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DEGENERATIONS FOR INDECOMPOSABLE MODULES AND TAME ALGEBRAS

BY ANDRZEJ SKOWROŃSKI AND GRZEGORZ ZWARA

ABSTRACT. – Let A be a finite dimensional algebra over an algebraically closed field K. We investigate connection between the representation type of A and existence (and structure) of indecomposable A-modules N which are degenerations of other A-modules. We prove that if there is a common bound on the length of chains $M_r <_{\deg} \dots <_{\deg} M_2 <_{\deg} M_1$ of degenerations of indecomposable A-modules then A is of tame representation type. For A strongly simply connected, we show that A is (tame) of polynomial growth if and only if any indecomposable A-module N is a (proper) degeneration of at most one (up to isomorphism) indecomposable A-module. © Elsevier, Paris

RÉSUMÉ. – Soit A une algèbre de dimension finie sur un corps algébriquement clos K. Nous nous intéressons au lien entre le type de représentation de A et l'existence (et la structure) de A-modules indécomposables obtenus à partir d'autres modules par dégénération. Nous montrons que A est de type docile si la longueur de chaînes $M_r <_{\deg} \dots <_{\deg} M_2 <_{\deg} M_1$ entre indécomposables est bornée. Pour A fortement simplement connexe, nous montrons que A est (docile) à croissance polynomiale si et seulement si tout A-module indécomposable N s'obtient à partir d'au plus un indécomposable (à isomorphisme près) par une dégénération propre. © Elsevier, Paris

1. Introduction and main results

Throughout the paper K denotes a fixed algebraically closed field of arbitrary characteristic. Let A be an associative finite dimensional K-algebra with basis $a_1 = 1$, $a_2, \dots, a_\alpha$, and the associated structure constants a_{ijk}. For any natural number d, the affine variety $\mathrm{mod}_A(d)$ of (unital) d-dimensional right A-modules consists in the α-tuples $m = (m_1, \dots, m_\alpha)$ of $d \times d$ matrices with coefficients in K such that m_1 is the identity and $m_i m_j = \sum m_k a_{kij}$ holds for all indices i and j. The general linear group $\mathrm{Gl}_d(K)$ acts on $\mathrm{mod}_A(d)$ by conjugation, and the orbits correspond to the isomorphism classes of d-dimensional A-modules. Denote by $\mathcal{O}(m)$ the orbit of a point m in $\mathrm{mod}_A(d)$ and by M the A-module on K^d given by m. By abuse of notation we also write M for the isomorphism class of M. Then N is a degeneration of M if n belongs to the closure of $\mathcal{O}(m)$ in the Zariski topology, and we denote this fact by $M \leq_{\deg} N$. It is not clear how

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to characterize the partial order $\leq_{\mathrm{deg}}$ on the set of isomorphism classes of d-dimensional modules in terms of representation theory.

There has been a work by S. Abeasis and A. del Fra, K. Bongartz, Ch. Riedtmann, and others (see [1], [2], [12], [13], [14], [26], [37], [39]) connecting $\leq_{\mathrm{deg}}$ with other partial orders $\leq_{\mathrm{ext}}$, $\leq_{\mathrm{virt}}$ and $\leq$ on the isomorphism classes of A-modules which are defined as follows:

- $M \leq_{\mathrm{ext}} N \Leftrightarrow$ there are modules M_i, U_i, V_i and exact sequences $0 \to U_i \to M_i \to V_i \to 0$ such that $M = M_1$, $M_{i+1} = U_i \oplus V_i$ and $N = M_s$ are true for some natural number s.
- $M \leq_{\mathrm{virt}} N \Leftrightarrow M \oplus X \leq_{\mathrm{deg}} N \oplus X$ for some A-module X.
- $M \leq N \Leftrightarrow [M, X] \leq [N, X]$ holds for all A-modules X.

Here and later on we abreviate $\dim_K \mathrm{Hom}_A(Y, Z)$ by $[Y, Z]$. Note that $\leq$ is a partial order on the isomorphism classes of modules by a result of M. Auslander (see [8], [11]). For an A-module M we denote by $[M]$ the image of M in the Grothendieck group $K_0(A)$ of A. Thus $[M] = [N]$ if and only if M and N have the same simple composition factors including the multiplicities. Observe that, if M and N have the same dimension and $M \leq N$, then $[M] = [N]$. If $[M] = [N]$ then M. Auslander and I. Reiten have shown in [10] for all non-injective indecomposable U the formula $[N, U] - [M, U] = [\mathrm{Tr}\, DU, N] - [\mathrm{Tr}\, DU, M]$. Hence it follows that $M \leq N$ is also equivalent to the inequalities $[X, M] \leq [X, N]$ for all modules X. For modules M and N the following implications hold:

$$M \leq_{\mathrm{ext}} N \Rightarrow M \leq_{\mathrm{deg}} N \Rightarrow M \leq_{\mathrm{virt}} N \Rightarrow M \leq N.$$

Unfortunately, the converse implications are not true in general, and it is interesting to find out when they are. It is known that $\leq_{\mathrm{ext}}$ and $\leq$ coincide for modules over representation-finite algebras whose Auslander-Reiten quiver has no oriented cycles (hence for representations of Dynkin quivers) and for representations of the Kronecker quiver (see [12]). It is also the case for modules from the additive categories of standard quasi-tubes [37]. Recently it was shown that $\leq_{\mathrm{deg}}$ and $\leq$ coincides for all representations of tame quivers [14] and modules from the additive categories of standard multicoils [39]. Finally, we know also that $\leq_{\mathrm{virt}}$ and $\leq$ coincide for all modules over representation-finite algebras [26], or more generally for all modules from the additive categories of generalized standard components of Auslander-Reiten quivers of arbitrary algebras [39].

It is clear that if $M <_{\mathrm{ext}} N$ then N is decomposable. In particular, for representation-finite algebras without oriented cycles in the Auslander-Reiten quivers, the indecomposable modules are not (proper) degenerations of another modules. On the other hand there are very simple representation-finite algebras for which there exist indecomposable modules M and N such that $M <_{\mathrm{deg}} N$. For example, if A is the bound quiver algebra KQ/I given by the quiver

$$Q: \quad \begin{tikzcd} \bullet \arrow[r, "\alpha"] & \bullet \arrow[loop below, "\beta"] \end{tikzcd}$$

and ideal I generated by β^2, then it is easy to see that, for indecomposable representations

$$M: \quad \begin{tikzcd} K \arrow[r, "{\begin{bmatrix}1\\0\end{bmatrix}}"] & K^2 \arrow[loop below, "{\begin{bmatrix}0&0\\1&0\end{bmatrix}}"] \end{tikzcd}$$

and

$$N: \quad \begin{tikzcd} K \arrow[r, "{\begin{bmatrix}0\\1\end{bmatrix}}"] & K^2 \arrow[loop below, "{\begin{bmatrix}0&0\\1&0\end{bmatrix}}"] \end{tikzcd}$$

of A, M degenerates to N.

In this article we are interested in the following two questions:

- When $M <_{\text{deg}} N$ for indecomposable modules M and N over an algebra A?
- When, for a given algebra A, there is a common bound on the length of chains

$$M_r <_{\text{deg}} \dots <_{\text{deg}} M_2 <_{\text{deg}} M_1$$

of degenerations of indecomposable A-modules $M_1, \dots, M_r$?

We show that there is a close connection between the above questions and the representation type of an algebra A and its geometric and homological properties.

Recall that by Drozd's remarkable Tame and Wild Theorem [16] the class of K-algebras may be divided into two disjoint classes. One class consists of wild algebras whose representation theory is as complicated as the study of finite dimensional vector spaces together with two non-commuting endomorphisms for which the classification is a well-known unsolved problem. The second class is formed by the tame algebras for which the indecomposable modules occur in each dimension in a finite number of discrete and a finite number of one-parameter families. More precisely, an algebra A is tame if, for any dimension d, there exists a finite number of $K[x]$-A-bimodules Q_i, $1 \leq i \leq n_d$, which are finitely generated and free as left $K[x]$-modules and all but a finite number of isomorphism classes of indecomposable A-modules of dimension d are of the form $K[x]/(x-\lambda) \otimes_{K[x]} Q_i$ for some $\lambda \in K$ and some i. Let $\mu_A(d)$ be the least number of $K[x]$–A-bimodules satisfying the above conditions for d. Then A is said to be of polynomial growth if there exists a positive integer m such that $\mu_A(d) \leq d^m$ for any $d \geq 1$. From the validity of the second Brauer-Thrall conjecture we know that an algebra A is representation-finite if and only if $\mu_A(d) = 0$ for any $d \geq 1$. The path algebras of Dynkin and extended Dynkin (Euclidean) quivers are respectively representation-finite and representation-infinite of polynomial growth.

Our first result is the following theorem.

THEOREM 1. – *Let A be an algebra and assume that there is an integer t such that for any sequence*

$$M_r <_{\deg} \ldots <_{\deg} M_2 <_{\deg} M_1$$

of degenerations of indecomposable A-modules $M_1, \ldots, M_r$, the inequality $r \leq t$ holds. Then A is tame.

As a direct consequence we get

COROLLARY 2. – *Let A be an algebra such that, for any degeneration $M <_{\deg} N$ of A modules, N is decomposable. Then A is tame.*

Recall that following [17] an algebra A is called quasi-tilted if gl.dim $A \leq 2$ and for any indecomposable A-module X we have $\operatorname{proj.dim} X \leq 1$ or $\operatorname{inj.dim} X \leq 1$. The class of quasi-tilted algebras is wide and contains hereditary algebras, canonical algebras and their tilts (see [17], [18], [19], [20], [22], [27]). Recently all tame quasi-tilted algebras have been described by the first named author in [35]. Using this description we prove the following characterization of tame quasi-tilted algebras in terms of degenerations of modules.

THEOREM 3. – *Let A be a quasi-tilted algebra. Then the following conditions are equivalent:*

(i) *A is tame.*

(ii) *A is of polynomial growth.*

(iii) *For any degeneration $M <_{\deg} N$ of A-modules, the module N is decomposable.*

We shall show also in the paper existence of many tame algebras (of polynomial growth and non-polynomial growth) of global dimension 2 having many degenerations $M <_{\deg} N$ with M and N indecomposable modules.

Following [29] an algebra A is said to be strongly simply connected if it is triangular (the Gabriel quiver of A has no oriented cycles) and for any convex subcategory C of A the first Hochschild cohomology group $H^1(C, C)$ vanishes. Tame strongly simply connected algebras are presently extensively investigated. In particular, in [34] a representation theory of polynomial growth strongly simply connected algebras has been established. Our main result of this paper is the following theorem.

THEOREM 4. – *Let A be a strongly simply connected algebra. The following conditions are equivalent:*

(i) *A is of polynomial growth.*

(ii) *For A-modules M, M', N such that $M <_{\deg} N$, $M' <_{\deg} N$ and N indecomposable, $M \simeq M'$ and is indecomposable.*

(iii) *There exists an integer t such that for any sequence*

$$M_r <_{\deg} \ldots <_{\deg} M_2 <_{\deg} M_1$$

with $M_1, \ldots, M_r$ indecomposable A-modules, the inequality $r \leq t$ holds.

In the course of our proof of Theorem 4 we describe in fact all pairs M and N of indecomposable modules over polynomial growth strongly simply connected algebras for which $M <_{\deg} N$. In particular, if there is such a pair then there are infinitely many of them (see Proposition 4.7). Moreover, we will obtain also the following interesting fact.

COROLLARY 5. – *Let A be a strongly simply connected algebra of polynomial growth, and M, N be two indecomposable A-modules.*

The following conditions are equivalent:

(i) $M <_{\deg} N$.

(ii) *There exists a nonsplittable short exact sequence of A-modules*

$$0 \to Z \to Z \oplus M \to N \to 0$$

with Z indecomposable.

(iii) *There exists a nonsplittable short exact sequence of A-modules*

$$0 \to N \to M \oplus Z \to Z \to 0$$

with Z indecomposable.

The following theorem gives a characterization of strongly simply connected algebras for which every proper degeneration of modules is decomposable.

THEOREM 6. – *Let A be a strongly simply connected algebra. The following conditions are equivalent:*

(i) *For any degeneration $M <_{\deg} N$ of A-modules, the module N is decomposable.*

(ii) *A is of polynomial growth and for any Auslander-Reiten sequence of A-modules*

$$0 \to M \to P \oplus X \to N \to 0$$

with P an indecomposable projective non-injective module lying on an oriented cycle of irreducible maps, the module X is indecomposable or zero.

(iii) *A is of polynomial growth and for any Auslander-Reiten sequence of A-modules*

$$0 \to M \to Y \oplus I \to N \to 0$$

with I an indecomposable injective non-projective module lying on an oriented cycle of irreducible maps, the module Y is indecomposable or zero.

As a direct concequence of [34, (4.1)] and the above theorem we get the following fact.

COROLLARY 7. – *Let A be an algebra whose Gabriel quiver is a tree. Then A is of polynomial growth if and only if for any degeneration $M <_{\deg} N$ of A-modules, the module N is decomposable.*

We mention that there is a rather efficient criterion for a strongly simply connected algebra to be of polynomial growth. Namely, it is shown in [34] that a strongly simply connected algebra A is of polynomial growth if and only if A does not contain a convex subcategory which is hypercritical or pg-critical. All hypercritical and pg-critical algebras are completely classified by quivers and relations in [38] and [25].

We conclude the introduction with a characterization of tame algebras with directing indecomposable projective modules in terms of degenerations of modules. Recall that an indecomposable module X is called directing [27] if it does not lie on a cycle

$X = X_0 \to X_1 \to \cdots \to X_r = X_0$, $r \geq 1$, of nonzero nonisomorphisms in $\operatorname{ind} A$. Applying the main results of [36] we prove the following theorem.

THEOREM 8. – *Let A be an algebra such that every indecomposable projective A-module is directing. Then A is tame if and only if for any degeneration $M <_{\mathrm{deg}} N$ of A-modules, the module N is decomposable.*

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2. Preliminaries on modules

2.1. Throughout the paper, by an algebra A is meant a finite dimensional K-algebra, which we shall assume (without loss of generality) to be basic and connected. An algebra A can be written as a bound quiver algebra $A \simeq KQ/I$ where $Q = Q_A$ is the Gabriel quiver of A and I is an admissible ideal in the path algebra KQ of Q. As is usual in representation theory, every algebra $A = KQ/I$ will equally be considered as a K-category of which the object class is the set of vertices of Q, and the set of morphisms from x to y is the K-vector space $KQ(x, y)$ of all linear combinations of paths in Q from x to y modulo the subspace $I(x, y) = I \cap KQ(x, y)$. We shall call an algebra A triangular whenever its quiver Q_A has no oriented cycle. A full subcategory C of A is called convex (in A) if any path in Q_A with source and target in Q_C lies entirely in Q_C.

2.2. For an algebra A, we denote by $\operatorname{mod} A$ the category of finite dimensional right A-modules, by $\operatorname{ind} A$ the full subcategory of $\operatorname{mod} A$ formed by the indecomposable modules, by $\operatorname{rad}(\operatorname{mod} A)$ the Jacobson radical of $\operatorname{mod} A$, and by $\operatorname{rad}^\infty(\operatorname{mod} A)$ the intersection of all powers $\operatorname{rad}^i(\operatorname{mod} A)$, $i \geq 1$, of $\operatorname{rad}(\operatorname{mod} A)$. By an A-module is meant an object of $\operatorname{mod} A$. Further, we denote by Γ_A the Auslander-Reiten quiver of A and by $\tau = \tau_A$ and $\tau^- = \tau_A^-$ the Auslander-Reiten translations $D\mathrm{Tr}$ and $\mathrm{Tr}\, D$, respectively. We shall agree to identify the vertices of Γ_A with the corresponding indecomposable modules. For an A-module M we denote by $[M]$ the image of M in the Grothendieck group $K_0(A)$ of A. Thus $[M] = [N]$ if and only if the modules M and N have the same composition factors including the multiplicities. For A-modules X and Y, we abbreviate $\dim_K \operatorname{Hom}_A(X, Y)$ by $[X, Y]$. By a component of Γ_A we mean a connected component of Γ_A. A component $\mathcal{C}$ of Γ_A is called standard if the full subcategory of $\operatorname{mod} A$ formed by all modules lying in $\mathcal{C}$ is equivalent to the mesh-category $K(\mathcal{C})$ of $\mathcal{C}$ (see [15], [27]). Following [31] a component $\mathcal{C}$ of Γ_A is said to be generalized standard if $\operatorname{rad}^\infty(X, Y) = 0$ for all module X and Y from $\mathcal{C}$. It is shown in [24] that every standard component is generalized standard. Finally, for a translation subquiver Γ of Γ_A, we denote by $\operatorname{add}(\Gamma)$ the additive category given by Γ, that is, the full subcategory of $\operatorname{mod} A$ formed by all modules isomorphic to the direct sums of modules from Γ.

2.3. Following [26], for A-modules M and N, we set $M \leq N$ if and only if $[X, M] \leq [X, N]$ for all A-modules X. The fact that $\leq$ is a partial order on the isomorphism classes of A-modules follows from a result by M. Auslander (see [8], [11]). M. Auslander and I. Reiten have shown in [10] that, if $[M] = [N]$ for A-modules M and N, then for all nonprojective indecomposable A-modules X and all noninjective indecomposable A-modules Y the following remarkable formulas hold:

$$[X,M]-[M,\tau X]=[X,N]-[N,\tau X],$$
$$[M,Y]-[\tau^{-}Y,M]=[N,Y]-[\tau^{-}Y,N].$$

Hence, if $[M]=[N]$, then $M\le N$ if and only if $[M,X]\le[N,X]$ for all A-modules X.

2.4. Let M and N be A-modules with $[M]=[N]$, and

$$\Sigma:\ 0\to D\to E\to F\to 0$$

be an exact sequence in $\operatorname{mod} A$. Following [26] we define the additive functions $\delta_{M,N}$, $\delta'_{M,N}$, δ_Σ, δ'_Σ on A-modules X as follows:

$$\begin{aligned} \delta_{M,N}(X) &= [N,X]-[M,X],\\ \delta'_{M,N}(X) &= [X,N]-[X,M],\\ \delta_\Sigma=\delta_{E,D\oplus F} &= [D\oplus F,X]-[E,X],\\ \delta'_\Sigma=\delta'_{E,D\oplus F} &= [X,D\oplus F]-[X,E]. \end{aligned}$$

From the Auslander-Reiten formulas (2.3) we get the following useful equalities:

$$\delta_{M,N}(X)=\delta'_{M,N}(\tau^{-}X),\qquad \delta_{M,N}(\tau X)=\delta'_{M,N}(X)$$

and

$$\delta_\Sigma(X)=\delta'_\Sigma(\tau^{-}X),\qquad \delta_\Sigma(\tau X)=\delta'_\Sigma(X)$$

for all A-modules X. Observe also that $\delta_{M,N}(I)=0$ for any injective A-module I, and $\delta'_{M,N}(P)=0$ for any projective A-module P. In particular, we get that the following conditions are equivalent:

(1) $M\le N$.

(2) $\delta_{M,N}(X)\ge 0$ for all X in $\operatorname{ind} A$.

(3) $\delta'_{M,N}(X)\ge 0$ for all X in $\operatorname{ind} A$.

2.5. Let Γ be a connected component of Γ_A. For modules M and N in $\operatorname{add}(\Gamma)$ with $[M]=[N]$ we set

$$M\le_\Gamma N \Leftrightarrow [X,M]\le[X,N]\qquad \text{for all } X \text{ in } \Gamma.$$

By the Auslander-Reiten formulas (2.3) we conclude again that $M\le_\Gamma N\Leftrightarrow[M,X]\le[N,X]$ for all X in Γ. It is shown in [37, (2.8)] that $\le_\Gamma$ is a partial order on the isomorphism classes of modules in $\operatorname{add}(\Gamma)$ having the same composition factors. Moreover, if M and N belong to $\operatorname{add}(\Gamma)$ and $[M]=[N]$ then the following conditions are equivalent:

(1) $M\le_\Gamma N$.

(2) $\delta_{M,N}(X)\ge 0$ for all X in Γ.

(3) $\delta'_{M,N}(X)\ge 0$ for all X in Γ.

2.6. For an A-module M and an indecomposable A-module Z, we denote by $\mu(M,Z)$ the multiplicity of Z as a direct summand of M. For a nonprojective indecomposable A-module U, we denote by $\Sigma(U)$ an Auslander-Reiten sequence

$$\Sigma(U):\ 0 \to \tau U \to E(U) \to U \to 0$$

in $\operatorname{mod} A$. For U indecomposable projective, we set $E(U) = \operatorname{rad} U$. We than have the following lemma.

LEMMA. – *Let M be an A-module and U an indecomposable A-module. Then*

(i) *If U is noninjective, then $\delta_{\Sigma(\tau^{-}U)}(M) = \mu(M,U)$.*

(ii) *If U is nonprojective, then $\delta'_{\Sigma(U)}(M) = \mu(M,U)$.*

Proof. – See [37, (2.5)].

2.7. LEMMA. – *Let M and N be A-modules with $[M] = [N]$. Then for any indecomposable A-modules U the following equality holds:*

$$\mu(N,U) - \mu(M,U) = \delta_{M,N}(U) + \delta'_{M,N}(U) - \delta_{M,N}(E(U)).$$

Proof. – If U is nonprojective, then the Auslander-Reiten sequence $\Sigma(U)$ induces an exact sequence

$$0 \to \operatorname{Hom}_A(M,\tau U) \to \operatorname{Hom}_A(M,E(U)) \to \operatorname{rad}(M,U) \to 0,$$

and hence we get

$$[M,\tau U \oplus U] - [M,E(U)] = [M,U] - \dim_K \operatorname{rad}(M,U) = \mu(M,U).$$

Similarly, we have

$$[N,\tau U \oplus U] - [N,E(U)] = \mu(N,U).$$

Then we obtain the equalities

$$\begin{aligned}\mu(N,U) - \mu(M,U) &= ([N,\tau U \oplus U] - [M,\tau U \oplus U]) - ([N,E(U)] - [M,E(U)])\\ &= \delta_{M,N}(\tau U) + \delta_{M,N}(U) - \delta_{M,N}(E(U))\\ &= \delta_{M,N}(U) + \delta'_{M,N}(U) - \delta_{M,N}(E(U)).\end{aligned}$$

Assume now that U is projective. Then $\operatorname{Hom}_A(M,\operatorname{rad} U) \simeq \operatorname{rad}(M,U)$, and so

$$[M,U] - [M,\operatorname{rad} U] = \mu(M,U).$$

Similarly, we have

$$[N,U] - [N,\operatorname{rad} U] = \mu(N,U).$$

Therefore, we get

$$\begin{aligned}\mu(N,U)-\mu(M,U) &= ([N,U]-[M,U])-([N,\operatorname{rad}U]-[M,\operatorname{rad}U])\\ &= \delta_{M,N}(U)-\delta_{M,N}(\operatorname{rad}U)\\ &= \delta_{M,N}(U)+\delta'_{M,N}(U)-\delta_{M,N}(E(U)),\end{aligned}$$

because $\delta'_{M,N}(U) = 0$.

2.8. LEMMA. – *Let* Γ *be a generalized standard component of an Auslander-Reiten quiver* Γ_A *and* M, N *be modules from* $\operatorname{add}(\Gamma)$ *such that* $[M]=[N]$ *and* $M \leq_\Gamma N$. *Then*

(i) *If* $X_{s+1} \to X_s \to \cdots \to X_1 \to X_0$ *is a sectional path in* Γ *and* $\delta'_{M,N}(X_0) - \delta_{M,N}(X_1) > \delta'_{M,N}(X_s) - \delta_{M,N}(X_{s+1})$, *then there is* $1 \leq i \leq s$ *such that* X_i *is a direct summand of* M.

(ii) *If* $\cdots \to X_2 \to X_1 \to X_0$ *is an infinite sectional path in* Γ *and* $\delta'_{M,N}(X_0) > \delta_{M,N}(X_1)$, *then there is* $i \geq 1$ *such that* X_i *is a direct summand of* M *and* $\delta'_{M,N}(X_j) > \delta_{M,N}(X_{j+1})$ *for all* $0 \leq j < i$.

Proof. – (i) For each $1 \leq j \leq s$, the module $X_{j+1} \oplus \tau X_{j-1}$ is a direct summand of $E(X_j)$. Then, by (2.7), we get

$$\begin{aligned}\mu(N,X_j)-\mu(M,X_j) &\leq \delta_{M,N}(X_j)+\delta'_{M,N}(X_j)-\delta_{M,N}(X_{j+1}\oplus\tau X_{j-1})\\ &= (\delta_{M,N}(X_j)-\delta'_{M,N}(X_{j-1}))-(\delta_{M,N}(X_{j+1})-\delta'_{M,N}(X_j)).\end{aligned}$$

Hence, by our assumption, we have

$$\sum_{j=1}^{s}(\mu(N,X_j)-\mu(M,X_j)) \leq (\delta_{M,N}(X_1)-\delta'_{M,N}(X_0))-(\delta_{M,N}(X_{s+1})-\delta'_{M,N}(X_s)) < 0$$

Consequently, $\mu(M,X_i) > 0$ and X_i is a direct summand of M, for some $1 \leq i \leq s$.

(ii) We know from [39, (4.2)] that there is $k \geq 1$ such that $\delta'_{M,N}(X_k) = \delta_{M,N}(X_{k+1}) = 0$. Take $s \geq 1$ minimal such that $\delta'_{M,N}(X_s) \leq \delta_{M,N}(X_{s+1})$. Applying (i) to the sectional path $X_{s+1} \to X_s \to \cdots \to X_1 \to X_0$, we infer that there is $1 \leq i \leq s$ such that X_i is a direct summand of M and $\delta'_{M,N}(X_j) > \delta_{M,N}(X_{j+1})$ for all $0 \leq j < i$, by our choice of s.

2.9. We shall need also the following Lemma $(3+3+2)$ from [4, (2.1)] and its direct consequence.

LEMMA. – *Let*

$$\Sigma_1 : 0 \longrightarrow M_1 \xrightarrow{\left[\begin{smallmatrix}u_1\\ f_1\end{smallmatrix}\right]} M_2 \oplus N_1 \xrightarrow{[f_2,u_2]} N_2 \longrightarrow 0,$$

$$\Sigma_2 : 0 \longrightarrow M_2 \xrightarrow{\left[\begin{smallmatrix}v_1\\ f_2\end{smallmatrix}\right]} M_3 \oplus N_2 \xrightarrow{[f_3,v_2]} N_3 \longrightarrow 0.$$

be short exact sequences in $\operatorname{mod}A$. *Then the sequence*

$$\Sigma_3 : 0 \longrightarrow M_1 \xrightarrow{\left[\begin{smallmatrix}v_1u_1\\ f_1\end{smallmatrix}\right]} M_3 \oplus N_1 \xrightarrow{[f_3,-v_2u_2]} N_3 \longrightarrow 0$$

is exact. Moreover, we have

$$\delta_{\Sigma_3} = \delta_{\Sigma_1} + \delta_{\Sigma_2} \quad \text{and} \quad \delta'_{\Sigma_3} = \delta'_{\Sigma_1} + \delta'_{\Sigma_2}.$$

2.10. An essential role in our investigations will be played by the following result due to Ch. Riedtmann [26, (3.4)].

PROPOSITION. – *Let*

$$0 \to Z \to Z \oplus X \to Y \to 0$$

be an exact sequence of A*-modules. Then* $X \leq_{\deg} Y$.

3. Algebras with many indecomposable degenerations

3.1. Let B be an algebra and R a B-module. The one-point extension of B by R is the algebra

$$B[R] = \begin{bmatrix} K & R \\ 0 & B \end{bmatrix}$$

with the usual addition and multiplication of matrices. The quiver of $B[R]$ contains Q_B as a convex subquiver and there is an additional (extension) vertex which is a source. The $B[R]$-modules are usually identified with the triples (V, M, φ), where V is a K-vector space, M a B-module and $\varphi : V \to \mathrm{Hom}_B(R, M)$ is a K-linear map. A $B[R]$-linear map $(V, M, \varphi) \to (V', M', \varphi')$ is thus a pair (f, g) where $f : V \to V'$ is K-linear and $g : M \to M'$ is B-linear such that $\varphi' f = \mathrm{Hom}_B(R, g)\varphi$. One defines dually the one-point coextension $[R]B$ of B by R (see [27]).

3.2. Following [27] a translation quiver $\mathcal{T}$ of the form $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ is called a stable tube of rank r. By a ray tube is meant a translation quiver Γ obtained from a stable tube $\mathcal{T}$ by insertion of finitely many rays. It has been proved in [23] that if B is an algebra and $\mathcal{C}$ a component of Γ_B without injective modules, then $\mathcal{C}$ contains an oriented cycle if and only if $\mathcal{C}$ is a ray tube. Moreover, it is known (see [32]) that if a ray tube $\mathcal{C}$ in Γ_B is generalized standard then it is also standard. An indecomposable B-module R in a standard ray tube $\mathcal{C}$ of Γ_B is said to be a 2-ray module if the support $\mathcal{S}(R)$ of the functor $\mathrm{Hom}_B(R, -)|_{\mathcal{C}}$ is the K-linear category of the partially ordered set

$$\begin{array}{ccccccccc}
Y_1 & \longrightarrow & Y_2 & \to \cdots \to & Y_{i+1} & \longrightarrow & Y_{i+2} & \to \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & \\
R = X_0 & \longrightarrow & X_1 & \to \cdots \to & X_i & \longrightarrow & X_{i+1} & \to \cdots
\end{array}$$

given by two parallel infinite sectional paths of $\mathcal{C}$. Here, by the support $\mathcal{S}(R)$ we mean the quotient category $\mathcal{H}_R/\mathcal{I}_R$, where $\mathcal{H}_R$ is the full subcategory of $\mathrm{ind}\, B$ consisting of modules M in $\mathcal{C}$ such that $\mathrm{Hom}_B(R, M) \neq 0$, and $\mathcal{I}_R$ is the ideal of $\mathcal{H}_R$ consisting of the morphisms $f : M \to N$ (with M, N in $\mathcal{H}_R$) such that $\mathrm{Hom}_B(R, f) = 0$.

3.3. Proposition. – *Let B be an algebra, $\mathcal{C}$ a standard ray tube in Γ_B, R an indecomposable 2-ray module in $\mathcal{C}$, $A = B[R]$, and Γ the component of Γ_A containing R. Take an arbitrary integer $r \geq 2$. Then there exist exact sequences*

$$0 \to N_t \to N_t \oplus M_{t+1} \to M_t \to 0$$

$1 \leq t \leq r-1$, *in* $\operatorname{mod} A$, *where $N_1, \ldots, N_{r-1}$ and $M_1, \ldots, M_r$ are pairwise nonisomorphic indecomposable modules from Γ. Moreover, there is a sequence of degenerations*

$$M_r <_{\deg} M_{r-1} <_{\deg} \cdots <_{\deg} M_2 <_{\deg} M_1$$

in $\operatorname{mod}_A(d)$, *where* $d = \dim_K M_1$.

Proof. – Since $\mathcal{C}$ is standard, for all arrows $\xi : X \to Y$ in $\mathcal{C}$ we may choose irreducible maps $f_\xi : X \to Y$ in such a way that, if

$$\begin{tikzcd} & V \arrow[dr, "\gamma"] & \\ U \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & T \\ & W \arrow[ur, "\sigma"'] & \end{tikzcd}$$

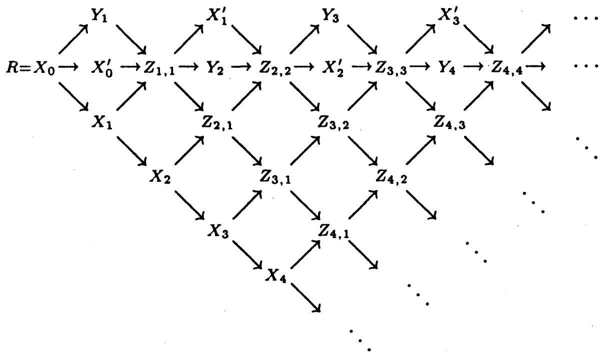
is a complete mesh in $\mathcal{C}$, then $f_\gamma f_\alpha = f_\sigma f_\beta$. Further, it follows from our assumption on R that $\mathcal{C}$ admits a full translation subquiver

$$\begin{tikzcd} Y_1 \arrow[r, "\gamma_1"] & Y_2 \arrow[r] & \cdots \arrow[r] & Y_{i+1} \arrow[r, "\gamma_{i+1}"] & Y_{i+2} \arrow[r] & \cdots \\ R = X_0 \arrow[u, "\beta_0"'] \arrow[r, "\alpha_0"] & X_1 \arrow[u, "\beta_1"'] \arrow[r] & \cdots \arrow[r] & X_i \arrow[u, "\beta_i"'] \arrow[r, "\alpha_i"] & X_{i+1} \arrow[u, "\beta_{i+1}"'] \arrow[r] & \cdots \end{tikzcd}$$

with Y_1 lying on the mouth of $\mathcal{C}$. Hence, $\operatorname{Hom}_B(R, X_i)$, $i \geq 0$, and $\operatorname{Hom}_B(R, Y_j)$, $j \geq 1$, are one dimensional generated by $u_0 = 1_R$, $u_i = f_{\alpha_{i-1}} \cdots f_{\alpha_0}$, for $i \geq 1$, and $v_1 = f_{\beta_0}$, $v_j = f_{\gamma_{j-1}} \cdots f_{\gamma_1} f_{\beta_0}$, $j \geq 2$, respectively. Moreover, $\operatorname{Hom}_B(R, h) = 0$ for any $h : Y_j \to X_i$, $j \geq 1$, $i \geq 0$, and $\operatorname{Hom}_B(R, g) = 0$ for any $g : X_i \to Y_j$ with $1 \leq j \leq i$. Then we get the following indecomposable A-modules

$$Z_{ij} = (K, X_i \oplus Y_j, \Delta_{ij}), \qquad 1 \leq j \leq i,$$

where $\Delta_{ij} : K \to \operatorname{Hom}_B(R, X_i \oplus Y_j) = \operatorname{Hom}_B(R, X_i) \oplus \operatorname{Hom}_B(R, Y_j)$ assigns to $1 \in K$ the pair (u_i, v_j). Consider also the indecomposable A-modules $X_i' = (K, X_i, \eta_i)$, $i \geq 0$, where $\eta_i(1) = u_i$ for each $i \geq 0$. Observe that X_0' is the new indecomposable projective A-module whose radical is R. Applying now [27, (2.5)], and calculating the corresponding cokernels, we infer that Γ admits a full translation subquiver $\mathcal{D}$ of the form



formed by the modules X_i, X'_i, $i \geq 0$, Y_j, $j \geq 1$, and $Z_{i,j}$, $1 \leq j \leq i$, which is closed under successors in Γ_A. We shall find the required modules $N_1, \ldots, N_{r-1}, M_1, \ldots, M_r$ among the modules $Z_{i,j}$ in $\mathcal{D}$.

Denote by Σ the sectional path $X_0 \to X_1 \to \cdots \to X_i \to X_{i+1} \to \cdots$ and by Ω the sectional path $Y_1 \to Y_2 \to \cdots \to Y_{i+1} \to Y_{i+2} \to \cdots$. Fix a module Y_m on Ω lying on an oriented cycle in $\mathcal{C}$, and consider the infinite sectional path Θ in $\mathcal{C}$ with target Y_m, consisting of arrows pointing to the mouth of $\mathcal{C}$. Then there is a sequence $m = i_1 < i_2 < i_3 < \ldots$ such that

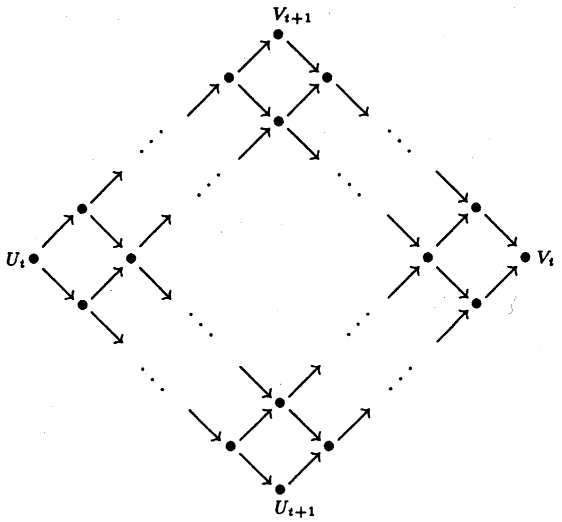
(1) $\Omega \cap \Theta$ consists of the modules $Y_{i_1}, Y_{i_2}, Y_{i_3}, \ldots$,

(2) $\Sigma \cap \Theta$ consists of the modules $X_{i_1-1}, X_{i_2-1}, X_{i_3-1}, \ldots$.

Consider now the indecomposable B-modules

$$U_t = X_{i_{r+t}-1} \quad \text{and} \quad V_t = Y_{i_{r-t+1}}, \qquad \text{for } 1 \leq t \leq r.$$

It is easy to see that, for each $1 \leq t \leq r-1$, we have in $\mathcal{C}$ a rectangle $\mathcal{R}_t$



whose border is formed by the corresponding sectional paths. Observe also that, for each $1 \leq t \leq r-2$, $\mathcal{R}_t$ is a proper full translation subquiver of $\mathcal{R}_{t+1}$. Fix t, $1 \leq t \leq r-1$. We know that, for each mesh

$$\begin{tikzcd}[column sep=small, row sep=small] & V \arrow[dr, "\gamma"] & \\ U \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & T \\ & W \arrow[ur, "\sigma"'] & \end{tikzcd}$$

in $\mathcal{R}_t$, there exists an exact sequence

$$0 \longrightarrow U \xrightarrow{\left[\begin{smallmatrix} f_\alpha \\ f_\beta \end{smallmatrix}\right]} V \oplus W \xrightarrow{[f_\gamma, -f_\sigma]} T \longrightarrow 0$$

in $\operatorname{mod} B$. Denote by $a_t : U_t \to U_{t+1}$ the composition of irreducible maps f_α corresponding to the arrows α of the sectional path in $\mathcal{R}_t$ from U_t to U_{t+1}, by $b_t : U_t \to V_{t+1}$ the composition of irreducible maps f_ξ corresponding to the arrows ξ of the sectional path in $\mathcal{R}_t$ from U_t to V_{t+1}, by $c_t : U_{t+1} \to V_t$ the composition of irreducible maps f_β corresponding to the arrows β forming the sectional path in $\mathcal{R}_t$ from U_{t+1} to V_t, and by $d_t : V_{t+1} \to V_t$ the composition of irreducible maps f_γ corresponding to the arrows γ forming the sectional path in $\mathcal{R}_t$ from V_{t+1} to V_t. Applying now Lemma 2.3 we get an exact sequence

$$(*) \qquad 0 \longrightarrow U_t \xrightarrow{\left[\begin{smallmatrix} a_t \\ b_t \end{smallmatrix}\right]} U_{t+1} \oplus V_{t+1} \xrightarrow{[c_t, -d_t]} V_t \longrightarrow 0.$$

Define now indecomposable A-modules M_t, $1 \leq t \leq r$, and N_t, $1 \leq t \leq r-1$, as follows:

$$M_t = Z_{i_{r+t}-1, i_{r-t+1}} \qquad \text{and} \qquad N_t = Z_{i_{r+t}-1, i_{r-t}}.$$

Observe that $U_t \oplus V_t$ is the restriction of M_t to B, and $U_t \oplus V_{t+1}$ the restriction of N_t to B. Further, for $1 \leq t \leq r$, define $\xi_t : R \to U_t$ and $\rho_t : R \to V_t$ by:

$$\xi_t = u_{i_{r+t}-1} \qquad \text{and} \qquad \rho_t = v_{i_{r-t+1}}.$$

Then $\xi_{t+1} = a_t \xi_t$, $\rho_t = d_t \rho_{t+1}$, $b_t \xi_t = 0$ and $c_t \xi_{t+1} = 0$, for any $1 \leq t \leq r-1$. We then get A-homomorphisms

$$\varphi_t : \ N_t \to N_t \oplus M_{t+1}, \qquad 1 \leq t \leq r-1,$$

given by the canonical monomorphism $K \to 0 \times K \hookrightarrow K^2$ and the maps

$$U_t \oplus V_{t+1} \xrightarrow{\left[\begin{smallmatrix} 0 & 0 \\ b_t & 0 \\ a_t & 0 \\ 0 & 1 \end{smallmatrix}\right]} U_t \oplus V_{t+1} \oplus U_{t+1} \oplus V_{t+1},$$

and A-homomorphisms

$$\psi_t : N_t \oplus M_{t+1} \to M_t, \qquad 1 \le t \le r-1,$$

given by the canonical epimorphism $K^2 \to K \times 0 \to K$, and the maps

$$U_t \oplus V_{t+1} \oplus U_{t+1} \oplus V_{t+1} \xrightarrow{\left[\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & d_t & -c_t & 0 \end{smallmatrix}\right]} U_t \oplus V_t.$$

A direct checking shows that the sequences

$$0 \to N_t \xrightarrow{\varphi_t} N_t \oplus M_{t+1} \xrightarrow{\psi_t} M_t \to 0, \qquad 1 \le t \le r-1,$$

are exact. Applying Proposition 2.10 we get also degenerations $M_{t+1} <_{\deg} M_t$, $1 \le t \le r-1$, because clearly the modules $M_1, \ldots M_r$ are pairwise nonisomorphic and have the same dimension (in fact the same composition factors). It finishes the proof.

3.4. Following [25] an algebra A is said to be pg-critical (polynomial growth critical) if the following conditions are satisfied:

(1) A is one of the forms

$$B[M] = \begin{bmatrix} K & M \\ 0 & B \end{bmatrix}, \qquad B[N,t] = \begin{bmatrix} K & K & \dots & K & K & K & N \\ & K & \dots & K & K & K & 0 \\ & & \ddots & \vdots & \vdots & \vdots & \vdots \\ & & & K & K & K & 0 \\ & & & & K & 0 & 0 \\ & 0 & & & & K & 0 \\ & & & & & & B \end{bmatrix}$$

where B is a representation-infinite tilted algebra of the type $\widetilde{\mathbb{D}}_m$, $m \ge 4$, with a complete slice in the preinjective component of Γ_B, and M (respectively, N) is a 2-ray module (respectively, 1-ray module) lying in a (standard) ray tube $\mathcal{T}$ of Γ_B having $m-2$ rays, and $t+1 \ge 3$ is the number of objects of $B[N,t]$ which are not in B.

(2) Every proper convex subcategory of A is of polynomial growth.

Then a pg-critical algebra A is tame minimal non-polynomial growth and $\operatorname{gl.dim} A = 2$. The pg-critical algebras has been classified in [25] by quivers and relations. There are only 31 frames of such algebras. It follows from this classification that every pg-critical algebra A is simply connected (in the sense of [3]) and the opposite algebra A^{op} of A is also pg-critical. Moreover, among the 31 frames of pg-critical algebras, only 16 are strongly simply connected. For more details on the module category of a pg-critical algebra we refer to [25]. Our interest in pg-critical algebras is motivated by the following criterion for the polynomial growth of strongly simply connected algebras proved by the first named author in [34]. Namely, a strongly simply connected algebra A is of polynomial growth if and only if A is tame and does not contain a convex subcategory which is pg-critical.

Since by an APR-tilting module [9] one can reduce a pg-critical algebra of the form $B[N,t]$ to a pg-critical algebra $B'[M]$ (see [25]), we get the following consequence of Proposition 3.3.

COROLLARY. – *Let A be a pg-critical algebra and Γ a unique component of Γ_A containing both a projective and an injective module. Then, for each $r \geq 2$, there exists a family of exact sequences*

$$0 \to N_t \to N_t \oplus M_{t+1} \to M_t \to 0, \qquad 1 \leq t \leq r-1,$$

given by pairwise nonisomorphic indecomposable A-modules $N_1, \ldots, N_{r-1}, M_1, \ldots, M_r$ from Γ, and hence a sequence of degenerations

$$M_r <_{\deg} M_{r-1} <_{\deg} \ldots <_{\deg} M_2 <_{\deg} M_1.$$

3.5. THEOREM. – *Let Λ be an algebra. Assume that there is a common bound on the length of chains*

$$M_r <_{\deg} M_{r-1} <_{\deg} \ldots <_{\deg} M_2 <_{\deg} M_1$$

of degenerations of indecomposable modules in the varieties $\operatorname{mod}_A(d)$, $d \geq 1$. *Then Λ is tame.*

Proof. – Suppose that Λ is not tame. Then by the Drozd's Theorem [16] Λ is wild, that is, denoting by $K\langle x, y\rangle$ the free K-algebra in two non-commuting variables x and y, there exists a $K\langle x, y\rangle$–Λ-bimodule M free and finitely generated as a left $K\langle x, y\rangle$-module such that the functor $F = - \otimes_{K\langle x,y\rangle} M : \operatorname{mod} K\langle x, y\rangle \to \operatorname{mod} \Lambda$, where $\operatorname{mod} K\langle x, y\rangle$ is the category of finite dimensional right $K\langle x, y\rangle$-modules, preserves indecomposability and isomorphism classes of indecomposable modules. Let A be a pg-critical algebra. Then it is well-known that there exists a full exact embedding $G : \operatorname{mod} A \to \operatorname{mod} K\langle x, y\rangle$. Hence we get a faithful functor $FG : \operatorname{mod} A \to \operatorname{mod}\Lambda$ which preserves indecomposability and isomorphism classes of indecomposable modules. We know from Corollary 3.4 that, for each $r \geq 2$, there exists a family of exact sequences

$$0 \to N_t \to N_t \oplus M_{t+1} \to M_t \to 0, \qquad 1 \leq t \leq r-1,$$

in $\operatorname{mod} A$ given by pairwise nonisomorphic indecomposable modules $N_1, \ldots, N_{r-1}, M_1, \ldots, M_r$. Applying FG we get the family

$$0 \to FG(N_t) \to FG(N_t) \oplus FG(M_{t+1}) \to FG(M_t) \to 0, \qquad 1 \leq t \leq r-1,$$

of exact sequences in $\operatorname{mod}\Lambda$ given by pairwise nonisomorphic indecomposable modules $FG(N_1), \ldots, FG(N_{r-1}), FG(M_1), \ldots, FG(M_r)$. Then, by Proposition 2.10, we get a sequence of degenerations

$$FG(M_r) <_{\deg} FG(M_{r-1}) <_{\deg} \ldots <_{\deg} FG(M_2) <_{\deg} FG(M_1)$$

of indecomposable modules in $\operatorname{mod}_\Lambda(d)$, where $d = \dim FG(M_1)$. This contradicts our assumption on the common bound of length of chains of degenerations of indecomposable Λ-modules. Therefore, Λ is tame.

4. Indecomposable degenerations in multicoils

4.1. In the representation theory of polynomial growth strongly simply connected algebras a crucial role is played (see [34]) by a class of translation quivers (Auslander-Reiten components) called multicoils. A multicoil consists of a finite set of coils glued together by some directed parts. Moreover, a coil is a translation quiver obtained from a stable tube by a sequence of admissible operations. We refer to [5], [6], [30] for details concerning the admissible operations, coils and multicoils. We shall fix now some notation.

For a coil Γ its full translation subquiver Γ_γ formed by all vertices lying on oriented cycles in Γ is also a coil (see [6, (3.3)]). Moreover, Γ_γ is cofinite in Γ. If $\Gamma = \Gamma_\gamma$, the coil Γ is said to be proper. For each vertex $x \in \Gamma_\gamma$ there there exists a unique infinite sectional path

$$x \to \psi x \to \psi^2 x \to \psi^3 x \to \cdots$$

with source x, and a unique infinite sectional path

$$\cdots \to \varphi^3 x \to \varphi^2 x \to \varphi x \to x$$

with target x. Then $\varphi^i x = \tau^i \psi^i x$ and $\varphi^i \psi^j x = \psi^j \varphi^i x$ for all $i, j \geq 0$. In particular, for $i, j \geq 0$, $\varphi^{i+1}\psi^j x$ is not injective and $\varphi^i \psi^{i+1} x$ is not projective. Further, for each $k \geq 0$, we have $\varphi^{kp(\Gamma)} x = \psi^{kq(\Gamma)} x$, where $p(\Gamma)$ is the number of pairwise disjoint infinite sectional paths in Γ of the form $y \to \psi y \to \psi^2 y \to \cdots$, and $q(\Gamma)$ is the number of pairwise disjoint infinite sectional paths in Γ of the form $\cdots \to \varphi^2 y \to \varphi y \to y$. We note also that any vertex x in Γ has at most three immediate predecessors and at most three immediate successors.

4.2. LEMMA. – *Let A be an algebra, and assume that Γ_A contains a proper coil Γ as a (generalized) standard full translation subquiver. Moreover, let X be an arbitrary A-module in Γ such that $\varphi^i X$, for some $i \geq 0$, has three immediate predecessors. Then $[X, \tau_A X] = 0$.*

Proof. – Apply induction on the number of admissible operations leading from a stable tube to Γ, and the constructions of modified translation quivers (see [6, (2.1)] or [30, Section 2]).

4.3. LEMMA. – *Let A be an algebra, Γ a proper coil in Γ_A, M and N A-modules from* $\mathrm{add}(\Gamma)$*, and U an indecomposable A-module. Then*

(i) $\mu(N,U) - \mu(M,U) = \delta_{M,N}(U) + \delta'_{M,N}(U) - \delta'_{M,N}(\psi U) - \sum_{\psi V = U} \delta_{M,N}(V)$.

(ii) $\mu(N,U) - \mu(M,U) = \delta_{M,N}(U) + \delta'_{M,N}(U) - \delta_{M,N}(\varphi U) - \sum_{\varphi V = U} \delta'_{M,N}(V)$.

Proof. – From Lemma 2.7 it is enough to show that

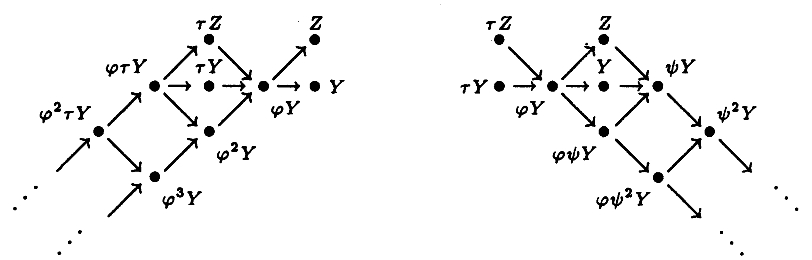
$$\delta_{M,N}(E(U)) = \delta'_{M,N}(\psi U) + \sum_{\psi V = U} \delta_{M,N}(V) = \delta_{M,N}(\varphi V) + \sum_{\varphi V = U} \delta'_{M,N}(V).$$

But $E(U) = \left(\bigoplus_{\psi V = U} V\right) \oplus \varphi U$, and hence

$$\delta_{M,N}(E(U)) = \delta_{M,N}(\varphi U) + \sum_{\psi V = U} \delta_{M,N}(V).$$

Since for any A-module X, projective A-module P and injective A-module I, we have $\delta_{M,N}(X) = \delta'_{M,N}(\tau^- X)$, $\delta'_{M,N}(P) = 0$, $\delta_{M,N}(I) = 0$, we get $\delta_{M,N}(\varphi U) = \delta'_{M,N}(\psi U)$ and $\sum_{\psi V = U} \delta_{M,N}(V) = \sum_{\varphi V = U} \delta'_{M,N}(V)$. Therefore the required equalities (i) and (ii) hold.

4.4. Let Γ be a proper coil in an Auslander-Reiten quiver Γ_A and X an indecomposable A-module from Γ. Then X belongs to at most two maximal infinite sectional paths in Γ consisting of arrows pointing to infinity. Hence there are numbers $0 \leq r \leq s$ and modules Y and Z in Γ such that $X = \psi^r Y = \psi^s Z$ and $Y \to \psi Y \to \psi^2 Y \to \cdots$ and $Z \to \psi Z \to \psi^2 Z \to \cdots$ are unique maximal infinite sectional paths pointing to infinite and containing X. Moreover, these two paths coincide if and only if $Y = Z$. If $Y \neq Z$ then Y or Z is not projective, and if $r < s$ then Y is projective. Thus we may assume that Z is not projective. Finally, in the case when $Y \neq Z$ and Y is also not projective, $r = s$ and we have in Γ mesh-complete subquivers of the form



LEMMA. — *In the above notations, there exist in* $\operatorname{mod} A$ *the following exact sequences and, for an A-module W, the formulas below hold:*

(i) $\Sigma_1 : \ 0 \to \varphi\psi^i Y \to \psi^i Y \oplus \varphi X \to X \to 0$ *for any* $0 < i < r$, *and* $i = 0$ *if* $Y = Z$. *Moreover,*

$$\delta'_{\Sigma_1}(W) = \sum_{k=i+1}^{r} \mu(W, \psi^k Y).$$

(ii) $\Sigma_2 : \ 0 \to \varphi\psi^i Z \to \psi^i Z \oplus Y \oplus \varphi X \to X \to 0$ *for any* $0 \leq i \leq s - r$ *provided* $Y \neq Z$. *Moreover,*

$$\delta'_{\Sigma_2}(W) = \sum_{k=i+1}^{s} \mu(W, \psi^k Z).$$

(iii) $\Sigma_3 : \ 0 \to \tau Y \to \varphi X \to X \to 0$ *if* $Y = Z$ *and is not projective. Moreover,*

$$\delta'_{\Sigma_3}(W) = \sum_{k=0}^{r} \mu(W, \psi^k Y).$$

(iv) $\Sigma_4 : \ 0 \to \tau Z \to Y \oplus \varphi X \to X \to 0$ *if* $Y \neq Z$. *Moreover,*

$$\delta'_{\Sigma_4}(W) = \sum_{k=0}^{s} \mu(W, \psi^k Z).$$

(v) $\Sigma_5: \ 0 \to \tau Y \to Z \oplus \varphi X \to X \to 0$,

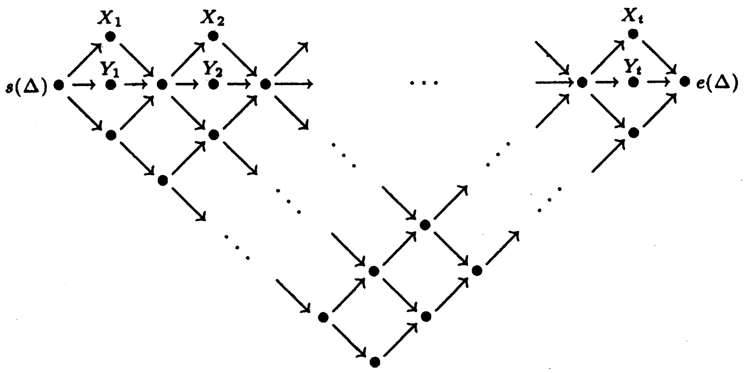
$\Sigma_6: \ 0 \to \tau Y \oplus \tau Z \to \varphi Y \oplus \varphi X \to X \to 0$,

$\Sigma_7: \ 0 \to \varphi^j \tau Y \to \varphi^{j+1} Y \oplus \varphi X \to X \to 0$ *for any* $j \geq 1$, *if* $Y \neq Z$ *and both nonprojective. Moreover, we have the equalities:*

$$\begin{aligned}
\delta'_{\Sigma_5}(W) &= \sum_{k=0}^{r} \mu(W, \varphi^k Y),\\
\delta'_{\Sigma_6}(W) &= \delta'_{\Sigma_5}(W) + \mu(W, Z),\\
\delta'_{\Sigma_7}(W) &= \sum_{k=1}^{j} \mu(W, \varphi^k Y) + \mu(W, Y) + \mu(W, Z) + \sum_{k=1}^{r} \mu(W, \psi^k Y).
\end{aligned}$$

Proof. – It is a direct application of Lemmas 2.9, 2.6 and 4.3.

4.5. By an exceptional triangle in a proper coil Γ of Γ_A we mean a mesh-complete translation subquiver Δ of Γ of the form



where $t = t(\Delta) \geq 2$. In this case $s(\Delta)$ and $e(\Delta)$ are said to be the starting and ending vertex of Δ, respectively. Observe that $\psi^t s(\Delta) = \varphi^t e(\Delta)$. Moreover, for $i \geq 0$ and $0 \leq j < t$, there are in Γ meshes of the form

$$\begin{tikzcd}[column sep=small]
 & \varphi^i\psi^j s(\Delta) \arrow[dr] & \\
\varphi^{i+1}\psi^j s(\Delta) \arrow[ur] \arrow[dr] & & \varphi^i\psi^{j+1} s(\Delta), \\
 & \varphi^{i+1}\psi^{j+1} s(\Delta) \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & \varphi^j\psi^i e(\Delta) \arrow[dr] & \\
\varphi^{j+1}\psi^i e(\Delta) \arrow[ur] \arrow[dr] & & \varphi^j\psi^{i+1} e(\Delta). \\
 & \varphi^{j+1}\psi^{i+1} e(\Delta) \arrow[ur] &
\end{tikzcd}$$

Further, for any $0 \leq i, j \leq t$ and $k \geq 1$, we have

$$\varphi^{kp(\Gamma)-t+i}\psi^j s(\Delta) = \varphi^i \psi^{kq(\Gamma)-t+j} e(\Delta),$$

and we denote this module (vertex) by $\Delta(k, i, j)$.

4.6. PROPOSITION. – *Let A be an algebra, Γ a generalized standard proper coil in Γ_A, and M, N be A-modules such that $M \in \operatorname{add}(\Gamma)$, $N \in \Gamma$, $[M] = [N]$ and $M <_\Gamma N$. Then there exist an exceptional triangle Δ in Γ and a number $k \geq 1$ such that $M = \Delta(k, t(\Delta) - 1, t(\Delta) - 1)$ and $N = \Delta(k, 0, 0)$. In particular, M is indecomposable and M, N do not lie on a common sectional path in Γ.*

Proof. – Since $M <_\Gamma N$ we have, for each indecomposable direct summand X of M, $0 < [X, M] \leq [X, N]$, $0 < [M, X] \leq [N, X]$, and hence a short cycle $X \to N \to X$. Then, X and N lie on an oriented cycle in Γ, because Γ is (generalized) standard. Therefore, $M \in \operatorname{add}(\Gamma)$.

We shall prove our claim by induction on $\sum_{X \in \Gamma} \delta_{M,N}(X) = \sum_{X \in \Gamma} \delta'_{M,N}(X) \geq 0$. We know from [39, (4.2)] that the above sum is finite. We will find a nonsplittable exact sequence

$$\Sigma : \ 0 \to U \to W \oplus N' \to N \to 0$$

given by modules from $\operatorname{add}(\Gamma)$ such that $\delta'_\Sigma \leq \delta'_{M,N}$, $M = W \oplus V$ for some module V, and N' is indecomposable. Then for $M' = U \oplus V$ we have $[M'] = [N']$, $M' \leq_\Gamma N'$ and

$$\sum_{X \in \Gamma} \delta'_{M',N'}(X) = \sum_{X \in \Gamma} \delta'_{M,N}(X) - \sum_{X \in \Gamma} \delta'_\Sigma(X) < \sum_{X \in \Gamma} \delta'_{M,N}(X).$$

By inductive assumption, we then infer that either $M' = N'$ or there exists an exceptional triangle Δ' in Γ and a number $k \geq 1$ such that $M' = \Delta'(k, t(\Delta') - 1, t(\Delta') - 1)$, $N' = \Delta'(k, 0, 0)$ and M', N' do not lie on a common sectional path in Γ. We have several cases to consider.

Assume that $\delta'_{M,N}(N) > \delta_{M,N}(\varphi N)$. Then $\delta_{M,N}(\tau N) = \delta'_{M,N}(N) > 0$, and so $[N, \tau N] > 0$. From Lemma 4.3 we conclude that, for each $j \geq 0$, the vertex $\varphi^j N$ has at most two immediate predecessors in Γ. Applying now Lemma 2.8(ii) to the sectional path $\cdots \to \varphi^2 N \to \varphi N \to N$ we infer that, for some $i \geq 1$, $M = \varphi^i N \oplus M_1$ and $\delta'_{M,N}(\varphi^j N) > 0$ for all $0 \leq j < i$. Consider now the short exact sequences given by the meshes in Γ with the end terms $\varphi^j N$, $0 \leq j < i$. Applying Lemma 2.9, we get an exact sequence

$$\Sigma : \ 0 \to \tau(\varphi^{i-1} N) \to \varphi^i N \oplus M_2 \to N \to 0$$

where $M_2 = 0$ or M_2 is indecomposable with $\psi M_2 = N$. Since $[\varphi^i N] + [M_1] = [M] = [N] \leq [\varphi^i N] + [M_2] - [\tau(\varphi^{i-1} N)]$, we get $[M_2] \geq [\tau(\varphi^{i-1} N)] > 0$, and consequently $M_2 \neq 0$. Moreover, by Lemma 2.6(ii), we have also $\delta'_\Sigma(X) = \sum_{j=0}^{i-1} \mu(X, \varphi^j N)$, for any A-module X. Hence, $\delta'_\Sigma \leq \delta'_{M,N}$. For $M' = \tau(\varphi^{i-1} N) \oplus M_1$ and $N' = M_2$ we have $M' \not\simeq N'$ and the modules $\tau(\varphi^{i-1} N)$ and N' lie on a common sectional path, a contradiction with our inductive assumption.

Therefore, we may assume that $\delta'_{M,N}(N) \leq \delta_{M,N}(\varphi N)$, and by duality $\delta_{M,N}(N) \leq \delta'_{M,N}(\psi N)$. We set $N' = \varphi N$. Applying Lemma 4.3(i) we obtain the equalities

$$1 = \mu(N, N) - \mu(M, N) = \delta_{M,N}(N) + \delta'_{M,N}(N) - \delta'_{M,N}(\psi N) - \sum_{\psi V = N} \delta_{M,N}(V).$$

Thus $\delta'_{M,N}(N) > \sum_{\psi V=N} \delta_{M,N}(V)$. As in (4.4) we consider unique maximal sectional paths in Γ consisting of arrows pointing to infinity

$$Y \to \psi Y \to \psi^2 Y \to \cdots \qquad \text{and} \qquad Z \to \psi Z \to \psi^2 Z \to \cdots$$

(not necessarily different) such that $X = \psi^r Y = \psi^s Z$ for some numbers $0 \le r \le s$.

We assume first that there is $0 \le k < s$ such that $\delta'_{M,N}(\psi^k Z) \le \sum_{\psi V=\psi^k Z} \delta_{M,N}(V)$. Let k be the maximal number with this property. Then

$$\delta'_{M,N}(\psi^{k+1} Z) > \sum_{\psi V=\psi^{k+1} Z} \delta_{M,N}(V) \ge \delta_{M,N}(\psi^k Z).$$

Applying Lemma 4.3(i) we get

$$\begin{aligned} \mu(N, \psi^k Z) - \mu(M, \psi^k Z) = {} & \delta_{M,N}(\psi^k Z) + \delta'_{M,N}(\psi^k Z) \\ & - \delta'_{M,N}(\psi^{k+1} Z) - \sum_{\psi V=\psi^k Z} \delta_{M,N}(V) < 0. \end{aligned}$$

Hence $\psi^k Z$ is a direct summand of M. Applying Lemma 4.4 we get a short exact sequence

$$\Sigma:\ 0 \to \varphi\psi^k Z \to \psi^k Z \oplus N' \to N \to 0$$

if $Y = Z$ or $s - r < k < s$, and

$$\Sigma:\ 0 \to \varphi\psi^k Z \to \psi^k Z \oplus Y \oplus N' \to N \to 0$$

if $Y \ne Z$ and $0 \le k \le s-r$. Moreover, by our choice of k, $\delta'_{M,N}(\psi^i Z) > 0$ for $k < i \le s$, and consequently $\delta'_\Sigma \le \delta'_{M,N}$. In the case when $Y = Z$ or $s - r < k < s$, we get then a contradiction with our inductive assumption on the modules $M' = \varphi\psi^k Z \oplus M_2$ and N', because $\varphi\psi^k Z$ and N' lie on a common sectional path. If $Y \ne Z$, Y is a direct summand of M and $0 \le k \le s - r$ we get a contradiction with our inductive assumption applied for the modules $M' = \varphi\psi^k Z \oplus M_3$ and N', where M_3 is given by $M = \psi^k Z \oplus Y \oplus M_3$. Hence, Y is not a direct summand of M. Applying now Lemma 4.3(i) we get

$$\begin{aligned} 0 &= \mu(N,Y) - \mu(M,Y) \\ &= \delta_{M,N}(Y) + \delta'_{M,N}(Y) - \delta'_{M,N}(\psi Y) - \sum_{\psi V=Y} \delta_{M,N}(V) \\ &= \delta_{M,N}(Y) + \delta'_{M,N}(Y) - \delta'_{M,N}(\psi Y) \\ &\le \sum_{\psi V=\psi^{s-r+1} Z} \delta_{M,N}(V) - \delta'_{M,N}(\psi^{s-r+1} Z) + \delta'_{M,N}(Y). \end{aligned}$$

Since $k < s - r + 1 \le s$, we have $\delta'_{M,N}(Y) > 0$, and consequently Y is not projective (see (2.4)). This implies $r = s$ (see (4.4)), $k = 0$ and $M = Z \oplus M_4$ for some A-module M_4. Hence, by Lemma 4.4(v), there exists an exact sequence

$$\Sigma:\ 0 \to \tau Y \to Z \oplus N' \to N \to 0$$

such that $\delta'_\Sigma = \sum_{k=0}^{r} \mu(-, \varphi^k Y) \leq \delta'_{M,N}$. Also in this case we get a contradiction with our inductive assumption for the modules $M' = \tau Y \oplus M_4$ and N', because τY and N' are not isomorphic and lie on a common sectional path. Hence,

$$\delta'_{M,N}(\psi^k Z) > \sum_{\psi V = \psi^k Z} \delta_{M,N}(V) \geq 0 \qquad \text{for all } 0 \leq k \leq s.$$

Applying now Lemma 4.4(iii), (iv) we get an exact sequence

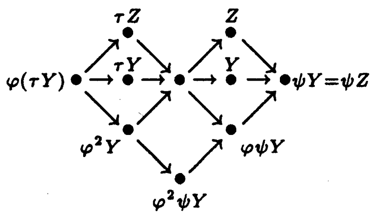
$$\Sigma : \ 0 \to \tau Z \to N' \to N \to 0$$

provided $Y = Z$, and an exact sequence

$$\Sigma : \ 0 \to \tau Z \to Y \oplus N' \to N \to 0$$

otherwise. Moreover, $\delta'_\Sigma = \sum_{k=0}^{s} \mu(-, \psi^k Z) \leq \delta'_{M,N}$.

In the same way as above we can show that Y is not projective and not isomorphic to Z. Hence, there is an exceptional triangle Δ of the form



where $s(\Delta) = \varphi(\tau Y) = \varphi(\tau Z)$, $e(\Delta) = \psi Y = \psi Z$, $t(\Delta) = 2$. Applying Lemma 4.4(v) we obtain an exact sequence

$$\Sigma : \ 0 \to \tau Y \to Z \oplus N' \to N \to 0$$

such that $\delta'_\Sigma = \sum_{k=0}^{r} \mu(-, \varphi^k Y) \leq \delta'_{M,N}$. Assume that $M = Z \oplus M_3$, and set $M' = \tau Y \oplus M_3$. Then we get a contradiction with inductive assumption because N' and τY lie on a common sectional path and are not isomorphic. Hence, Z is not a direct summand of M, and applying Lemma 4.3(i) we get

$$\begin{aligned} 0 &= \mu(N, Z) - \mu(M, Z) = \delta_{M,N}(Z) + \delta'_{M,N}(Z) - \delta'_{M,N}(\psi Z), \\ 0 &= \mu(N, Y) - \mu(M, Y) = \delta_{M,N}(Y) + \delta'_{M,N}(Y) - \delta'_{M,N}(\psi Z). \end{aligned}$$

From the inequality $\delta'_{M,N}(\psi^k Z) > \sum_{\psi V = \psi^k Z} \delta_{M,N}(V)$, applied for $k = 1$, we get

$$\delta'_{M,N}(\psi Z) > \delta_{M,N}(Y) + \delta_{M,N}(Z).$$

Thus

$$\begin{aligned} \delta'_{M,N}(Z) &= \delta'_{M,N}(\psi Z) - \delta_{M,N}(Z) > 0, \\ \delta'_{M,N}(Y) &= \delta'_{M,N}(\psi Z) - \delta_{M,N}(Y) > 0, \end{aligned}$$

and finally

$$\delta'_{M,N}(Y) + \delta'_{M,N}(Z) = 2\delta'_{M,N}(Z) - \delta_{M,N}(\psi Z) - \delta_{M,N}(Y) > \delta'_{M,N}(\psi Z).$$

Applying now Lemma 4.4(v) we obtain a short exact sequence

$$\Sigma:\ 0 \to \tau Y \oplus \tau Z \to \varphi Y \oplus N' \to N \to 0$$

such that

$$\delta'_{\Sigma} = \sum_{k=0}^{r} \mu(-, \varphi^k Y) + \mu(-, Z) \leq \delta'_{M,N}.$$

If $M = \varphi Y \oplus M_3$ for some A-module M_3, then we get a contradiction with our inductive assumption for the modules $M' = \tau Y \oplus \tau Z \oplus M_3$ and N', since τY and N' lie on a common sectional path. Hence, $\varphi Y = \varphi Z$ is not a direct summand of M. Applying Lemma 4.3(ii) we obtain

$$\begin{aligned}
0 &= \mu(N, \varphi Y) - \mu(M, \varphi Y) \\
&= \delta_{M,N}(\varphi Y) + \delta'_{M,N}(\varphi Y) - \delta_{M,N}(\varphi^2 Y) - \delta'_{M,N}(Y) - \delta'_{M,N}(Z) \\
&= \delta'_{M,N}(\varphi Y) - \delta_{M,N}(\varphi^2 Y) + \delta'_{M,N}(\tau^-(\varphi Y)) - \delta'_{M,N}(Y) - \delta'_{M,N}(Z) \\
&= \delta'_{M,N}(\varphi Y) - \delta_{M,N}(\varphi^2 Y) + (\delta'_{M,N}(\psi Y) - \delta'_{M,N}(Y) - \delta'_{M,N}(Z)) \\
&< \delta'_{M,N}(\varphi Y) - \delta_{M,N}(\varphi^2 Y).
\end{aligned}$$

Now, applying Lemma 2.8(ii) to the sectional path $\cdots \to \varphi^3 Y \to \varphi^2 Y \to \varphi Y$, we get a number $l \geq 1$ such that $\varphi^{l+1} Y$ is a direct summand of M and $\delta'_{M,N}(\varphi^i Y) > \delta_{M,N}(\varphi^{i+1} Y)$ for all $1 \leq i \leq l$. Applying Lemma 4.4(v) we get an exact sequence

$$\Sigma:\ 0 \to \varphi^l(\tau Y) \to \varphi^{l+1} Y \oplus N' \to N \to 0$$

such that

$$\delta'_{\Sigma}(W) = \sum_{k=1}^{l} \mu(W, \varphi^k Y) + \mu(W, Y) + \mu(W, Z) + \sum_{k=1}^{r} \mu(W, \psi^k Y)$$

for any A-module W. We know that, for any $i, j \geq 1$, the modules $\varphi^i Y$ and $\psi^j Y$ coincide if and only if $i = hp(\Gamma)$, $j = hq(\Gamma)$ for some $h \geq 1$, and then $\varphi^i Y = \psi^j Y = \Delta(h, 0, 1)$.

Let $\mathcal{Z} = \{\varphi^i Y, Y, Z, \psi^j Y;\ 1 \leq i \leq l, 1 \leq j \leq r\}$ and $p = p(\Gamma)$, $q = q(\Gamma)$. Then

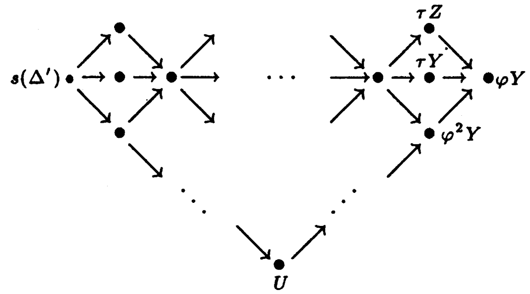
$$\delta'_{\Sigma}(W) = \begin{cases} 2 & W = \varphi^{hp} Y,\ h \geq 1,\ hp \leq l,\ hq \leq r, \\ 1 & \text{for other } W \in \mathcal{Z}, \\ 0 & W \in \Gamma_A \setminus \mathcal{Z}. \end{cases}$$

Observe that $\delta'_{M,N}(W) \geq 1$ for all $W \in \mathcal{Z}$. Take now $h \geq 1$ such that $hp \leq l$ and $hq \leq r$. Since $\psi^{hq-1} Y = \Delta(h, 0, 0) = \varphi^{hp-1}(\tau Y)$, we have

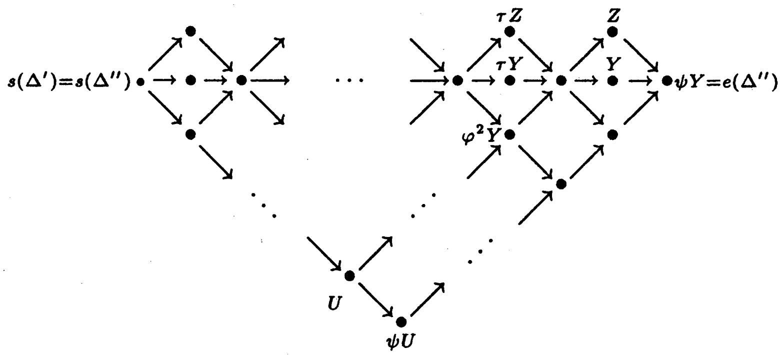
$$\begin{aligned}
\delta'_{M,N}(\varphi^{hp} Y) &= \delta'_{M,N}(\psi^{hq} Y) > \sum_{\psi V = \psi^{hq} Y} \delta_{M,N}(V) \\
&\geq \delta_{M,N}(\psi^{hq-1} Y) = \delta_{M,N}(\varphi^{hp-1}(\tau Y)) \\
&= \delta'_{M,N}(\tau^- \varphi^{hp-1}(\tau Y)) = \delta'_{M,N}(\varphi^{hp-1} Y) \\
&> \delta_{M,N}(\varphi^{hp} Y) \geq 0.
\end{aligned}$$

Hence, $\delta'_{M,N}(\varphi^{hp}Y) \geq 2$ and consequently $\delta'_\Sigma \leq \delta'_{M,N}$. We know that $M = \varphi^{l+1}Y \oplus M_4$ for some A-module M_4. Let $M' = \varphi^l(\tau Y) \oplus M_4$. Then, by our inductive assumption, M' is indecomposable. Thus $M_4 = 0$ and $M' = \varphi^l(\tau Y)$. If $M' \simeq N'$, then $\varphi^l(\tau Y) \simeq \varphi\psi^r Y = \psi^{r+1}(\tau Y)$ and there exists $k \geq 1$ such that $l = kp$, $r + 1 = kq$. This leads to the equalities $M = \varphi^{l+1}Y = \Delta(k,1,1) = \Delta(k, t(\Delta) - 1, t(\Delta) - 1)$ and $N = \psi^r Y = \Delta(k,0,0)$.

Now we may assume that $M' <_\Gamma N'$ and there exist a number $k \geq 1$ and an exceptional triangle Δ' of the form



such that $N' = \Delta'(k,0,0)$ and $M' = \Delta'(k, t(\Delta') - 1, t(\Delta') - 1)$. We may extend Δ' to a new exceptional triangle Δ'' of the form



such that $t(\Delta'') = t(\Delta') + 1$. Clearly, $N = \Delta''(k,0,0)$ and $M = \psi M' = \Delta'(k, t(\Delta') - 1, t(\Delta')) = \Delta''(k, t(\Delta'') - 1, t(\Delta'') - 1)$. This finishes the proof.

4.7. Following [6, (3.4)] a component $\mathcal{C}$ of Γ_A is said to be a multicoil if it contains a full translation subquiver $\mathcal{C}'$ such that

(i) $\mathcal{C}'$ is a disjoint union of coils.

(ii) No point of $\mathcal{C} \setminus \mathcal{C}'$ lies on an oriented cycle in $\mathcal{C}$.

The following proposition describes all degenerations to indecomposable modules in the additive categories of generalized standard multicoils.

PROPOSITION. – *Let A be an algebra and $\mathcal{C}$ be a generalized standard multicoil in Γ_A. Let M and N be A-modules such that $M \in \operatorname{add}(\mathcal{C})$, $N \in \mathcal{C}$ and $[M] = [N]$. Then $M <_{\deg} N$ if and only if there exists an exceptional triangle Δ in $\mathcal{C}$ and a number $k \geq 1$ such that $M = \Delta(k, t(\Delta) - 1, t(\Delta) - 1)$ and $N = \Delta(k, 0, 0)$.*

Proof. – Assume $M <_{\deg} N$. Then $M <_{\mathcal{C}} N$. Hence, for each indecomposable direct summand X of M, we get $[X, N] \geq [X, M] > 0$, $[N, X] \geq [M, X] > 0$, and so a short cycle $X \to N \to X$. Since $\mathcal{C}$ is generalized standard and $X, N \in \mathcal{C}$, we infer that X and N lie on an oriented cycle in $\mathcal{C}$. Therefore, there is a proper coil Γ of $\mathcal{C}$ such that $M \in \operatorname{add}(\Gamma)$ and $N \in \Gamma$. Applying Proposition 4.6 we conclude that there exists an exceptional triangle Δ in Γ (hence in $\mathcal{C}$) and $k \geq 1$ such that $M = \Delta(k, t(\Delta)-1, t(\Delta)-1)$ and $N = \Delta(k, 0, 0)$. Conversely, assume that $M = \Delta(k, t(\Delta)-1, t(\Delta)-1)$ and $N = \Delta(k, 0, 0)$ for some $k \geq 1$ and an exceptional triangle Δ in $\mathcal{C}$. We use the notation of (4.5) and set $t = t(\Delta)$. We shall prove that $\Delta(k, t-1, t-1) <_{\deg} \Delta(t, 0, 0)$. Applying Lemma 2.9 we get exact sequences

$$0 \to \varphi^i \psi^j s(\Delta) \to \varphi^i \psi^{j+1} s(\Delta) \oplus X_{j+1} \to Y_{j+2} \to 0, \tag{1}$$

$$0 \to \varphi^i \psi^j s(\Delta) \to \varphi^i \psi^{j+1} s(\Delta) \oplus Y_{j+1} \to X_{j+2} \to 0, \tag{2}$$

for any $i \geq 0$ and $0 \leq j \leq t-2$. Applying Lemma 2.9 again, we get exact sequences

$$0 \to \varphi^i s(\Delta) \to \varphi^i \psi^{t-1} s(\Delta) \oplus X_1 \to X_t \to 0, \tag{3}$$

for $i \geq 0$ and t odd, and exact sequences

$$0 \to \varphi^i s(\Delta) \to \varphi^i \psi^{t-1} s(\Delta) \oplus X_1 \to Y_t \to 0, \tag{4}$$

for $i \geq 0$ and t even.

Similarly, we get exact sequences

$$0 \to X_1 \to X_t \oplus \varphi^{t-1} \psi^j e(\Delta) \to \psi^j e(\Delta) \to 0 \tag{5}$$

for $j \geq 0$ and t odd, and exact sequences

$$0 \to X_1 \to Y_t \oplus \varphi^{t-1} \psi^j e(\Delta) \to \psi^j e(\Delta) \to 0, \tag{6}$$

for $j \geq 0$ and t even. Further, applying Lemma 2.9 to the exact sequences (3) and (5), for t odd, and to the exact sequences (4) and (6), for t even, we get the exact sequences

$$0 \to \varphi^i s(\Delta) \to \varphi^i \psi^{t-1} s(\Delta) \oplus \varphi^{t-1} \psi^j e(\Delta) \to \psi^j e(\Delta) \to 0 \tag{7}$$

for all $i, j \geq 0$. Recall that, for $p = p(\Gamma)$ and $q = q(\Gamma)$, we have

$$\begin{aligned}
\Delta(k, 0, 0) &= \varphi^{kp-t} s(\Delta) = \psi^{kq-t} e(\Delta), \\
\Delta(k, t-1, 0) &= \varphi^{kp-1} s(\Delta) = \varphi^{t-1} \psi^{kq-t} e(\Delta), \\
\Delta(k, t-1, t-1) &= \varphi^{kp-1} \psi^{t-1} s(\Delta) = \varphi^{t-1} \psi^{kq-1} e(\Delta).
\end{aligned}$$

Hence, letting $i = kp - 1$, $j = kq - t$ in (7) we get an exact sequence

$$0 \to \Delta(k, t-1, 0) \to \Delta(k, t-1, t-1) \oplus \Delta(k, t-1, 0) \to \Delta(k, 0, 0) \to 0.$$

Finally, by Proposition 2.10, we then infer that $\Delta(k, t-1, t-1) \leq_{\mathrm{deg}} \Delta(k, 0, 0)$. Clearly, $\Delta(k, t-1, t-1) \neq \Delta(k, 0, 0)$. Therefore, $\Delta(k, t-1, t-1) <_{\mathrm{deg}} \Delta(k, 0, 0)$, and this finishes the proof.

4.8. Following [5] an algebra A is said to be a multicoil algebra if, for any cycle $M_0 \to M_1 \to \cdots \to M_r = M_0$, $r \geq 1$, of nonzero nonisomorphisms in $\operatorname{ind} A$, the indecomposable modules M_i belong to one standard coil of a multicoil in Γ_A. It is known [5] that every multicoil algebra A is of polynomial growth. We have the following consequence of Proposition 4.7.

COROLLARY. – *Let A be a multicoil algebra, M an A-module, N an indecomposable A-module, and $[M] = [N]$. Then $M <_{\mathrm{deg}} N$ if and only if there exists an exceptional triangle Δ in a coil $\mathcal{C}$ of Γ_A and a natural number $k \geq 1$ such that $M = \Delta(k, t(\Delta) - 1, t(\Delta) - 1)$ and $N = \Delta(k, 0, 0)$.*

Proof. – Observe, if $M <_{\mathrm{deg}} N$, then $M < N$. Hence, for any indecomposable direct summand X of M, we have a short cycle $X \to M \to X$. Since A is a multicoil algebra, it follows that there is a standard multicoil $\mathcal{C}$ containing N and all indecomposable direct summands of M. The claim now is a direct consequence of Proposition 4.7.

4.9. We conclude this section with some remarks on algebras whose Auslander-Reiten quiver admits families of standard coils. Recall from [27, (3.1)] that a family $\mathcal{T} = (\mathcal{T}_i)_{i \in I}$ of stable tubes of an Auslander-Reiten quiver Γ_Λ is called separating if the remaining indecomposable Λ-modules fall into two classes $\mathcal{P}$ and $\mathcal{Q}$ such that the following conditions are satisfied: (1) the tubes $\mathcal{T}_i$, $i \in I$, are pairwise orthogonal and standard, (2) $\operatorname{Hom}_\Lambda(\mathcal{T}, \mathcal{P}) = \operatorname{Hom}_\Lambda(\mathcal{Q}, \mathcal{T}) = \operatorname{Hom}_\Lambda(\mathcal{Q}, \mathcal{P}) = 0$, and (3) given any map from $\mathcal{P}$ to $\mathcal{Q}$, and any $i \in I$, then this map can be factored through $\mathcal{T}_i$. Recently it has been proved in [21] (see also [33]) that an Auslander-Reiten quiver Γ_Λ admits a separating tubular family $\mathcal{T} = (\mathcal{T}_i)_{i \in I}$ if and only if Λ is a concealed-canonical algebra [20], that is, $\Lambda \simeq \operatorname{End}_C(\Sigma)$, where C is a canonical algebra (in the sense of [27, (3.7)] and Σ is a tilting C-module being a direct sum of indecomposable C-modules of positive rank. Moreover, if it is the case, then I is the projective line $\mathbb{P}_1(K)$. Let Λ be a fixed concealed-canonical algebra and $\mathcal{T} = (\mathcal{T}_\lambda)_{\lambda \in \mathbb{P}_1(K)}$ a separating family of stable tubes in Γ_Λ. Consider now an arbitrary coil enlargement B of Λ in the sense of [7, (2.2)] using modules from $\mathcal{T}$. This means that there exists a sequence of algebras $\Lambda = \Lambda_0, \Lambda_1, \ldots, \Lambda_m = B$ such that, for each $0 \leq j < m$, Λ_{j+1} is obtained from Λ_j by an admissible operation with pivot either on a stable tube of $\mathcal{T}$ or on a coil of Γ_{Λ_j} obtained from a stable tube of $\mathcal{T}$ by means of the sequence of admissible operations done so far. It is shown in [7, (2.7)] that Γ_B admits a (weakly) separating family $\mathcal{C} = (\mathcal{C}_\lambda)_{\lambda \in \mathbb{P}_1(K)}$ of pairwise orthogonal standard coil. In particular, for any short cycle $X \to N \to X$ in $\operatorname{ind} B$ and $\lambda \in \mathbb{P}_1(K)$, $N \in \mathcal{C}_\lambda$ if and only if $X \in \mathcal{C}_\lambda$. Hence, all degenerations $M <_{\mathrm{deg}} N$ with N from $\mathcal{C}$ are described according to Proposition 4.7. We know also by [7, (4.2)] that B is tame if and only if B is a multicoil algebra. If this is the case, then unique exceptional triangles in Γ_B are those in the family $\mathcal{C}_\lambda$, $\lambda \in \mathbb{P}_1(K)$. On the other hand, if B is wild, then, by Theorem 1, there are arbitrary long chains $M_r <_{\deg} \dots <_{\deg} M_2 <_{\deg} M_1$ of degenerations of indecomposable B-modules, obviously formed by modules which do not belong to $\mathcal{C}$.

5. Proofs of Theorems 3, 4, 6 and 8

5.1. *Proof of Theorem* 3. – Let A be a quasi-tilted algebra. It is shown in [35] that if A is tame then A is a multicoil algebra (hence of polynomial growth) and any multicoil in Γ_A do not contain exceptional triangles. Hence, by Corollary 4.8, for any degeneration $M <_{\deg} N$ of A-modules, the module N is decomposable. Finally, if for any degeneration $M <_{\deg} N$ of A-modules, N is decomposable, then A is tame, by Corollary 2. Since, every polynomial growth algebra is tame, the proof of Theorem 3 is complete.

5.2. *Proof of Theorem* 4. – Let A be a strongly simply connected algebra. Assume that A is of polynomial growth. Then, by [34, (4.1)], A is a multicoil algebra. Therefore, by Proposition 4.8, if M, M', N are A-modules such that $M <_{\deg} N$, $M' <_{\deg} N$ and N is indecomposable, then $M \simeq M'$ and is indecomposable. Hence (i) implies (ii). The implication (ii)⇒(iii) is trivial. Assume that there is an integer t such that for any sequence

$$M_r <_{\deg} \dots <_{\deg} M_2 <_{\deg} M_1$$

with $M_1, \dots, M_r$ indecomposable A-modules, $r \le t$ holds. Then, by (3.4) and (3.5), A is tame and does not contain a convex subcategory which is pg-critical. Applying now [34, (4.2)] we conclude that A is of polynomial growth.

5.3. *Proof of Theorem* 6. – Let A be a strongly simply connected algebra. If follows from Theorem 4 that if for any degeneration $M <_{\deg} N$ of A-modules, N is decomposable, then A is of polynomial growth. We know also that A is of polynomial growth if and only if A is a multicoil algebra [34, (4.1)]. Then each of conditions (i), (ii) and (iii) is equivalent to the condition: A is a multicoil algebra and there is no (proper) coil in Γ_A which contains an exceptional triangle.

5.4. *Proof of Theorem* 8. – Assume that A is an algebra such that every indecomposable projective A-module is directing. It is shown in [36, (4.1)] that A is tame if and only if A is a multicoil algebra. Observe also that if a (proper) coil Γ of Γ_A contains an exceptional triangle then Γ contains both a nondirecting indecomposable projective modules and a nondirecting indecomposable injective module. Therefore, A is tame if and only if, for degeneration $M <_{\deg} N$ of A-modules, N is decomposable.

5.5. We end the paper with an example showing that there exist simply connected algebras of polynomial growth having arbitrary long sequences of degenerations of indecomposable modules. Let $A = KQ/I$ be the algebra given by the quiver

$$Q: \quad \begin{tikzcd} & 6 \arrow[dl, "\xi"'] \arrow[dr, "\sigma"] & & 7 \arrow[dl, "\gamma"] \\ 3 \arrow[ddr, "\eta"'] & & 5 \arrow[d, "\beta"] & \\ & & 4 \arrow[dl, "\alpha"] & \\ & 2 \arrow[d, "\varphi"] & & \\ & 1 & & \end{tikzcd}$$

and the ideal I in the path algebra KQ generated by $\varphi\eta\xi - \varphi\alpha\beta\sigma$ and $\alpha\beta\gamma$. Then A is simply connected (in the sense of [3]) of global dimension 2 and tame (even $\mu_A(d) \leq 1$ for any $d \geq 1$). Moreover, A is the one-point extension $B[R]$ where B is the representation-infinite tilted algebras of type $\widetilde{\mathbb{D}}_5$ given by the vertices $1, \ldots, 6$, and R is an indecomposable 2-ray module in the unique stable tube of rank 3 in Γ_B (containing the simple modules given by the vertices 4 and 5). Applying now Proposition 3.3 we infer that, for each $r \geq 1$, there exists a sequence $M_r <_{\text{deg}} \ldots <_{\text{deg}} M_2 <_{\text{deg}} M_1$ of degenerations of indecomposable A-modules. Observe that A is not strongly simply connected, because it contains a hereditary convex subcategory C of type $\widetilde{\mathbb{A}}_n$ for which $H^1(C, C) \simeq \mathbb{Z}$.

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