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WEIGHTS IN RIGID COHOMOLOGY APPLICATIONS TO UNIPOTENT F-ISOCRYSTALS

BY BRUNO CHIARELLOTTO (*)

ABSTRACT. – Let X be a smooth scheme defined over a finite field k. We show that the rigid cohomology groups $H^{\bullet}_{rig}(X)$ are endowed with a weight filtration with respect to the Frobenius action. This is the crystalline analogue of the étale or classical theory. We apply the previous result to study the weight filtration on the crystalline realization of the mixed motive "(unipotent) fundamental group". We then study unipotent F-isocrystals endowed with weight filtration. © Elsevier, Paris

RÉSUMÉ. – Soit X un schéma lisse défini sur un corp fini k. On montre que les groupes de cohomologie rigide, $H^{\bullet}_{rig}(X)$, admettent une filtration des poids par rapport à l'action du Frobenius. On va utiliser le résultat précédent pour étudier la filtration des poids dans la réalisation cristalline du motif mixte « groupe fondamental (unipotent) ». On s'intéresse ensuite aux F-isocristaux unipotents qui ont une filtration des poids. © Elsevier, Paris

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Introduction

Let X be a smooth scheme defined over a finite field k of characteristic $p > 0$ and assume that $k = \mathbf{F}_{p^a}$. We use K to denote a complete discrete valuation field with valuation ring $\mathcal{V}$ and residue field k. The a-th iterate, F^a of the absolute Frobenius morphism on X is k-linear.

Let l be a rational prime distinct from p. The properties and importance of F^a acting on the l-adic étale cohomology groups of X have long been known. It is therefore natural to ask what happens in the case of a suitable p-adic cohomology. Does F^a retain similar properties? All the expected properties have been formulated by Deligne in [DE2,II] and called the *"petit camarade cristallin"* to the étale theory.

One candidate for an appropriate p-adic cohomology is the rigid cohomology of X. The recent proofs of its finiteness ([BER1], [CH-M]), its equivalence with crystalline cohomology when X is proper and smooth, and with the Monsky-Washnitzer formal cohomology for smooth affine X all strengthen the belief that rigid cohomology is in fact the appropriate crystalline companion. Indeed the rigid cohomology groups are K-vector spaces on which F^a induces a K-linear endomorphism. Moreover one has the following theorem:

THEOREM. – (Ch.I, 2.2). *Let X be a smooth scheme of finite type over k. Then under the action of F^a the cohomology groups $H^i_{rig}(X/K)$ are mixed F-isocrystals of integral weights in* $[i, 2i]$.

One also has an analogous result for rigid cohomology with support in a closed subscheme (*cf.* the purity statement for rigid cohomology in [BER1, 5.7]):

THEOREM. – (Ch.I, 2.3). *Let Z be a closed k-subscheme of a smooth k-scheme Y and let $cod(Z, Y)$ be its codimension. Then, under the action of F^a, the cohomology groups $H^i_{Z,rig}(Y/K)$ are mixed F-isocrystals of integral weights in* $[i, 2(i - cod(Z, Y))]$.

The proofs of these results closely follows Berthelot's proof of the finiteness and purity of rigid cohomology. In section 2.4 of chapter I we show that a particular case of the Gysin isomorphism respects the Frobenius structure and in section 3.1 we establish, again in a particular case, the existence of an adapted Frobenius in the Monsky-Washnitzer setting.

Chapter II. Let X be a smooth, geometrically connected scheme of finite type over k and assume X to be open in the special fiber of a flat and proper $\mathcal{V}$-formal scheme P of finite type which is smooth around X (In view of [LS-C1] this technical hypothesis can be removed). We assume that X has a rational point $x \in X(k)$. We introduce the unipotent rigid fundamental group

$$\pi_1^{rig,un}(X, x),$$

which is constructed as the fundamental group of the tannakian category of the unipotent overconvergent isocrystals on X [LS-C]. If we place ourselves in the situation studied by Deligne [DE3, §11] (in which X is the special fiber of a smooth $\mathcal{V}$-scheme $X_{\mathcal{V}}$ which is the complement of a normal crossings divisor with respect to $\mathcal{V}$ in a smooth and proper scheme), then our $\pi_1^{rig,un}(X, x)$, coincides with Deligne's definition for the crystalline realization [LS-C]. To the fundamental group $\pi_1^{rig,un}(X, x)$, we associate the K-algebra obtained as the completion (by the augmentation ideal) of the enveloping algebra of the Lie algebra of $\pi_1^{rig,un}(X,x)$. We denote the algebra so constructed by

$$\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x)),$$

and show that it admits a K-linear isomorphism F_* arising from the a-th iterate of the absolute Frobenius. In section 3.3.1 of Chapter II we show that under the action of $\overline{F} = F_*^{-1}$ the algebra $\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))$ admits an increasing weight filtration W_j consisting of ideals stable under the map $\overline{F}$ and such that

$$Gr_j^W(\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x)))$$

are pure of weight j (in fact, they are different from zero only for $j \leq 0$).

Remark. – The results of Chapter II are the transposition to the p-adic setting of constructions made by Joerg Wildeshaus [W] for the unipotent fundamental group for the de Rham and l-adic realizations. This is an improvement with respect to [DE3].

Chapter III. We study unipotent F-K-isocrystals on X [LS-C], namely pairs (E,ϕ) where E is a unipotent overconvergent K-isocrystal on X and ϕ is an K-isomorphism of E with its Frobenius transform F^*E. We introduce a notion of integral weight filtration for a unipotent overconvergent F-K-isocrystal ([FA], [CH-M], [CR]). For any $f \in \mathbf{N}$ we denote by k_f the finite extension of k such that $[k_f : k] = f$, and by K_f an unramified extension of K with residue field k_f.

DEFINITION. – (Ch.III, 1.3,1.4). *For $n \in \mathbf{Z}$, a unipotent overconvergent F-K-isocrystal on X, (M,ϕ), is said to be pure of weight n (briefly n-pure), if for each $f \in \mathbf{N}$ and for each $y \in X(k_f)$, the F-K_f-isocrystal (M_y, ϕ_y^f) is pure of weight n relative to k_f. It is said to be mixed with integral weights, if it admits a finite (increasing) filtration by sub-F-K-isocrystals $\mathcal{W}_j$ ($j \in \mathbf{Z}$) on X, such that $(Gr^j_{\mathcal{W}}M, Gr^j_{\mathcal{V}}\phi)$ is a j-pure F-K-isocrystal on X.*

We could have given a more general definition by requiring only that the graded parts be pure without any hypotheses on the index. It will be shown in §4 (4.1.10) that every unipotent F-K-isocrystal which is mixed with integral weights (even without hypotheses on the filtration) admits a structure as in the definition given above. For the étale case, see [DE2, II, 3.4.1 (ii)]. The category of unipotent overconvergent F-K-isocrystals which are mixed with integral weights should play the role filled by the category of good graded-polarized unipotent variations of $\mathbf{Q}$-mixed Hodge structure in the complex case [H-Z]. The admissibility condition for a variation defined over a complex variety $X_\mathbf{C}$ suggests the possibility of extending the Hodge filtration to a compactification $\overline{X}_\mathbf{C}$ of $X_\mathbf{C}$ and the possibility of good behaviour of the weight filtration with respect to the monodromy operator around each point of $\overline{X}_\mathbf{C} \setminus X_\mathbf{C}$. In characteristic zero, we may think of $\overline{X}_\mathbf{C} \setminus X_\mathbf{C}$ as the complement of a normal crossing divisor. In the p-adic case the recent work of Christol-Mebkhout [CH-M] shows that it is possible (at least in the case of a curve C, whose compactification will be denote by $\overline{C}$) to extend an overconvergent isocrystal satisfying the Robba condition at each point of $C \setminus \overline{C}$ (with some additional hypotheses about "non-Liouvilleness" on the exponents) to $\overline{C}$, and to do it in such a way that there is at most one regular singularity in the residual class of each point of $\overline{C} \setminus C$. Unipotent overconvergent isocrystals satisfy the Robba condition and, obviously, the condition for the exponents. Using an extension of this type we hope to introduce a monodromy operator.

Finally we give a p-adic analogue of the theorem of Hain and Zucker [H-Z1]. Assuming the existence of $x \in X(k)$, we show that the category of unipotent mixed overconvergent F-K-isocrystals on X is equivalent to the category of triples (V, ϕ, ρ), where V is a K-vector space which is a mixed isocrystal with respect to the K-linear Frobenius ϕ and ρ is a homomorphism of mixed F-K-pro-isocrystals which is also a morphism of algebras [Ch.III, 3.1.1]:

$$\rho : (\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x)), \overline{F}) \longrightarrow (End(V), Ad(\phi)),$$

(The Frobenius structure on $End(V)$ is given by $Ad(\phi) = \phi \circ - \circ \phi^{-1}$).
As in the classical case the key point is a rigidity result which actually gives the good definition of mixed unipotent overconvergent F-K-isocrystals on X:

THEOREM. – (Ch.III, 5.3.2.). *A unipotent overconvergent F-K-isocrystal on X, (E, ϕ), is mixed with integral weights if and only if it exists a closed point y of X such that its fiber at y, $(E_y, \phi_y^{\deg y})$, is mixed with integral weights relative to $q^{\deg y}$.*

In §4 and §5 of Chapter III we also study the structure of a generic unipotent F-K-isocrystal having real weights, but without any hypothesis on the filtration, and in (4.1.3) we introduce the notion of ι-mixed unipotent F-K-isocrystal. The results we obtain are analogous to those in [DE3,II] and [FA]. In particular we will show in (4.1.10) that under this general definition every mixed with integral weights unipotent F-K-isocrystal admits a mixed structure as in definition given before (ch.III, 1.3 and 1.4).

Moreover, since $\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))$ acts on itself via left multiplication, we can associate a pro-unipotent mixed F-isocrystal on X, $\mathcal{G}en_x$ to the map of F-pro-isocrystals

$$(\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x)), \overline{F}) \longrightarrow (End(\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))), Ad(\overline{F})).$$

As in [W] we call $\mathcal{G}en_x$ the generic unipotent sheaf on X. This sheaf $\mathcal{G}en_x$ will allow us to conclude in §6 of Chapter III that each unipotent isocrystal on X is a quotient of a unipotent F-isocrystal and hence by duality also a subobject of such an isocrystal.

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Notation

We use k to denote a finite field with $q = p^a$ elements and we consider k to be the residue field of a complete discrete valuation ring $\mathcal{V}$ with maximal ideal $\mathcal{M}$ and field of fraction K. On k the a-th iteration of the Frobenius isomorphism is the identity and we lift it as the identity on K.

Given a scheme X defined over k, we can consider the a-th iterate of the absolute Frobenius acting on X. It is a k-linear endomorphism of X, which we denote F.

An ordered pair (H, ψ) is said an F-K-isocrystal if H is a K-vector space and ψ is a K-linear isomorphism. For each $n \in \mathbf{N}$ and for each F-K-isocrystal (H, ψ) we use $(H(-n), \psi(-n))$ to denote the F-K-isocrystal which has H as underlying vector space and Frobenius map given by ψ multiplied by q^n.

Chapter I

Mixed Weight Filtration on rigid cohomology

§1. Preliminaries

Let X be a smooth scheme of finite type over a finite field k of characteristic p. To X one can associate rigid cohomology K-vector spaces

$$H^i_{rig}(X/K).$$

Recently Berthelot [BER1] and Christol-Mebkhout [CH-M] have independently shown that these K-vector spaces are finite dimensional.

1.1. We summarize the properties of rigid cohomology. For the proofs the reader may consult [BER1]. We note first that rigid cohomology commutes with finite extensions of the field K. If K' is a finite extension of K with ring of integers $\mathcal{V}'$ and residual field k', and if we denote by X' the k'-scheme obtained by scalar extension from k to k', then we have

$$H^i_{rig}(X_{k'}/K') \simeq H^i_{rig}(X/K) \otimes K'.$$

Moreover if, in addition to satisfying the above hypotheses, X is proper then rigid and crystalline cohomologies are isomorphic. More precisely, one has

$$H^i_{rig}(X/K) \simeq H^i_{crys}(X/W) \otimes K,$$

where W is the Witt ring of k.

1.2. There is also a rigid cohomology analogue for the following result about cohomology with support on a closed set [BER1]. If $Z \subseteq X$ is a closed subscheme and $U = X \setminus Z$ its complementary open, we have the usual long exact sequence of cohomology

$$\cdots \to H^i_{Z,rig}(X/K) \to H^i_{rig}(X/K) \to H^i_{rig}(U/K) \to \cdots \tag{1.2.1}$$

Again, formation of $H^i_{Z,rig}(X/K)$ commutes with finite extensions of K.

Proposition 1.2.2. – *Let X be a scheme of finite type, $Z \subseteq X$ a closed subscheme.*
(i) If X' is an open set of X containing Z, then

$$H^i_{Z,rig}(X/K) \simeq H^i_{Z,rig}(X'/K).$$

(*ii*) *If* $Z = Z_1 \cup Z_2$ *and* $Z_1 \cap Z_2 = \emptyset$, *the canonical homomorphism*

$$H^i_{Z_1,rig}(X/K) \oplus H^i_{Z_2,rig}(X/K) \to H^i_{Z,rig}(X/K)$$

is an isomorphism.

PROPOSITION 1.2.3. – *Let* Y *be a scheme of finite type, and let* $T \subset Z \subset Y$ *be two closed subschemes. Let* $Y' = X \setminus T$ *and* $Z' = Z \setminus T$. *Then there exists a natural excision long exact sequence:*

$$\cdots \to H^i_{T,rig}(Y/K) \to H^i_{Z,rig}(Y/K) \to H^i_{Z',rig}(Y'/K) \to \cdots$$

Remark 1.2.4. – $H^i_{Z,rig}(X/K)$ depends only on Z_{red}, the reduced closed subscheme associated to Z.

§2. Main Theorem

2.0. The absolute Frobenius on X induces a K-linear map on $H^i_{rig}(X/K)$:

$$F \, : \, H^i_{rig}(X/K) \longrightarrow H^i_{rig}(X/K) .$$

We wish to show that $H^i_{rig}(X/K)$ has a mixed weight filtration with integral weights with respect to F.

A F-K-isocrystal (H, ϕ), is *pure of integral weight n relative to* k ($n \in \mathbf{N}$) if all the eigenvalues of ϕ are Weil numbers of weight n relative to q. Recall that an algebraic number α is said to be a Weil number of weight n relative to q if α and all its conjugates have archimedean absolute value equal to $q^{\frac{n}{2}}$. We say that (H, ϕ) is *mixed with integral weights relative to* k if it admits a finite increasing K-filtration respected by ϕ whose associated graded module has graded parts which are pure of integral weights. We have that (H, ϕ) is pure (resp. mixed) relative to q if and only if (H, ϕ^m) is pure (resp. mixed) relative to q^m for some $m \in \mathbf{N}$. Note that to say that (H, ϕ) (resp. (H, ϕ^m)) is mixed is equivalent to the assertion that all the roots of the characteristic polynomial of ϕ (resp. ϕ^m) are Weil numbers. Indeed if (H, ϕ) is an F-isocrystal which is mixed with integral weights then H can be decomposed in a unique way into a direct sum of pure integral weight F-isocrystals. It follows that on H one can construct an increasing finite filtration T_j, $j \in \mathbf{Z}$, such that each associated $gr^T_j H$ is pure of weight j. In the following we refer to such a filtration for K-isocrystal (H, ϕ) which is mixed with integral weight.

2.1 One can also make a scalar extension k' of k, $f = [k' : k]$, consider a complete valuation field K' whose residual field is k', and suppose it to be a finite extension of K. If one takes the extension of scalars $X_{k'}$, and acts via the af-th iterate of the absolute Frobenius on $X_{k'}$, which is k'-linear, one obtains an action F' on the K'-rigid cohomology groups:

$$F' \, : \, H^i_{rig}(X_{k'}/K') \longrightarrow H^i_{rig}(X_{k'}/K').$$

But the af-th iterate of the Frobenius on $X_{k'}$ can be viewed as the scalar extension to $X_{k'}$ of the af-th iterate of the Frobenius on X. For this reason we conclude from 1.1 that $F^f \otimes K' = F'$. As a consequence one has that F' is mixed if and only if F is mixed.

It follows that proving

$$F \,:\, H^i_{rig}(X/K) \longrightarrow H^i_{rig}(X/K)\,,$$

to be mixed relative to k is equivalent to checking that, after a finite scalar extension k' of k (hence K' of K), the extended Frobenius

$$F' \,:\, H^i_{rig}(X_{k'}/K') \longrightarrow H^i_{rig}(X_{k'}/K')\,,$$

is mixed relative to k'.

We wish to prove the following result.

THEOREM 2.2. – *If X is a smooth k-scheme of finite type, then $H^i_{rig}(X/K)$ is a mixed isocrystal with integral weights in the interval $[i, 2i]$.*

In fact we will prove an analogous result for rigid cohomology with support in a closed subscheme:

THEOREM 2.3. – *Let Z be a closed subscheme of a smooth scheme of finite type Y. Let $cod(Z, Y)$ be its codimension. Then $H^i_{Z,rig}(Y/K)$ is a mixed isocrystal with weights in the interval $[i, 2(i - cod(Z, Y))]$.*

We prove the previous theorems in (2.5). In order to do so, we shall need another result whose proof will be given in (3.3). It is a result of Gysin's type in a particular case, which will allow us to compare two Frobenius structures.

THEOREM 2.4. – *Suppose we have a closed immersion*

$$Z_{\mathcal{V}} \to Y_{\mathcal{V}}$$

of two smooth affine schemes over $\mathcal{V}$, and let the closed subscheme $Z_{\mathcal{V}}$ be given by global sections $f_1, \ldots, f_r$ of $Y_{\mathcal{V}}$ which are local coordinates of $Y_{\mathcal{V}}$ over $\mathcal{V}$. Then, for the special fiber $Z \to Y$, the Gysin isomorphism

$$H^i_{Z,rig}(Y/K) \simeq H^{i-2r}_{rig}(Z/K)(-r),$$

respects the two Frobenius structures, for all i in $\mathbf{N}$.

2.5. *Proof of 2.2 and 2.3.* – We will use techniques similar to those introduced by Berthelot in his proof of finiteness for rigid cohomology [BER1]. In fact for a large part of the proof the words *"finite dimensional"* have to be replaced by *"mixed"*, with some care for the values of the weights.

We will make double induction on the following results ($n \in \mathbf{N}$)

$(a)_n$ If $\dim X = n$, then for each i, $H^i_{rig}(X/K)$ is a mixed isocrystal of integral weights in $[i, 2i]$.

$(b)_n$ For a closed immersion $Z \to Y$, where $\dim Z = n$ and Y is smooth, for each i, $H^i_{Z,rig}(Y/K)$ is a mixed isocrystal of integral weights in $[i, 2(i - cod(Z, Y))]$.

The starting point is $(a)_0$: since we are allowed to make finite scalar extensions (2.1), we may suppose that $H^0_{rig}(X/K)$ is a finite dimensional K-vector space (of dimension equal to the number of geometrically connected components of X [BER2]) where the Frobenius is the identity map. There are no other non-zero cohomological spaces [BER1].

Consider now $(b)_0$. Using a finite extension of the base field k and by (1.2.4), we can suppose that Z is just the union of a finite number of points which are rational over k. Using the excision lemmas of 1.2, we can limit ourselves to the case of a reduced point. In that situation, by localisation on Y and applying the Gysin type result 2.4 we only obtain the information

$$H^{2\mathrm{dim}Y}_{Z,rig}(Y/K) \simeq H^0_{rig}(Z/K)(-\mathrm{dim}Y).$$

Then using $(a)_0$ for $H^0_{rig}(Z/K)$, we have that the action of the Frobenius on $H^{2\mathrm{dim}Y}_{Z,rig}(Y/K)(-\mathrm{dim}Y)$ is exactly the multiplication by $q^{\mathrm{dim}Y}$, so of weight $2\mathrm{dim}Y$.

Continuing the induction, we now show that $(b)_{n-1}$ implies $(a)_n$. Let X be a smooth scheme of finite type of dimension n. We may take a finite extension of scalars (which we indicate again by k) and suppose that X is connected, hence, irreducible. By another finite extension (which we indicate again by k), we can arrange to have a de Jong alteration, i.e. a connected, projective and smooth k-scheme X', an open subcheme $U \subset X'$ and a proper, surjective, generically étale morphism

$$\phi : U \longrightarrow X.$$

Then $H^i_{rig}(X'/K) \simeq H^i_{crys}(X') \otimes K$ and by a result of Katz-Messing [KA-ME], we know it is pure of weight i (See 2.1).

But ϕ is generically étale, so $\mathrm{dim}X' = n$; for every open set U_1 ($\neq \emptyset$) of X', one can apply induction on the closed set $X' \setminus U_1$ and by the long exact sequence of cohomology with support 1.2.1

$$\cdots \to H^i_{rig}(X'/K) \to H^i_{rig}(U_1/K) \to H^i_{X'\setminus U_1,rig}(X'/K) \to \cdots$$

one obtains the result for U_1. On the other hand, the images of the points of U where ϕ is not étale over X is a closed subset of X which does not contain the generic point of X. We then have an open affine $X_1 \subset X$ such that $U_1 = \phi^{-1}(X_1) \to X_1$ is étale, and, by properness, it is finite. Again by 1.2.1, we conclude that $H^i_{rig}(X/K)$ is mixed with integral weights in $[i, 2i]$ if and only if the same result holds for $H^i_{rig}(X_1/K)$.

But the injection [BER1]

$$H^i_{rig}(X_1/K) \longrightarrow H^i_{rig}(U_1/K)$$

respects the Frobenius and we know that U_1 is mixed of integral weights in $[i, 2i]$, so we can conclude that the same holds for $H^i_{rig}(X_1/K)$, too.

We now wish to show that $(b)_{n-1}$ and $(a)_n$ imply $(b)_n$. We will make an implicit use of the purity theorem [BER1, 5.7] for rigid cohomology. Hence, consider $Z \to Y$ a k-closed immersion, where $\mathrm{dim}Z = n$ and Y is smooth. We may first consider the scheme we obtain by extension of scalars to the algebraic closure of k, $\overline{Z}$. There exists a finite extension of k over which the reduced sub-scheme $\overline{Z}_{red}$ is defined and, moreover, there is an open smooth subscheme of $\overline{Z}_{red}$ which contains all the generic points of all the irreducible components of $\overline{Z}_{red}$. Because $H^i_{Z,rig}(Y/K)$ depends only on Z_{red}, we can make this finite scalar extension and use Z_{red} instead of Z. It follows that in Z there exists a closed $T \subset Z$, $\dim T < n$, such that $Z \setminus T$ is smooth. We may write $Y' = Y \setminus T$, $Z' = Z \setminus T$ ($cod(Z', Y') \geq cod(Z, Y)$) and by 1.2.3 we have a long exact sequence

$$\cdots \to H^i_{T,rig}(Y/K) \to H^i_{Z,rig}(Y/K) \to H^i_{Z',rig}(Y'/K) \to H^{i+1}_{T,rig}(Y/K) \to \cdots.$$

We conclude, using $(b)_{n-1}$, that it suffices to establish the result for Z' and Y'. So, we are reduced to proving the result for

$$Z \longrightarrow Y$$

where $\dim Z = n$ and Z, Y are smooth.

If U is open in Y, such that $Z \cap U \neq \emptyset$, we again have a long exact sequence for $Z \setminus (Z \cap U) \subset Z \subset Y$ (1.2.3)

$$\cdots \to H^i_{Z\setminus(Z\cap U),rig}(Y/K) \to H^i_{Z,rig}(Y/K) \to H^i_{(Z\cap U),rig}(Y \setminus (Z \setminus (Z \cap U))/K) \to \cdots$$

and by induction we will be able to conclude if we know the result for $H^i_{(Z\cap U),rig}(Y \setminus (Z \setminus (Z \cap U))/K)$. Finally, we may suppose that Y, Z are affine and smooth and Z is defined by a sequence of global sections of Y, $f_1, \ldots, f_r$, which are local coordinates. We can write $Y = SpecA$, $S = Spec\mathcal{V}$. By Elkik's theorem [EL], we know that there exists an affine S-scheme $Y' = SpecA'$ which lifts Y. Let $f'_1, \ldots, f'_r$ be liftings in A' of $f_1, \ldots, f_r$ and Z' the subcheme defined by $f'_1, \ldots, f'_r$ in Y'. Because Z' is smooth over S at Z, we can find an open affine set $U' \subset Y'$ such that $Z' \cap U'$ is not empty and $Z' \cap U'$ is smooth over S [SGA1]. By excision we are reduced to the Gysin type Theorem 2.4. This concludes the proof. **Q.E.D.**

§3. A Gysin type isomorphism. Proof of theorem 2.4

In order to prove theorem 2.4, we need some preliminary results.

3.1. The first result is about Monsky-Washnitzer algebras. We recall the situation:

$$Z_{\mathcal{V}} \longrightarrow Y_{\mathcal{V}}$$

is a closed immersion of smooth affine $\mathcal{V}$-schemes, where $Z_{\mathcal{V}}$ is defined by sections $f_1, \ldots, f_r$ which are local coordinates of $Y_{\mathcal{V}}$.

There exists a commutative diagram [SGA1, II, 4.10]

$$\begin{array}{ccc} Z_{\mathcal{V}} & \longrightarrow & Y_{\mathcal{V}} \\ \downarrow & & \downarrow \\ Spec\frac{\mathcal{V}[t_1,\ldots,t_n]}{(t_1,\ldots t_r)} & \longrightarrow & Spec\mathcal{V}[t_1,\ldots,t_n] \end{array}$$

where each vertical map is étale and the local coordinates $f_1, \ldots, f_r, \ldots, f_n$ are the images of $t_1, \ldots, t_r, \ldots, t_n$. We suppose $Y_{\mathcal{V}} = SpecA$. We have an étale map $g : \mathcal{V}[t_1, \ldots, t_n] \to A$, and one can take the weak completion of g to obtain

$$g^\dagger : \mathcal{V}[t_1, \ldots, t_n]^\dagger \longrightarrow A^\dagger$$

which is again étale. In $\mathcal{V}[t_1, \ldots, t_n]^\dagger$ we have a natural $\mathcal{V}$-linear Frobenius given by

(3.1.1) $$t_i \longmapsto t_i^q$$

which we will denote by ϕ. We want to show the following lemma.

LEMMA 3.1.1. – *In the previous notation, there exists a $\mathcal{V}$-linear Frobenius F (i.e. lifting the natural Frobenius in $A/\mathcal{M}A$) in $A^\dagger$ such that the following diagram is commutative*

$$\begin{array}{ccc} A^\dagger & \xrightarrow{F} & A^\dagger \\ \uparrow g^\dagger & & \uparrow g^\dagger \\ \mathcal{V}[t_1,\ldots,t_n]^\dagger & \xrightarrow{\phi} & \mathcal{V}[t_1,\ldots,t_n]^\dagger \end{array}.$$

Proof. – We show first the existence of such a map formally, i.e. for the $\mathcal{M}$-adic completion of A. It is clear that we have a commutative diagram $mod\mathcal{M}$:

$$\begin{array}{ccc} A/\mathcal{M} & \xrightarrow{\overline{F}} & A/\mathcal{M} \\ \uparrow \overline{g} & & \uparrow \overline{g} \\ \mathcal{V}[t_1,\ldots,t_n]/\mathcal{M} & \xrightarrow{\overline{\phi}} & \mathcal{V}[t_1,\ldots,t_n]/\mathcal{M} \end{array},$$

induced by the Frobenius in characteristic p. Using the fact that the map $\mathcal{V}[t_1,\ldots,t_n] \to A$ is étale, one gets a commutative diagram

$$\text{(3.1.3)} \qquad \begin{array}{ccc} \hat{A} & \xrightarrow{\hat{F}} & \hat{A} \\ \uparrow \hat{g} & & \uparrow \hat{g} \\ \mathcal{V}[\![t_1,\ldots,t_n]\!] & \xrightarrow{\hat{\phi}} & \mathcal{V}[\![t_1,\ldots,t_n]\!] \end{array}$$

where "^" indicates the $\mathcal{M}$-adic completion. At the level of Monsky-Washintzer algebras we have the diagram

$$\text{(3.1.4)} \qquad \begin{array}{ccc} A^\dagger & & A^\dagger \\ \uparrow g^\dagger & & \uparrow g^\dagger \\ \mathcal{V}[t_1,\ldots,t_n]^\dagger & \xrightarrow{\phi^\dagger} & \mathcal{V}[t_1,\ldots,t_n]^\dagger \end{array}$$

whose $\mathcal{M}$-adic completion can be completed to a commutative diagram by (3.1.3). We then apply [VdP, 2.4.3] to conclude that (3.1.4) can be completed to a commutative diagram of Monsky-Washintzer algebras as well. **Q.E.D.**

Remark 3.1.5. – After the previous lemma, it is clear that there exists a Frobenius map $F : A^\dagger \longrightarrow A^\dagger$ respecting the ideal which defines the subscheme $Z_\mathcal{V}$, $(f_1,\ldots,f_r)$. Actually f_i is sent by F to f_i^q. We then have a commutative diagram

$$\text{(3.1.5.1)} \qquad \begin{array}{ccc} A^\dagger & \xrightarrow{F} & A^\dagger \\ \uparrow g^\dagger & & \uparrow g^\dagger \\ A^\dagger/(f_1,\ldots,f_r) & \xrightarrow{F_Z} & A^\dagger/(f_1,\ldots,f_r) \end{array}$$

where $A^\dagger/(f_1,\ldots,f_r)$ is the Monsky-Washintzer algebra of $Z_\mathcal{V}$ and the induced F_Z is a Frobenius map on it. The Frobenius F on $Y_\mathcal{V}$ can be seen as a Frobenius *adapted to* $Z_\mathcal{V}$.

3.2. We need a few further observation about sheaves on algebraic and rigid analytic varieties. Consider a finitely generated K-algebra B. To the affine scheme $SpecB = X$ one can associate the rigid analytic space X^{an} (for the strong topology [BGR]). There is a map of ringed spaces

$$\epsilon : X^{an} \longrightarrow X.$$

We now consider a B-module M. To M one can associate the sheaf $\tilde{M}$ of modules on X. It is clear that we can take $\tilde{M}^{an}$: the $\mathcal{O}_{X^{an}}$-module on X^{an} given by inverse image under ϵ.

LEMMA 3.2.1. – *The sheaf $\tilde{M}^{an}$ is the sheaf associated to the presheaf*

$$U \longmapsto M \otimes_B \mathcal{O}_{X^{an}}(U)$$

for each admissible open U of X^{an}.

Proof. – We use the weak Grothendieck topology on X^{an} [BGR] and follow the construction of Fresnel [FRE]. For that topology, the admissible open sets are $U \subset X$ such that there exists an open affine $Y \subset X$ and $g_1, \ldots, g_r$ sections of $\mathcal{O}_X(Y)$, such that $\mathcal{O}_X(Y) = K[g_1, \ldots, g_r]$ and

$$U = \{x \in Y \mid |g_i(x)| \leq 1,\ 1 \leq i \leq r\}.$$

An admissible covering of an admissible open U is a covering $\{\mathcal{U}_i\}_{i \in I}$ such that, for each admissible $V \subset U$, there exists a finite subset $I_V \subset I$ such that

$$V \subset \bigcup_{i \in I_V} \mathcal{U}_i.$$

Using this definition X becomes a rigid analytic space for the weak topology. We denote it by X_w^{an}. A property of this topology is that for an admissible U associated to the affine open Y, then a cofinal directed subset in the directed set of Zariski open sets of X which contain Y is given by the principal open sets of Y. On the other hand there is no continuous map between X_w^{an} endowed with such a weak Grothendieck topology and X, while there does exist a continuous map

$$\epsilon : X^{an} \longrightarrow X \tag{3.2.2}$$

for the strong topology. But starting with a sheaf $\mathcal{M}$ on X, we can associate a presheaf on X_w^{an}, $\epsilon_w^{-1}(\mathcal{M})$, such that its associated sheaf for the strong topology is the inverse image sheaf of $\mathcal{M}$ using ϵ in (3.2.2). The definition is as follows.

For each admissible open U of X_w^{an}, we set

$$\epsilon_w^{-1}(\mathcal{M})(U) = \varinjlim_{U \subset V} \mathcal{M}(V)$$

where the limit is taken over all Zariski open sets which contain U. It is clear that such a presheaf is the restriction to the weak topology of the analogous inverse image presheaf using the strong topology and ϵ (3.2.2). Then if one considers the sheaf associated for the weak topology to $\epsilon_w^{-1}(\mathcal{M})$ and takes the associated strong topology sheaf, we obtain the inverse image sheaf for the strong topology, $\epsilon^{-1}(\mathcal{M})$ [BGR, 9.2.2].

If $\mathcal{M}$ is associated to a B-module M, i.e. $\mathcal{M} = \tilde{M}$, we have the exact sequence:

$$B^J \longrightarrow B^I \longrightarrow M \longrightarrow 0$$

If we take $V \subset X$ affine, we have again an exact sequence for the associated module $\tilde{M}$

$$\mathcal{O}_X^J(V) \longrightarrow \mathcal{O}_X^I(V) \longrightarrow \tilde{M}(V) \longrightarrow 0$$

which corresponds to

$$B^J \otimes \mathcal{O}_X(V) \longrightarrow B^I \otimes \mathcal{O}_X(V) \longrightarrow M \otimes \mathcal{O}_X(V) \longrightarrow 0.$$

If one takes the limit over all the affine V containing an admissible U, exactness still holds. Thus we obtain an exact sequence of presheaves on X^{an}:

$$\epsilon_w^{-1}(\mathcal{O}_X^J) \longrightarrow \epsilon_w^{-1}(\mathcal{O}_X^I) \longrightarrow \epsilon_w^{-1}(\tilde{M}) \longrightarrow 0.$$

If $\mathcal{O}_{X_w^{an}}$ denotes the structural sheaf on X_w^{an}, there is an exact sequence of presheaves

$$B^J \otimes \mathcal{O}_{X_w^{an}} \longrightarrow B^I \otimes \mathcal{O}_{X_w^{an}} \longrightarrow M \otimes \mathcal{O}_{X_w^{an}} \longrightarrow 0.$$

Furthemore, if we consider $\tilde{M}$, we have for each admissible $U \subset X_w^{an}$

$$\epsilon_w^{-1}(\tilde{M})(U) = \varinjlim_{U \subset V} M \otimes \mathcal{O}_X(U) = M \otimes \varinjlim_{U \subset V} \mathcal{O}_X(U) \longrightarrow M \otimes \mathcal{O}_{X_w^{an}}(U)$$

The same holds for the pre-sheaves $\epsilon_w^{-1}(\mathcal{O}_X^J) = \epsilon_w^{-1}(\tilde{B}^J)$ and $\epsilon_w^{-1}(\mathcal{O}_X^I) = \epsilon_w^{-1}(\tilde{B}^I)$, so we finally obtain a map between two exact sequences of presheaves for the weak topology:

$$\begin{array}{ccccccc} \epsilon_w^{-1}(\mathcal{O}_X^J) & \longrightarrow & \epsilon_w^{-1}(\mathcal{O}_X^I) & \longrightarrow & \epsilon_w^{-1}(\tilde{M}) & \longrightarrow 0 \\ \downarrow & & \downarrow & & \downarrow & \\ B^J \otimes \mathcal{O}_{X_w^{an}} & \longrightarrow & B^I \otimes \mathcal{O}_{X_w^{an}} & \longrightarrow & M \otimes \mathcal{O}_{X_w^{an}} & \longrightarrow 0 \end{array} . \tag{3.2.3}$$

All the vertical maps are $\epsilon_w^{-1}(\mathcal{O}_X)$-linear. Diagram (3.2.3) holds once again if we now take first the associated sheaves for the weak topology and then the strong associated sheaves. In fact we will have a commutative diagram with exact rows

$$\begin{array}{ccccccc} \epsilon^{-1}(\mathcal{O}_X^J) & \longrightarrow & \epsilon^{-1}(\mathcal{O}_X^I) & \longrightarrow & \epsilon^{-1}(\tilde{M}) & \longrightarrow 0 \\ \downarrow & & \downarrow & & \downarrow & \\ \mathcal{O}_{X^{an}}^J & \longrightarrow & \mathcal{O}_{X^{an}}^I & \longrightarrow & (M \otimes \mathcal{O}_{X_w^{an}})^{st} & \longrightarrow 0 \end{array} \tag{3.2.4}$$

(where by $(M \otimes \mathcal{O}_{X_w^{an}})^{st}$ we indicate the sheaf associated to the presheaf $(M \otimes \mathcal{O}_{X_w^{an}})$ for the strong topology), and the vertical arrows are $\epsilon^{-1}(\mathcal{O}_X)$-linear. Tensoring the first line in (3.2.4) by $\mathcal{O}_{X^{an}}$ as sheaves for the strong topology we get a map between two exact sequences of sheaves for the strong topology in X^{an}

$$\begin{array}{ccccccc} \epsilon^{-1}(\mathcal{O}_X^J) \otimes_{\epsilon^{-1}(\mathcal{O}_X)} \mathcal{O}_{X^{an}} & \longrightarrow & \epsilon^{-1}(\mathcal{O}_X^I) \otimes_{\epsilon^{-1}(\mathcal{O}_X)} \mathcal{O}_{X^{an}} & \longrightarrow & \epsilon^{-1}(\tilde{M}) \otimes_{\epsilon^{-1}(\mathcal{O}_X)} \mathcal{O}_{X^{an}} & \longrightarrow 0 \\ \downarrow & & \downarrow & & \downarrow & \\ \mathcal{O}_{X^{an}}^J & \longrightarrow & \mathcal{O}_{X^{an}}^I & \longrightarrow & (M \otimes \mathcal{O}_{X_w^{an}})^{st} & \longrightarrow 0 \end{array}$$

The first two vertical maps each being the identity, we conclude that we have an isomorphism for the strong topology of the following sheaves

$$\epsilon^{-1}(\tilde{M}) \otimes_{\epsilon^{-1}(\mathcal{O}_X)} \mathcal{O}_{X^{an}} \simeq (M \otimes \mathcal{O}_{X_w^{an}})^{st}.$$

But the presheaf $(M \otimes \mathcal{O}_{X_w^{an}})$ is the restriction for the weak topology of the presheaf for the strong topology given by $M \otimes \mathcal{O}_{X^{an}}$. Hence the sheaf associated to $M \otimes \mathcal{O}_{X^{an}}$ for the strong topology is isomorphic to the sheaf which is obtained from $(M \otimes \mathcal{O}_{X_w^{an}})$ by taking first the associated sheaf for the weak topology and then the associated sheaf for the strong topology [BGR, 9.2.3]. **Q.E.D.**

COROLLARY 3.2.5. – *Let $X = SpecB$, and let M a B-module. Then for $\mathcal{M} = \tilde{M}^{an}$ on X^{an} and for each affinoid V, we have*

$$H^i(V, \mathcal{M}_{|V}) = 0 \quad if\ i > 0.$$

Proof. – We have that $\mathcal{M}_{|V}$ is an associated module. By Tate's acyclicity theorem, V being affinoid, we conclude. **Q.E.D.**

3.3. We are now ready to prove our Gysin type result.

Proof. – (*Theorem 2.4*) We first start with the algebraic setting. We take the generic fiber of the immersion of $Z_\mathcal{V}$ in $Y_\mathcal{V} = SpecB$:

$$Z_K \longrightarrow Y_K.$$

This is a closed immersion of smooth affine schemes of $\mathrm{codim}_{Y_K}(Z_K) = r$, and Z_K is defined by sections of $Y_K = SpecB_K$ which are local coordinates: $f_1, \ldots, f_r$. We may define

$$\mathcal{H}^r_{Z_K}(Y_K) \simeq \frac{\mathcal{O}_{Y_K}[\frac{1}{f_1 \ldots f_r}]}{\sum_i \mathcal{O}_{Y_K}[\frac{1}{f_1 \ldots \hat{f}_i \ldots f_r}]}$$

It is the sheaf associated to the module

$$\frac{B_K[\frac{1}{f_1 \ldots f_r}]}{\sum_i B_K[\frac{1}{f_1 \ldots \hat{f}_i \ldots f_r}]}. \tag{3.3.1}$$

We have a natural quasi-isomorphism on Y_K given by

$$\Omega^\bullet_{Z_K} \longrightarrow \mathcal{H}^r_{Z_K}(Y_K) \otimes \Omega^\bullet_{Y_K}[r],$$

defined by

$$\omega \longmapsto \tilde{\omega} \frac{df_1}{f_1} \wedge \ldots \wedge \frac{df_r}{f_r}$$

where $\tilde{\omega}$ is a section lifting ω to $\Omega^\bullet_{Y_K}$. By [BER1, 5.4.1] this induces an isomorphism on the associated rigid analytic sheaves in Y_K^{an}:

$$\Omega^\bullet_{Z_K^{an}} \longrightarrow \mathcal{H}^r_{Z_K}(Y_K)^{an} \otimes \Omega^\bullet_{Y_K^{an}}[r].$$

Note that, by (3.2.1), $\mathcal{H}^r_{Z_K}(Y_K)^{an}$ is an associated module in Y_K^{an}, associated to (3.3.1). Since each $\Omega^j_{Y_K^{an}}$ is also an associated module, we conclude by (3.2.5), that for each affinoid V in Y_K^{an} and for each j

$$H^i(V, \mathcal{H}^r_{Z_K}(Y_K)^{an} \otimes \Omega^j_{Y_K^{an}}) = 0$$

if $i > 0$. The same holds for $\Omega^\bullet_{Z_K^{an}}$.

For a suitable n we will have the embedding (Z and Y are the special fibers of $Z_{\mathcal{V}}$ and $Y_{\mathcal{V}}$)

$$Z \longrightarrow Y \xrightarrow{j} \mathbf{A}^n_{\mathcal{V}}.$$

One finds that Y_K^{an} is a strict neighborhood of $]Y_k[_{\mathbf{A}^n_{\mathcal{V}}}$, and we can define an isomorphism [BER1, 5.4]

$$j^\dagger \Omega^\bullet_{Z_K^{an}} \longrightarrow j^\dagger(\mathcal{H}^r_{Z_K}(Y_K)^{an} \otimes \Omega^\bullet_{Y_K^{an}}[r])$$

as sheaves on $\overline{Y}_K^{an}$, where $\overline{Y}_{\mathcal{V}}$ is a compactification of $Y_{\mathcal{V}}$ in $\mathbf{P}^n_{\mathcal{V}}$. The cohomology of the left hand side is the rigid cohomology of Z, while on the right we have the groups $H^{\bullet+2r}_{Z,rig}(Y/K)$ [BER1]. If V_ϵ are the closed polydisks of radius $\epsilon > 1$ in the affine space $\mathbf{A}^{n\ an}_K$ which is thought of as lying in $\mathbf{P}^{n\ an}_K$, then

$$H^\bullet_{rig}(Z/K) = \mathbf{H}^\bullet(\overline{Y}_K^{an}, j^\dagger \Omega^\bullet_{Z_K^{an}}) = \lim_{\rightarrow} \mathbf{H}^\bullet(V_\epsilon \cap Z_K^{an}, \Omega^\bullet_{Z_K^{an}}{}_{|V_\epsilon \cap Z_K^{an}}),$$

and each sheaf is an associated sheaf whence acyclic, $V_\epsilon \cap Z_K^{an}$ being an affinoid [BER1, 1.10]. In a similar way

$$\begin{aligned} H^{\bullet+2r}_{Z,rig}(Y/K) &= \mathbf{H}^\bullet(\overline{Y}_K^{an}, j^\dagger(\mathcal{H}^r_{Z_K}(Y_K)^{an} \otimes \Omega^\bullet_{Y_K^{an}}[r])) \\ &= \lim_{\rightarrow} \mathbf{H}^\bullet(V_\epsilon \cap Y_K^{an}, \mathcal{H}^r_{Z_K}(Y_K)^{an} \otimes \Omega^\bullet_{Y_K^{an}}{}_{|V_\epsilon \cap Y_K^{an}}[r]) \end{aligned}$$

and again by (3.2.5) each element in the complex is acyclic on the affinoid set $V_\epsilon \cap Y_K^{an}$.

Finally the hyper-cohomology of the two complexes and the map between them is calculated by the cohomology of the two complexes and by the map in the following diagram

$$\frac{B_K^\dagger}{(f_1, \dots, f_r)} \otimes \Omega^\bullet_{B_K} \longrightarrow \frac{B_K^\dagger[\frac{1}{f_1 \dots f_r}]}{\sum_i B_K^\dagger[\frac{1}{f_1 \dots \hat{f_i} \dots f_r}]} \otimes \Omega^\bullet_{B_K}[r]$$

It is then obvious, if we use our adapted Frobenius of (3.3.1), that there is the expected relationship among the Frobenius maps. **Q.E.D.**

Chapter II

Application to the rigid unipotent fundamental group: weight filtration

§1. Preliminaries

1.1. In a earlier paper [LS-C], we studied the following situation. Let X be a smooth, geometrically connected k-scheme of finite type, which is an open subscheme of the special fiber of a proper, flat and smooth around X, $\mathcal{V}$-formal scheme of finite type P:

$$X \longrightarrow P.$$

(After [LS-C1], this technical hypothesis can be removed). In this setting we proved that $H^1_{rig}(X/K)$ parametrizes the extensions of the trivial isocrystal by itself.

We introduced the tannakian category $\mathcal{U}n(X)$ of unipotent isocrystals on X. Such isocrystals are necessarily overconvergent . The fiber functor over a point $x \in X(k)$, makes $\mathcal{U}n(X)$ a neutral tannakian category over K [CR, lemma 1.8]. When the situation can be lifted to caracteristic 0, we have shown [LS-C] that this category is equivalent to the algebraic category introduced by Deligne [DE, §11]. We consider on X the a-th iterate F of the absolute Frobenius (where as in Chapter I, $q = p^a$). If $E \in ob(\mathcal{U}n(X))$, then its inverse image by Frobenius

$$F^*E$$

is again an object of $\mathcal{U}n(X)$. The Frobenius F is K-linear and the following theorem summarizes these facts.

THEOREM 1.2. – ([LS-C, 2.4.2]) *$\mathcal{U}n(X)$ is a neutral Tannakian K-category on which the Frobenius induces a K-linear autoequivalence.*

We assume that there exists a rational point $x \in X(k)$. We can then introduce the Tannakian fundamental group of $\mathcal{U}n(X)$ endowed with the fiber functor associated to $x \in X(k)$ which we denote by

$$\pi_1^{rig,un}(X, x).$$

Then $\pi_1^{rig,un}(X, x)$ is a proalgebraic group. Moreover since every object of $\mathcal{U}n(X)$ is an iterated extension of the trivial object, $\pi_1^{rig,un}(X, x)$ is prounipotent [SA, ch.II, §4.3]. By functoriality the Frobenius F induces an isomorphism

$$F_* : \pi_1^{rig,un}(X, x) \longrightarrow \pi_1^{rig,un}(X, x).$$

Remark 1.2.1. – F_* coincides with the Frobenius appearing in [DE3,11.11.3].

In this paragraph we introduce a weight filtration in the completion of the universal enveloping algebra of the Lie algebra of $\pi_1^{rig,un}(X, x)$. The idea underlying the construction are influenced by [W].

1.3. Let $\mathrm{Rep}_K(\pi_1^{rig,un}(X, x))$ the category of finite dimensional representations of $\pi_1^{rig,un}(X, x)$ over K. There is an equivalence of categories

$$\mathrm{Rep}_K(\pi_1^{rig,un}(X, x)) \simeq \mathcal{U}n(X),$$

given by the fiber functor in x, and, in particular, if $E \in ob(\mathcal{U}n(X))$ is associated to

$$\pi_1^{rig,un}(X, x) \xrightarrow{\rho} GL(E_x),$$

then F^*E is associated to

$$\pi_1^{rig,un}(X, x) \xrightarrow{F_*} \pi_1^{rig,un}(X, x) \xrightarrow{\rho} GL(E_x).$$

1.4. We recall that under our hypotheses [LS-C]

PROPOSITION 1.4.1. – *The classes of extensions of the trivial overconvergent isocrystal by itself are naturally isomorphic to $H^1_{rig}(X/K)$ and hence form a finite dimensional vector space.*

Thus, if we denote by $Ext(\mathcal{O}^\dagger, \mathcal{O}^\dagger)$ the extension of the trivial isocrystal by itself in the category of isocrystals on X, we have a natural isomorphism

$$Ext(\mathcal{O}^\dagger, \mathcal{O}^\dagger) \simeq H^1_{rig}(X/K)$$

respecting the Frobenius action [LS-C, §2]. We remark that in [LS-C1] it is proved that this parametrization of the extensions of the trivial isocrystal by itself via $H^1_{rig}(X/K)$ actually holds without the seemingly technical hypotheses on the space X that are imposed here.

§2. Structure of $\pi_1^{rig,un}(X, x)$ and its associated pro-Lie-algebra.

In this paragraph, for notation and proofs, we follow [W]. To the fundamental group $\pi_1^{rig,un}(X, x)$ one associates the unipotent completion of the universal enveloping algebra associated to its Lie-algebra. We denote it by

$$\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X, x)).$$

Its construction is as follows: we introduce the Lie algebra of $\pi_1^{rig,un}(X, x)$, $Lie\pi_1^{rig,un}(X, x)$, which can be written as the projective limit of its finite dimensional quotient Lie-algebras m_α. Thus

$$Lie\pi_1^{rig,un}(X, x) = \varprojlim_{\alpha} m_\alpha$$

and

$$\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X, x)) = \varprojlim_{\alpha} \hat{\mathcal{U}}(m_\alpha)$$

where $\hat{\mathcal{U}}(m_\alpha)$ is the completion of the universal enveloping algebra of m_α with respect to its augmentation ideal a_α. Since the characteristic of K is 0, we have an equivalence of categories [D-G, IV §2, cor.4.5.b]:

$$\mathrm{Rep}_K(\pi_1^{rig,un}(X, x)) \simeq Mod^{fin}_{\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))}$$

where on the right hand side we denote the category of finite dimensional K-vector spaces which are modules over $\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X, x))$, such that the module action is continuous with respect to the discrete topology on the module and the inverse limit topology on $\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X, x))$. The algebra $\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X, x))$ acts on itself by left multiplication: it is a pro-object of $Mod^{fin}_{\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))}$.

Proposition 2.1. – *$\pi_1^{rig,un}(X, x)$ can be constructed as a countable inverse limit of algebraic (unipotent) groups (with surjective transition maps).*

Proof. – The proof can be found in [W, ch.I, 1.5], using 1.4.1. **Q.E.D.**

2.2. In view of the previous proposition we can write

$$\pi_1^{rig,un}(X, x) = \varprojlim_{j\in\mathbf{N}} \pi_1^{rig,un}(X, x)_j$$

and

$$\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X, x)) = \varprojlim_{j\in\mathbf{N}} \hat{\mathcal{U}}(m_j)$$

where $\hat{\mathcal{U}}(m_j)$ is the completion of the universal enveloping algebra of the Lie-algebra m_j of $\pi_1^{rig,un}(X,x)_j$ with respect to the augmentation ideal a_j. We define the augmentation ideal of $\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))$ by

$$a = \lim_{\substack{\leftarrow \\ j \in \mathbf{N}}} \hat{a}_j$$

($\hat{a}_j$ is the ideal generated by a_j in $\hat{\mathcal{U}}(m_j)$).

2.3. Since $\mathrm{Lie}(\pi_1^{rig,un}(X,x))^{ab} = a/a^2$ it follows from the proof of proposition 2.1 that to give a linear map

$$a/a^2 \longrightarrow K$$

is equivalent to giving an extension in $Ext_{\mathrm{Rep}_K(\pi_1^{rig,un}(X,x))}(K,K)$. Moreover, this correspondence is natural. (For an explicit proof of this result and others see [LS-C1].) Furthemore, if we then use the natural identification between $Ext_{\mathrm{Rep}_K(\pi_1^{rig,un}(X,x))}(K,K)$ and $H^1_{rig}(X/K)$ (1.4.1), we obtain a natural linear isomorphism

$$(a/a^2)^\vee \simeq H^1_{rig}(X/K).$$

LEMMA 2.4. – *$\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))$ endowed with the projective limit topology is complete for the a-adic topology.*

Proof. – We shall show that the inverse limit topology coincides with the a-adic topology. The topology of $\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))$is given by a basis of open sets which are the inverse image in

$$\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x)) \longrightarrow \hat{\mathcal{U}}(m_j)$$

of powers $\hat{a}_j^k$ of the augmentation ideal. It follows that each open set of $\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))$ contains a power of a. Hence the map

$$\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))_{a-adic\,top.} \longrightarrow \hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))_{proj.limit\,top}$$

is continuous. To prove the equivalence it suffices to show that any power a^k is not merely contained in, but is actually equal to the full pre-image of $\hat{a}_j^k$ for a suitable j. But our hypothesis tells us that $\hat{\mathcal{U}}(m_j)/\hat{a}_j^k$ is increasing and that the transition maps are surjective, hence stationary, since the dimension is bounded by

$$\dim(\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))/a^k).$$

The latter is finite because it is bounded by

$$\sum_{i=0}^{k} dim(a/a^2)^i = \sum_{i=0}^{k} (dim H^1_{rig}(X/K))^i.$$

Indeed we have a natural surjection

$$\bigotimes^i (a/a^2) \longrightarrow a^i/a^{i+1}.$$

and we apply 2.3. **Q.E.D.**

§3. Weight filtration on the fundamental group

We wish to give a pro-mixed integral weight structure to the K-vector space

$$\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x)).$$

First we must introduce the Frobenius isomorphism. We know that in the pro-unipotent algebraic group $\pi_1^{rig,un}(X,x)$ there is a K-linear isomorphism F_*, induced by the Frobenius. We wish to show that it will also induce an isomorphism in $\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))$, too.

Our proof works in a general setting: given a pro-unipotent group W, one can always carry out the same construction as that of $\pi_1^{rig,un}(X,x)$, and so associate to W a K-algebra, $\hat{\mathcal{U}}(LieW)$.

PROPOSITION 3.1. – *If W is a pro-unipotent algebraic group, then*

$$W \longrightarrow \hat{\mathcal{U}}(LieW)$$

is a functor from the category of pro-unipotent algebraic groups over K to the category of augmented K-algebras.

Proof. – We must prove that if $W \xrightarrow{\varphi} V$ is a K-morphism between two pro-unipotent K-algebraic groups, then we can construct a natural K-linear morphism

$$\hat{\mathcal{U}}(\varphi) : \hat{\mathcal{U}}(LieW) \longrightarrow \hat{\mathcal{U}}(LieV).$$

If W and V are both algebraic, then this is certainly the case. Now suppose that W is a pro-unipotent algebraic group and that V is again algebraic. Then one can factor φ as

$$W \xrightarrow{\pi} W' \xrightarrow{\overline{\varphi}} V$$

where W' is an algebraic quotient of W. But W can be written as the inverse limit of all its algebraic quotients: so the natural projection map $W \longrightarrow W'$ composed with the natural map between two algebraic groups will induce a map $\hat{\mathcal{U}}(LieW) \to \hat{\mathcal{U}}(LieV)$. Finally in the general case, one can write V as the inverse limit of all its algebraic quotient groups and apply the previous constructions to each term. It is also clear that $\hat{\mathcal{U}}(\varphi)$ respects the augmentation ideals: such ideals are associated with the trivial representation. **Q.E.D.**

Remark 3.2. – If we apply the previous results to the Frobenius isomorphism, F_* (1.2.1), in $\pi_1^{rig,un}(X,x)$, we obtain a K-isomorphism of augmented algebras

$$F_* : \hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x)) \longrightarrow \hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x)).$$

3.3. Now that we have defined the Frobenius K-linear isomorphism on $\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))$, we can introduce a weight filtration which will make it a pro-mixed isocrystal.

We first recall that $Mod^{fin}_{\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))}$ is equivalent to our category $\mathcal{U}n(X)$ of unipotent isocrystals (1.3). To each $E \in ob\mathcal{U}n(X)$, we associate its monodromy representation

$$\rho : \pi_1^{rig,un}(X,x) \longrightarrow GL(E_x)$$

and, of course, the derived map

$$\tilde{\rho} : \hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x)) \longrightarrow End(E_x).$$

The Frobenius transform, F^*E is then associated to

$$\pi_1^{rig,un}(X,x) \xrightarrow{F_*} \pi_1^{rig,un}(X,x) \xrightarrow{\rho} GL(E_x),$$

hence to

$$\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x)) \xrightarrow{F_*} \hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x)) \xrightarrow{\tilde{\rho}} End(E_x).$$

In particular, we get a Frobenius action on $\mathrm{Rep}_K(\pi_1^{rig,un}(X,x))$ and the induced Frobenius action in $Ext_{\pi_1^{rig,un}(X,x)}(K,K)$. As above, the Frobenius, F_*, respects the augmentation ideal a of $\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))$. Thus we have a natural action of the Frobenius on a/a^2. We then have a natural isomorphism

$$(a/a^2)^\vee \longrightarrow Ext_{\pi_1^{rig,un}(X,x)}(K,K) \simeq Ext(\mathcal{O}^\dagger,\mathcal{O}^\dagger).$$

If we wish our isomorphism to respect the Frobenius structures we must use the map on a/a^2 induced by the inverse $\overline{F} = F_*{}^{-1}$ of F_*. However by Proposition 1.4.1, $Ext(\mathcal{O}^\dagger,\mathcal{O}^\dagger)$ is naturally isomorphic to $H^1_{rig}(X/K)$ and this natural isomorphism respects the Frobenius structure [LS-C]:

$$H^1_{rig}(X/K) \longrightarrow Ext(\mathcal{O}^\dagger,\mathcal{O}^\dagger).$$

Finally there exists a natural K-isomorphism respecting the Frobenius

$$(H^1_{rig}(X/K),F)^\vee \simeq (a/a^2,\overline{F})$$

(Here we use F to denote the usual Frobenius on $H^1_{rig}(X/K)$, *cf* Ch.I. Note too that on the left hand side we are using not the dual Frobenius, but rather the contragradient Frobenius i.e. the inverse of the dual).

Remark. – This coincides with the expected crystalline realization in [DE3, 13.13].

After these introductory remarks we are able to establish

PROPOSITION 3.3.1. – *The infinite dimensional vector space*

$$\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))$$

under the action of $\overline{F} = F_*^{-1}$, is a pro-F-isocrystal, i.e. a projective limit of F-isocrystals. It has a natural pro-mixed (integral) weight structure given by an increasing sequence of sub-vector spaces

$$\ldots \subset W_{-2} \subset W_{-1} \subset W_0 = \hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))$$

each of which is stable under the action of Frobenius and such that the quotient W_{-j}/W_{-j-1} is a pure (finite dimensional) isocrystal of weight $-j$.

Proof. – For the first assertion it is enough to note that for each positive integer k the ideal a^k is stabilized by Frobenius and that there is a naturally surjective map of finite dimensional K-vector spaces

$$\bigotimes^k (a/a^2) \longrightarrow a^k/a^{k+1}.$$

and we conclude that

$$\frac{\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))}{a^k}$$

is an F-isocrystal.

For the weight filtration W_j in $\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))$, we define

$$W_0 = \hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))$$
$$W_{-1} = a.$$

Observe now that, as we noted just before 3.3.1, there exists a natural K-isomorphism respecting the Frobenius

$$H^1_{rig}(X/K)^\vee \simeq a/a^2.$$

By the result of chapter I, $H^1_{rig}(X/K)^\vee$ is mixed of weight -1 and -2. Let

$$p : a \longrightarrow a/a^2,$$

be the natural projection, and set

$$W_{-2} = p^{-1}(W_{-2}(a/a^2)).$$

We then construct the filtration generated by W_{-2} and W_{-1}. We define W_{-n} to be spanned by the products of the form $\alpha_1 \cdots \alpha_m \beta_1 \cdots \beta_l$ where $\alpha_i \in W_{-1}$ and $\beta_j \in W_{-2}$ and $m + 2l \geq n$. Then one has

$$a^n \subseteq W_{-n}$$

for all $n \in \mathbf{N}$. Moreover, with this definition the canonical surjection

$$H^1_{rig}(X/K)^{\otimes(-n)} \longrightarrow a^n/a^{n+1}$$

respects the weight filtration for all n and of course the graded part is pure. **Q.E.D.**

The filtration we have introduced is a filtration by ideals, hence each module of this filtration is stable under left multiplication by $\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))$. Hence we have:

COROLLARY 3.3.2. – *Consider $\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))$ acting on itself by multiplication. It is a pro-object of $Mod^{fin}_{\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))}$. Each W_{-n} of the weight filtration introduced in 3.3.1 is stable under the action of $\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))$ and by the Frobenius. The action of $\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))$ on each graded part*

$$Gr_n^W(\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x)))$$

is trivial.

Proof. – By 3.3.1, it is enough to note that $a = W_{-1}$. And that an action is trivial if the augmentation ideal acts trivially. **Q.E.D.**

Chapter III

Mixed weight filtration on unipotents F-isocrystals.

§1. Definitions. Preliminaries

We use the same hypotheses as in Chapter II.

1.1.

Assume that X, a smooth, geometrically connected scheme, has a point x defined over $X(k)$. We denote the closed points of X by $|X|$. The Frobenius on X will be the a-th iterate of the absolute Frobenius.

An overconvergent F-K-isocrystal on X is a couple (M, ϕ) where M is an overconvergent K-isocrystal on X and ϕ is an horizontal isomorphism

$$\phi : F^* M \xrightarrow{\sim} M,$$

with $F^* M$ the Frobenius transform of M. On each $x \in X(k)$, such an isomorphism induces a K-linear map on the fiber $M_x = F^* M_x$:

$$\phi_x : M_x \xrightarrow{\sim} M_x,$$

hence (M_x, ϕ_x) is a K-F-isocrystal.

We are interested in unipotent isocrystals, and begin by studying such objects endowed with a Frobenius structure.

DEFINITION 1.2. – *We say that (E, ϕ) is a unipotent F-K-isocrystal on X, if $E \in ob\,\mathcal{U}n(X)$ and (E, ϕ) is an overconvergent F-K-isocrystal on X.*

1.3. For each extension k_f of k of degree f we denote by K_f the unramified extension of K (of degree f), which has k_f as residue field. For each closed point $y \in X(k_f)$, one obtains a fiber functor on overconvergent F-K-isocrystals (M, ϕ). The fiber at y will be indicated by M_y: it is a K_f-vector space endowed with a Frobenius map ϕ_y which is not K_f-linear. Of course, if we consider ϕ_y^f, it will induce a K_f-linear isomorphism in M_y. For (M_y, ϕ_y^f) we will use the definition of chapter I.

DEFINITION 1.4. – *An overconvergent F-K-isocrystal (E, ϕ) on X is said to be pure of weight n ($n \in \mathbf{Z}$), if for each $y \in X(k_f)$, the F-K_f-isocrystal (E_x, ϕ_y) is pure of weight n relative to k_f.*

DEFINITION 1.5. – *A unipotent F-K-isocrystal (E, ϕ) on X is said to be mixed with integral weights, if it admits an increasing finite filtration by sub-F-K-isocrystals on X, $\mathcal{W}_j$ ($j \in \mathbf{Z}$), such that $(Gr^j_{\mathcal{W}} E, Gr^j_{\mathcal{W}} \phi)$ is a j-pure F-K-isocrystal on X.*

Remark. – In the previous definition we could have required only that E admit a finite filtration whose graded parts are pure: we will see in (4.1.10) that, even under this weaker condition, a unipotent F-K-isocrystal on X endowed with a finite filtration having pure integral weight graded parts always admits a structure as in Definition 1.5. The analogous statement in the étale case is [DE2,II,3.4.1, (*ii*)].

Moreover, if E is associated to the morphism

$$\tilde{\rho} : \hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x)) \longrightarrow End(E_x)$$

then F^*E is associated to the morphism

$$\tilde{\rho} \circ F_* : \hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x)) \longrightarrow End(E_x).$$

Such an E is a unipotent F-K-isocrystal on X if and only if there exists an isomorphism $\phi_x : E_x \to E_x$, such that the following diagramm commutes

$$\begin{array}{ccc} \hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x)) & \xrightarrow{\tilde{\rho}\circ F_*} & End(E_x) \\ \downarrow{\scriptstyle \overline{F}} & & \downarrow{\scriptstyle Ad(\phi_x)} \\ \hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x)) & \xrightarrow{\tilde{\rho}\circ F_*} & End(E_x) \end{array}$$

where $Ad(\phi_x) = \phi \circ - \circ \phi^{-1}$, and $\tilde{\rho} \circ F_* \circ \overline{F}$ ($\overline{F} = F_*^{-1}$, cf 3.3) is the map $\tilde{\rho}$ which represents E. This condition is equivalent to requiring that $\tilde{\rho}$ be a morphism respecting the Frobenius structures

$$(\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x)), F_*^{-1} = \overline{F}) \longrightarrow (End(E_x), Ad(\phi)).$$

We give some results on unipotent F-K-isocrystals on X.

PROPOSITION 1.6. – *Let $E \in ob\mathcal{U}n(X)$ be endowed with a Frobenius structure ϕ_1. Suppose E admits another Frobenius structure ϕ_2 such that there exists $x \in X(k)$ on which $\phi_{2x} = \phi_{1x}$. Then $\phi_1 \equiv \phi_2$.*

Proof. – The fiber functor at $x \in X(k)$ of the Tannakian category $\mathcal{U}n(X)$ is faithful and $\phi_1 \circ (\phi_2)^{-1} = \varphi$ is a an element of $Hom_{\mathcal{U}n(X)}(E,E)$ which is the identity on the fiber at x. **Q.E.D.**

COROLLARY 1.7. – *Let E be a constant K-isocrystal on X, endowed with the Frobenius structure given by ϕ. Then ϕ is constant.*

Proof. – We can write $E = E_x \otimes \mathcal{O}^\dagger$ (where $\mathcal{O}^\dagger$ is the trivial isocrystal), and then $(E_x \otimes \mathcal{O}^\dagger, \phi_x \otimes id_{\mathcal{O}^\dagger})$ is another Frobenius structure on E which coincides with ϕ on x. Now it suffices to apply 1.6. **Q.E.D.**

We have (cf Ch.I, 2.0)

COROLLARY 1.8. – *If (E,ϕ) is a constant F-isocrystal on X, such that for some point $x \in X(k)$, ϕ_x has only Weil numbers as eigenvalues on E_x, then (E,ϕ) is mixed with integral weights.*

Remark. – The category of overconvergent isocrystals on X, $Overc(X)$, is a neutral Tannakian K-category once we have chosen a point $x \in X(k)$ ([CR], [BER2, 2.3.9]). It turns out that if an object M of $Overc(X)$, has two Frobenius structures, ϕ_1 and ϕ_2, which coincide at x, then $\phi_1 \equiv \phi_2$. We are not able to prove that the Frobenius induces an auto-equivalence in $Overc(X)$, although in the case $dimX = 1$ there are some results on the Frobenius antecedent [CH-M].

§2. Mixed weight filtration in the generic pro-unipotent isocrystal

2.1. Let X be as in the previous sections. We consider a point $x \in X(k)$. In Chapter II we have associated to the Tannakian category $\mathcal{U}n(X)$ the completion of the enveloping algebra of its Lie algebra $\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))$, endowed with a K-linear isomorphism, $\overline{F}$. In particular $\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))$ is an infinite dimensional vector space which is a module over itself by left multiplication.

By filtering $\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))$ by the powers of the augmentation ideal a, which are stabilized by the multiplication by $\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))$, one finds that the graded parts are finite dimensional vector spaces on which $\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))$ acts trivially. We can associate a pro-unipotent isocrystal $\mathcal{G}en_x$ to $\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))$. The isocrystal $\mathcal{G}en_x$ is called the *generic pro-unipotent isocrystal.*

On the other hand, $\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))$ admits a K-linear isomorphism F_*, the Frobenius. So, if we let m denote multiplication on $\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))$, and if in the diagram of §1 we replace E_x by $\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))$, and ϕ_x by the isomorphism $\overline{F}$ (The inverse of F_* as in 3.3), we obtain a commutative diagram

$$\begin{array}{ccc} \hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x)) & \xrightarrow{m\circ F_*} & End(\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))) \\ \downarrow{\scriptstyle \overline{F}} & & \downarrow{\scriptstyle Ad(\overline{F})} \\ \hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x)) & \xrightarrow{m\circ F_*} & End(\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))) \end{array}.$$

By means of the increasing filtration W_j in $\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))$ and invoking Corollary 3.3.2 of Chapter II, one finds that $\mathcal{G}en_x$ is a pro-unipotent F-K-isocrystal on X which admits an increasing filtration by sub-pro-unipotent F-K-isocrystals, $\mathcal{W}_j$, such that each associated graded part

$$Gr_j^{\mathcal{W}}\mathcal{G}en_x$$

is a constant F-isocrystal on X (i.e. a direct sum of trivial isocrystals endowed with a Frobenius which, a priori, is non constant). We denote by $\overline{F}$ the Frobenius on $\mathcal{G}en_x$.

2.2. The next proposition gives the structure of each

$$Gr_j^{\mathcal{W}}\mathcal{G}en_x.$$

PROPOSITION 2.2.1. – *The pro-unipotent F-isocrystal on X, $\mathcal{G}en_x$, is a pro-mixed F-isocrystal on X whose weights lie in the set* $\{0,-1,-2,-3,\ldots\}$.

Proof. – By 2.1, we know that the weight filtration on $\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))$ induces a filtration on $\mathcal{G}en_x$, $\mathcal{W}_j$, whose graded pieces are constant F-K-isocrystals on X of weights j. We then use corollary 1.8. **Q.E.D.**

Remark 2.2.2. – The weight filtration, $\mathcal{W}_j$ in $\mathcal{G}en_x$, is such that $\mathcal{W}_j = \mathcal{G}en_x$, for $j \geq 0$.

§3. Results of the type Hain-Zucker [HZ]

We consider an example coming from the complex case. Let $X_{\mathbf{C}}$ be a smooth algebraic connected variety over $\mathbf{C}$ (or, more generally, a Zariski-open set of a compact Kähler manifold). Let $x \in X_{\mathbf{C}}$. For each $r \in \mathbf{N}$, it is possible to define a graded polarized $\mathbf{Q}$-mixed Hodge structure on

$$\mathbf{Q}[\pi_1(X_{\mathbf{C}}, x)]/J^r$$

where J is the augmentation ideal of the group algebra $\mathbf{Q}[\pi_1(X_{\mathbf{C}}, x)]$. These structures are all compatible and so we may conclude that

$$\varprojlim_{r\in\mathbf{N}} \mathbf{Q}[\pi_1(X_{\mathbf{C}}, x)]/J^r = \mathbf{Q}[\pi_1(X_{\mathbf{C}}, x)]^\wedge$$

admits a pro-graded polarized $\mathbf{Q}$-mixed Hodge structure. On the other hand one can introduce the Tannakian category of unipotent representations of $\pi_1(X_{\mathbf{C}}, x)$, $\mathcal{U}n(X_{\mathbf{C}})$ (which are the unipotent local systems), and apply the same techniques as in Chapter II to obtain an algebra that we denote by

$$\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X_{\mathbf{C}}, x)).$$

Wildeshaus has proved ([W], lemma 2.1) that

$$\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X_{\mathbf{C}}, x)) \equiv \mathbf{Q}[\pi_1(X_{\mathbf{C}}, x)]^\wedge.$$

On the other hand one can introduce in $X_{\mathbf{C}}$, the category of admissible graded-polarized variations of $\mathbf{Q}$-mixed Hodge structure on $X_{\mathbf{C}}$. For the definition see [H-Z1,2], [BR-Z]. Admissibility is sometimes referred to as a "good graded-....". This admissibility condition expresses the possibility of extending the Hodge filtration to some compactification $\overline{X}_{\mathbf{C}}$ of $X_{\mathbf{C}}$ and also ensures good behavior of the weight filtration with respect to the monodromy filtration at the points of $\overline{X}_{\mathbf{C}} \setminus X_{\mathbf{C}}$. Thus, if we restrict to a unipotent admissible graded-polarized variation of $\mathbf{Q}$-mixed Hodge structure on $X_{\mathbf{C}}$, $\mathbf{V}$, we can construct a map obtained by linearity from the representation associated to $\mathbf{V}$ at x:

$$\rho_x : \mathbf{Q}[\pi_1(X_{\mathbf{C}}, x)]^\wedge \longrightarrow End_{\mathbf{Q}}(\mathbf{V}_x).$$

Both the source and the target have a mixed Hodge structure [C]. Then

THEOREM 3.1.1. – ([H-Z1], [H-Z2]) *The functor*

$$\mathbf{V} \longrightarrow (\rho_x, \mathbf{V}_x)$$

is an equivalence of category between the categories of admissible graded-polarized unipotent variations of $\mathbf{Q}$-mixed Hodge structures on $X_{\mathbf{C}}$ and the category of graded polarized $\mathbf{Q}$-mixed Hodge structures, $\mathcal{V}$, endowed with a morphism of $\mathbf{Q}$-mixed Hodge structures

$$\mathbf{Q}[\pi_1(X_{\mathbf{C}}, x)]^\wedge \longrightarrow End_{\mathbf{Q}}(\mathcal{V}_x),$$

which also respects the algebra structure.

3.2. We are going to prove a similar result in our setting. The category of graded-polarized unipotent variations of $\mathbf{Q}$-mixed Hodge structure on $X_{\mathbf{C}}$ is replaced by the category of unipotent mixed F-K-isocrystals on X. Recent works of Christol and Mebkhout [CH-M] have shown that for overconvergent isocrystals satisfying the *Robba condition* defined on a curve (and, more generically, for an isocrystal which is *"soluble"*) it is possible to obtain an extension to the residue class (at least for curves), similar to that requested by the "admissibility" condition. Unipotent isocrystals fall into this setting. We should expect a monodromy operator (given by the residue at infinity of the associated differential module) which respects the weight filtration.

3.3. One of the most important tools in the proof of Hain and Zucker (see also [SC], [B-Z], [H-Z1,2]) is a rigidity theorem.

PROPOSITION 3.3.1. – *Consider a unipotent F-K-isocrystal on X, E, and $x \in X(k)$. If E admits a mixed weight filtration, then $H^0_{rig}(X, E)$ is mixed and the map*

$$H^0_{rig}(X, E) \otimes \mathcal{O}^\dagger \longrightarrow E$$

is a morphism of unipotent F-K-isocrystals on X endowed with a mixed weight filtration.

Proof. – By [LS-C], we know that $H^0_{rig}(X, E)$ is stabilized by the Frobenius. Moreover, there is an obvious inclusion of unipotent F-K-isocrystals

$$H^0_{rig}(X, E) \otimes \mathcal{O}^\dagger \longrightarrow E. \tag{3.3.2}$$

Taking the fiber at x we have the obvious functorial inclusion

$$H^0_{rig}(X, E) \longrightarrow E_x$$

of F-K-isocrystals. We conclude that $H^0_{rig}(X, E)$ is mixed. Then $H^0_{rig}(X, E) \otimes \mathcal{O}^\dagger$ is mixed (1.8), and the map (3.3.2) is an inclusion of unipotent mixed F-K-isocrystals on X. **Q.E.D.**

We now give a theorem concerning the compatibility of filtrations on a mixed with integral weights unipotent F-K-isocrystal on X.

THEOREM 3.3.3. – *Consider a unipotent F-K-isocrystal E on X, with Frobenius isomorphism ϕ. If it admits an integral weights filtration, then it is unique.*

Proof. – Suppose it admits two integral filtrations $T^1_\bullet$ and $T^2_\bullet$ each of which makes a E into a mixed unipotent F-isocrystal on X. Then at each $x \in X(k)$ one sees that $T^1_{\bullet\, x}$ and $T^2_{\bullet\, x}$ coincide. Moreover, the element $id_E \in H^0_{rig}(X, \mathcal{H}om(E, E))$ is fixed by the Frobenius action. The unipotent F-K-isocrystal on X, $\mathcal{H}om(E, E)$, can be viewed as a mixed isocrystal using the filtration $T^1_\bullet$ on the first and $T^2_\bullet$ on the second. Then

$$id_E \in W_0(H^0_{rig}(X, \mathcal{H}om(E, E)) \otimes \mathcal{O}^\dagger_X).$$

Applying 3.3.1 to the inclusion

$$H^0_{rig}(X, \mathcal{H}om(E, E)) \otimes \mathcal{O}^\dagger \longrightarrow \mathcal{H}om(E, E)$$

of mixed unipotent F-isocrystals, we find that id_E is an element of $W_0(\mathcal{H}om(E,E))$ and so $T^1_\bullet \subseteq T^2_\bullet$. We again apply the same technique to $\mathcal{H}om(E,E)$, endowed with the weight filtration using $T^2_\bullet$ in the first and $T^1_\bullet$ in the second. **Q.E.D.**

3.4. We are ready to state our theorem on equivalence of categories in the Hain-Zucker sense [H-Z1], [H-Z2]. We recall that in a mixed with integral weights K-isocrystal (H,ϕ) we use the filtration such that $gr_j H$ are pure of weight $j \in \mathbf{Z}$ (cfr. ChI, 2.0).

THEOREM 3.4.1. – *Let X be a smooth scheme over k, and $x \in X(k)$. Then the fiber functor at x*

$$E \longrightarrow E_x$$

induces an equivalence between the category of unipotent mixed with integral weights F-K-isocrystals (E,ϕ) on X and the category of mixed F-K-isocrystals (E_x,ϕ_x) endowed with a K-algebra morphism respecting Frobenius.

$$(\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x)),\overline{F}) \longrightarrow (End(E_x), Ad\phi_x).$$

Proof. – Given a unipotent F-K-isocrystal on X, (E,ϕ), the morphism

$$\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x)) \longrightarrow End(E_x).$$

respects the Frobenius structures, hence the image is in $W_0(End(E_x))$ (Note that here the mixed structure on $End(E_x)$ is given by $Ad(\phi_x)$). In fact $\mathcal{W}_0 = \hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))$ and the action of $\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))$ respects the weight filtration in E_x. By the rigidity results of 3.3, it is enough to prove that to each mixed integral F-K-isocrystal (H,ψ) which admits a morphism

$$\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x)) \longrightarrow End(H). \tag{3.4.2}$$

respecting the Frobenius (for the mixed structure see Ch.I, 2.0) one can associate a unipotent mixed F-K-isocrystal on X, (E,ϕ), with filtration $\mathcal{W}_\bullet$ and satisfying

1.) the fibers at x satisfy $(E_x,\phi_x) = (H,\psi)$, and

2.) the filtration induced by $\mathcal{W}_\bullet$ on E_x coincides with the weight filtration of H.

By the results of Chapter II, we know that from the morphism (3.4.2), we obtain a unipotent F-K-isocrystal (E,ϕ), whose fiber at x coincides with (H,ψ). We must show that it admits a mixed weight filtration whose fiber at x is the weight filtration of H. But, the homomorphism (3.4.2) respects the mixed structures, and it induces $(W_0 = \hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x)))$

$$\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x)) \longrightarrow W_0(End(H)). \tag{3.4.3}$$

Thus the weight filtration on H is compatible with the action of $\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))$. Furthermore, again for the same reason, the action on the graded parts (which are pure) is trivial: the weights of $\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))$ are all ≤ 0 and W_{-1} is the augmentation ideal a. In this way (E,ϕ) receives an increasing filtration by unipotent F-K-isocrystals on X which we denote by $\mathcal{W}_\bullet$. The graded pieces $Gr_j^{\mathcal{W}}(E)$ associated to this filtration are constant F-K-isocrystals on X. The Frobenius is constant (§1) and they are pure of weight j, since they are pure in a fiber. **Q.E.D.**

COROLLARY 3.4.4. – *Let X be a smooth scheme over k, and $x \in X(k)$. Let (E, ϕ) be a unipotent overconvergent F-K-isocrystal on X, and let (E_x, ϕ_x) be the fiber of (E, ϕ) at x. Then (E, ϕ) is mixed if and only if the eigenvalues of ϕ_x are Weil numbers.*

Proof. – Starting from (E_x, ϕ_x), one can decompose E_x into stable subspaces relative to the irreducible factors of the characteristic polynomial (Ch.I, 2.0). These subspaces are stable under the action ϕ_x and the eigenvalues of ϕ_x on each of these factors are Weil numbers of the same weight (1.8). If we take the subspace relative to the smallest weight, then it is stabilized by the action of $\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))$, and in fact the action is trivial. We then go up on the weights. **Q.E.D.**

Remark 3.4.5. – Although the category of overconvergent F-K-isocrystals on X is a Tannakian category, we need further information on its Tannakian fundamental group (weight filtration ...) to obtain the same result in that category. One can give a unicity statement for such a category, but not a construction.

§4. ι-Weight filtration for a unipotent F-K-isocrystal on X

Our definition of integral mixed unipotent F-K-isocrystal on X, (E, ϕ), seems to be somewhat restrictive. Indeed we require an increasing filtration $\mathcal{W}_j$, $j \in \mathbf{Z}$, such that the graded parts $Gr_j^{\mathcal{W}} E$ are pure of weight exactly j. In this paragraph we show that our definition is actually completely general: we will work without hypotheses on the filtration (4.1.4) and, at the same time, we will deal with **real** weights.

First we choose an imbedding ι

$$K \to \overline{\mathbf{Q}}_p \xrightarrow{\iota} \mathbf{C},$$

and we give a general definition

DEFINITION. – *A K-vector space of finite dimension H endowed with a K-linear map, Φ, the Frobenius, is an ι-pure K-isocrystal of weight λ ($\lambda \in \mathbf{R}$), if for each eigenvalue $\zeta \in \overline{\mathbf{Q}}_p$ of Φ, its image $\iota(\zeta)$ has absolute value $q^{\lambda/2}$. We say that H is an ι-mixed isocrystal if it admits an increasing filtration whose graded parts are ι-pure.*

4.1. We begin with some generic results concerning pro-unipotent F-isocrystals on X. Consider first a unipotent F-K-isocrystal on X, (E, ϕ). One can associate to it the morphism

$$\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x)) \xrightarrow{\tilde{\rho}} End(E_x) \tag{4.1.1}$$

which commutes with the Frobenius action (using $\overline{F}$ on the left hand side) (II.3.3.1). On the other hand we can interpretate (4.1.1) as the map

$$E_x \times \hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x)) \longrightarrow E_x \tag{4.1.2}$$

which sends (e, ψ) to $\psi(e)$. This map is surjective. We let $\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))$ act on the source of the map (4.1.2) by the trivial action on the first factor and the usual multiplication on the second, while $\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))$ acts on the target E_x via (4.1.1). It follows that (4.1.2) commutes with these actions, and so we obtain a surjective map of pro-unipotent isocrystals on X

$$E_x \otimes \mathcal{G}en_x \longrightarrow E.$$

Furthemore, we may introduce a Frobenius action on $E_x \times \hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))$ given by $\phi_x \otimes \overline{F}$. This action is compatible with the Frobenius on $\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))$. This structure is compatible with the morphism (4.1.2), once we have defined ϕ_x in the target. Finally one obtains a surjective map of pro-unipotent F-K-isocrystals

$$(E_x \otimes \mathcal{G}en_x, \phi_x \otimes \overline{F}) \longrightarrow (E,\phi).$$

It is then clear that at each point $y \in X(k)$ the eigenvalues of ϕ_y in E_y are related to the eigenvalues of E_x and to those of $\mathcal{G}en_x$.

Using the previous notation, we give (cf.1.5) some new definitions.

DEFINITION 4.1.3. – *An overconvergent F-K-isocrystal (M,ϕ) on X is said ι-pure of weight λ ($\lambda \in \mathbf{R}$), if for each $f \in \mathbf{N}$ and for each $y \in X(k_f)$, the F-isocrystal (M_y, ϕ_y^f) is pure of ι-weight λ relative to k_f.*

DEFINITION 4.1.4. – *An overconvergent F-K-isocrystal (M,ϕ) on X is said ι-mixed, if it admits an increasing filtration by sub-F-K -isocrystals on X, $\mathcal{W}_\bullet$, such that the associated graded parts are ι-pure F-K-isocrystal on X.*

PROPOSITION 4.1.5. – *Let (E,ϕ) be a unipotent F-K-isocrystal on X. If (E,ϕ) is ι-pure at a point $x \in X(k)$, then it is constant.*

Proof. – Let λ be the weight of (E,ϕ). Since E is unipotent, we have as before a surjective map (4.1.2):

$$E_x \times \hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x)) \longrightarrow E_x.$$

If n is the index of unipotency of E, we can actually find a surjective map

$$E_x \times \hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X,x))/a^n \longrightarrow E_x$$

and the latter is a map of F-K-isocrystals. In particular, in the first F-K-isocrystals, the ι-weights are the sums of the weights of the F-K-isocrystals in the tensor product. Hence they are of the type $\lambda + i$ ($i \in \mathbf{N}$). We can also study the following action

$$E_x \times a/a^n \longrightarrow E_x$$

which is again a map of F-K-isocrystals. But in this case there are no common eigenvalues for the Frobenius (on the left we have weights of the type $\lambda + i$ with $i \neq 0$). Hence the map must be 0 and so the action is trivial. **Q.E.D.**

PROPOSITION 4.1.6. – *Let (E,ϕ) be a constant F-isocrystal which is ι-pure of weight λ. Then the ι-weights of the Frobenius acting on*

$$H^1_{rig}(X/K, E)$$

are of the type $\lambda + 1$, $\lambda + 2$.

Proof. – Since E is constant,we can write $V = E_x$ and $E = V \otimes \mathcal{O}^\dagger$ The Frobenius here is merely the extension of the Frobenius

$$\varphi : V \longrightarrow V$$

of the K-vector space V. Then

$$H^1_{rig}(X/K,E) = H^1_{rig}(X/K) \otimes V$$

and here the Frobenius action here is just the tensor of the Frobenius on $H^1_{rig}(X/K)$ with φ. **Q.E.D.**

COROLLARY 4.1.7. – *Let (E_1,ϕ_1),(E_2,ϕ_2) be two constant F-isocrystals which are (resp.) λ_1 and λ_2 ι-pure. If the difference $\lambda_1 - \lambda_2$ is not integral or ≥ 0 , then all the unipotent F-isocrystals which are extensions of (E_1,ϕ_1) by (E_2,ϕ_2) are trivial.*

Proof. – By the results [LS-C] the extensions in question are characterized by the Frobenius invariant in

$$H^1_{rig}(X/K,\mathcal{H}om_{\mathcal{O}^\dagger}(E_2,E_1)). \tag{4.1.8}$$

But $\mathcal{H}om_{\mathcal{O}^\dagger}(E_2,E_1) = Hom_K(V_2,V_1) \otimes \mathcal{O}^\dagger$, If $E_1 = V_1 \otimes \mathcal{O}^\dagger$ and $E_2 = V_2 \otimes \mathcal{O}^\dagger$. On the other hand, by 1.1.7, the two Frobenius maps are constant and are given by φ_1 and φ_2 on V_1 and V_2 respectively. On $Hom_K(V_2,V_1)$ we have a Frobenius structure, φ, ι-pure of weight $\lambda_1 - \lambda_2$. Using the proof of 4.1.6, we have that $H^1_{rig}(X/K,\mathcal{H}om_{\mathcal{O}^\dagger}(E_2,E_1)) = H^1_{rig}(X/K) \otimes Hom_K(V_2,V_1)$ and the Frobenius is given by the tensor product of the Frobeniuses on $H^1_{rig}(X/K)$ and on $Hom_K(V_2,V_1)$. The Frobenius will have no invariant if the difference $\lambda_1 - \lambda_2$ is ≥ 0 or not integral. **Q.E.D.**

Remark 4.1.9. – Under the previous hypotheses, if the difference of the weights is non integral or is strictly greater than 0, then there will a unique splitting which respects the Frobenius. In fact the number is characterized by the F-invariant in

$$H^0_{rig}(X/K,\mathcal{H}om_{\mathcal{O}^\dagger}(E_2,E_1)) = H^0_{rig}(X/K) \otimes Hom_K(V_2,V_1).$$

We can now give the structure of an ι-mixed unipotent overconvergent F-K-isocrystal on X, (M,ϕ).

PROPOSITION 4.1.10. – *Let (M,ϕ) be a unipotent overconvergent ι-mixed F-K-isocrystal on X. Then (M,ϕ) admits a decomposition by sub-ι-mixed F-K-isocrystals on X*

$$M = \bigoplus_{\mathbf{b}\in\mathbf{R}/\mathbf{Z}} M_{\mathbf{b}}$$

where each $M_{\mathbf{b}}$ has ι-weights in the same class of $b \in \mathbf{R}/\mathbf{Z}$. Moreover, if $M_{\mathbf{0}}$ is not zero, then when endowed with the induced Frobenius ϕ, it has a natural structure of ι-mixed unipotent F-K-isocrystal of ι-integral weights, and admits an increasing filtration T_j ($j \in \mathbf{Z}$) such that the associated graded parts $Gr^T_j(M_{\mathbf{b}})$ are pure of ι-weight j.

Proof. – The proof, using the results established above, is as in [DE2, II, 3.4.7] (see also [FA]). **Q.E.D.**

§5. Relation with other points, $y \in |X|$

5.1. In this paragraph we will study the behaviour of the unipotent fundamental group in relation with scalar extensions. Our previous notation and terminology remains in force. We choose a compatible lifting σ of the q-power isomorphism to each K_f. In the Tannakian ([DE3, §4], [S]) setting we can introduce the category

$$\mathcal{U}n(X) \otimes K_f.$$

Its objects are unipotent K-isocrystals on X endowed with a K_f-module structure. On the other hand we may also take the scalar extension X_{k_f} and consider the category $\mathcal{U}n(X_{k_f}))$ of unipotent K_f-isocrystals on X_{k_f}. There is an obvious functor associated with the extension of scalars $a : X_{k_f} \to X$

$$a^* : \mathcal{U}n(X) \longrightarrow \mathcal{U}n(X_{k_f}) \tag{5.1.1}$$

which is exact [BER2, 2.3.3]. On the other hand one can prove that a^* is essentially surjective and faithfull. In fact, for each $E \in ob\, \mathcal{U}n(X)$,and for each $i \in \mathbf{N}$ we have

$$H^i_{rig}(X_{k_f}, a^*(E)) \simeq H^i_{rig}(X, E) \otimes K_f. \tag{5.1.2}$$

This can be proven by induction on the order of unipotency and by using the base change theorem in Ch.I, 1.1. Hence our functor is essentially surjective ($i = 1$). We obtain the faithfulness by noticing that the two categories have internal hom and using 5.1.2 (for $i = 0$).

The map $a : X_{k_f} \to X$ induces a direct image functor

$$a_* : \mathcal{U}n(X_{k_f})) \longrightarrow \mathcal{U}n(X) \otimes K_f$$

PROPOSITION 5.1.3. – *The functor a_* is an equivalence of categories.*

Proof. – We have another natural functor

$$U : \mathcal{U}n(X) \otimes K_f \longrightarrow \mathcal{U}n(X_{k_f}).$$

Every object of $\mathcal{U}n(X) \otimes K_f$, E, is realized by a rigid integrable unipotent connection (with a structure of K_f-module) defined in some strict neighborhood of X in P_K. Considering it as a K_f-vector space (we recall that K_f is a finite extension of K, so, we don't need to complete), we obtain a rigid integrable unipotent connection in some strict neighborhood of X_{k_f} in P_{K_f}, (i.e. an object of $\mathcal{U}n(X_{k_f})$) which we denote by $U(E)$. This functor is the quasi-inverse of a_*: in fact one can construct on P_{K_f} a fundamental system of strict neighborhoods of X which is an extension (by K_f) of a fundamental system of strict neighborhoods of $]X[_P$ (hence, defined on K) [BER2, 1.2.4(i)]. **Q.E.D.**

Remark 5.1.4. – This is compatible with [DE3, §4 and §10].

5.2. Let x be a closed point in $X(k)$. We have introduced the K-fiber functor ω_x for the Tannakian K-category $\mathcal{U}n(X)$, and we have associated to it the unipotent fundamental group $\pi_1^{rig,un}(X, x)$, which is a K-proalgebraic group. Of course we can extend of the previous fiber functor to a K_f-fiber functor $\omega_x \otimes K_f$

$$\omega_x \otimes K_f : \mathcal{U}n(X) \otimes K_f \longrightarrow Vect(K_f),$$

Its Tannakian fundamental group is just the scalar extension to K_f of $\pi_1^{rig,un}(X,x)$ ([S] and [DM]). By the results of 5.1 we deduce

PROPOSITION 5.2.1. – *Under the previous hypotheses, the scalar extension $\pi_1^{rig,un}(X,x)\otimes K_f$ is the Tannakian fundamental group of the K_f-tensor category of the unipotent K_f-isocrystals $\mathcal{U}n(X_{k_f})$ on X_{k_f} by the fiber functor $\omega_x\otimes K_f$, i.e.*

$$\pi_1^{rig,un}(X,x)\otimes K_f=\pi_1^{rig,un}(X_{k_f},x).$$

5.3. Consider $\mathcal{U}n(X)$ and its fundamental group $\pi_1^{rig,un}(X,x)$ at $x\in X(k)$. There is a linear action of Frobenius, which now is denoted by F_x, on $\pi_1^{rig,un}(X,x)$. As in [DE2,II], [CR, §5] and [B-O] we can define the Weil-group of $\mathcal{U}n(X)$ at x, $W(\mathcal{U}n(X),x)$ as the semi-direct product of $\pi_1^{rig,un}(X,x)$ and $\mathbf{Z}$, where the action of $n\in\mathbf{Z}$ on $\pi_1^{rig,un}(X,x)$ is given by F_x^{-n}. Consider, now, a point $y\in X(k_f)$. It induces a fiber functor

$$\omega_y\,:\,\mathcal{U}n(X)\longrightarrow Vect(K_f),$$

We can extend to

$$\omega_y\,:\,\mathcal{U}n(X)\otimes K_f\simeq\mathcal{U}n(X_{k_f})\longrightarrow Vect(K_f),$$

though of course on $\mathcal{U}n(X)\otimes K_f\simeq\mathcal{U}n(X_{k_f})$ the action of the a-th iterate of Frobenius ($q=p^a$, $N(k)=q$) is no longer K_f-linear and so we must take the af-th iterate of Frobenius. We indicate the associated fundamental group $\pi_1^{rig,un}(X_{k_f},y)$ and the Frobenius F_y^{-1}. Of course we can do the same with the K_f-scalar extension of ω_x. We again introduce the Weil fundamental group $W(\mathcal{U}n(X_{k_f})),x)$, which fits into the exact sequence (Here we denote with the same symbol, F_x^{-f}, an element of the Weil group and its action on the fundamental group)

$$0\to\pi_1^{rig,un}(X_{k_f},x)\to W(\mathcal{U}n(X_{k_f})),x)\to<F_x^{-f}>\to 0.$$

On the other hand we have an inclusion [CR, 5.4]:

$$W(\mathcal{U}n(X_{k_f},x)(K_f)\to W(\mathcal{U}n(X),x)(K_f)$$

(arising via push-out from the inclusion $<F_x^{-f}>\to<F_x^{-1}>$). Starting from an isomorphism between $\pi_1^{rig,un}(X_{k_f},x)$ and $\pi_1^{rig,un}(X_{k_f},y)$, finally we have that F_y^{-1} induces a Frobenius conjugacy class in $W(\mathcal{U}n(X),x)(K_f)$, given by $Frob_y$ [CR, 5.4].Thus we find that [CR, §5] $Frob_y=F_x^{-f}g$ where $g\in\pi_1^{rig,un}(X,x)(K_f)$.

Consider now an object of $\mathcal{U}n(X)$ which has a Frobenius structure (E,ϕ). Again we can consider the fiber E_x at $x\in X(k)$ and the group of automorphisms $Aut(E_x)$. We can consider also the semidirect product $W(E_x)$ of $Aut(E_x)$ and $<\phi_x>$ where the action of ϕ_x on $Aut(E_x)$ is given by conjugation. We then obtain a morphism

$$\overline{\rho}\,:\,W(\mathcal{U}n(X),x)\to W(E_x)\tag{5.3.1}$$

associated to the representation $\rho\,:\,\pi_1^{rig,un}(X,x)\to Aut(E_x)$ and such that $\overline{\rho}(F_x^{-1})=\phi_x$. Of course we may then consider $y\in X(k_f)$, and (E_y,ϕ_y^f), E_y is a K_f-vector space.

We can also extend the morphism (5.3.1) to K_f, and finally we find that there is an isomorphism between (E_y, ϕ_y^f) and $(E_x \otimes K_f, \overline{\rho}(F_x^{-f} g))$. But $\overline{\rho}$ is a morphism and we have $\overline{\rho}(F_x^{-f} g) = \overline{\rho}(F_x^{-f})\rho(g)$, where $\overline{\rho}(F_x^{-f})$ is $\phi_x^f \otimes K_f$ [CR, 5.5]. We may state

THEOREM 5.3.2. – *The unipotent F-K-isocrystal (E, ϕ) is mixed if and only if there exists a $f \in \mathbf{N}$ and a point $y \in X(k_f)$ such that (E_y, ϕ_y^f) is a mixed K_f-isocrystal relative to k_f.*

Proof. – We have seen that (E_y, ϕ_y^f) is isomorphic to $(E_x \otimes K_f, \overline{\rho}(F_x^{-f} g))$. But E is unipotent, and so we can find a filtration of E stable by Frobenius whose graded pieces are trivial K-isocrystals (it is the natural filtration of [LS-C]: we start with $H^0_{rig}(X, E)$, we take the associated constant F-isocrystal $H^0_{rig}(X, E) \otimes \mathcal{O}^\dagger = E'$, we consider E/E', and we take $H^0_{rig}(X, E/E')$...). Then $\rho(g)$ acts trivially on the graded pieces of that filtration and $\phi_x^f \otimes K_f$ respects the associated filtration. The eigenvalues of $\overline{\rho}(F_x^{-f} g)$ are the eigenvalues of $\overline{\rho}(F_x^{-f}) = \phi^f \otimes K_f$. So they are Weil numbers if and only if the eigenvalues of ϕ_x be Weil numbers (Ch.I, §2). It now suffices to apply Corollary 3.4.4. **Q.E.D.**

Remark 5.3.3. – We have an analogous statement for ι-mixed. Note that in Theorem 5.3.2 it would have been enough to require that the eigenvalues of ϕ_y^f be Weil numbers (3.4.4).

5.4. In this subsection we will introduce an ι-weight filtration for a generic unipotent F-K-isocrystal on X at the cost of extending the coefficients.

PROPOSITION 5.4.1. – *Let (E, ϕ) be an object in $\mathcal{U}n(X)$. Then there exists a finite extension K' of K with residue field k' of degree f over k such that the extension of (E, ϕ) to an object of $\mathcal{U}n(X_{k'})$), $(E_{k'}, \phi^f)$, is ι-mixed.*

Proof. – We again use the natural filtration of E introduced in the proof of theorem 5.3.2. We then apply corollary 1.7. Each graded part $(gr_\bullet E, gr_\bullet \phi)$ is constant. We take the fiber (E_x, ϕ_x) at $x \in X(k)$. If it is not mixed, we can take a finite field extension K' (of residue field k') in which ϕ_x can be decomposed according to its eigenvalues. We then take $(E_x \otimes K', \phi_x^f \otimes K')$: it is ι-mixed. **Q.E.D.**

§6. Remark on unipotent isocrystals

In this paragraph we prove a result which we conjectured in [LS-C]. Independently, Deligne and Crew have also given a positive answer.

PROPOSITION 6.1. – *Every unipotent isocrystal E on X is a quotient of a unipotent F-isocrystal on X.*

Proof. – Given E, we know that there exists a morphism

$$\hat{\mathcal{U}}(Lie\pi_1^{rig,un}(X, x)) \longrightarrow End(E_x)$$

As in §4, we can extend such a map to a surjective map of unipotent isocrystals on X

$$E_x \times (\mathcal{G}en_x/a^n) \longrightarrow E$$

for a suitable n connected with the index of unipotency of E. On $\mathcal{G}en_x/a^n$ we have a natural induced Frobenius structure which we will again indicate by $\overline{F}$. Finally, $id_{E_x} \otimes \overline{F}$ gives a Frobenius structure on $E_x \times (\mathcal{G}en_x/a^n)$. **Q.E.D.**

REFERENCES

[BER1] P. BERTHELOT, Finitude et pureté en cohomologie rigide. Inv. Math. 128, 1997, pp. 329-377.

[BER2] P. BERTHELOT, Cohomologie rigide et cohomologie rigide à support propre. Prepubl. 96-03. I.R.M.A.R., Rennes 1996.

[B-O] P. BERTHELOT and A. OGUS, F-isocrystals and the de Rham cohomology. Inv. Math. 72, 1983, pp. 159-199.

[BGR] S. BOSCH, U., GUNTZER and R. REMMERT, *Non archimedean analysis*. Grundlehren der Math. Wissenshaften 261. Springer-Verlag 1984.

[BR-Z] J. L. BRYLINSKI and S. ZUCKER, An overview of recent advances in Hodge theory. In Encyclopaedia of Math. Sciences vol.69. *Several Complex Variables VI* (Barth W., Narasimhan R. Eds.) Springer-Verlag 1990, pp. 39-142.

[C] K.-T. CHEN, Iterated path integral. Bull.AMS 83, 1977, pp. 831-879.

[CH-M] G. CHRISTOL and Z. MEBKHOUT, Sur le théorème d'indice des équations différentielles p-adiques III. Preprint 1994.

[CR] R. CREW, F-isocrystals and their monodromy groups. Ann. Scient. Éc. Norm. Sup. Ser.IV, tome 25, 1992, pp. 429-464.

[DE1] P. DELIGNE, Théorie de Hodge I. Actes , Congr. Intern. Math. Nice 1970, pp. 425-430. II, Publ. Math. I.H.E.S. **40**, 1971, pp. 5-58. III, Publ. Math. I.H.E.S. **44**, 1974, pp. 5-77.

[DE2] P. DELIGNE, La conjecture de Weil I. Publ. Math. I.H.E.S. **43**, 1974, pp. 273-308. II, Publ. Math. I.H.E.S. **52**, 1980, pp. 137-252.

[DE3] P. DELIGNE, Le groupe fondamental de la droite projective moins trois points. In Ihara, Y., Ribet, K., Serre, J.P. (eds.): *Galois groups over* **Q**, Math. M.S.R.I. Publ.16, Springer-Verlag 1989, pp. 79-297.

[DM] P. DELIGNE, and J. S. MILNE, Tannakian categories. In *Hodge cycles, Motives and Shimura varieties* (Deligne, P., Milne, J.S., Ogus, A., Shih, K.Y. Eds.) L.N.M. 900, Springer-Verlag 1982. pp. 101-228.

[D-G] M. DEMAZURE and P. GABRIEL, *Groupes algébriques*. Tome I. North-Holland 1970.

[E] R. ELKIK, Solutions d'équations à coefficients dans un anneau hensélien. Ann. Scient. Ec. Norm. Sup. **6**, 1973, pp. 553-604.

[FA] G. FALTINGS, F-isocrystals on open varieties. Results and conjectures. In Cartier, P. et al. (eds.) *The Grothendieck Festschrift*, vol. II. Birkhauser 1990, pp. 219-248.

[FRE] J. FRESNEL, Géometrie analytique rigide. Université de Bordeaux I. Math. Pures 1983/84.

[H-Z1] R. HAIN and S. ZUCKER, Unipotent variations of mixed Hodge structure. Inv. Math. **88**, 1987, pp. 83-124.

[H-Z2] R. HAIN and S. ZUCKER, A guide to unipotent variations of mixed Hodge structure. In Hodge Theory Proceedings, Sant Cugat 1985. LNM vol. 1246, Springer, 1987, pp. 92-106.

[K-M] N. M. KATZ and W. MESSING, Some consequences of the Riemann hypothesis for varieties over finite field. Inv. Math. **23**, 1974, pp. 73-77.

[LS-C] B. LE STUM and B. CHIARELLOTTO, F-Isocristaux unipotents. Prepubl. 96-19, I.R.M.A.R., Rennes 1996. To appear in Comp. Math.

[LS-C1] B. LE STUM and B. CHIARELLOTTO, Pentes en cohomologie rigide et F-isocristaux unipotents. Preprint, 1998.

[SC] W. SCHMID, Variation of Hodge structure: the singularities of the period mapping. Inv. Math. **22**, 1973, pp. 211-319.

[S] N. SAAVEDRA, *Catégories tannakiennes* . L.N.M.265, Springer-Verlag 1972.

[SGA1] A. GROTHENDIECK et al., *Revêtements étales et groupe fondamental*. L.N.M.224. Springer-Verlag 1971.

[VdP] M. Van der PUT, The cohomology of Monsky-Washnitzer. In Introduction aux cohomologies p-adiques. Memoires de la Société Math. de France. N.23, 1986, pp. 33-60.

[W] J. WILDESHAUS, *Realizations of polylogarithms*. L.N.M. 1650. Springer-Verlag 1997.

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