}^{k-1} \times \mathrm{C}) \tag{4.6}$$

separates X into two connected components, $X_- = \mathrm{S}^{k-1} \times \mathrm{C} \times \mathrm{T}^1 \times B^{n-k-1}$ and X_+ (for which no product structure is available). Let $\mathrm{T}^2 = \mathrm{C} \times \mathrm{T}^1$ and $L = \mathrm{S}^{k-1} \times \mathrm{S}^{n-k-2}$. We choose a fixed metric on X in such a way that T^2 becomes the unit square torus (with both C and T^1 of unit length), while the product $Y \times L$ has a direct sum metric. We now insert a 'cylinder'

$$\Sigma \times I = Y \times L \tag{4.7}$$

where $Y = \mathrm{T}^2 \times I$, and use the metrics g_j of Lemma 2.1 on Y, to obtain the manifold

$$X_j = X_- \cup (Y_j \times L) \cup X_+ \tag{4.8}$$

diffeomorphic to X. The quadratic (in j) lower bound for the $(n-k)$-systole of X_j is obtained by calibration using the form $\beta = \phi_j \alpha \wedge \mathrm{vol}_{\mathrm{S}^{n-k-2}}$ (*cf.* formula (3.4)). Proposition 4.2 now follows from the following lemma.

LEMMA 4.3. – *The k-systole of X_j is uniformly bounded from below in j.*

Proof. – Recall that the metrics (Y, g_j) of Lemma 2.1 are periodic, while the metric on X outside $Y \times L$ is fixed. In the case $k = 1$ this was sufficient to obtain a lower bound for the 1-systole. However, for $k \geq 2$, the diameter of a small k-cycle is not necessarily small. We therefore have to give some proofs using the coarea (Eilenberg's) inequality. Let z be a k-cycle in X_j representing a nonzero homology class.

Case 1. If z lies in $X_- \cup X_+ \subset X_j$ then it may be viewed as a cycle in (X, g). Hence $\mathrm{vol}\,(z) \geq \mathrm{sys}_k(g)$.

Case 2. Suppose z avoids $X_- \cup X_+$, then $z \subset Y_j \times L \subset X_j$. Let $\pi : Y_j \times L \to \mathrm{S}^{k-1}$ be the projection to the first factor of L. Since we have a projection of a Riemannian product, the coarea inequality applies. Assuming transversality, we can write

$$\mathrm{vol}\,(z) \geq \int_{\mathrm{S}^{k-1}} \mathrm{length}\,(z \cap \pi^{-1}(t))\mathrm{d}t \geq \int_{\mathrm{S}^{k-1}} \mathrm{sys}_1(Y_j)\mathrm{d}t \geq \int_{\mathrm{S}^{k-1}} \pi\,\mathrm{sys}_1(N)\mathrm{d}t, \tag{4.9}$$

again a uniform lower bound (*cf.* formula (2.6)). To justify the last inequality, we apply a relative version of Lemma 4.1 with $D = \mathrm{S}^{k-1} \times \mathrm{C}$, $E = Y_j \times \mathrm{S}^{n-k-2}$, and $F = \mathrm{C}$. We conclude that the cycle $z \cap \pi^{-1}(t) \subset Y_j \times \{t\} \times \mathrm{S}^{n-k-2}$ also represents a nonzero multiple of $[\mathrm{C}] \in \mathrm{H}_1(\Sigma \times I)$ and therefore participates in the evaluation of $\mathrm{sys}_1(Y_j \times L) = \mathrm{sys}_1(Y_j)$.

General case. The idea is to cut z at a narrow place and split it into the sum of 2 cycles which fall into the 2 special cases above. We argue by contradiction. Suppose the area of the k-cycle z_j in X_j tends to 0 as $j \to \infty$. Let d be the distance function from the subset $X_- \sqcup X_+ \subset X_j$ (on the inserted cylinder, d equals $\min(x, 2j - x)$). By the coarea inequality, we find $x_0 \in [0,1]$ such that $\mathrm{vol}_{k-1}(z \cap d^{-1}(x_0)) \to 0$. Note that the subset $d^{-1}([0, x_0]) \subset X_j$ admits a continuous retraction with a fixed Lipschitz constant to $(X_- \sqcup X_+, g)$, and its complement, to $\Sigma \times I = Y_j \times L$. Let $\gamma = z \cap d^{-1}(x_0)$. Since $\mathrm{vol}_{k-1}(\gamma) \to 0$, it is homologous to 0 in $\Sigma = \mathrm{S}^{k-1} \times \mathrm{C} \times \mathrm{T}^1 \times \mathrm{S}^{n-k-2}$ (product of four spheres). Hence γ can be filled in by a k-chain D with $\mathrm{vol}_k(D) \to 0$, so that $\partial D = \gamma$. This follows from the isoperimetric inequality for small cycles in products of spheres, proved in [38], p. 203 (the generalisation to the case of 4 spheres instead of 2 is straightforward). This isoperimetric inequality is an immediate consequence of the isoperimetric inequality of Federer and Fleming [24]. It is also a special case of [31], Sublemma 3.4.B'. Now we let $a = (z \cap d^{-1}([0, x_0])) - D$ and $b = z - a$, and apply the two special cases discussed above, obtaining

$$\mathrm{vol}_k(z) \geq \min(\mathrm{vol}_k(a), \mathrm{vol}_k(b)) - \mathrm{vol}_k(D) \geq \min(\mathrm{sys}_k(g), \pi\,\mathrm{sys}_1(N)) - o(1). \tag{4.10}$$

This proves Lemma 4.3 and Proposition 4.2.

5. Construction in general codimension and real Bott periodicity

LEMMA 5.1. – *Let X be an n-dimensional orientable manifold and let $k < \frac{n}{2}$. Assume that $\mathrm{H}_{k-1}(X)$ is torsion-free. Then X is $(k, n-k)$-free if a suitable multiple of each k-dimensional homology class contains a representative A with trivial normal bundle such that either A splits off a circle in a Cartesian product (*i.e.* $A = B \times \mathrm{C}$ where C is a circle), or A is a sphere S^k.*

Proof. – By Poincaré duality $\mathrm{H}_{n-k}(X) = \mathrm{H}^k(X)$. By the universal coefficient formula (*cf.* [29], p. 194, ex. (23.40)), the torsion of this group equals that of $\mathrm{H}_{k-1}(X)$ which vanishes by hypothesis. Let M_i be an integer basis for $\mathrm{H}_{n-k}(X)$. Let A_l, a dual 'basis' of $\mathrm{H}_k(X)$, in the sense that the intersection numbers satisfy $M_i \cdot A_l = \delta_{il}$ (*cf.* section 3). Recall that the intersection numbers are well defined for each pair of cycles of complementary dimensions by [44], p. 30-31. By hypothesis, a suitable multiple of each class A_l contains a representative to which we can apply the construction of Proposition 3.1. These multiples are fixed once and for all. Therefore a systolic lower bound for the multiple implies a lower bound for the classes A_l themselves. If $b_k = 1$, we carry out the construction of Proposition 3.1, with $L = \mathrm{S}^{n-3}$ replaced by $L = B \times \mathrm{S}^{n-k-2}$, as in the proof of Proposition 4.2. For arbitrary b_k we proceed as in the proof of Theorem 1. If $A = \mathrm{S}^k$, we add a homologically trivial handle to replace S^k by $\mathrm{S}^{k-1} \times \mathrm{C}$, where C is a circle, and argue as before.

Proof of Theorem 2. – An oriented rank $n-2$ bundle ν over an oriented surface A is determined by the identification of a pair of trivialisations over a small disk and its complement, along the circle which is their common boundary, *i.e.* by an element of $\pi_1(\mathrm{SO}_{n-2}) = \mathbf{Z}_2$ since $n-2 \geq 3$. This $\mathbf{Z}_2$ information is also contained in the second Stiefel-Whitney class $w_2(\nu) \in \mathrm{H}^2(A, \mathbf{Z}_2) = \mathbf{Z}_2$, or equivalently the second Stiefel-Whitney number $w_2[\nu] = w_2(\nu)[A] \in \mathrm{H}_0(A, \mathbf{Z}_2) = \mathbf{Z}_2$ (*cf.* [41], p. 50). Now let $g : A \to X$ be an imbedding, and let $\nu = \nu(A \subset X)$ be its normal bundle in X. Since $g^*(TX) = TA + \nu$, we have the following identity in $\mathrm{H}^2(A)$ by [41], p. 38:

$$g^* w_2(TX) = w_2(TA) + w_1(TA) w_1(\nu) + w_2(\nu) = w_2(\nu) \tag{5.1}$$

since A is spin. If X is spin, then ν is automatically trivial. Otherwise we assume that $g(A)$ represents an even multiple of A_0, an integer class: $g_*[A] = 2[A_0] \in \mathrm{H}_2(X)$. Then

$$w_2[\nu] = w_2(\nu)[A] = g^* w_2(TX)[A] = w_2(TX)(g_*[A]) = w_2(TX)(2[A_0]) = 0. \tag{5.2}$$

5.2 Use of Thom's theorem. Every 2-dimensional homology class in codimension at least 3 can be represented, up to a multiple, by an imbedded surface A (*cf.* [46] and [22], p. 434). If $A \subset X$ has genus $g \geq 2$ then it can be cut along nullhomologous curves into pieces of genus 1. Since by hypothesis $\pi_1(X)$ is abelian, the curves are nullhomotopic. Hence A is homologous to a union of tori. An imbedded 2-sphere in X can also be turned into a torus by adding a homologically insignificant handle. Thus every 2-dimensional class can be represented by a union of tori. Furthermore, the torus admits a self-map of degree 2, which can be perturbed in this codimension to be made into an imbedding. The analysis above now implies that every class in $\mathrm{H}_2(X)$ can be represented, up to a multiple, by a union of imbedded tori with trivial normal bundles. We apply Lemma 5.1 to complete the proof of Theorem 2.

EXAMPLE 5.3. – The manifold $X = \mathbf{C}\mathrm{P}^m$ is $(2, 2m-2)$-free for all $m \geq 3$.

Proof of Theorem 3. – We first treat two special cases $k = 3$ and $k = 4$. Let $k = 3$. The normal bundle of every imbedded 3-sphere in X is trivial because X and S^3 are oriented so that the normal bundle is orientable, and an oriented bundle of rank $n-3$ over S^3 is determined by the identification of the trivialisations over two hemispheres along their common boundary S^2, *i.e.* by an element of $\pi_2(\mathrm{SO}_{n-3})$. The latter group is trivial by a theorem of Elie Cartan [15], [19]. To give a quick proof of the vanishing of this group, note that $\pi_2(\mathrm{SO}_3) = \pi_2(\mathrm{Spin}_3) = 0$. The long exact sequence of the fibration $\mathrm{SO}_m \to \mathrm{SO}_{m+1} \to \mathrm{S}^m$ gives $\pi_2(\mathrm{SO}_m) \to \pi_2(\mathrm{SO}_{m+1}) \to \pi_2(\mathrm{S}^m)$ and therefore $\pi_2(\mathrm{SO}_{n-3}) = 0$ for all $n \geq 7$. Since X is 2-connected, every 3-dimensional class can be represented by the map of the 3-sphere, by Hurewicz's theorem (*cf.* [25], p. 101), and we apply Lemma 5.1 to complete the proof in the case $k = 3$.

EXAMPLE 5.4. – The manifold $\mathrm{SU}(3)$ is 2-connected and therefore (3,5)-free. Is this still true if the competing metrics of formula (1.2) are required to be left-invariant?

Let $k = 4$ and $p_1(X) = 0$. A calculation similar to (5.1) shows that the first Pontrjagin class $p_1(\nu)$ of the normal bundle ν (of rank $n - 4 \geq 5$) of an imbedded 4-sphere vanishes modulo 2-torsion (*cf.* [41], Theorem 15.3, p. 175). Since $\mathrm{H}^4(\mathrm{S}^4) = \mathbf{Z}$ is torsion-free, we have $p_1(\nu) = 0$. We have

$$\pi_4(\mathrm{BSO}(n-4)) = \pi_3(\mathrm{SO}(n-4)) = \mathbf{Z} \text{ for } n \geq 9 \tag{5.3}$$

(*cf.* [22], vol. 4, p. 1745, Appendix A, Table 6.VII). The map from $\pi_3(\mathrm{SO}(n-4)) = \mathbf{Z}$ to $\mathrm{H}^4(\mathrm{S}^4) = \mathbf{Z}$, defined by taking the first Pontrjagin class of the corresponding bundle, is a nontrivial additive homomorphism (*cf.* [41], p. 246, 145), hence injective. It follows that ν is the trivial bundle. Now suppose k is not divisible by 4. Since $(n-k) - (k-1) \geq 2$, we have $\pi_{k-1}(\mathrm{SO}(n-k)) = \pi_{k-1}(\mathrm{SO})$ (the stable group; *cf.* [21], chapter 6, paragraph 24, p. 233). By real Bott Periodicity, $\pi_{k-1}(\mathrm{SO})$ is either 0 or $\mathbf{Z}_2$ if k is not divisible by 4 (*cf.* [12], p. 315). The theorem now follows from the following lemma.

LEMMA 5.5. – *Let $g : \mathrm{S}^k \to X$ be a sphere representing an even multiple of a class in $\mathrm{H}_k(X)$, where k is not a multiple of 4 and $2k < n$. Then its normal bundle ν is trivial.*

Proof. – Recall that for such k, real Bott periodicity implies that the only nontrivial bundle, if it exists, is distinguished by the k-th Stiefel-Whitney class. Now

$$g^* w_k(TX) = w_k(T\mathrm{S}^k) + w_{k-1}(T\mathrm{S}^k) w_1(\nu) + \ldots + w_k(\nu) = w_k(\nu) \tag{5.4}$$

since the total Stiefel-Whitney class of the sphere is trivial: $w(\mathrm{S}^k) = 1$. Writing $g_*[A] = 2[A_0] \in \mathrm{H}_k(X, Z)$, we obtain

$$w_k[\nu] = w_k(\nu)[\mathrm{S}^k] = g^* w_k(TX)[\mathrm{S}^k] = w_k(TX)(g_*[\mathrm{S}^k]) = w_k(TX)(2[A_0]) = 0. \tag{5.5}$$

Hence the normal bundle of $g(\mathrm{S}^k)$ is trivial. Lemma 5.5 and Theorem 3 are proved.

Remark 5.6. – The simplest manifold not covered by Theorem 3 is the quaternionic projective space $\mathbf{H}\mathrm{P}^3$. Its (4,8) systolic freedom cannot be established by our methods as $p_1(\mathbf{H}\mathrm{P}^3) = 4u \in \mathrm{H}^4(\mathbf{H}\mathrm{P}^3)$ is nonzero, where u is a generator (*cf.* [41], p. 248). Note that the systolic 4-freedom of $\mathbf{H}\mathrm{P}^2$ follows by a method similar to Lemma 6.4 below, starting with free metrics on $\mathrm{S}^4 \times \mathrm{S}^4$, *cf.* [38] and [2]. We find ourselves in the awkward situation of being able to establish the systolic freedom of $\mathbf{C}\mathrm{P}^3$ (Theorem 2) and $\mathbf{H}\mathrm{P}^2$, but not of $\mathbf{C}\mathrm{P}^2$ or $\mathbf{H}\mathrm{P}^3$.

6. Systolic freedom in four dimensions

The main goal of this section is the proof of Theorem 4. This theorem follows from Propositions 6.2-6.4.

LEMMA 6.1. – *Let X and X' be manifolds of the same dimension n. Suppose X' admits a continuous map to the mapping cone C_f, where $f : S \to X$ is an imbedding of a sphere of codimension at least 2 in X. If the continuous map induces monomorphism in homology in the relevant dimensions, then the freedom of X implies that of X'.*

Proof. – Let $f : S \hookrightarrow X$ be the imbedding. Let $I = [0, \ell]$ with $\ell >> 1$ to be determined. Let $\mathrm{Cyl}_f = X \cup_{f \times \{0\}} (S \times I)$ be the mapping cylinder and $W = \mathrm{Cyl}_f \cup_{id \times \{\ell\}} D$ the mapping cone, where D is a cell of dimension $\dim(S) + 1 \leq n - 1$. Let g be a metric on X. Let $h_0 = f^*(g)$ be the induced metric on the sphere S. Let h_1 be the metric of a round sphere of sufficiently large radius r so that $h_1 \geq h_0$. We endow the cylinder $S \times I$ with the metric $(1 - x)h_0 + xh_1 + \mathrm{d}x^2$ for $0 \leq x \leq 1$ and $h_1 + \mathrm{d}x^2$ for $1 \leq x \leq \ell$. We thus obtain a metric on the complex W. Denote the resulting metric space by $W(g, \ell)$. Let $p : W(g, \ell) \to I$ be the map extending the projection to the second factor $S \times I \to I$ on the cylinder, while $p(X) = 0$ and $p(D) = \ell$. Let $q \leq n - 1$ and let z be a q-cycle in W. The complex W is not a manifold, but its subspace $S \times I$ is a manifold and we can apply the coarea inequality just in this part of W. By the coarea inequality, $\mathrm{vol}_q(z) \geq \int_1^l \mathrm{vol}_{q-1}(z \cap p^{-1}(x))\mathrm{d}x$. Hence we can find an $x_0 \in I$ such that

$$\mathrm{vol}_{q-1}(z \cap p^{-1}(x_0)) \leq \frac{1}{\ell - 1} \mathrm{vol}_q(z). \tag{6.1}$$

Let us show that if X admits a systolically free sequence of metrics g_j, then so does W. Here the volume of W is by definition the sum of the volumes of all cells of maximal dimension. Suppose a sequence of cycles z_j in $W(g_j, \ell)$ satisfies $\mathrm{vol}_q(z_j) = o(\mathrm{sys}_q(g_j))$ ($o(\cdot)$ having the usual meaning, $o(1)$ meaning in particular a function which tends to zero as j tends to infinity). Choosing $\ell = \ell(j) \geq \mathrm{sys}_q(g_j)$, we obtain $\lim_{j\to\infty} \mathrm{vol}_{q-1}(z \cap p^{-1}(x_0)) = 0$. By the isoperimetric inequality for small cycles (*cf.* [31], Sublemma 3.4.B'), this $(q-1)$-cycle can be filled by a q-chain B^q of volume which also tends to 0. Here we must choose ℓ big enough as a function of the metric g_j so that the isoperimetric inequality would apply. Let $a = (z \cap d^{-1}([0, x_0])) - B^q$ and $b = z - a$. Note that $[b] = 0$ and so $[a] = [z] \neq 0$. The cycle a lies in the mapping cylinder which admits a distance-decreasing projection to (X, g_j), hence $\mathrm{vol}_q(a) \geq \mathrm{sys}_q(X)$ and so $\mathrm{vol}_q(z) \geq \mathrm{vol}_q(a) - \mathrm{vol}_q(B) \geq \mathrm{sys}_q(X) - o(1)$. This shows that the systoles of W are not significantly diminished when compared to those of X.

Now choose a simplicial structure on W. By the cellular approximation theorem, a continuous map from X' to W can be deformed to a simplicial map. As in [1], we can replace it by a map which has the following property with respect to suitable triangulations of X' and W: on each simplex of X', it is either a diffeomorphism onto its image or the collapse onto a wall of positive codimension. Let p be the maximal number of n-simplices of X' mapping diffeomorphically to an n-simplex of $X \subset W$. Since the $(n - 1)$-cell does not contribute to n-dimensional volume, the pullback of the metric on W is a positive quadratic form on X' whose n-volume is at most p times that of X. This form is piecewise smooth and satisfies natural compatibility conditions along the common face of each pair of simplices.

Note that if a smooth compact n-manifold X admits systolically free piecewise smooth metrics, then it also admits systolically free smooth metrics. To construct a smooth metric from a piecewise smooth one, we proceed as in [1]. Given a piecewise smooth metric g, compatible along the common face of each pair of adjacent simplices, we choose a smooth metric h on X such that $h > g$ at every point (in the sense of lengths of all tangent vectors). Let N be a regular neighbourhood of small volume of the $(n-1)$-skeleton of the triangulation. Choose an open cover of X consisting of N and the interiors U_i of all n-simplices. Using a partition of unity subordinate to this cover, we patch together the metrics $g|_{U_i}$ and $h|_N$. The new metric dominates g for each tangent vector to M. In particular, the volume of a cycle is not decreased. Meanwhile, n-dimensional volume is increased no more than the volume of the regular neighbourhood.

The piecewise smooth metric on X' may *a priori* not be compatible with its smooth structure, since the triangulation may not be smooth. To clarify this point, denote the triangulation by s, and the piecewise smooth metric by g. Consider a smooth triangulation s', and approximate the identity map of X' by simplicial map with respect to the two triangulations s' and s. Now we pull the metric g back to s'. This gives a metric g' adapted to the smooth triangulation s', to which we may apply the argument with the regular neighbourhood N.

We have thus obtained a smooth positive form on X'. We make it definite without significantly increasing its volume by adding a small multiple of a positive definite form. The lower bounds for the systoles are immediate from the injectivity of the map $X' \to W$ on the relevant homology groups.

PROPOSITION 6.2. – *Suppose* $\mathbf{C}\mathrm{P}^2$ *and* $\mathrm{S}^2 \times \mathrm{S}^2$ *admit 2-free metrics. Then so does every closed simply connected 4-manifold* X.

Proof. – Let $b = b_2(X)$. Let W be the 4-skeleton of the Cartesian product $\mathbf{C}\mathrm{P}^2 \times \ldots \times \mathbf{C}\mathrm{P}^2$ (b times), where $\mathbf{C}\mathrm{P}^2$ comes with its standard cell structure. Note that W is the 4-skeleton of the standard model of the Eilenberg-Maclane space $K(Z^b, 2)$. Choose a CW structure on (the homotopy type of) X whose 2-skeleton is the wedge of b copies of S^2, with a single 4-cell attached. The identification of the 2-skeleta of X and W extends across the 4-cell since $\pi_3(W) = \pi_3(\mathrm{CP}^2)^b = 0$ from the long exact sequence of the Hopf fibration over $\mathbf{C}\mathrm{P}^2$. The freedom of W and then that of X is established along the lines of the proof of Lemma 6.1, using the fact that two distinct 4-cells of W meet along cells of codimension at least 2. For details, see section 5 of [49].

PROPOSITION 6.3. – *Every closed simply connected* 4*-manifold* X *admits a map to a connected sum of copies of* $\mathbf{C}\mathrm{P}^2$ *with either orientation, which induces a monomorphism in homology of dimension* 2.

Proof. – Let $f(\overline{x}, \overline{y}) = \sum_{i,j=1}^{b_2} f_{i,j} x_i y_j$ be the intersection form of X, where $b_2 = b_2(X)$. Let $\overline{x}, \overline{y} \in \mathrm{H}_2(X, Z)$ and let $\sigma = \sigma(X)$ be the signature of X. Let $p = (b_2 + \sigma)/2$ and $q = (b_2 - \sigma)/2$ and define a manifold N by setting

$$N = N(b_2, \sigma) = (\mathrm{CP}^2 \# \ldots \# \mathrm{CP}^2) \# (\overline{\mathrm{CP}}^2 \# \ldots \# \overline{\mathrm{CP}}^2), \tag{6.2}$$

the connected sum of p copies of $\mathbf{C}\mathrm{P}^2$ and q copies of $\overline{\mathrm{CP}}^2$. The intersection form of N is

$$g(\overline{u}, \overline{v}) = u_1 v_1 + \ldots + u_p v_p - u_{p+1} v_{p+1} - \ldots - u_{b_2} v_{b_2}. \tag{6.3}$$

If $pq \neq 0$, then f and g are rationally equivalent by the classification theorem of integer unimodular forms [45]. If, say, $q = 0$, we consider the pair of forms $f \dotplus \{-1\}$ and $g \dotplus \{-1\}$ and use the 'Witt lemma' [16] to the effect that if two forms are rationally equivalent after adding a common summand, then they are rationally equivalent. The rational equivalence entails the existence of a non-singular integer matrix C such that

$$f(C\overline{u}, C\overline{v}) = \Delta^2 g(\overline{u}, \overline{v}), \tag{6.4}$$

where Δ is a nonzero integer. It is well known [40] that every simply connected 4-manifold X with intersection form $f = (f_{ij})$ is homotopy equivalent to the complex

$$\vee_{i=1}^{b_2} \mathrm{S}_i^2 \cup_\phi D^4, \tag{6.5}$$

where the glueing map $\phi : \mathrm{S}^3 \to (\vee_{i=1}^{b_2} \mathrm{S}_i^2)$ represents the homotopy class

$$\frac{1}{2} \sum_{i,j=1}^{b_2} f_{i,j}[e_i, e_j]. \tag{6.6}$$

Here the $e_i \in \pi_2(\vee_{i=1}^{b_2} \mathrm{S}_i^2)$ are the canonical generators, while the Whitehead products $[e_i, e_j]$ for $i < j$ together with $\frac{1}{2}[e_i, e_i]$ form a basis for $\pi_3(\vee_{i=1}^{b_2} \mathrm{S}_i^2)$ by the Hilton-Milnor theorem (*cf.* [37], [47]). Let us consider the map

$$\xi_C : \vee_{i=1}^{b_2} \mathrm{S}_i^2 \to \vee_{i=1}^{b_2} \mathrm{S}_i^2 \tag{6.7}$$

defined by the matrix C introduced above. The usual calculation shows that

$$\begin{aligned} \xi_{C*}(\{\phi\}) &= \xi_{C*}(1/2 \textstyle\sum_{i,j=1}^{b_2} f_{i,j}[e_i, e_j]) \\ &= 1/2 \textstyle\sum_{i,j=1}^{b_2} f_{i,j}[\xi_{C*}(e_i), \xi_{C*}(e_j)] \\ &= \Delta^2/2([\overline{e_1}, \overline{e_1}] + \ldots + [\overline{e_p}, \overline{e_p}] - [\overline{e_{p+1}}, \overline{e_{p+1}}] - \ldots - [\overline{e_{b_2}}, \overline{e_{b_2}}]), \end{aligned} \tag{6.8}$$

where the $\overline{e_i} \in \pi_2(N)$ are the canonical generators defined by the inclusion

$$\vee_{i=1}^{b_2} \mathrm{S}_i^2 \subset N. \tag{6.9}$$

The computation shows that ξ_C can be extended to a map

$$\Xi_C : X =_{\mathrm{hom}} \vee_{i=1}^{b_2} \mathrm{S}_i^2 \cup_\phi D^4 \to N, \tag{6.10}$$

moreover $\deg \Xi_C = \Delta^2$.

Note that Proposition 6.3 provides an alternative proof, not involving CW complexes, of the fact that if $\mathbf{C}\mathrm{P}^2$ is free then every simply connected 4-manifold X is free. Here we apply Lemma 6.1 to the mapping cone of the empty map.

PROPOSITION 6.4. – *The following three assertions are equivalent:* (i) $\mathbf{C}\mathrm{P}^2$ *is 2-free;* (ii) $\mathbf{C}\mathrm{P}^2 \# \overline{\mathbf{C}P}^2$ *is 2-free;* (iii) $\mathrm{S}^2 \times \mathrm{S}^2$ *is 2-free.*

Proof. – *(i)* $\Rightarrow$ *(ii)*: If $\mathbf{C}\mathrm{P}^2$ has free metrics then so does $\overline{\mathbf{C}P}^2$. Taking a connected sum by a thin long tube produces a metric on $\mathbf{C}\mathrm{P}^2 \# \overline{\mathbf{C}P}^2$ which admits distance-decreasing projections of degree 1 to each of the summands. Since $\mathrm{H}_2(\mathbf{C}\mathrm{P}^2 \# \overline{\mathbf{C}P}^2) = \mathrm{H}_2(\mathbf{C}\mathrm{P}^2) + \mathrm{H}_2(\overline{\mathbf{C}P}^2) = \mathbf{Z} + \mathbf{Z}$, the implication follows.

$(ii) \Rightarrow (iii)$: Recall that $\mathbf{C}\mathrm{P}^2\#\overline{\mathbf{C}P}^2$ is the nontrivial 2-sphere bundle over S^2. Let $f : \mathrm{S}^2 \to \mathrm{S}^2$ be a degree 2 map. The pullback of the nontrivial bundle by f is the trivial one (*cf.* the w_2 discussion preceding (5.1)). We thus obtain a map $f_* : \mathrm{S}^2\times\mathrm{S}^2 \to \mathbf{C}\mathrm{P}^2\#\overline{\mathbf{C}P}^2$ inducing an injective homomorphism $\mathrm{H}_2(\mathrm{S}^2 \times \mathrm{S}^2) \to \mathrm{H}_2(\mathbf{C}\mathrm{P}^2\#\overline{\mathbf{C}P}^2)$. We approximate this map by a simplicial one as in [1], and pull back the free metrics from $\mathbf{C}\mathrm{P}^2\#\overline{\mathbf{C}P}^2$ to $\mathrm{S}^2 \times \mathrm{S}^2$ (*cf.* 6.5), proving this implication.

$(iii) \Rightarrow (i)$: We cannot map $\mathbf{C}\mathrm{P}^2$ to $\mathrm{S}^2\times\mathrm{S}^2$ in such a way as to induce a monomorphism in H_2. The obstruction lies in the group $\pi_3(\mathrm{S}^2 \times \mathrm{S}^2) = \mathbf{Z} + \mathbf{Z}$. We can, however, kill two birds with one stone, or more precisely with one 3-cell, eliminating 3-dimensional homotopy (at least rationally). We glue in a 3-ball B^3 to $\mathrm{S}^2 \times \mathrm{S}^2$ along the diagonal sphere to obtain a CW complex W which admits a map from $\mathbf{C}\mathrm{P}^2$ inducing an injective homomorphism on 2-dimensional homology. The 2-systole of W obeys the same asymptotics as that of the area-rich metrics on $\mathrm{S}^2 \times \mathrm{S}^2$. The coveted metrics on $\mathbf{C}\mathrm{P}^2$ are pulled back from W by the map $\mathbf{C}\mathrm{P}^2 \to W$.

Let $W = (\mathrm{S}^2 \times \mathrm{S}^2) \cup B^3$ where the 3-ball is glued in along the imbedded diagonal 2-sphere representing the element $(1,1) \in \pi_2(\mathrm{S}^2\times\mathrm{S}^2) = \mathbf{Z}+\mathbf{Z}$. Let $g : \mathrm{S}^2\times\mathrm{S}^2 \to W$ be the inclusion. We will specify a map $f : \mathbf{C}\mathrm{P}^1 \to \mathrm{S}^2\times\mathrm{S}^2$ such that the induced homomorphism $(g\circ f)_2 : \pi_2(\mathbf{C}\mathrm{P}^1) \to \pi_2(W)$ is injective while $(g\circ f)_3 : \pi_3(\mathbf{C}\mathrm{P}^1) \to \pi_3(W)$ is zero. Recall that $\mathbf{C}\mathrm{P}^2 = \mathbf{C}\mathrm{P}^1 \cup_h B^4$, where h is the generator of $\pi_3(\mathbf{C}\mathrm{P}^1)$ (in fact, W is homotopy equivalent to $\mathrm{S}^2 \cup_{2h} B^4$, whence the existence of the map extending a degree 2 map on the 2-sphere). Therefore the map $g\circ f$ extends to a map $\mathbf{C}\mathrm{P}^2 \to W$ which induces an injective homomorphism in π_2, and hence in H_2 by the Hurewicz theorem (*cf.* [25]). The property $(g\circ f)_3 = 0$ follows from the fact that $\mathrm{Image}\,(g_3)$ is 2-torsion while $\mathrm{Image}\,(f_3)$ is even (see Lemma 6.5 below). The proof is completed by applying Lemma 6.1 to the mapping cone of the inclusion of the diagonal in $\mathrm{S}^2 \times \mathrm{S}^2$.

LEMMA 6.5. – *Let $f : \mathbf{C}\mathrm{P}^1 \to \mathrm{S}^2 \times \mathrm{S}^2$ be a map sending $\mathbf{C}\mathrm{P}^1$ to the first factor with degree 2. Let $W = (\mathrm{S}^2\times\mathrm{S}^2)\cup B^3$ where the 3-ball is glued in along the imbedded diagonal 2-sphere representing the element $(1,1) \in \pi_2(\mathrm{S}^2 \times \mathrm{S}^2) = \mathbf{Z} + \mathbf{Z}$. Let $g : \mathrm{S}^2 \times \mathrm{S}^2 \to W$ be the inclusion. Then the map $g \circ f$ extends to a map $\mathbf{C}\mathrm{P}^2 \to W$.*

Proof. – How can we extend the map when the attaching map is not a multiple of a Whitehead product? The idea is to use torsion freeness of $\pi_3(\mathrm{S}^2)$. A finer version of this argument for other rank one symmetric spaces appears in [2], Remark 3.4. Let e_1, e_2 be the standard generators of $\pi_2(\mathrm{S}^2 \times \mathrm{S}^2) = \mathbf{Z} + \mathbf{Z}$. The group $\pi_3(\mathrm{S}^2 \times \mathrm{S}^2) = \mathbf{Z}+\mathbf{Z}$ is generated by elements h_1 and h_2 satisfying $2h_i = [e_i, e_i]$ for $i = 1, 2$, where the brackets denote the Whitehead product (*cf.* [25], p. 74 and [47], p. 495, theorem 2.5). Here $[e_1, e_2] = 0$ in $\pi_3(\mathrm{S}^2\times\mathrm{S}^2)$ by definition of the Whitehead product. The induced homomorphism g_2 satisfies $g_2(e_1 + e_2) = 0 \in \pi_2(W)$ by definition of W. Now let $e \in \pi_2(\mathbf{C}\mathrm{P}^1)$ and $h \in \pi_3(\mathbf{C}\mathrm{P}^1)$ be the respective generators, so that $[e, e] = 2h$. Consider the class $2e_1 \in \pi_2(\mathrm{S}^2 \times \mathrm{S}^2)$, and take a representative $f \in 2e_1$. Then $f_2(e) = 2e_1 \in \pi_2(\mathrm{S}^2 \times \mathrm{S}^2)$. By naturality of the Whitehead product,

$$(6.12)\quad 2f_3(h) = [f_2(e), f_2(e)] = [2e_1, 2e_1] = 4[e_1, e_1] = 8h_1 \in \pi_3(\mathrm{S}^2 \times \mathrm{S}^2) = \mathbf{Z} + \mathbf{Z}.$$

In a free abelian group we can divide by 2, obtaining

$$(6.13)\qquad f_3(h) = 4h_1 \in \pi_3(\mathrm{S}^2 \times \mathrm{S}^2).$$

Now the lemma follows from the fact that $\mathbf{C}P^2 = \mathbf{C}P^1 \cup_h B^4$ and the following calculation:

$$\begin{aligned}(g \circ f)_3(h) &= g_3(4h_1) = 2g_3(2h_1) = 2[g_2(e_1), g_2(e_1)] \\ &= 2[g_2(e_1+e_2), g_2(e_1)] = 2[0, g_2(e_1)] = 0.\end{aligned} \tag{6.14}$$

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