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DEGENERATED SINGULAR CYCLES OF INCLINATION-FLIP TYPE (*)

BY C. A. MORALES AND M. J. PACÍFICO

ABSTRACT. – In the present article we show that the elements of certain class of vector field cycles, called singular cycles, yield through two-parameter families a transition from simple to more complex systems. Our cycles here are more degenerate than the ones first described in [BLMP], due to a further nontransversal intersection between the invariant manifolds of the elements in the cycle. Bifurcation diagrams are provided, as well as hyperbolicity is proved to be prevalent for generic codimension-two perturbations of these degenerated cycles. © Elsevier, Paris

RÉSUMÉ. – Dans le présent article, nous montrons que les éléments de certaines classes de cycles de champs de vecteurs, appelés cycles singuliers, permettent de passer à l'aide d'une famille à deux paramètres d'un système simple à un plus complexe. Nos cycles sont plus dégénérés que ceux décrits dans [BLMP], à cause d'une intersection non transversale de plus entre les variétés invariantes des éléments du cycle. Les diagrammes de bifurcations sont donnés, et on montre que l'hyperbolicité est prévalente pour des perturbations de codimension deux génériques de ces cycles. © Elsevier, Paris

1. Introduction

The objective of this paper is to present new results in bifurcation of vector fields. This is achieved by further exploring a 3-dimensional bifurcating structure for vector fields, called *singular cycle*, through which a parametrized system may evolve from a simple to a highly nontrivial dynamic. Recall that a singular cycle, for a vector field, is a finite set of hyperbolic periodic orbits and at least one singularity, which are linked in a cyclic way by orbits in the intersection of the stable and unstable manifolds of its periodic orbits and singularities. Here the singularity is unique (a saddle) and it is expanding, i.e., its expanding eigenvalue is stronger than the weakest contracting one. This kind of singular cycle is called *expanding*. Otherwise, it is called *contracting*.

The unfolding of expanding singular cycles was studied in [BLMP] and the corresponding study for the contracting ones in [PR] and [S]. In both cases it was showed that hyperbolicity is a prevalent phenomenon: for generic families X_μ of vector fields on R^3 passing through a vector field presenting a singular cycle, the set of parameters corresponding to hyperbolic vector fields has total Lebesgue measure. More than this, the complement of this set has

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zero limit capacity when the cycle is expanding. In these papers, besides the hyperbolicity of the critical elements in the cycle, it was assumed that the cycle is of codimension one, that is, there was only one nontransversal orbit in the intersection of the stable and unstable manifolds of its elements. Here we shall analyze the unfolding of a codimension two expanding singular cycle, characterized by the existence of two nontransversal orbits in the intersection of the stable and unstable manifolds of the critical elements in the cycle.

Let us now present the precise statements of our results. Let M be a compact and boundaryless 3-manifold and let $\mathcal{X}^r(M)$ be the Banach space of C^r vector fields on M endowed with the C^r-topology. If $X \in \mathcal{X}^r(M)$, a *singular cycle* for X is a set $\Gamma = \{\sigma_0, \sigma_1, \gamma_0, \gamma_1\}$ satisfying:

a. σ_0 is a hyperbolic saddle-type singularity of X, such that the eigenvalues $\lambda_1, -\lambda_2, -\lambda_3$ of the derivative matrix $DX(\sigma_0)$ are real and satisfy $-\lambda_2 < -\lambda_3 < 0 < \lambda_1$;

b. σ_1 is a hyperbolic saddle periodic orbit of X. If λ and σ are the eigenvalues of the linear part of the corresponding Poincaré map Π induced by X, then they are real and satisfy $0 < \lambda < 1 < \sigma$;

c. γ_0 is a regular orbit of X lying in the intersection of the stable manifold $W^s(\sigma_1)$ and the unstable manifold $W^u(\sigma_0)$ of σ_1 and σ_0 respectively;

d. γ_1 is a regular orbit of X belonging to the intersection of the stable manifold $W^s(\sigma_0)$ and the unstable manifold $W^u(\sigma_1)$ of σ_0 and σ_1 respectively;

e. The cycle Γ is *isolated*, that is, it has an isolating block. Recall that an isolating block for an invariant set Γ of a vector field X is an open set $U \subset M$ such that $\cap_{t\in R} X_t(U) = \Gamma$ where X_t is the flow induced by X.

DEFINITION. – *A singular cycle $\Gamma = \{\sigma_0, \sigma_1, \gamma_0, \gamma_1\}$ of a vector field X is called inclination-flip whenever $W^s(\sigma_0)$ and $W^u(\sigma_1)$ have a quadratic nondegenerated intersection along γ_1, each center-unstable manifold (see [HPS]) W^{cu} passing through σ_0 is transversal to $W^s(\sigma_1)$ along γ_0 and $\lambda_3 < \lambda_1$* (see *Figure 1). The cycle will be called expansive (contractive) if $\lambda\sigma > 1$ ($\lambda\sigma < 1$) (see notation in (b) above).*

Our motivation to study this kind of degenerated singular cycles comes from [BLMP], as well as from the study of degenerated loops associated to hyperbolic (or nonhyperbolic) saddle singularities (see [HKK], [N], [M], and [Ry]). Indeed, our cycle is obtained inserting in a convenient way, a hyperbolic closed orbit in the loop studied at [HKK]. As we shall see below, the dynamical behavior arising from the unfolding of an inclination-flip singular cycle Γ depends on certain conditions on the way the singularity and the periodic orbit are linked. These conditions yield eight cases. We shall analyze in detail one of then which we call *inward type*. For the others, we indicate how to proceed to obtain the corresponding results at Remark 3.1.

Theorems A and B explain the bifurcation diagram for the unfolding of an inward inclination-flip type singular cycle (see Figures 2(a) and 2(b) below). As a consequence, we obtain that hyperbolicity is a prevalent phenomenon for generic two-parameter families passing through a vector field presenting an inclination-flip cycle of either expanding or contracting type (*see* Theorem C). We shall denote by W_e and W_c the subset of $\mathcal{X}^r(M)$ consisting of vector fields that exhibit an inclination-flip singular cycle of expansive and contractive type, respectively.

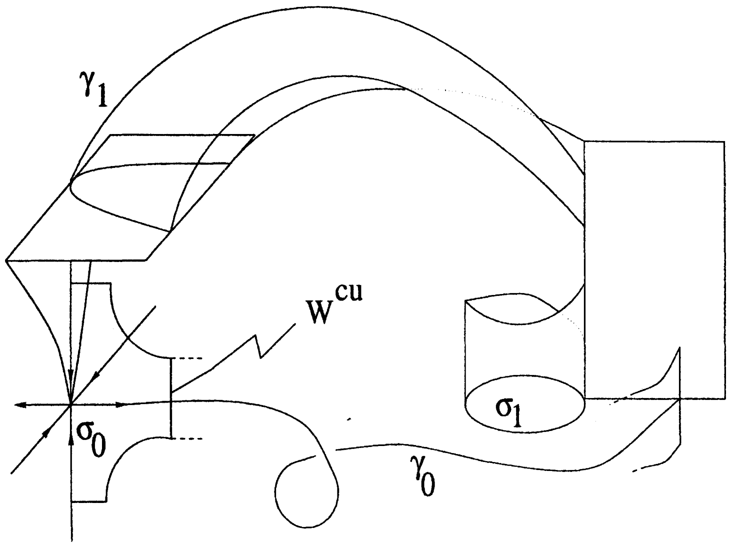

Fig. 1

To state our results, we use the following notation. Let $X = \{X_\mu\}_{\mu \in R}$ be a parametrized family of vector fields such that X_0 has a singular cycle Γ. If U is a fixed isolating block of Γ, then $\Omega_\mu(U)$ denotes the nonwandering set of X_μ in U. $H(U, X)$ denotes the set of parameter values μ for which $\Omega_\mu(U)$ is hyperbolic (see [PT]). $H_s(U, X)$ will denote those parameter values μ in $H(U, X)$ for which $\Omega_\mu(U)$ is just the continuations of the singularity and the periodic orbit of Γ. $H_h(U, X)$ denotes the subset of $H(U, X)$ formed by parameter values μ such that $\Omega_\mu(U)$ has at least one nontrivial basic set. Recall that B is a basic set for a vector field X if it is hyperbolic, transitive and coincides with the closure of its periodic orbits. The Lebesgue measure for subsets in R will be denoted by m.

THEOREM A. – *Let $X = X_{(\mu,\eta)} \in \mathcal{X}^r(M)$ be a two-parameter family crossing W_c transversally at $(\mu, \eta) = (0, 0)$. Suppose that U is a fixed isolating block of the cycle at $(0, 0)$. Assume that the cycle at $(0, 0)$ is inward. Then, through a smooth change of parameter, there are two curves $\eta = K_c^+(\mu) \geq 0$ and $\eta = K_c^-(\mu) \leq 0$ for $\mu \geq 0$ such that if $\epsilon > 0$ is a small positive number, then the properties below hold* (see *Figure 2(a)):*

1. *$H_s(U, X)$ includes the parameter regions $\{(\mu, \eta) : -\epsilon < \mu < 0, |\eta| < \epsilon\}$ and $\{(\mu, \eta) : \epsilon > \mu > 0, \eta > K_c^+(\mu)\}$;*

2. *$m(\{(\mu, \eta) : \epsilon > \mu > 0, -\epsilon < \eta < K_c^-(\mu), (\mu, \eta) \notin H_h(U, X)\}) = 0$;*

3. *the derivatives $(K_c^+)'(0)$ and $(K_c^-)'(0)$ are equal to zero. Also the curves $K_c^+(\mu)$ and $K_c^-(\mu)$ vanish at $\mu = 0$ only.*

Next theorem deals with two-parameter families bifurcating through a cycle of an element in W_e.

THEOREM B. – *Let $X = X_{(\mu,\eta)} \in \mathcal{X}^r(M)$ be a two-parameter family crossing W_e transversally at $(\mu, \eta) = (0, 0)$. Suppose that U is a fixed isolating block of the cycle at $(0, 0)$. Assume that the cycle at $(0, 0)$ is inward. Then, through a smooth change of parameter, there are $\epsilon_0 > 0$ and four curves $\eta = K_e^{0,+}(\mu) \geq 0$, $\eta = K_e^{1,+}(\mu) \geq 0$,*

$\eta = K_e^{2,+}(\mu) \geq 0$, $\eta = K_e^-(\mu) \leq 0$ *for* $\mu \geq 0$ *which satisfy the properties below* (see *Figure 2(b) below):*

1. $H_s(U,X)$ *includes the parameter regions* $\{(\mu,\eta) : -\epsilon_0 < \mu < 0\}$ *and* $\{(\mu,\eta) : \epsilon_0 > \mu > 0, \eta > K_e^{0,+}(\mu)\}$*;*

2. $m(\{(\mu,\eta) : \epsilon_0 > \mu > 0, -\epsilon_0 < \eta < K_e^-(\mu), (\mu,\eta) \notin H_h(U,X)\}) = 0$*;*

3. *if we define* $B^*(\epsilon) = \{(\mu,\eta) : \mu > 0, K_e^{2,+}(\mu) < \eta < K_e^{1,+}(\mu)\} \cap B(\epsilon)$ *for* $\epsilon \in (0,\epsilon_0)$, *then the equality below holds:*

$$\lim_{\epsilon \to 0^+} \frac{m(H(U,X) \cap B^*(\epsilon))}{m(B^*(\epsilon))} = 1;$$

4. $(K_e^{i,+})'(0) = \infty$ *for* $i = 0,1$ *and* $(K_e^{2,+})'(0) = (K_e^-)'(0) = 0$. *In addition,* $K_e^{i,+}(\mu)$ $(i = 0,1,2)$ *and* $K_e^-(\mu)$ *vanish at* $\mu = 0$ *only.*

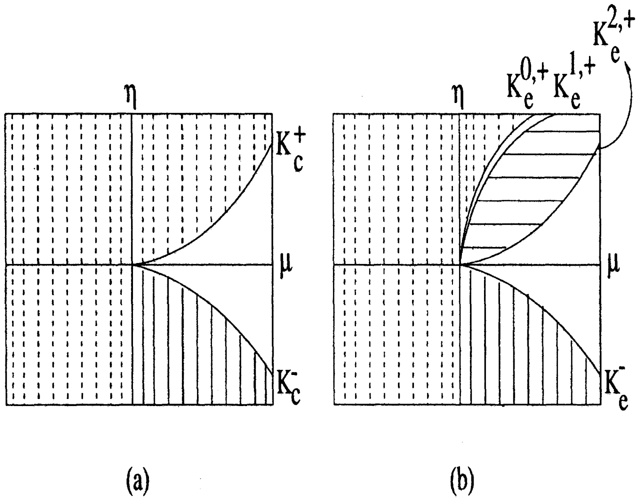

Fig. 2

Finally, we state a general result.

THEOREM C. – *If* $X = X_{(\mu,\eta)} \in \mathcal{X}^r(M)$ *is a two-parameter family crossing transversally* $W_e \cup W_c$ *at* $(\mu,\eta) = (0,0)$ *and* U *is a fixed isolating block of the cycle at* $(0,0)$*, then:*

$$\lim_{\epsilon \to 0^+} \frac{m(H(U,X) \cap B(\epsilon))}{\epsilon^2} = 1$$

Observe that theorems A and B give a good picture of the dynamics for two-parameter perturbations of an inclination-flip singular cycle. Comparing with the results in [BLMP] and [PR], we point out that the two-dimensional measure of the set of parameters outside $H(U,X)$ is not zero. This follows from well known facts relating chaotic dynamics with the unfolding of a homoclinic tangency associated to a hyperbolic periodic motion. See [PT] for an accurate description of such chaotic phenomena.

This paper is divided as follows. In §2 we make an initial reduction of the problem using the expression for the Poincaré map associated to the given bifurcating cycle. Next, two-parameter families $X_{(\mu,\eta)}$ are considered. Technical and straightforward reductions lead us to consider the right half-plane $\{\mu > 0\}$ only. Later, we use graph transform techniques to find invariant foliations in order to reduce the problem to a one-dimensional map, at least in the parameter region $\{(\mu,\eta) : \mu > 0 > \eta\}$, when the bifurcating cycle is inward. This implies parameter exclusions, but the remaining set will be large in terms of Lebesgue measure near the bifurcating parameter value. This is a common part of the Theorems A and B, and it is the subject of §2. The proof of the Theorem A will be completed using Proposition 3.2 in §3. Theorem B will be proved by making more parameter exclusions. Theorem C will follow from theorems A and B together with Remark 3.1 at the end of the paper.

2. The Poincaré map

Let us consider a vector field $X \in \mathcal{X}^r(M)$ having an inclination-flip singular cycle Γ. In this section we study the Poincaré map Π_Y, induced by Y close to X, in a fixed isolating block of Γ. Consider an isolating block U of the cycle and let (x_0, y_0, z_0) and (x, y) be C^2-linearizing coordinates around $\sigma_1(Y)$ and $\sigma_0(Y)$ respectively. Then the transversal sections $\Sigma = \{(x,y) : ||(x,y)|| \le \Delta\}$ and $\Sigma_0 = \{(x_0, y_0, 1) : ||(x_0,y_0)|| \le 1\}$ are defined inside the corresponding coordinate systems. The number Δ is chosen so that $\Delta < 1 < \Delta\sigma$. Now, by assumption, the Poincaré map Π_{0Y} (see the notation §1) is a linear map, namely $\Pi_{0Y}(x,y) = (\lambda x, \sigma y)$. Here the point $(0,0)$ is the periodic orbit $\sigma_1(Y)$.

On the other hand, the flow generated by Y nearby $\sigma_0(Y)$ is given by the solution of the linear system $(x_0, y_0, z_0) \to (\lambda_1 x_0, -\lambda_2 y_0, -\lambda_3 z_0)$ with λ_i depending smoothly on the parameters; $\sigma_0(Y)$ is the origin $(0,0,0)$ of the coordinate system (x_0, y_0, z_0). A first hit map Π_{LY} from $\Sigma_0^+ = \{(x_0, y_0, 1) \in \Sigma_0 : x_0 \ge 0\}$ to $\Sigma_1 = \{(1, y_0, z_0) : ||(y_0, z_0)|| \le 1\}$ is obtained by solving the above linear system and it has the form $\Pi_{LY}(x_0, y_0) = (y_0|x_0|^\beta, |x_0|^\alpha)$ with $\beta = \lambda_2/\lambda_1$ and $\alpha = \lambda_3/\lambda_1$. Both α and β depend smoothly on the parameters and $\alpha < 1$ everywhere by hypothesis (*see* Figure 3 below)

Thus a Poincaré map Π_Y defined on a subset of Σ is obtained as:

$$\Pi_Y(x,y) = \begin{cases} (\lambda x, \sigma y) & \text{if } (x,y) \in R_Y \\ (\Pi_{2Y} \circ \Pi_{LY} \circ \Pi_{1Y})(x,y) & \text{if } (x,y) \in R_Y^0 \end{cases}$$

where:

1. $\Pi_{1Y} : \mathrm{Dom}(\Pi_{1Y}) \subset \Sigma^+ = \{(x,y) : x, y \ge 0\} \to \Sigma_0$ is a smooth flow-induced diffeomorphism $\Pi_{1Y}(x,y) = (\phi(x,y,Y), A(x,y,Y))$ satisfiying $\phi(0,1,X) = A(0,1,X) = \partial_y\phi(0,1,X) = 0$, $\partial_{yy}\phi(0,1,X) \ne 0$. Here $(0,1) \in \Sigma$ corresponds to the orbit γ_1 of Σ;

2. $\Pi_{2Y} : \mathrm{Dom}(\Pi_{2Y}) \subset \{(1, y_0, z_0) : ||(z_0, y_0)|| \le 1\} \to \Sigma$ is a smooth flow-defined map $\Pi_{2Y}(x_0, y_0) = (C(y_0, z_0, Y), B(y_0, z_0, Y))$ with $B(0,0,X) = 0$, $C(0,0,X) = c_X \in (\lambda, 1)$ and $\partial_{z_0} B(0,0,X) < 0$. As before, the point $(1,0,0)$ corresponds to the regular orbit $\gamma_0 \in \Gamma$;

3. R_Y is the square $\{(x,y) \in \Sigma : -\Delta \le y \le \sigma^{-1}\Delta \ , \ x \in [-\Delta, \Delta]\}$;

4. R_Y^0 is the set of points (x,y) such that $(\Pi_{2Y} \circ \Pi_{LY} \circ \Pi_{1Y})(x,y) \in \Sigma^+$. Of course this set may be empty for some parameter values.

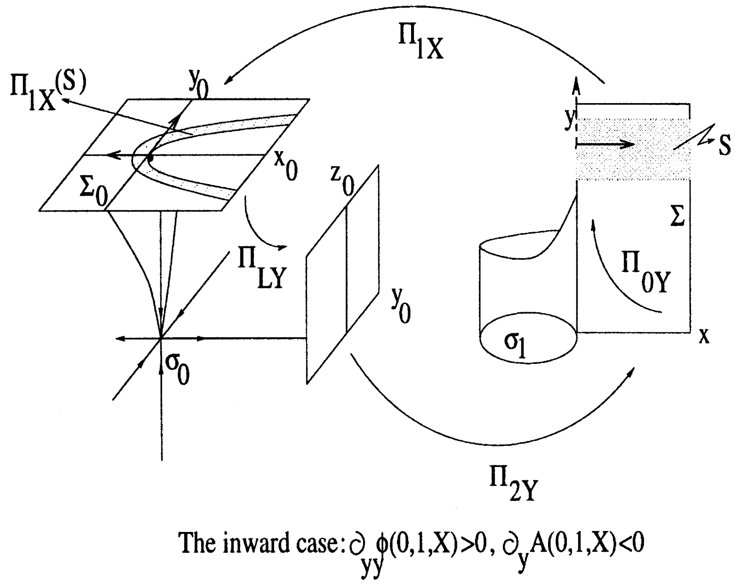

Fig. 3

Remark 2.1.

(a) Property (1) above is a consequence of the quadratic contact of $W^s(\sigma_0)$ and $W^u(\sigma_1)$ along γ_1. In particular, four cases arise from the inequalities $\partial_{yy}\phi(0,1,X) \neq 0$ and $\partial_y A(0,1,X) \neq 0$ and they describe the shape of the parabolic-like region determinate by the image set of Π_{1Y}. We keep one of these cases: an inclination-flip singular cycle is called *inward* when $\partial_{yy}\phi(0,1,X) > 0$ and $\partial_y A(0,1,X) < 0$. In what follows, we study the dynamical behavior arising from generic codimension-two perturbations of an inward inclination-flip singular cycle. In the next section, we explain how the bifurcation diagrams of the remaining cases can be obtained using the techniques developed below (see Remark 3.1 in the next section).

(b) Property (2) is a consequence of the definition of an inclination-flip singular cycle given in §1. In particular, $\partial_{z_0} B(0,0,X) < 0$ because the cycle is isolated.

Now, we work with two-parameter families $X = X_{(\mu,\eta)} \in \mathcal{X}^r(M)$ crossing $W_e \cup W_c$ transversally at the parameter value $(0,0)$, and with the corresponding cycle at $(0,0)$ being inward. We shall denote by $\Pi_{\mu,\eta}$, $\phi(x,y,\mu,\eta)$, $A(x,y,\mu,\eta)$, etc. the corresponding functions for the vector field $X_{(\mu,\eta)}$. Through a smooth change of parameters we assume that $\partial_y\phi(0,1,\mu,\eta) = 0$, $\phi(0,1,\mu,\eta) = \eta$ and $B(0,0,\mu,\eta) = \mu$ for all (μ,η). In a similar way $C(0,0,\mu,\eta) = c_0 \in (\lambda, 1)$ does not depend on the parameters. The remainer of this section is occupied with the study of the dynamics of $X_{(\mu,\eta)}$ close to the vector field $X_{(0,0)}$, for parameter values (μ,η) with $\eta < 0$. Note that if $\mu < 0$, then the vector field

$(\mu, \eta) \in H_s(U, X)$ (see the notations in §1). So we consider parameters (μ, η) with $\mu > 0$ only. To start with, choose a fixed isolating block U of the cycle at $(0, 0)$.

LEMMA 2.1. – *Suppose that U is an isolating block of the cycle at $(0, 0)$. Then for all $\epsilon > 0$ there exist a neighborhood $U_\epsilon \subset U$ and a positive number δ such that:*

1. *$\Omega_{\mu,\eta}(U) \subset U_\epsilon$ for all $(\mu, \eta) \in B(\delta)$;*

2. *all the partial derivatives of ϕ and A up to the order r at $(x, y, \mu, \eta) \in U_\epsilon \cap R^0_{\mu,\eta}$ are ϵ-close to the corresponding values at $(0, 1, 0, 0)$*

The notation $R^0_{\mu,\eta}$ stands for $R^0_{X(\mu,\eta)}$.

Proof. – Let Γ be the cycle of $X_{(0,0)}$. It is well-known that the function $(\mu, \eta) \to \Omega_{\mu,\eta}(U)$ is upper semicontinuous with respect to the Hausdorff metric. This is proved using filtration arguments, because Γ is isolated (see, for instance, the appendix in [PT]). Thus it is suffice to find, for each $\epsilon > 0$, a neigborhood U_ϵ satisfying (2) for (μ, η) close to $(0, 0)$. To do so, fix $\epsilon > 0$ and take a small band $[-\epsilon_0, \epsilon_0] \times [-1, 1] \subset \Sigma_0$ with $\epsilon_0 > 0$ depending on ϵ. If $\epsilon_0 > 0$ s taken small then $\Pi^{-1}_{1,0,0}([-\epsilon_0, \epsilon_0] \times [-1, 1]) \subset \Sigma_0$ is a narrow parabolic-like region enclosing the curve $\Pi^{-1}_{1,0,0}(\{0\} \times [-1, 1])$. Now, shrinking Σ along the ox axis if nessesary, one can arrange the bounds (2) for $(\mu, \eta) = (0, 0)$. Finally, (2) holds for parameters (μ, η) close to $(0, 0)$ because the family $X_{(\mu,\eta)}$ is continuous.

Throughout the paper, U will mean U_ϵ with ϵ small enough. The set U_ϵ comes from the above lemma. The theorem below state that for a large set of parameters in $\{(\mu, \eta) : \eta < 0, \mu > 0\}$, the dynamics of $X_{(\mu,\eta)}$ is reduced to the dynamics of certain one-dimensional map to be exhibited later (*see* Figure 5).

THEOREM 2.2. – *Let $X(\mu, \eta)$ be a two-parameter family of vector fields crossing $W_c \cup W_e$ transversally at $(\mu, \eta) = (0, 0)$. Then there exist $K > 0$, $\delta > 0$ such that for (μ, η) close to $(0, 0)$, $\mu > 0$ and $\eta \leq -K\mu^{(1+\delta)}$ the following hold:*

1. *there is a C^1 contractive foliation $F^{ss}_{\mu,\eta}$ defined on Σ^+, invariant by $\Pi_{\mu,\eta}$;*

2. *the local stable manifold $W^s_{loc}(\sigma_1(\mu, \eta)) = \Sigma^+ \cap W^s(\sigma_1(\mu, \eta)) = \{(x, 0) : x \geq 0\}$ is a leave of $F^{ss}_{\mu,\eta}$.*

Proof. – In what follows we choose $\delta \in (0, min\{\frac{\beta}{\alpha} - 1, 2\delta_0 + 1\})$ with $\delta_0 = \frac{\log \lambda^{-1}}{\log \sigma}$, and $K = 2 \sup_{(x,y,\mu,\eta)} |\partial_x \phi(x, y, \mu, \eta)|$. Recall $\beta > \alpha$ because of **a** in §1. Notice that $K > 0$ because $\partial_x \phi(0, 1, 0, 0)$ is not zero (see Lemma 2.1). Suppose that (μ, η) is a parameter with $\eta < -K\mu^{(1+\delta)}$. Let us consider a cone field $C(0)$ in $R_\Delta = [\Delta, \sigma\Delta] \times [0, 1]$ around the horizontal direction, with openness angle equal to $K/\sqrt{-\eta}$. Let $N = N(\mu, \eta)$ be the first integer such that $\Delta < \mu.\sigma^N \leq \Delta.\sigma$. Consider the cone field $C(i)$, in $\Pi^{-i}_{\mu,\eta}(R_\Delta) \cap \Sigma^+$, obtained by iteration, under $D\Pi^{-i}_{\mu,\eta}$, of the one defined above on R_Δ, *i.e.* $C(i) = D\Pi^{-i}_{\mu,\eta}(C(0))$ (*see* Figure 4 below).

Let ϵ_N be the openness angle of $C(N)$. On $[0, 1] \times [0, \sigma^{-N}\Delta]$ we set the cone field with openness angle ϵ_N. Denote this cone field by C and by $C(p)$ the cone of C at $p \in \Sigma^+$. We shall prove that C is positively invariant by $\Pi^{-1}_{\mu,\eta}$. First observe that $\epsilon_N = \lambda^N \sigma^{-N} K/\sqrt{-\eta}$. Thus $\epsilon_N \leq K'\mu^{(1+\delta_0)}/\sqrt{-\eta} \leq K'\mu^{(\delta_0+(-\delta+1)/2)}$. Therefore ϵ_N is a small positive number for $\mu > 0$ close to zero. Now $\Pi_{\mu,\eta}/R^0_{\mu,\eta}$ coincides with (f, g), where $f = f_{\mu,\eta}(x, y)$ and $g = g_{\mu,\eta}(x, y)$ satisfy:

– $\partial_x f = \{\partial_{y_0} C\phi^{(\beta-\alpha+1)}(\partial_x A/\partial_x \phi) + \partial_{y_0} C\beta\phi^{(\beta-\alpha)} A + \partial_{z_0} C\alpha\}\phi^{(\alpha-1)}\partial_x \phi;$

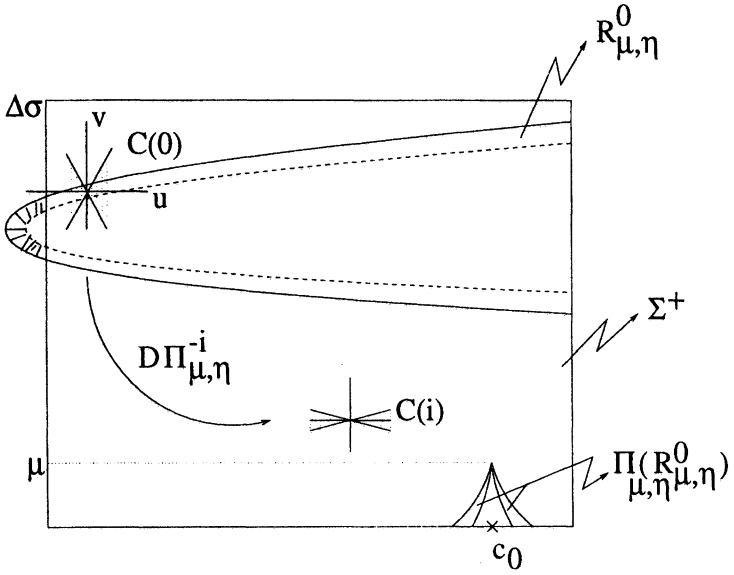

Fig. 4

– $\partial_y f = \{\partial_{y_0} C \partial_y A(\phi^{(\beta-\alpha+1)}/\partial_y\phi) + \partial_{y_0} C\beta\phi^{(\beta-\alpha)}A + \partial_{z_0} C\alpha\}\phi^{(\alpha-1)}\partial_y\phi$;
– $\partial_x g = \{\partial_{y_0} B\phi^{(\beta-\alpha+1)}(\partial_x A/\partial_x\phi) + \partial_{y_0} B\beta\phi^{(\beta-\alpha)}A + \partial_{z_0} B\alpha\}\phi^{(\alpha-1)}\partial_x\phi$;
– $\partial_y g = \{\partial_{y_0} B\partial_y A(\phi^{(\beta-\alpha+1)}/\partial_y\phi) + \partial_{y_0} B\beta\phi^{(\beta-\alpha)}A + \partial_{z_0} B\alpha\}\phi^{(\alpha-1)}\partial_y\phi$.

Let $(1,\varphi)$ be a vector in $C(\Pi_{\mu,\eta}(q))$, where $\Pi_{\mu,\eta}(q) \in [0,1]\times[0,\sigma^{-N}\Delta]$ and $q \in R^0_{\mu,\eta}$. Take $(u,v) = D\Pi^{-1}_{\mu,\eta}(q)(1,\varphi)$. Then, using the above relations, we obtain:

$$(*) \qquad |v/u| \le \left|\frac{M_1\phi^{(\beta-\alpha+1)} + M_2\phi^{(\beta-\alpha)} + \varphi M_3 + M_4}{M_5(\phi^{(\beta-\alpha+1)}/\partial_y\phi) + M_6\phi^{(\beta-\alpha)} + M_4 + \varphi M_3}\right|\left|\frac{\partial_x\phi}{\partial_y\phi}\right|$$

where the functions M_i are bounded ($i \in \{1,...,6\}$) and $|M_4|$ is bounded away from zero. Now, it follows easly from the definitions that $|\phi(x,y,\mu,\eta)| \le K'\mu^{1/\alpha}$ for $(x,y) \in R^0_{\mu,\eta}$ with K' being a positive fixed constant. In addition, we claim that the quotient $\frac{|\partial_y\phi(x,y,\mu,\eta)|}{\sqrt{-\eta}}$ is bounded away from zero. Indeed, we have:

$$0 \le \phi(x,y,\mu,\eta) = \phi(x,\hat{y},\mu,\eta) + \partial_{yy}\phi(x,\hat{\xi},\mu,\eta)(\xi-\hat{y})(y-\hat{y})$$

where the point $\hat{\xi}$ lies between y and $\hat{y}$. The number $\hat{y} = \hat{y}(x,\mu,\eta)$ is defined by the equality $\partial_y\phi(x,\hat{y},\mu,\eta) = 0$ for all x,μ,η. We can also deduce $\phi(x,\hat{y},\mu,\eta) < \eta$ (here we use $\partial_y A(0,1,0,0) < 0$) and so $|y-\hat{y}| \ge \sqrt{-\eta}$. The claim is proved.

The proof of the invariance of the cone field C is achieved because (*) above and the claim imply $|v/u| \le (1+\epsilon)K/\sqrt{-\eta}$ for $\mu > 0$ small. To continue with the proof we use graph transformed techniques (*see* [HPS]) as follows. First we consider the set of continuous maps φ, defined in the square Σ^+, for which the vector $(1,\varphi)$ belongs to the cone field C. Choose a smooth function Z, defined outside the interior of $\Sigma^+\backslash(R^0_{\mu,\eta}\cup R_{\mu,\eta})$ (at this time (μ,η) can be taken fixed), in a way that: $(1,Z(q))$ belongs to $C(q)$ for all q

and $(1, Z(q'))$ belongs to the tangent direction of the boundary of $\Sigma^+ \setminus (R^0_{\mu,\eta} \cup R_{\mu,\eta})$ for q' in such a boundary set. Now, we define the graph transformed operator $\Gamma(\varphi)$ as the slope of the vector $D\Pi^{-1}_{\mu,\eta}(\Pi(q))(1, \varphi(\Pi(q))$ if $q, \Pi(q) \in \Sigma^+$, and $Z(q)$ otherwise. The following bounds hold:

$$\frac{|detD\Pi_{\mu,\eta}|}{|[\partial_y g - (\varphi \circ \Pi_{\mu,\eta})\partial_y f][\partial_y g - (\varphi' \circ \Pi_{\mu,\eta}\partial_y f]|} \leq c_0 \mu^{(\frac{\beta-\alpha+1}{\alpha} - (1+\delta))}$$

and

$$\frac{|\det D\Pi_{\mu,\eta}|.||D\Pi_{\mu,\eta}||}{(\partial_y g - (\varphi \circ \Pi_{\mu,\eta})\partial_y f)^2} \leq c_1 \mu^{(\frac{\beta}{\alpha} - (1+\delta))}$$

for every φ, φ'. Here $||.||$ stands for matrix norm and c_1, c_2 are positive constants. Using these relations, it can be proved that Γ has a fixed point φ_0 which is a C^1 map (*see* appendix in [BLMP]). The foliation $\mathcal{F}^{ss}_{\mu,\eta}$ is then obtained integrating the function φ_0. The theorem is proved. □

The following proposition says that most parameter values in $N(\epsilon) = \{(\mu, \eta) : 0 < \mu < \epsilon, 0 < \eta < K^-(\mu)\}$ are in fact contained in $H(U, X)$ (recall notations in §1), where $K^-(\mu) = -K\mu^{(1+\delta)}$ with ϵ is a small positive fixed number and U, δ, K coming from theorem above.

PROPOSITION 2.3. – *Let $X = X_{(\mu,\eta)}$ be a two-parameter family of vector field as the one in the Theorem 2.2. Then $m(N(\epsilon) \cap H(U, X)) = m(N(\epsilon))$ for all $\epsilon > 0$ small.*

Proof. – Using Theorem 2.2 one can see that the dynamic of $\Pi_{\mu,\eta}$ in Σ^+ is given by an one-dimensional map $\pi_{\mu,\eta}$ as below (*see* Figure 5):

$$\pi_{\mu,\eta}(y) = \begin{cases} \sigma y & \text{if } y \in [0, \sigma^{-1}\Delta] \\ \pi^i_{\mu,\eta}(y) & \text{if } y \in I^i_{\mu,\eta}, \ i = r, l \\ \mu & \text{if } y \in I^c_{\mu,\eta} \end{cases}$$

where the sets $I^l_{\mu,\eta}$, $I^r_{\mu,\eta}$ and $I^c_{\mu,\eta}$ are intervals and the functions $\pi^l_{\mu,\eta}$ and $\pi^r_{\mu,\eta}$ are expanding maps with $(\pi^r_{\mu,\eta})'(y) < -\sigma$ and $(\pi^l_{\mu,\eta})'(y) > \sigma$ for all $y \in I^l_{\mu,\eta}$ and $y \in I^r_{\mu,\eta}$ respectively.

In addition, the union $I_{\mu,\eta} = I^l_{\mu,\eta} \cup I^r_{\mu,\eta} \cup I^c_{\mu,\eta}$ is a closed connected interval with $|I^{(l,r)}_{\mu,\eta}| \leq c_1 \mu^{1/\alpha}$, $|I^c_{\mu,\eta}| \leq c_2\sqrt{-\eta}$ for some positive constants c_1 and c_2.

Now it can be assumed that 1 is the right boundary point of $I^r_{\mu,\eta}$. Let us denote by $a_{\mu,\eta}$ the left boundary point of $I^l_{\mu,\eta}$. Fix $\eta < 0$ small and consider $\mu \in [0, (1/K)(-\eta)^{1/(1+\delta)}]$. Define the set $B_\eta(\epsilon)$ as those $\mu \in (0, \epsilon)$ for which $\mu \in (0, \epsilon)$ and $\pi^n_{\mu,\eta}(\mu) \in [0, \sigma^{-1}\delta] \cup I_{\mu,\eta}$ for all $n \geq 0$. We claim that $B_\eta(\epsilon)$ has limit capacity close to zero for ϵ close to zero (see [PT] for definitions). For that we proceed as follows. Take $\epsilon_0 > 0$ such that $\sigma - \epsilon_0 > 1$. Let us define $A = \sup_{x,\mu \in [0,1]}\{|\partial_\mu \pi(x, \mu)|\}$ where $\pi(x, \mu)$ stands for $\pi_{\mu,\eta}(x)$ (recall η is fixed). Choose a decreasing sequence (μ_n) converging to zero such that $\pi^n_{\mu_n,\eta}(\mu_n) = 1$ and $\pi^i_{\mu_n,\eta}(\mu_n) \in [0, \sigma^{-1}\Delta]$ for all $0 \leq i \leq n-1$. Fix n_0 satisfying $A/\sigma^n < \epsilon_0$ if $n > n_0$. Define $G_m : [0, \mu_n] \to R$ inductively by the relation $G_m(\mu) = \pi(G_{m-1}(\mu), \mu)$ for all m, n. We shall prove the inequality $|G'_m(\mu)| \geq (\sigma - \epsilon_0)^m$ if $\mu \in [0, \mu_n]$ for all m, n. Indeed the following formula holds:

$$G'_m(\mu) = \partial_x \pi(G_{m-1}(\mu), \mu) G'_{m-1}(\mu) + \partial_\mu \pi(G_{m-1}(\mu), \mu)$$

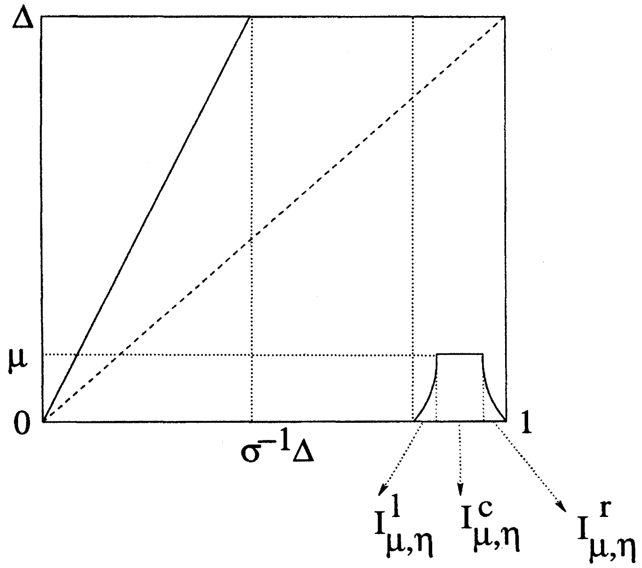

Fig. 5

thus we have $G_0'(\mu) = 1$, $G_1'(\mu) = \sigma$,..., $G_n'(\mu) = \sigma^n$. In addition, the formula implies:

$$G'_{n+1}(\mu) = \left[\partial_x \pi(G_n(\mu), \mu) + \frac{\partial_\mu \pi(G_n(\mu), \mu)}{G'_n(\mu)}\right] G'_n(\mu),$$

therefore $|G'_{n+1}(\mu)| \geq (\sigma - \epsilon_0)\sigma^n \geq (\sigma - \epsilon_0)^{n+1}$ and $|G'_{n+1}(\mu)| \geq \sigma^n$. Now for any $k > 1$ we have:

$$G'_{n+k}(\mu) = \left[\partial_x \pi(G_{n+k-1}(\mu), \mu) + \frac{\partial_\mu \pi(G_{n+k-1}(\mu), \mu)}{G_{n+k-1}(\mu)}\right] G'_{n+k-1}(\mu)$$

Hence an inductive argument shows that:

$$|G'_{n+k}(\mu)| \geq (\sigma - A/\sigma^n)(\sigma - \epsilon_0)^{n+k-1} \geq (\sigma - \epsilon_0)^{n+k}$$

and so $|G'_{n+k}(\mu)| \geq \sigma^n$. The claim is proved. To finish we observe that the one-dimensional map $\pi_{\mu,\eta}$ is a hyperbolic map for $\mu \notin B_\eta(\epsilon)$. From this and Theorem 2.2 we conclude that $\Omega_{(\mu,\eta)}(U)$ is a hyperbolic set if $\mu \notin B_\eta(\epsilon)$, completing the proof of the proposition. □

3. Proof of theorems A and B

In this section we shall prove theorems A and B at §1. The results in the preceding section imply that we can restrict the parameter region to $\{(\mu, \eta) : \mu, \eta > 0\}$. The proofs go as follows. Theorem A will be consequence of an estimate of the region where the new nonwandering points are located. This estimate, as we shall see below, improves the one found in the last section (*see* Lemma 2.1). For the proof of Theorem B, we still use the above estimate and make two parameter exclusions. For this we introduce *quasihyperbolic* and the *free cusp* parameters. We shall prove that these kind of parameters correspond to hyperbolic flows and that the density (respect to the Lebesgue measure in R^2) of the remaining parameters go to zero as we approach $(0,0)$.

Let us start with an improvement on Lemma 2.1 in §2. Denote $\hat{\Omega}_{\mu,\eta}(U) = \Omega_{\mu,\eta}(U) \setminus \{\sigma_0(\mu,\eta), \sigma_1(\mu,\eta)\}$.

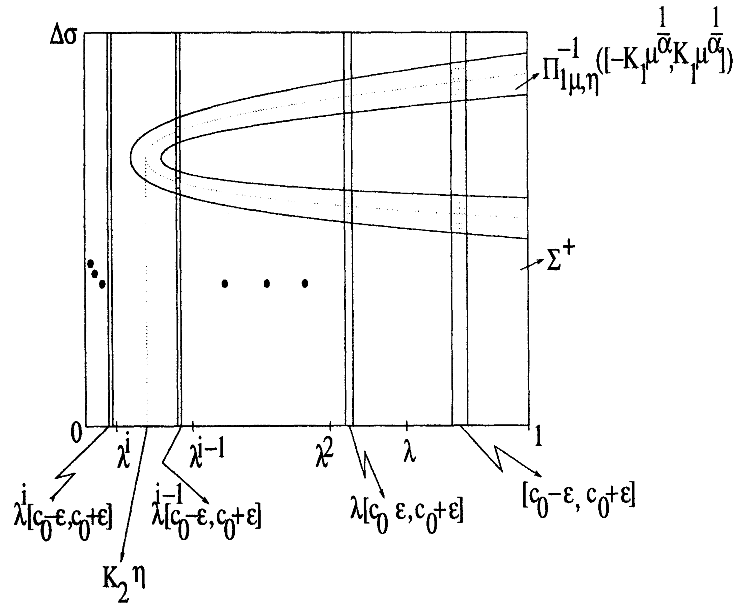

Fig. 6

LEMMA 3.1. – *For each parameter* (μ, η) *close to* $(0,0)$, *we can choose a positive number* $\epsilon = \epsilon(\mu,\eta)$ *with the following property: there is a positive constant* $K_1 > 0$ *such that every orbit belonging to* $\hat{\Omega}_{\mu,\eta}(U)$ *has at least one point into the set:*

$$\Pi^{-1}_{1\mu,\eta}([-K_1\mu^{1/\alpha}, K_1\mu^{1/\alpha}] \times [-1,1]) \cap (\bigcup_{i\in N} \lambda^i[c_0-\epsilon, c_0+\epsilon] \times [0, \sigma\Delta])$$

The constant K_1 *does not depend on the parameter. Furthermore* ϵ *goes to zero when* (μ,η) *goes to* $(0,0)$.

The proof of this lemma can be done using similar arguments as the ones in the proof of Lemma 2.1 (*see* Figure 6).

For $\mu, \eta > 0$, we define the integers $L = L(\mu,\eta)$ and $N = N(\mu,\eta)$ by the inequalities $\lambda < P_x(\lambda^L P_1(\mu,\eta)) \leq 1$ and $\Delta < P_y(\sigma^N P_2(\mu,\eta)) \leq \Delta\sigma$ respectively, where P_x (P_y) is the orthogonal projection on the x-axis (y-axis). $P_2(\mu,\eta)$ is the point (c_0,μ) (thus $P_y(\sigma^N P_2(\mu,\eta) = \sigma^N\mu$, recall §2) and $P_1(\mu,\eta)$ is the critical point of the curve $\Pi^{-1}_{1\mu,\eta}(\{0\}\times[-1,1])$. It is clear that $L < 0$ and $N > 0$. Theorem A is proved using the proposition below and the results in §2. Recall the definition of $H(U,X)$ given in §1.

PROPOSITION 3.2. – *Let $X = X_{\mu,\eta} \in \mathcal{X}^r(M)$ be a two-parameter family crossing W_c transversally at $(\mu,\eta) = (0,0)$. Then there exist $K', \delta' > 0$ and an isolating block of the cycle Γ at $(0,0)$ such that $H(U,X)$ contains the set $\{(\mu,\eta) : \mu,\eta > 0, \eta \geq K'\mu^{(1+\delta')}\}$.*

Proof. – It can be proved that $-L(\mu,\eta) > N(\mu,\eta)$ implies $(\mu,\eta) \in H(U,X)$ because of Lemma 3.1. Now, $-L > N$ holds when:

$$\frac{\log(\sigma^{-1}\Delta) - \log\mu}{\log\sigma} > \frac{\log(\eta - \mu^{1/\alpha})}{\log\lambda}.$$

But this inequality holds if $\eta > c\mu^{\delta_0} + \mu^{1/\alpha}$ for some positive constant c. Thus the proof follows because the cycle Γ is contractive and so $\delta_0 > 1$. □

Now we start with the proof of Theorem B. In what follows, a $D\Pi_{\mu,\eta}^{-1}$-invariant cone field in $\Omega_{(\mu,\eta)}(U)$ is called a *stable cone field* for $\Pi_{\mu,\eta}$. We state and prove

LEMMA 3.3. – *There exist $\delta'', K_2, K_3 > 0$ with the following property, for $(\mu,\eta) \in R^+ \times R^+$ close to $(0,0)$ and ϵ as in Lemma 3.1: if the distance between $\cup_{i\in N}\lambda^i[c_0-\epsilon, c_0+\epsilon]$ and $K_2\eta$ is bigger than $K_3\mu^{(1+\delta'')}$, then there exists a stable cone field for $\Pi_{\mu,\eta}$ whose vectors are uniformly contracted by the linear map $D\Pi_{\mu,\eta}$.*

Proof. – We use the arguments of the proof of Theorem 2.2. First recall that the map $\hat{y} = \hat{y}(x,\mu,\eta)$ is defined by the relation $\partial_y\phi(x,\hat{y},\mu,\eta) = 0$. We choose the constants K_2, K_3, and δ'' by the relations:

1) $0 < \delta'' < \inf\{\frac{\beta}{\alpha} - 1, 1 + 2\delta_0\}$;
2) $K_2 > \frac{1}{\inf_{(x,\mu,\eta)}\{|\partial_x\rho(x,\mu,\eta)|\}}$, where $\rho(x,\mu,\eta) = \phi(x,\hat{y}(x,\mu,\eta),\mu,\eta)$. It follows that $\partial_x\rho(x,\mu,\eta)$ is close to $\partial_x\phi(0,1,0,0)$ and so this quantity is everywhere negative (*see* Lemma 2.1);
3) $K_3 = 2\sup_{(x,y,\mu,\eta)}\{|\partial_x\phi(x,y,\mu,\eta)|\}$.

Let us choose a vector (u,v) inside the cone with openness angle $\frac{K_3}{\sqrt{\mu^{(1+\delta'')}}}$ around the horizontal direction. Using (1), we conclude that if $(u',v') = (\lambda^{-N}u, \sigma^{-N}v)$, then we have that $|u'/v'| \leq \text{cnt}.\mu^\gamma$ for some $\gamma > 0$. We claim that the quotient:

$$\frac{\partial_y\phi(x,y,\mu,\eta)}{\sqrt{\mu^{(1+\delta)}}}$$

is bounded away from zero, for $(x,y) \in \hat{\Omega}_{\mu,\eta}(U)$. Indeed, for those points (x,y) we have:

$$0 \leq \phi(x,y,\mu,\eta) = \phi(x,\hat{y},\mu,\eta) + \partial_y\phi(x,\hat{\xi},\mu,\eta)(\xi - \hat{y})(y - \hat{y})$$

for some ξ in between y and $\hat{y}$. It follows that $\phi(x,\hat{y},\mu,\eta) = \eta + \partial_x\rho(\xi_0,\mu,\eta)x$ for some ξ_0 and thus:

$$0 \leq \eta + \partial_x\rho(\xi_0,\mu,\eta)x + \partial_{yy}\phi(x,\hat{\xi},\mu,\eta)(\xi - \hat{y})(y - \hat{y}).$$

Now we assume that the distance between $H(\mu,\eta)$ and $K_2\eta$ is bigger than $K_3\mu^{(1+\delta'')}$. As $\partial_x\phi(0,1,0,0)$ is negative (because of $\partial_y A(0,1,0,0)<0$), we have $K_2\eta+K_3\mu^{(1+\delta'')}<x$ for all $(x,y)\in\hat{\Omega}_{\mu,\eta}(U)$. This together with 2) imply:

$$|\partial_x\rho(\xi_0,\mu,\eta)|K_3\mu^{(1+\delta)}>\eta+\partial_x\rho(\xi_0,\mu,\eta)x.$$

Thus

$$|y-\hat{y}|>\sqrt{\frac{K_3\mu^{(1+\delta)}}{K_2\sup|\partial_{yy}\phi|}}.$$

The claim follows using the equality $\partial_y\phi(x,y,\mu,\eta)=\partial_{yy}(x,\xi_1,\mu,\eta)(y-\hat{y})$ for some ξ_1. Now one can proceed as in the proof of Theorem 2.2, proving the result. □

In what follows $H(\mu,\eta)$ denotes the set $\cup_{i\in N}\lambda^i[c_0-\epsilon,c_0+\epsilon]$ where ϵ is as in Lemma 3.1. Lemma 3.3 motivates the following definition.

DEFINITION. – *A parameter value (μ,η) is called quasihyperbolic whenever the inequality below hold:*

$$dist(H(\mu,\eta),K_2\eta)>K_3\mu^{(1+\delta'')}$$

where δ'', K_2 and K_3 are as in the above lemma. Here $dist$ means distance. If $(\mu,\eta)\in R^+\times R^+$, we define the interval $I(\mu,\eta)$ as the convex hull, in the oy axis in Σ^+, of the orthogonal projection over this axis of the set

$$(\lambda^{N(\mu,\eta)}[c_0-\epsilon,c_0+\epsilon]\times[0,\sigma\Delta])\bigcap\Pi^{-1}_{1\mu,\eta}([-K_1\mu^{1/\alpha},K_1\mu^{1/\alpha}]\times[-1,1]).$$

A parameter (μ,η) as above is called free cusp if $\sigma^{N(\mu,\eta)}\mu$ does not belong to $I(\mu,\eta)$.

Observe that the interval $I(\mu,\eta)$ may be empty. The following technical lemma will be used for the measure estimate in the Proposition 3.7.

LEMMA 3.4. – *There are positive constants K_5, K_6, and K_7 such that if $\eta\geq K_7\mu^{1/\alpha}$, then $I(\mu,\eta)$ is contained in the interval:*

$$[1-\sqrt{K_5\lambda^{N(\mu,\eta)}+K_6\sigma^{-\frac{N(\mu,\eta)}{\alpha}}},1+\sqrt{K_5\lambda^{N(\mu,\eta)}+K_6\sigma^{-\frac{N(\mu,\eta)}{\alpha}}}].$$

Proof. – We write $\phi(x,y,\mu,\eta)=\eta+\partial_x\phi x+(\frac{1}{2}\partial_{yy}\phi+\epsilon_0)(y-1)^2$, where ϵ_0 denotes a function close to zero. Now, for

$$(x,y)\in(\lambda^{N(\mu,\eta)}[c_0-\epsilon,c_0+\epsilon]\times[0,\sigma\Delta])\cap\Pi^{-1}_{\mu,\eta}([-K_1\mu^{1/\alpha},K_1\mu^{1/\alpha}],$$

we have

$$\left(\inf\left\{\frac{1}{2}\partial_{yy}\phi+\epsilon_0\right\}\right)|y-1|^2\leq K_1\mu^{1/\alpha}+|\eta+\partial_x\phi x|.$$

Since $|\eta+\partial_x\phi x|=-\eta-\partial_x\phi x$ at least for $\eta>K_7\mu^{1/\alpha}$ for some positive constant K_7, we conclude that

$$K_1\mu^{1/\alpha}-\eta\leq(K_1-K_7)\mu^{1/\alpha}\leq(K_1-K_7)\lambda^{N(\mu,\eta)}$$

if $K_7\in(0,K_1)$. This finishes the proof of the lemma. □

Let I_i $(i \in N)$ be defined as

$$I_i = [1 - \sqrt{K_5\lambda^i + K_6\sigma^{-\frac{i}{\alpha}}}, 1 + \sqrt{K_5\lambda^i + K_6\sigma^{-\frac{i}{\alpha}}}]$$

and

$$H_i = \{(\mu,\eta) : \eta > 0, \mu \in (0, \Delta\sigma^{-i}) \text{ and } \mu \notin \cup_{M \geq i}\sigma^{-M} I_i\}.$$

Denote $H = \cup_{N \geq 2} H_i$.

LEMMA 3.5. – *If* $(\mu,\eta) \in H_i$ *for some positive integer* i, *then* (μ,η) *is a free cusp parameter.*

Proof. – If $(\mu,\eta) \in H_i$ then $\mu \in (0, \Delta\sigma^{-i})$ and $\mu \notin \sigma^{-M} I_i$ for all $M \geq N$. Thus $N(\mu,\eta) \geq i$ and $I_{N(\mu,\eta)} \subset I_i$. Therefore $\mu \notin \sigma^{-N(\mu,\eta)} I_{N(\mu,\eta)}$ and we are done. □

Next lemma deals with dynamical properties of $X_{(\mu,\eta)}/\Omega_{(\mu,\eta)}(U)$ when (μ,η) is quasihyperbolic and free cusp.

LEMMA 3.6. – *If* (μ,η) *is quasihyperbolic and free cusp then* $\Omega_{(\mu,\eta)}(U)$ *is a hyperbolic set of* $X_{(\mu,\eta)}$.

This lemma is proved using Lemma 3.3 and the estimates as the ones considered in the proof of Theorem 2.2. We finish the proof of Theorem B with the following proposition.

PROPOSITION 3.7. – *The set of parameter values* (μ,η) *which are both quasihyperbolic and free cusp has Lebesgue density one in* $R^+ \times R^+$ *at* $(0,0)$.

Proof. – It is enough to prove that both quasihyperbolic and free cusp parameters accumulate the origin $(0,0)$ with full Lebesgue density in $R^+ \times R^+$. We deal with free cusp parameters. Let us prove that for all ϵ there exist $\delta_0 > 0$ and $N_0 > 0$ such that $(1/\delta^2)m(H_N \cap ([0,\delta] \times [0,\delta])) > 1 - \epsilon$ if $0 < \delta < \delta_0$ and $N \geq N_0$. First observe that

$$H_N^c = (R^+ \times R^+) \setminus H_N \subset (\bigcup_{M \geq N} \sigma^{-M} I_N) \times R^+.$$

Let $M_0 > 0$ be such that $\sigma^{-M_0} < \delta \leq \sigma^{-M_0+1}$. Thus

$$H_N^c \cap ([0,\delta] \times [0,\delta]) \subset (\cup_{M \geq M_0} \sigma^{-M} I_N) \times [0, \sigma^{-M_0+1}]$$

and so

$$(1/\delta^2)m(H_N^c \cap ([0,\delta] \times [0,\delta])) \leq |I_N| \left(\left(\frac{\sigma^{-M_0}}{\delta} \right)^2 (\Sigma_{M \geq 0} \sigma^{-M}) \right) \leq K|I_N|,$$

where K is some positive constant and $|.|$ stands for the length of I_N. A similar computation can be performed for quasihyperbolic parameters. The proof is completed. □

Remark 3.1.

As was pointed out in §1 and Remark 2.1 (a), the bifurcation diagrams derived from the unfolding of an inclination-flip singular cycle depends heavily on the nature of the numbers $\partial_{yy}\phi(0,1,X)$ and $\partial_y A(0,1,X)$. The ones we have exhibited (Figures 2(a) and 2(b)) correspond to what we called *inward type*, i.e. $\partial_{yy}\phi(0,1,X) > 0$ and $\partial_y A(0,1,X) < 0$. Here we indicate how the bifurcating diagram varies, according to the numbers $\partial_{yy}\phi(0,1,X)$ and $\partial_y A(0,1,X)$. For instance, if $\partial_{yy}\phi(0,1,X) > 0$ and $\partial_y A(0,1,X) > 0$ we have the following picture for the cycle:

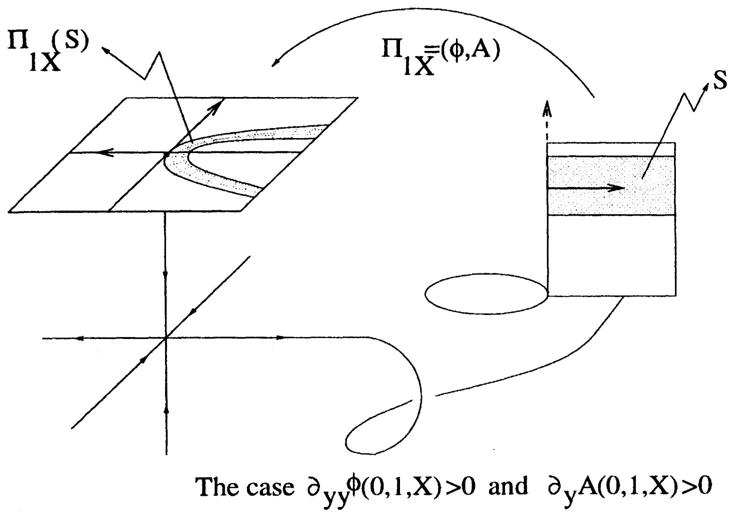

Fig. 7

The bifurcation diagrams for this case are:

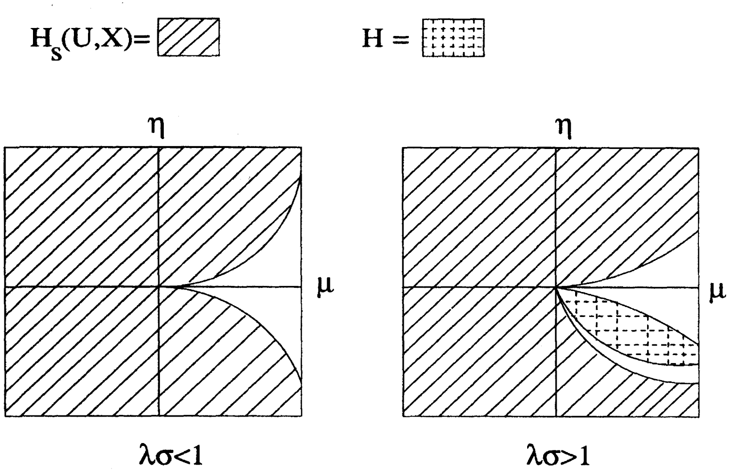

Fig. 8

Here $H_s(U,X)$ is defined as above, and H is a region enclosed by two curves with perpendicular contact at $(0,0)$. In a similar way we can prove that $H_h(U,X) \cap H$ accumulates $(0,0)$ with full Lebesgue two-dimensional measure (recall the definitions in

§1), as well as the complete description of the bifurcation diagrams for the remaining cases. For instance, the bifurcating diagrams in Figure 8 follows from an appropriate version of the Proposition 3.2, together with a parameter exclusion as the one explained in this section. Using this we obtain the proof of Theorem C.

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