ANNALES SCIENTIFIQUES DE L'É.N.S.

FRANCIS BONAHON

Geodesic laminations with transverse Hölder distributions

Annales scientifiques de l'É.N.S. 4e série, tome 30, nº 2 (1997), p. 205-240

<http://www.numdam.org/item?id=ASENS_1997_4_30_2_205_0>

Ann. scient. Éc. Norm. Sup.,
4e série, t. 30, 1997, p. 205 à 240.

GEODESIC LAMINATIONS WITH TRANSVERSE HÖLDER DISTRIBUTIONS

BY FRANCIS BONAHON

ABSTRACT. – To interpolate between isotopy classes of simple closed curves on a surface S, Thurston introduced the space $\mathcal{ML}(S)$ of geodesic laminations with transverse measures on S. The main purpose of this paper is to develop a differential calculus on $\mathcal{ML}(S)$. This space is a piecewise linear manifold, but does not admit any natural differentiable structure. We give an analytic interpretation of the combinatorial tangent vectors to $\mathcal{ML}(S)$, as geodesic laminations with a certain type of transverse distributions. As an illustration, we apply this technique to determine the derivative of the length function associated to a hyperbolic 3-manifold.

Consider a compact connected surface S of negative Euler characteristic, possibly with boundary. To interpolate between isotopy classes of simple closed curves on S, W.P. Thurston introduced the notion of geodesic lamination with transverse invariant measure on S (*see* [Th1], [PeH] and §1; *see* also [Th3], [FLP] for the closely related notion of measured foliation, and [Le] for a 'dictionary' between measured foliations and measured laminations). He used these measured geodesic laminations as a tool to attack various geometric problems, notably the analysis of hyperbolic structures on surfaces and on 3-manifolds. The main motivation of this paper is to develop a differential calculus on the space $\mathcal{ML}(S)$ of measured geodesic laminations of S, so as to compute the variations of various quantities defined on this space. As a side benefit, this leads us to the discovery of certain transverse structures for geodesic laminations, which are not transverse measures and have interesting geometric applications.

The space $\mathcal{ML}(S)$ does not possess a natural differentiable structure, but Thurston exhibited a natural structure of piecewise linear manifold on $\mathcal{ML}(S)$. This leads to the abstract definition of tangent vectors of this space (although the tangent vectors at a given point of $\mathcal{ML}(S)$ do not necessarily form a vector space). However, these combinatorial tangent vectors are not always very easy to work with in practice. The main result of this paper is a geometric interpretation of tangent vectors to $\mathcal{ML}(S)$ as geodesic laminations with certain transverse distributions. This process, converting combinatorial data to analytic data, makes these tangent vectors easier to handle for geometric applications.

Mathematics subject Classification. 57 N 02, 53 C 22.
Key words and phrases. geodesic laminations, measured laminations.
This research was partially supported by N.S.F. grants DMS-8700642, DMS-8958665 and DMS-9001895.

Given a combinatorial vector tangent to $\mathcal{ML}(S)$ at α_0, we compute the associated geodesic lamination with transverse distribution in §§2-4. This geodesic lamination contains the geodesic lamination underlying α_0, but may be strictly larger. In particular, in Theorem 15, we give a formula which explicitly describes the transverse distribution in terms of the combinatorial data. This formula involves the sum of a series over the gaps of the geodesic lamination, and even its restriction to the case of a transverse measure seems to be new (Corollary 17).

There is a problem, however, namely that the notion of transverse distribution for a lamination or a foliation usually requires that we are given a transverse differentiable structure for this lamination or foliation. More precisely, our computations define a distribution on each arc transverse to the geodesic lamination, but we want to be able to say that this distribution is invariant under homotopy respecting the lamination. Unfortunately, there is in general no transverse differentiable structure for a geodesic lamination. If we fix a negatively curved metric on the surface, a geodesic lamination admits only a transverse Lipschitz structure, and even this structure depends on the choice of the metric. On the other hand, a geodesic lamination does admit a metric independent transverse Hölder structure. It luckily turns out that the distributions on transverse arcs provided by our computations are regular enough to extend to continuous linear forms on the space of Hölder continuous functions on this arc. In this way, we can interpret the analytic objects provided by our computations as transverse Hölder distributions, an object which is well defined because of the transverse Hölder structure of a geodesic lamination.

Having associated a transverse Hölder distribution for a certain geodesic lamination to each combinatorial tangent vector to $\mathcal{ML}(S)$, we push the analysis further in the companion paper [Bo4], where we analyze the vector space $\mathcal{H}(\lambda)$ of all transverse Hölder distributions with which a given geodesic lamination λ can be endowed. A geodesic lamination admits many more transverse Hölder distributions than transverse measures, but we show in [Bo4] that the dimension of $\mathcal{H}(\lambda)$ is finite and can be explicitly determined.

It turns out that the notion of transverse Hölder distribution is precisely adapted to our original goal, in the sense that 'most' geodesic laminations with transverse Hölder distributions come from tangent vectors to $\mathcal{ML}(S)$. Indeed, the main result of [Bo4] is that an element of $\mathcal{H}(\lambda)$ is characterized by certain combinatorial data, which can be seen as a partial converse to the results of §§3-4. In §5, we prove the complete converse result. We show that a transverse Hölder distribution for a geodesic lamination λ is associated to a tangent vector at $\alpha_0 \in \mathcal{ML}(S)$ if and only if it satisfies a certain positivity condition, namely if and only if it belongs to a certain convex cone bounded by finitely many faces in the vector space $\mathcal{H}(\lambda)$. In particular, under the generic assumption that the geodesic lamination λ_{α_0} underlying α_0 is maximal, in the sense that it cannot be enlarged to a larger geodesic lamination, the positivity condition is empty and there is a one-to-one correspondence between the combinatorial tangent vectors of $\mathcal{ML}(S)$ at α_0 and the transverse Hölder distributions for λ_{α_0}.

Finally, we conclude the paper with two examples of the applications we had in mind when starting this program.

In §6, we consider the length function $l_m : \mathcal{ML}(S) \to \mathbb{R}^+$ associated to a negatively curved metric m on the surface S. After the hard work done in §3, the simple observation that the definition of l_m makes sense for geodesic laminations with transverse Hölder distributions provides an expression for the differential of l_m, and proves that this differential is linear on the faces of the piecewise linear structure of $\mathcal{ML}(S)$ (Theorem 24 and Corollary 25). This last property had independently (although much earlier) been obtained by Thurston [Th4, §7], by different methods.

In §7, we generalize this to higher dimensions. If M is a hyperbolic manifold of any dimension and if $f : S \to M$ is a map inducing an injective homomorphism between the fundamental groups, there is a function $l_M : \mathcal{ML}(S) \to \mathbb{R}^+$ associating to each measured geodesic lamination the length of its realization in M; *see* [Th1], [CEG]. We use the same formalism of transverse Hölder distributions to determine the differential of l_M (Theorem 29). A consequence is that, if $\alpha \in \mathcal{ML}(S)$ is realized by a pleated surface and if m is the hyperbolic metric induced on S by this pleated surface, then the two functions l_M and l_m have the same differential at α (Proposition 32). As a consequence, the differential of l_M depends only on the pull back metric m, and not on the bending of the pleated surface. A similar analysis for the rotation number of the realization of a measured lamination is developed in [Bo5].

Other applications of the theory developed in this paper appear in [Bo5], [Bo6] and [Bo7]. In [Bo5], we show that transverse Hölder distributions can be used to describe the shearing of shear maps between hyperbolic surfaces, as well as the bending of pleated surfaces in hyperbolic 3-manifolds. This generalizes the cases of earthquake maps [Ke], [Th2], [EpM] and of locally convex pleated surfaces [Th1], [EpM], where the shearing and bending always occur in the same direction and are described by measured laminations. In [Bo6], [Bo7], we consider the variation of the geometry of the convex core of a hyperbolic 3-manifold under deformation of the metric. In particular, we obtain a Schläfli-type formula which expresses the variation of the volume of the convex core in terms of the length of the transverse Hölder distribution describing the variation of the pleating locus of this convex hull. See also [Bo3] for some related material.

A large part of this work was done while the author was visiting the University of California at Davis, and we would like to thank this institution for its beneficial hospitality. We would also like to thank the referee for carefully reading the manuscript, and Yair Minsky for pointing out to us the overlap between §6 and [Th4, §7].

1. Measured geodesic laminations

In this section, we review some properties of geodesic laminations. The proofs and details can be found in [Th1], [CaB], [PeH], for instance.

To define measured geodesic laminations, one starts by endowing the surface S with an auxiliary Riemannian metric m of negative curvature, for which the boundary ∂S is totally geodesic; such a metric exists because of our assumption that the Euler characteristic of S is negative. Then, a ***geodesic lamination*** of S is a partial foliation of S by m-geodesics, namely a closed subset $\lambda \subset S$ decomposed as a union of disjoint geodesics which are simple and do not transversely hit the boundary. Recall that a geodesic is simple if it does not cross itself; it may be closed or infinite. Also, note that components of ∂S are allowed as part of a geodesic lamination. A geodesic lamination $\lambda \subset S$ covers only a small part of S, in the sense that it has Lebesgue measure 0, and even Hausdorff dimension 1 [Th1, §8], [BiS], [Th4, §10]. In particular, the decomposition of the subset λ as a union of disjoint simple geodesics is unique; these geodesics are the ***leaves*** of λ.

A ***transverse (invariant) measure*** for a geodesic lamination λ is a measure defined on each arc k transverse to λ, such that every homotopy sending k to another arc k' while respecting λ sends the measure defined on k to the measure defined on k'. Throughout the paper, a measure will be a positive Radon measure, namely one which is defined on the σ-algebra of Borel sets and which assigns non-negative finite mass to each compact set; via the Riesz Representation Theorem, such a Radon measure is equivalent to a positive linear form on the space of continuous functions with compact support. Also, we adopt the convention that the end points of an arc transverse to a geodesic lamination λ are disjoint from λ. Observe that the invariance property implies that the support of the measure deposited on the transverse arc k is contained in $k \cap \lambda$; a transverse measure for λ is said to have ***full support*** if the support of the measure it induces on each transverse arc k is exactly $k \cap \lambda$. A ***measured geodesic lamination*** α consists of a geodesic lamination λ_α together with a full support transverse measure for λ_α.

An example of measured geodesic lamination is provided by a closed geodesic λ endowed with the transverse measure which, on each arc k transverse to λ, is the Dirac measure of weight $a > 0$ based at $k \cap \lambda$ (for which the mass of $A \subset k$ is a times the cardinal of $A \cap \lambda$); but 'generic' examples are more complex.

The space $\mathcal{ML}(S)$ of measured geodesic laminations can be topologized as follows. For simplicity, say that an arc $k \subset S$ is ***generic*** if it is transverse to all geodesic laminations of S. By [BiS], almost all geodesic arcs are generic, so that every arc can be arbitrarily approximated by a generic arc. Then, by definition, a sequence of measured geodesic laminations α_n converges to $\alpha \in \mathcal{ML}(S)$ if and only if, for every generic arc k, the mass $\alpha_n(k)$ of the measure deposited by α_n on k converges to $\alpha(k)$.

These $\alpha(k)$ also define the piecewise linear structure of $\mathcal{ML}(S)$. Indeed, Thurston showed that it is possible to find finitely many generic arcs $k_1, k_2, \ldots, k_p$ such that the map $\alpha \mapsto (\alpha(k_1), \alpha(k_2), \ldots, \alpha(k_p))$ defines an embedding of $\mathcal{ML}(S)$ in $\mathbb{R}^p$, whose image is a piecewise linear submanifold of $\mathbb{R}^p$ of dimension $-3\chi(S)$, where $\chi(S)$ is the Euler characteristic of S. In addition, for every generic arc k, the mass $\alpha(k)$ is a piecewise linear function of these $\alpha(k_i)$, which shows that the structure of piecewise linear manifold so defined on $\mathcal{ML}(S)$ does not depend on the choice of the parametrizing arcs k_i. As a piecewise linear manifold, $\mathcal{ML}(S)$ is isomorphic to $\mathbb{R}^{-3\chi(S)}$; *see* [PeH], [FLP].

As usual, this piecewise linear structure on $\mathcal{ML}(S)$ defines a space of tangent vectors at each $\alpha_0 \in \mathcal{ML}(S)$. Namely, such a tangent vector is associated to each path $t \mapsto \alpha_t \in \mathcal{ML}(S)$, with $t \in [0, t_0]$, whose image in $\mathbb{R}^n$ under a coordinate chart has a right derivative at $t = 0$. And two such paths define the same tangent vector when their images under a coordinate chart have the same right derivatives at $t = 0$. The fact that the changes of chart are piecewise linear guarantees that these properties do not depend on the coordinate charts considered. This space of tangent vectors at $\alpha_0 \in \mathcal{ML}(S)$ is endowed with a natural law of multiplication by real numbers, but does not necessarily possess any natural structure of a vector space.

The piecewise linear structure also enables us to define the ***differential*** of a function $f : \mathcal{ML}(S) \to \mathbb{R}$, if it exists. This differential associates to each vector tangent to a path $t \mapsto \alpha_t$ the derivative of $t \mapsto f(\alpha_t)$, if it exists. For instance, every piecewise linear function on $\mathcal{ML}(S)$ has such a well defined differential.

A priori, the above notion of measured geodesic lamination depends on the choice of a negatively curved metric on the surface S. However, there are various ways to make this definition independent of the choice of metric. Here is one, based on a more intrinsic description of measured geodesic laminations, which we will use extensively (*see* [Bo1], [Bo2] for details).

The universal covering $\widetilde{S}$ has a natural compactification by its ***boundary at infinity*** $\widetilde{S}_\infty$. One way to define this compactification is to choose a base point $\widetilde{x}_0$ in the interior of $\widetilde{S}$, and to abstractly add an end point to each infinite geodesic ray issued from $\widetilde{x}_0$. It can be shown that the space $\widetilde{S} \cup \widetilde{S}_\infty$ so obtained is topologically independent of the choice of the base point $\widetilde{x}_0$ and of the choice of the negatively curved metric on S; *see* for instance [Mo], [Fl], [Gr] for descriptions of this compactification using only the algebraic structure of the fundamental group $\pi_1(S)$. Let $G(\widetilde{S})$ be the space of bi-infinite geodesics of $\widetilde{S}$, namely of those unoriented geodesics of $\widetilde{S}$ which do not transversely hit the boundary $\partial\widetilde{S}$. Each geodesic of $G(\widetilde{S})$ is asymptotic to two distinct points in the boundary at infinity $\widetilde{S}_\infty$ and, conversely, any two distinct points in $\widetilde{S}_\infty$ are joined by a unique such geodesic. It follows that $G(\widetilde{S})$ can be identified to the set of unoriented pairs of distinct points in $\widetilde{S}_\infty$, namely $G(\widetilde{S}) \cong (\widetilde{S}_\infty \times \widetilde{S}_\infty - \Delta)/\mathbb{Z}_2$ where Δ denotes the diagonal and where $\mathbb{Z}_2$ acts by exchanging the two factors. For instance, when S has empty boundary, the boundary at infinity is topologically a circle (every geodesic ray issued from the base point of $\widetilde{S}$ is infinite) and the space $G(\widetilde{S})$ is homeomorphic to an open Möbius strip. When S has non-empty boundary, the spaces $\widetilde{S}_\infty$ and $G(\widetilde{S})$ both are Cantor sets.

Given a measured geodesic lamination α, the preimage $\widetilde{\lambda}_\alpha \subset \widetilde{S}$ of its underlying geodesic lamination $\lambda_\alpha \subset S$ is decomposed as a union of geodesics of $\widetilde{S}$, and therefore defines a closed subset of $G(\widetilde{S})$ which we will also denote by $\widetilde{\lambda}_\alpha$. Given a geodesic $g \in \widetilde{\lambda}_\alpha \subset G(\widetilde{S})$, choose a small arc $\widetilde{k} \subset \widetilde{S}$ cutting transversely g in its interior. Because the geodesics of $\widetilde{\lambda}_\alpha$ are pairwise disjoint, the elements of a neighborhood of g in $\widetilde{\lambda}_\alpha$ are parametrized by their intersection points with $\widetilde{k}$. The measure deposited by α on the projection $k \subset S$ of $\widetilde{k}$ then pulls back to a measure defined on this neighborhood of g. From the invariance of the transverse measure under homotopies respecting λ_α, it follows that this measure is independent of the choice of the arc $\widetilde{k}$, and that these measures defined on neighborhoods of elements of $\widetilde{\lambda}_\alpha$ fit together to define a measure over all of $\widetilde{\lambda}_\alpha$. Pushing forward this measure by the inclusion map $\widetilde{\lambda}_\alpha \to G(\widetilde{S})$, we have associated to α a measure on $G(\widetilde{S})$; note that this measure is invariant under the action of $\pi_1(S)$ on $G(\widetilde{S})$, and that its support is contained in $\widetilde{\lambda}_\alpha$.

In this way, we have defined an embedding of the space $\mathcal{ML}(S)$ of measured laminations on S into the space $\mathcal{C}(S)$ of $\pi_1(S)$-invariant measures on $G(\widetilde{S}) \cong (\widetilde{S}_\infty \times \widetilde{S}_\infty - \Delta)/\mathbb{Z}_2$. The elements of $\mathcal{C}(S)$ are called ***geodesic (measure) currents***. It is not hard to see that the image of $\mathcal{ML}(S)$ in $\mathcal{C}(S)$ consists exactly of those geodesic currents whose support forms a geodesic lamination of $\widetilde{S}$, namely such that no two geodesics of this support cross each other in $\widetilde{S}$ (*see* [Bo1], [Bo2], [Bo4]). Note that this description depends only on the space $\widetilde{S} \cup \widetilde{S}_\infty$, and not on a choice of metric or base point.

The space $\mathcal{C}(S)$ is endowed with the weak* topology, for which a sequence of geodesic currents α_n converges to $\alpha \in \mathcal{C}(S)$ if and only if the integrals $\alpha_n(\varphi) = \int \varphi \, d\alpha_n$ converge to $\gamma(\varphi)$ for every continuous function $\varphi : G(\widetilde{S}) \to \mathbb{R}$ with compact support. Then, the embedding $\mathcal{ML}(S) \to \mathcal{C}(S)$ is a homeomorphism onto its image (*see* [Bo2]).

In a similar vein, we can give a metric independent definition of geodesic laminations of S. Indeed, the preimage in $\widetilde{S}$ of a geodesic lamination λ of S defines a $\pi_1(S)$-invariant geodesic lamination $\widetilde{\lambda}$ of $\widetilde{S}$, whose leaves form a closed subset of $G(\widetilde{S})$. This establishes a one-to-one correspondence between geodesic laminations of S and $\pi_1(S)$-invariant closed subsets of $G(\widetilde{S}) \cong (\widetilde{S}_\infty \times \widetilde{S}_\infty - \Delta)/\mathbb{Z}_2$ consisting of pairwise disjoint geodesics. The property of whether or not two geodesics of $G(\widetilde{S})$ intersect depends only on a linking property of their end points in $\widetilde{S}_\infty$. Therefore, this description of geodesic laminations depends only on the action of $\pi_1(S)$ on $\widetilde{S} \cup \widetilde{S}_\infty$, and is metric independent.

In the paper, we will usually identify a measured geodesic lamination α with its image in $\mathcal{C}(S)$.

While we are discussing the space $G(\widetilde{S})$, we should mention that, even when $G(\widetilde{S})$ is homeomorphic to a Möbius strip, it does not admit an intrinsic differentiable structure. Indeed, if we fix a negatively curved metric on S and a base point $\widetilde{x}_0 \in \widetilde{S}$, the boundary at infinity $\widetilde{S}_\infty$ is in this case identified to the metric circle of directions at $\widetilde{x}_0$, and therefore inherits a differentiable structure. However, if we change our choice of metric or of base point, this differentiable structure on $\widetilde{S}_\infty$ will be modified, as well as the induced differentiable structure on $G(\widetilde{S}) \cong (\widetilde{S}_\infty \times \widetilde{S}_\infty - \Delta)/\mathbb{Z}_2$ (*see* [Gh] and references mentioned there).

Nevertheless, $\widetilde{S}_\infty$ and $G(\widetilde{S})$ have a well-defined ***Hölder structure***, namely a preferred metric defined up to the ***Hölder equivalence*** relation which identifies two metrics d_1 and d_2 when there are constants $\nu > 0$ and $K > 0$ such that $K^{-1} d_1(x,y)^{1/\nu} \leqslant d_2(x,y) \leqslant K d_1(x,y)^\nu$ for every x, y. Indeed, the choice of a negatively curved metric with totally geodesic boundary on S and of a base point $\widetilde{x}_0 \in \widetilde{S}$ identifies $\widetilde{S}_\infty$ to a subset of the circle of directions at $\widetilde{x}_0$. This induces a metric on $\widetilde{S}_\infty$ by restriction of the angle metric of this circle of directions. It can be shown that, if we vary the metric or the base point, the Hölder equivalence class of this metric is unchanged (*see* [Fl], [Gr, §7.2.M]). Therefore, this defines a natural Hölder structure on $\widetilde{S}_\infty$, and consequently on $G(\widetilde{S}) \subset (\widetilde{S}_\infty \times \widetilde{S}_\infty)/\mathbb{Z}_2$. In particular, it makes sense to talk of ***Hölder continuous*** functions $\varphi : G(\widetilde{S}) \to \mathbb{R}$, namely functions for which there exists two constants $A \geqslant 0$ and $\nu > 0$ such that $|\varphi(g) - \varphi(h)| \leqslant A d(g,h)^\nu$ for every g, $h \in G(\widetilde{S})$, where $d(\ ,\ )$ is any metric compatible with the Hölder structure of $G(\widetilde{S})$; if the metric $d(\ ,\ )$ is fixed, the number ν is the ***Hölder exponent*** of φ, and the ***Hölder norm of exponent*** ν of φ is

$$\|\varphi\|_\nu = \sup_g |\varphi(g)| + \sup_{g \neq h} |\varphi(g) - \varphi(h)| d(g,h)^{-\nu}.$$

Throughout the paper, we will assume that $G(\widetilde{S})$ is endowed with a metric compatible with this Hölder structure.

To analyze the local piecewise linear geometry of $\mathcal{ML}(S)$, a very convenient tool is provided by train tracks (*see* [Th1], [PeH]).

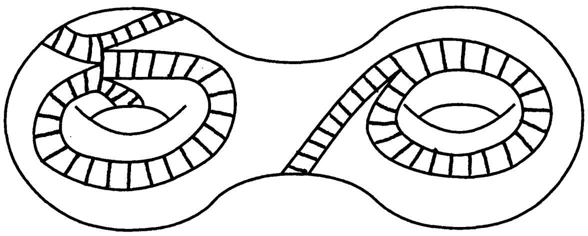

Fig. 1

A ***train track*** τ on the surface S consists of a finite family of 'long' rectangles e_i in S, each foliated by arcs parallel to the 'short' sides, and meeting as illustrated in Figure 1. Namely, two rectangles meet only along their short sides, and every point of the short side of a rectangle is contained in another short side of rectangle; note that this allows the two short sides of a same rectangle to meet along an arc. If a component of the boundary ∂S meets τ, then this whole component is contained in τ. In addition, we require a condition on the complement of τ: Observe that each component of $S - \tau$ has a certain number of spikes, corresponding to points belonging to three rectangles; we require that no component of $S - \tau$ is a disc with 0, 1 or 2 spikes or an annulus with no spike.

The rectangles e_i are the ***edges*** of the train track τ. The leaves of the foliation of τ induced by the foliation of the e_i by arcs parallel to the short sides are the ***ties*** of τ. The (finitely many) ties where several edges meet are the ***switches*** of τ. A tie which is not a switch is said to be ***generic***.

An m-geodesic lamination λ is said to be ***carried*** by the train track τ if it is contained in the interior of τ and if each leaf of λ is transverse to the ties of τ. Note that this depends on the negatively curved metric m.

When a measured geodesic lamination α is carried by a train track τ, the mass of the measure deposited by α on a tie of τ depends only on the edge e containing this tie, by invariance under homotopy respecting the geodesic lamination underlying α. Consequently, α associates a number $\alpha(e) \geqslant 0$ to each edge e of τ.

Given a train track τ and a negatively curved metric m, the set $\mathcal{ML}_m(\tau)$ consisting of those measured m-geodesic laminations which are carried by τ is a piecewise linear submanifold of $\mathcal{ML}(S)$. In addition, the map which associates the edge weights $\alpha(e)$ to the measured geodesic lamination $\alpha \in \mathcal{ML}(S)$ is piecewise linear. A remarkable fact is that it is also injective.

Thurston proved that, for every $\alpha \in \mathcal{ML}(S)$, there is a negatively curved metric m and a train track τ carrying α such that $\mathcal{ML}_m(\tau)$ is a neighborhood of α in $\mathcal{ML}(S)$. Thus, the interiors of such $\mathcal{ML}_m(\tau)$, parametrized by the corresponding edge weight maps, form a piecewise linear atlas for $\mathcal{ML}(S)$.

Since this paper is about tangent vectors, namely right derivatives of paths, we will often have to take the right derivative of a quantity a_t depending on a real parameter t, usually at

$t = 0$. To alleviate some otherwise cumbersome expressions, we will write $\dot{a}_0$ for the right derivative $\frac{\partial}{\partial t^+} a_{t|t=0}$ of a_t at 0. For instance, given a path $t \mapsto \alpha_t \in \mathcal{ML}(S) \subset \mathcal{C}(S)$, the right derivative of the edge weight $\alpha_t(e)$ is $\dot{\alpha}_0(e) = \frac{\partial}{\partial t^+}\alpha_t(e)_{|t=0}$ and the right derivative of the α_t-integral $\alpha_t(\varphi)$ of the function $\varphi : G(\widetilde{S}) \to \mathbb{R}$ for α_t is $\dot{\alpha}_0(\varphi) = \frac{\partial}{\partial t^+}\alpha_t(\varphi)_{|t=0}$.

2. The essential support of a family of measured geodesic laminations

Consider a 1-parameter family of measured geodesic laminations $\alpha_t \in \mathcal{ML}(S) \subset \mathcal{C}(S)$, depending continuously on $t \in [0, t_0]$. Let $\widetilde{\lambda}_t \subset G(\widetilde{S})$ denote the support of α_t, considered as a measure on $G(\widetilde{S})$. Define the ***essential support of*** α_t ***as*** t ***tends to*** 0^+ as the subset $\widetilde{\lambda}_{0+}$ of $G(\widetilde{S})$ defined as follows: A geodesic g is in $\widetilde{\lambda}_{0+}$ if and only if it admits arbitrary small neighborhoods U such that the infimum limit $\liminf_{t\to 0^+} \frac{1}{t}\alpha_t(U)$ is different from 0 (possibly infinite). In Lemma 3, we will relate this essential support to the tangent vector $\dot{\alpha}_0$ of $\mathcal{ML}(S)$ defined by the path $t \mapsto \alpha_t$, if it exists. We first investigate some elementary properties of this essential support.

Clearly, $\widetilde{\lambda}_{0+}$ is a closed subset of $G(\widetilde{S})$. Also, no two geodesics of $\widetilde{\lambda}_{0+}$ can cross each other since no two geodesics of the support of α_t cross each other. It follows that $\widetilde{\lambda}_{0+}$ defines a geodesic lamination of $\widetilde{S}$. By invariance under the action of $\pi_1(S)$ on $\widetilde{S}$, this geodesic lamination is the preimage of a geodesic lamination λ_{0+} of S.

Observe that the geodesic lamination λ_0 underlying α_0 is contained in λ_{0+}. Also, if t_n is a sequence converging to 0^+ such that λ_{t_n} has a limit λ'_{0+} for the Hausdorff topology on the space of closed subsets of S, this limit clearly contains λ_{0+}. The inclusions $\lambda_0 \subset \lambda_{0+} \subset \lambda'_{0+}$ can all be strict, for instance in the case where α_t consists of three fixed disjoint simple closed geodesics with Dirac transverse measures of respective weights 1, t and t^2.

The following elementary test will often be useful to search for geodesics of $\widetilde{\lambda}_{0+}$.

LEMMA 1 (Tracking Lemma). – *Let K be a compact subset of $G(\widetilde{S})$ such that $\liminf_{t\to 0^+} \alpha_t(K)/t > 0$. Then K contains at least one geodesic of $\widetilde{\lambda}_{0+}$.*

Proof. – By compactness, K can be written as the union of finitely many compact subsets of diameter less than $\frac{1}{2}$. For at least one of these, say K_1, we must have $\liminf_{t\to 0^+} \alpha_t(K_1)/t > 0$. Reapplying the same process to K_1, we can construct a sequence of nested compact subsets $K \supset K_1 \supset \ldots \supset K_n \supset \ldots$ such that the diameter of K_n is less than 2^{-n} and such that $\liminf_{t\to 0^+} \alpha_t(K_n)/t > 0$ for every n. The intersection of the K_n consists of a point $g \in G(\widetilde{S})$ which, by construction, must be in $\widetilde{\lambda}_{0+}$. □

Although easy to state, the above definition of $\widetilde{\lambda}_{0+}$ is not very easy to handle in practice. We want to give a more convenient description of $\widetilde{\lambda}_{0+}$ in terms of train tracks.

As indicated in §1, there is always a train track τ which carries the α_t for t sufficiently close to 0, for an appropriate choice of the negatively curved metric m. Consider the preimage $\widetilde{\tau}$ of τ in the universal covering $\widetilde{S}$. A compact oriented curve c carried by $\widetilde{\tau}$ traverses some oriented edges $e_1, e_2, \ldots, e_n$ of $\widetilde{\tau}$, in this order, where an orientation of an edge amounts to a coherent transverse orientation of its ties. Let an ***edge path*** be any ordered finite sequence $\gamma = \langle e_1, e_2, \ldots, e_n \rangle$ associated in this way to a compact oriented curve carried by $\widetilde{\tau}$. In this situation, we will say that the curve c ***follows*** the edge path γ. A curve c (possibly infinite) ***realizes*** the edge path γ if there is a compact subinterval c' of c which is carried by $\widetilde{\tau}$ and which follows γ. The ***length*** of the edge path $\gamma = \langle e_1, e_2, \ldots, e_n \rangle$ is the number n of edges in the sequence.

Given an edge path γ in $\widetilde{\tau}$ and a measured geodesic lamination α carried by τ, let $\alpha(\gamma)$ denote the α-mass of the subset G_γ of $G(\widetilde{S})$ consisting of those geodesics which realize γ.

LEMMA 2. – *For a fixed edge path γ of the train track $\widetilde{\tau}$, the number $\alpha(\gamma)$ is a piecewise linear function of the weights $\alpha(e)$ assigned to the edges e of τ by $\alpha \in \mathcal{ML}(S)$ carried by τ. In addition, the norm of the differential of this piecewise linear map is bounded by a constant times the length of γ.*

Proof. – We will argue by induction on the length n of the edge path $\gamma_n = \langle e_1, e_2, \ldots, e_n \rangle$, assuming that it starts at a fixed edge e_1.

Select a 'right' and a 'left' side for e_1. Then, this distinguishes a right and a left side for γ_n.

Consider those geodesics $g \in G(\widetilde{S})$ which are carried by $\widetilde{\tau}$ and which cross the edge e_n. For such a g, either g realizes γ_n, or it realizes some edge path $\langle e'_i, e_{i+1}, \ldots, e_n \rangle$ where $1 \leqslant i < n$ and where $e'_i \neq e_i$. Let $G_n^{\mathrm{l}} \subset G(\widetilde{S})$ consist of those geodesics which realize such an edge path $\langle e'_i, e_{i+1}, \ldots, e_n \rangle$ where e'_i branches in on the left of γ_n. Similarly, let G_n^{r} consist of those geodesics which branch in on the right of γ_n.

We will prove by induction on n that $\alpha(\gamma_n)$, $\alpha(G_n^{\mathrm{l}})$ and $\alpha(G_n^{\mathrm{r}})$ are piecewise linear functions of the weights $\alpha(e)$, and that the norms of their differentials are bounded by a constant times n.

The property is trivially true when $n = 1$. Assume as induction hypothesis that it holds for $n - 1$. We want to prove it for n.

Consider the switch s where e_{n-1} meets e_n. Let a_-^{l} (resp. a_-^{r}, a_+^{l}, a_+^{r}) denote the sum of the weights of the edges entering s on the same side as e_{n-1} (resp. e_{n-1}, e_n, e_n) and on the left (resp. right, left, right) side of $\langle e_1, e_2, \ldots, e_n \rangle$. Then, analyzing what can happen to the geodesics which are carried by $\widetilde{\tau}$ and pass through the switch s, we find that

$$\begin{aligned}
\alpha(G_n^{\mathrm{l}}) &= \min\{\alpha(e_n), \max\{\alpha(G_{n-1}^{\mathrm{l}}) + a_-^{\mathrm{l}} - a_+^{\mathrm{l}}, 0\}\} \\
\alpha(G_n^{\mathrm{r}}) &= \min\{\alpha(e_n), \max\{\alpha(G_{n-1}^{\mathrm{r}}) + a_-^{\mathrm{r}} - a_+^{\mathrm{r}}, 0\}\} \\
\alpha(\gamma_n) &= \alpha(e_n) - \alpha(G_n^{\mathrm{l}}) - \alpha(G_n^{\mathrm{r}}),
\end{aligned}$$

which clearly concludes the proof by induction. □

Assume that the path $t \mapsto \alpha_t$ in $\mathcal{ML}(S)$ has a tangent vector at α_0 for the piecewise linear structure of $\mathcal{ML}(S)$, namely that the derivative $\frac{\partial}{\partial t^+}\alpha_t(e)_{|t=0}$ exists for every edge e of τ. Then, Lemma 2 shows that there is a well defined right derivative $\dot{\alpha}_0(\gamma)$ for every edge path γ of $\widetilde{\tau}$. Note that it is always possible to enlarge τ a little bit so that the geodesic laminations λ_t underlying the α_t are contained in a compact subset of the interior of τ.

LEMMA 3. – *Consider a path $t \mapsto \alpha_t \in \mathcal{ML}(S)$ which has a tangent vector at α_0, and let $\widetilde{\lambda}_{0^+}$ be its essential support as t tends to 0^+. Assume that the α_t are carried by a train track τ, of preimage $\widetilde{\tau}$ in $\widetilde{S}$, and that their underlying geodesic laminations are contained* *in a compact subset of the interior of* τ. *Then, the geodesic* $g \in G(\widetilde{S})$ *is in the essential support* $\widetilde{\lambda}_{0+}$ *if and only if it is carried by* $\widetilde{\tau}$ *and if* $\alpha_0(\gamma) > 0$ *or* $\dot{\alpha}_0(\gamma) > 0$ *for every edge path* γ *of* $\widetilde{\tau}$ *that is realized by* g. *In particular,* $\widetilde{\lambda}_{0+}$ *depends only on* α_0 *and on the tangent vector* $\dot{\alpha}_0$, *and not on the specific family* α_t *tangent to* $\dot{\alpha}_0$ *at* α_0.

Proof. – The hypothesis that the geodesic laminations underlying the α_t are contained in a compact subset of the interior of τ guarantees that $\widetilde{\lambda}_{0+}$ is carried by $\widetilde{\tau}$.

As before, let G_γ denote the open subset of $G(\widetilde{S})$ consisting of those geodesics which realize the edge path γ. Because of the negative curvature, two geodesics of $\widetilde{S}$ which stay at bounded distance from each other for a long time are actually close to each other. It follows that, if $g \in G(\widetilde{S})$ is carried by $\widetilde{\tau}$, the G_γ where γ ranges over all edge paths realized by g form a basis of neighborhoods of g. As a consequence, a geodesic g carried by $\widetilde{\tau}$ is in $\widetilde{\lambda}_{0+}$ if and only if $\liminf_{t\to 0^+} \frac{1}{t}\alpha_t(\gamma) > 0$ for every edge path γ realized by g. Since $\dot{\alpha}_0(\gamma)$ exists, the condition that $\liminf_{t\to 0^+} \frac{1}{t}\alpha_t(\gamma) > 0$ is equivalent to the property that $\alpha_0(\gamma) > 0$ or $\dot{\alpha}_0(\gamma) > 0$, which concludes the proof. □

PROPOSITION 4. – *If the path* $t \mapsto \alpha_t \in \mathcal{ML}(S)$ *is piecewise linear, the essential support* λ_{0+} *is equal to the Hausdorff limit as* t *tends to* 0^+ *of the geodesic laminations* λ_t *underlying the* α_t.

Proof. – Let τ be a train track such that the λ_t are carried by τ and contained in a compact subset of the interior of τ. It clearly suffices to prove that $\widetilde{\lambda}_{0+} \subset G(\widetilde{S})$ is the Hausdorff limit of the supports $\widetilde{\lambda}_t$ of the α_t. Actually, it even suffices to prove that, for every generic tie k_0 of the preimage $\widetilde{\tau} \subset \widetilde{S}$, the set of those geodesics of $\widetilde{\lambda}_t$ crossing k_0 converges to the set of those geodesics of $\widetilde{\lambda}_{0+}$ crossing k_0.

For $r \geqslant 0$, let Γ_r denote the set of edge paths $\gamma = \langle e_{-r}, e_{-r+1}, \dots, e_{r-1}, e_r \rangle$ in $\widetilde{\tau}$ such that e_0 is the edge containing k_0. If $G(\widetilde{S})$ is endowed with a metric compatible with its Hölder structure, an easy geometric estimate provides two constants $A > 0$, $B > 1$, depending only on k_0 and on the lengths and widths of the edges of τ, such that the diameter of the set G_γ of geodesics realizing $\gamma \in \Gamma_r$ is bounded by AB^{-r}; indeed, it suffices to check this when $G(\widetilde{S}) = \left(\widetilde{S}_\infty \times \widetilde{S}_\infty - \Delta\right)/\mathbb{Z}_2$ is endowed with the product metric coming from the angle metric on $\widetilde{S}_\infty$ based at some point of k_0, in which case it follows from the negativity of the curvature of the metric of S.

By Lemma 3, $\widetilde{\lambda}_{0+}$ consists of those geodesics g carried by $\widetilde{\tau}$ such that $\alpha_0(\gamma) > 0$ or $\dot{\alpha}_0(\gamma) > 0$ for every edge path γ realized by g. By Lemma 2, the map $t \mapsto \alpha_t(\gamma)$ is piecewise linear. It follows that the above condition is equivalent to the condition that $\alpha_t(\gamma) > 0$ for every t sufficiently close to 0.

Since Γ_r is finite, this proves that, for t sufficiently close to 0, the $\gamma \in \Gamma_r$ for which G_γ meets $\widetilde{\lambda}_{0+}$ are exactly those for which G_γ meets $\widetilde{\lambda}_t$. As a consequence, the Hausdorff distance between the set of geodesics of $\widetilde{\lambda}_{0+}$ crossing k_0 and the set of geodesics of $\widetilde{\lambda}_t$ crossing k_0 is at most AB^{-r} for t sufficiently close to 0.

Letting r tend to ∞ then proves the result we wanted. □

3. Computing derivatives

Consider a 1-parameter family α_t, $t \in [0, t_0]$, of measured geodesic laminations carried by the train track τ. We assume that this path in $\mathcal{ML}(S)$ has a tangent vector at α_0 for the piecewise linear structure of $\mathcal{ML}(S)$, namely that the derivative $\frac{\partial}{\partial t^+}\alpha_t(e)_{|t=0}$ exists for every edge e of τ. Given a function $\varphi : G(\widetilde{S}) \to \mathbb{R}$, we want to evaluate the right derivative $\dot{\alpha}_0(\varphi) = \frac{\partial}{\partial t^+}\alpha_t(\varphi)_{|t=0}$.

Since all geodesics of the support of the α_t are carried by $\widetilde{\tau}$, we can restrict attention, using a suitable partition of unity, to the case where there is a generic tie k_0 of $\widetilde{\tau}$ which transversely meets every geodesic of the support of φ. Arbitrarily choose an orientation for k_0.

Enlarging τ a little bit if necessary, we can assume that the essential support $\widetilde{\lambda}_{0+}$ is carried by $\widetilde{\tau}$.

Let d be a component of $k_0 - \widetilde{\lambda}_{0+}$, and choose a base point $x_d \in d$. For every t, let $h_t(x_d)$ denote the 'α_t-height' of x_d, namely the α_t-mass of the component of $k_0 - x_d$ consisting of those points of k_0 which are below x_d for the orientation of k_0.

Clearly, $h_t(x_d)$ may depend on the choice of the base point x_d. However:

LEMMA 5. – *The right derivative $\dot{h}_0(x_d)$ exists and is independent of the choice of the base point x_d in the component d of $k_0 - \widetilde{\lambda}_{0+}$.*

Proof. – First consider the case where d is not one of the two components of $k_0 - \widetilde{\lambda}_{0+}$ that are adjacent to the ends of k_0. Then, there are two geodesics g_d^+ and g_d^- of $\widetilde{\lambda}_{0+}$ which pass through the end points of d, where $k_0 \cap g_d^-$ is below $k_0 \cap g_d^+$ for the orientation of k_0.

Since g_d^- and g_d^+ are distinct, they cannot realize the same bi-infinite edge path in $\widetilde{\tau}$. Therefore, g_d^+ and g_d^- respectively realize some edge paths $\langle e_0, e_1, \ldots, e_r, e_{r+1}\rangle$ and $\langle e_0, e_1, \ldots, e_r, e'_{r+1}\rangle$ with $e_{r+1} \neq e'_{r+1}$, where e_0 is the edge of $\widetilde{\tau}$ containing the tie k_0.

Let us consider what can happen to the geodesics of the support $\widetilde{\lambda}_t$ of α_t which hit k_0 below x_d, for the orientation of k_0.

First of all, observe that, for t sufficiently close to 0, there is a geodesic $g_t^+ \in \widetilde{\lambda}_t$ which is close to g_d^+; in particular, g_t^+ realizes $\langle e_0, e_1, \ldots, e_r, e_{r+1}\rangle$ and hits k_0 above x_d. Any geodesic $g \in \widetilde{\lambda}_t$ which hits k_0 below x_d must be disjoint from g_t^+. As a consequence, g must realize either $\langle e_0, e_1, \ldots, e_r, e_{r+1}\rangle$, or an edge path $\langle e_0, e_1, \ldots, e_i, f\rangle$ with $0 \leqslant i \leqslant r$ where f is different from e_{i+1} and branches out on the negative side of $\langle e_0, e_1, \ldots, e_r, e_{r+1}\rangle$ for the transverse orientation of this edge path determined by the orientation of k_0. Let f_1, f_2, $\ldots$, f_p be the collection of these edges f which branch out on the negative side of $\langle e_0, e_1, \ldots, e_r, e_{r+1}\rangle$, including $f_1 = e'_{r+1}$.

Conversely, note that, for t sufficiently small, there is also a geodesic $g_t^- \in \widetilde{\lambda}_t$ which is close to g_d^-, and in particular which realizes $\langle e_0, e_1, \ldots, e_r, e'_{r+1}\rangle$ and which hits k_0 below x_d. Therefore, for every geodesic $g' \in \widetilde{\lambda}_t$ which realizes $\langle f_j\rangle$ with $1 \leqslant j \leqslant p$, the fact that this geodesic g' is disjoint from g_t^- implies that it must, either hit k_0 below x_d, or realize an edge path $\langle f, e_i, \ldots, e_{k-1}, e_k, f_j\rangle$ where f is different from e_{i-1} and branches in on the negative side of $\langle e_0, e_1, \ldots, e_r, e_{r+1}\rangle$. Let f_{p+1}, f_{p+2}, $\ldots$, f_q be the collection of these edges f which branch in on the negative side of $\langle e_0, e_1, \ldots, e_r, e_{r+1}\rangle$.

In addition, the existence of the geodesic $g_t^- \in \widetilde{\lambda}_t$ shows that, for t sufficiently small, every geodesic $g'' \in \widetilde{\lambda}_t$ which realizes $\langle f_k \rangle$ with $p+1 \leqslant k \leqslant q$ must exit through one of the f_j with $1 \leqslant j \leqslant p$.

As a consequence,

$$h_t(x_d) = \alpha_t(X) + \sum_{j=1}^{p} \alpha_t(f_j) - \sum_{k=p+1}^{q} \alpha_t(f_k) - \alpha_t(X')$$

where X denotes the set of those oriented geodesics of $G(\widetilde{\tau})$ which realize $\langle e_0, e_1, \dots, e_r, e_{r+1} \rangle$ and which hit k_0 below x_d, and where X' denotes the set of those oriented geodesics of $G(\widetilde{\tau})$ which realize $\langle e_0, e_1, \dots, e_r, e'_{r+1} \rangle$ and which hit k_0 at or above x_d.

We claim that $\frac{1}{t}\alpha_t(X)$ tends to 0 as t tends to 0^+. Indeed, the Tracking Lemma 1 would otherwise provide a geodesic of g in the closure of X which is in $\widetilde{\lambda}_{0+}$. This $g \in \widetilde{\lambda}_{0+}$, hitting k_0 at or below x_d, must actually hit it at or below $k_0 \cap g_d^-$ by definition of g_d^-. Therefore, it cannot be on the same side of g_d^- as g_d^+. But this contradicts the fact that g realizes the edge path $\langle e_0, e_1, \dots, e_r, e_{r+1} \rangle$.

A similar argument shows that $\frac{1}{t}\alpha_t(X')$ tends to 0 as t tends to 0^+.

Passing to the derivatives, we conclude that $\dot{h}_0(x_d) = \sum_{j=1}^{p} \dot{\alpha}_0(f_j) - \sum_{k=p+1}^{q} \dot{\alpha}_0(f_k)$. Since this expression depends only on d, and not on the choice of the base point $x_d \in d$, this concludes the proof of Lemma 5 in the case which we were considering, namely when d is not one of the two components of $k_0 - \widetilde{\lambda}_{0+}$ that are adjacent to the ends of k_0.

The proof is very similar for the remaining two cases, and gives that $\dot{h}_0(d) = 0$ if d is adjacent to the negative end of k_0, and $\dot{h}_0(d) = \dot{\alpha}_0(e_0)$ if d is adjacent to the positive end of k_0. □

In view of Lemma 5, we will henceforth write $\dot{h}_0(d)$ for $\dot{h}_0(x_d)$.

Note that we proved a little more:

LEMMA 6. – *With the data of Lemma 5 and of its proof, if the component d of $k_0 - \widetilde{\lambda}_0^+$ is not adjacent to one of the two ends of k_0, then*

$$\dot{h}_0(d) = \sum_f \varepsilon(f)\dot{\alpha}_0(f)$$

where the sum is over all edges f that branch in or out on the negative side of the edge path $\langle e_0, e_1, \dots, e_r, e_{r+1} \rangle$, and where $\varepsilon(f) = -1$ or $\varepsilon(f) = +1$ according to whether the edge f branches in or out. If d is adjacent to one of the ends of k_0, then $\dot{h}_0(d)$ is equal to $\dot{\alpha}_0(e_0)$ for the positive end, and to 0 for the negative end. □

We can now state our main theorem. At this point, it may be useful to summarize what we have defined so far. Given the family α_t of measured geodesic laminations carried by τ, we consider the essential limit $\widetilde{\lambda}_{0+}$ of their supports as t tends to 0^+. If k_0 is an oriented tie of $\widetilde{\tau}$ and if d is a component of $k_0 - \widetilde{\lambda}_{0+}$, g_d^+ and g_d^- are the geodesics of $\widetilde{\lambda}_{0+}$ passing through the end points of d, where $k_0 \cap g_d^-$ is below $k_0 \cap g_d^+$ for the orientation of k_0. Only one of g_d^+ and g_d^- is defined when d is adjacent to an end of k_0, and we arbitrarily decide that, for a function $\varphi : G(\widetilde{S}) \to \mathbb{R}$, $\varphi(g_d^{\pm}) = 0$ when $g_d^{\pm}$ is not defined. Finally, we also associated to d a number $\dot{h}_0(d)$ which is defined by Lemmas 5 and 6.

THEOREM 7. – *Consider a 1-parameter family of measured geodesic laminations α_t, $t \in [0, t_0]$, carried by the train track τ such that, for every edge e of τ, the weight $\alpha_t(e)$ has a right derivative at $t = 0$. Assume that τ also carries the essential support $\widetilde{\lambda}_{0^+}$ of α_t as t tends to 0^+. Then, if $\varphi : G(\widetilde{S}) \to \mathbb{R}$ is a Hölder continuous function such that every geodesic of the support of φ transversely meets the generic tie k_0 of $\widetilde{\tau}$,*

$$\dot{\alpha}_0(\varphi) = \sum_d \dot{h}_0(d)\big(\varphi(g_d^-) - \varphi(g_d^+)\big)$$

where d ranges over all components of $k_0 - \widetilde{\lambda}_{0^+}$, and where $\dot{h}_0(d)$, $\varphi(g_d^+)$, $\varphi(g_d^-)$ are defined as above.

It is not too difficult to obtain the formula of Theorem 7 by formal computations, using for instance Corollary 17 (compare [Bo4, §4]). The justification of these formal computations however requires much more care. Our proof will be based on an analysis of the interplay between components of $k_0 - \widetilde{\lambda}_{0^+}$ and edge paths realized by leaves of $\widetilde{\lambda}_{0^+}$.

First, let us show that this formula makes sense.

For the tangent vector $\dot{\alpha}_0$, let $\|\dot{\alpha}_0\|_\tau$ denote the maximum of the absolute values of the weights $\dot{\alpha}_0(e)$, where e ranges over all edges of τ. Similarly, if the measured geodesic laminations α and β are carried by τ, let $\|\alpha - \beta\|_\tau$ denote the maximum of the $|\alpha(e) - \beta(e)|$.

PROPOSITION 8. – *Under the hypotheses of Theorem 7, the series*

$$D_{0^+}(\varphi) = \sum_d \dot{h}_0(d)\big(\varphi(g_d^-) - \varphi(g_d^+)\big)$$

is (absolutely) convergent. In addition, the linear functional $D_{0^+}(\varphi)$ is a continuous function of φ and $\dot{\alpha}_0$ in the sense that, if φ has Hölder exponent ν, there is a constant C independent of φ and $\dot{\alpha}_0$ such that $|D_{0^+}(\varphi)| \leqslant C\|\dot{\alpha}_0\|_\tau \|\varphi\|_\nu$.

Proof. – First consider the two exceptional components of $k_0 - \widetilde{\lambda}_{0^+}$ that are adjacent to the end points of k_0. The one which is adjacent to the negative end point of k_0 does not contribute anything to the above sum, since the corresponding coefficient $\dot{h}_0(d)$ is equal to 0. For the one adjacent to the positive end point, $\dot{h}_0(d) = \dot{\alpha}_0(e_0)$ where e_0 is the edge containing k_0, and its contribution is therefore bounded by $\|\dot{\alpha}_0\|_\tau \|\varphi\|_\nu$.

For our analysis, we can therefore restrict the above series to those components d of $k_0 - \widetilde{\lambda}_{0^+}$ which are not adjacent to an end of k_0. For such a d, let $r(d)$ be the maximal $r \geqslant 0$ for which g_d^+ and g_d^- both realize a common edge path $\gamma = (e_{-r}, e_{-r+1}, \ldots, e_{r-1}, e_r)$ in $\widetilde{\tau}$ with k_0 in e_0. As in the proof of Proposition 4, a geometric estimate shows that $d(g_d^+, g_d^-) \leqslant AB^{-r(d)}$ for some constants $A > 0$, $B > 1$, and therefore that $|\varphi(g_d^-) - \varphi(g_d^+)| \leqslant A\|\varphi\|_\nu B^{-\nu r(d)}$.

Also, it follows from Lemma 6 that $|\dot{h}_0(d)|$ is bounded by a constant times $(r(d) + 1)\|\dot{\alpha}_0\|_\tau$. Therefore, the series $\sum_d |\dot{h}_0(d)||\varphi(g_d^-) - \varphi(g_d^+)|$ is bounded by a constant times $\|\dot\alpha_0\|_\tau \|\varphi\|_\nu \sum_d (r(d)+1)B^{-\nu r(d)}$. We therefore only have to prove that the series $\sum_d r(d)B^{-\nu r(d)}$ is convergent.

Identify k_0 to the tie of τ it projects to, and let λ_{0+} be the geodesic lamination of S obtained by projecting $\widetilde{\lambda}_{0+}$. We can then identify components of $k_0 - \widetilde{\lambda}_{0+}$ in $\widetilde{S}$ and components of $k_0 - \lambda_{0+}$ in S.

If d is a component of $k_0 - \lambda_{0+}$ which is not adjacent to an end of k_0, the two leaves of λ_{0+} passing through the end points of d realize a common edge path $\gamma = \langle e_0, \dots, e_{r(d)} \rangle$ in τ with k_0 in e_0, and diverge at the end switch s of the edge $e_{r(d)}$. Observe that each of these two leaves of λ_{0+} is closest in s to the end point of a spike of $S - \tau$, namely can be connected to this end point by an arc contained in s and with interior disjoint from λ_{0+}. In particular, these two leaves of λ_{0+} are uniquely determined by the corresponding pair of spikes ending at s (possibly equal, and actually equal if λ_{0+} crosses every edge of τ). We can then go backwards, starting from s and following these two leaves so as to retrace the edge path γ backwards, until we eventually hit the tie k_0 at d after crossing $r(d)$ edges of τ. Consequently, d is uniquely determined by s, the two spikes ending at s, and $r(d)$.

From this observation, we conclude that the series $\sum_d (r(d)+1)B^{-\nu r(d)}$ is bounded by the sum of finitely many series $\sum_{r=1}^\infty (r+1)B^{-\nu r}$, one for each pair of (possibly equal) spikes of $S - \tau$ ending on the same side of a switch s. Since these series are convergent, this proves the convergence of $\sum_d (r(d)+1)B^{-\nu r(d)}$, and concludes the proof of Proposition 8. □

Note that the proof of Proposition 8 says something on the convergence of the series $D_{0^+}(\varphi)$. Namely, if $r(d)$ is defined as in that proof when the component d of $k_0 - \widetilde{\lambda}_{0+}$ is not adjacent to an end of k_0, and if $r(d) = 0$ by convention when d is adjacent to such an end:

COMPLEMENT 9. – *With the data of Proposition 8 and of its proof,*

$$D_{0^+}(\varphi) = \sum_{r(d)<r} \dot h_0(d)\big(\varphi(g_d^-) - \varphi(g_d^+)\big) + \|\dot\alpha_0\|_\tau \|\varphi\|_\nu O\big(rB^{-\nu r}\big)$$

for every $r > 0$. □

Here, we used the classical notation where $O(X)$ represents any quantity for which the absolute value of $O(X)/X$ is bounded.

Given $r \geqslant 1$, let Γ_r denote the set of edge paths $\gamma = \langle e_{-r}, e_{-r+1}, \dots, e_{r-1}, e_r \rangle$ of $\widetilde{\tau}$ which are realized by some geodesic g_γ, where e_0 is the edge containing k_0 and where we identify two edge paths when they differ by reversal of orientation. For every $\gamma \in \Gamma_r$, pick a geodesic g_γ realizing γ.

By the geometric estimate which we already used in the proof of Propositions 4 and 8, there are constants $A > 0$, $B > 1$ such that, for every two geodesics g and h which realize the same $\gamma \in \Gamma_r$, the distance from g to h is bounded by AB^{-r}. Since the G_γ with $\gamma \in \Gamma_r$ are pairwise disjoint, and have diameter at most AB^{-r} by the above estimate, we can use them to approximate by Riemann sums the integral of φ for the signed measure $(\alpha_t - \alpha_0)/t$. This gives

$$\frac{\alpha_t(\varphi) - \alpha_0(\varphi)}{t} = \lim_{r\to\infty} \sum_{\gamma\in\Gamma_r} \frac{\alpha_t(\gamma) - \alpha_0(\gamma)}{t}\varphi(g_\gamma)$$

for every $t > 0$.

We want to estimate the speed of convergence of this limit. For this, we first show that the number of terms contributing to the above sum has polynomial growth in r.

LEMMA 10. – *If the measured geodesic lamination α is carried by the train track τ, the number of edge paths $\gamma \in \Gamma_r$ for which $\alpha(\gamma) > 0$ is an $O(r^{n+2})$, where n is the number of edges of τ.*

Proof. – The main point is that such a γ is followed by an arc carried by $\widetilde{\tau}$ whose projection to τ is a simple curve. Up to isotopy, a simple arc carried by τ is uniquely determined by the number of times it crosses each edge of τ together with some information about where its end points are. In particular, such an arc of length r is determined by the number of times it crosses each edge up to an ambiguity of at most $4r^2$ times the square of the number of switches of τ (for each arc end, we have to specify on which switch it sits, in which direction it arrives, and where it sits with respect to the other pieces of the arc that cross that switch). It follows that the number of simple arcs carried by τ is an $O(r^{n+2})$. Since an arc carried by τ admits at most one lift to $\widetilde{S}$ for which its central edge is e_0, this completes the proof. □

The exponent for the growth rate given by Lemma 10 is far from optimal, but we will only need to know that this growth is polynomial.

For $\gamma \in \Gamma_r$, let $\sigma(\gamma) \in \Gamma_{r-1}$ be the edge path obtained from γ by removing its two end edges. Note that, if α is a measured geodesic lamination carried by τ, then $\alpha(\gamma') = \sum_{\sigma(\gamma)=\gamma'} \alpha(\gamma)$ for every $\gamma' \in \Gamma_{r-1}$. Also, since φ is Hölder continuous of exponent ν, $\varphi(g_\gamma) - \varphi(g_{\sigma(\gamma)}) = \|\varphi\|_\nu O(B^{-\nu r})$. In addition, by Lemma 2, $\alpha_t(\gamma) - \alpha_0(\gamma) = \|\alpha_t - \alpha_0\|_\tau O(r)$ if $\gamma \in \Gamma_r$. Combining these facts and regrouping terms, it follows that the difference

$$\sum_{\gamma\in\Gamma_r} \frac{\alpha_t(\gamma) - \alpha_0(\gamma)}{t}\varphi(g_\gamma) - \sum_{\gamma'\in\Gamma_{r-1}} \frac{\alpha_t(\gamma') - \alpha_0(\gamma')}{t}\varphi(g_{\gamma'})$$

is equal to

$$\sum_{\gamma\in\Gamma_r} \frac{\alpha_t(\gamma) - \alpha_0(\gamma)}{t}\big(\varphi(g_\gamma) - \varphi(g_{\sigma(\gamma)})\big) = \frac{\|\alpha_t - \alpha_0\|_\tau}{t}\|\varphi\|_\nu O\big(r^{n+3}B^{-\nu r}\big).$$

Summing these differences from $r+1$ to infinity, we conclude that

$$\begin{aligned}\frac{\alpha_t(\varphi) - \alpha_0(\varphi)}{t} &= \lim_{r\to\infty} \sum_{\gamma\in\Gamma_r} \frac{\alpha_t(\gamma) - \alpha_0(\gamma)}{t}\varphi(g_\gamma)\\ &= \sum_{\gamma\in\Gamma_r} \frac{\alpha_t(\gamma) - \alpha_0(\gamma)}{t}\varphi(g_\gamma) + \frac{\|\alpha_t - \alpha_0\|_\tau}{t}\|\varphi\|_\nu O\big(r^{n+3}B^{-\nu r}\big).\end{aligned}$$

We can then let t tend to 0^+. By finiteness of Γ_r, we conclude:

LEMMA 11. – *For every $r \geqslant 1$, the supremum limit and the infimum limit of $(\alpha_t(\varphi) - \alpha_0(\varphi))/t$ as t tends to 0^+ are both of the form*

$$\sum_{\gamma\in\Gamma_r} \dot\alpha_0(\gamma)\varphi(g_\gamma) + \|\dot\alpha_0\|_\tau\|\varphi\|_\nu O\big(r^{n+3}B^{-\nu r}\big).$$ □

Let us connect this estimate to the series of Proposition 8.

LEMMA 12.

$$\sum_{\gamma\in\Gamma_r} \dot\alpha_0(\gamma)\varphi(g_\gamma) = \sum_{r(d)<r} \dot h_0(d)\big(\varphi(g_d^-) - \varphi(g_d^+)\big) + \|\dot\alpha_0\|_r \|\varphi\|_\nu O\big(r^3 B^{-\nu r}\big).$$

Proof. – If $\dot\alpha_0(\gamma) \neq 0$ note that, by the Tracking Lemma 1, γ is realized by a geodesic of $\widetilde\lambda_{0+}$. Therefore, if Γ_r' consists of those $\gamma \in \Gamma_r$ which are realized by geodesics of $\widetilde\lambda_{0+}$,

$$\sum_{\gamma\in\Gamma_r} \dot\alpha_0(\gamma)\varphi(g_\gamma) = \sum_{\gamma\in\Gamma_r'} \dot\alpha_0(\gamma)\varphi(g_\gamma).$$

For every $\gamma \in \Gamma_r'$, the intersection points of k_0 with the geodesics of $\widetilde\lambda_{0+}$ realizing γ form a closed interval in $k_0 \cap \widetilde\lambda_{0+}$, for the ordering of $k_0 \cap \widetilde\lambda_{0+}$ induced by the orientation of k_0. It follows that there are two components d_γ^+ and d_γ^- of $k_0 - \widetilde\lambda_{0+}$ such that the geodesics of $\widetilde\lambda_{0+}$ realizing γ are exactly those which hit k_0 above d_γ^- and below d_γ^+.

Note that, for $d = d_\gamma^+$, the geodesic g_d^- realizes γ and g_d^+ does not; it follows that $r(d_\gamma^+) < r$. Similarly, $r(d_\gamma^-) < r$. Conversely, every component d of $k_0 - \widetilde\lambda_{0+}$ with $r(d) < r$ is equal to $d = d^-_{\gamma_d^+} = d^+_{\gamma_d^-}$ where γ_d^+ and γ_d^- are the edge paths in Γ_r respectively realized by the geodesics g_d^+ and g_d^- (if defined).

Choose base points $x_\gamma^+ \in d_\gamma^+$ and $x_\gamma^- \in d_\gamma^-$. Then, $h_t(x_\gamma^+) - h_t(x_\gamma^-)$ is the α_t-mass of those geodesics which hit k_0 below x_γ^+ and at or above x_γ^-. For every $\gamma' \in \Gamma_r$ different from γ, the Tracking Lemma 1 shows that the contribution to $h_t(x_\gamma^+) - h_t(x_\gamma^-)$ of those geodesics which realize γ' has right derivative 0 at 0; otherwise, there would be a geodesic of $\widetilde\lambda_{0+}$ which realizes γ' and hits k_0 between x_γ^+ and x_γ^-. For the same reason, the α_t-mass of those geodesics which realize γ and do not contribute to $h_t(x_\gamma^+) - h_t(x_\gamma^-)$ has right derivative 0 at 0, since no geodesic of $\widetilde\lambda_{0+}$ realizing γ hits k_0 above x_γ^+ or below x_γ^-. It follows that

$$\dot h_0(d_\gamma^+) - \dot h_0(d_\gamma^-) = \dot h_0(x_\gamma^+) - \dot h_0(x_\gamma^-) = \dot\alpha_0(\gamma).$$

As a consequence,

$$\begin{aligned}
\sum_{\gamma\in\Gamma_r} \dot\alpha_0(\gamma)\varphi(g_\gamma) &= \sum_{\gamma\in\Gamma_r} \dot h_0(d_\gamma^+)\varphi(g_\gamma) - \sum_{\gamma\in\Gamma_r} \dot h_0(d_\gamma^-)\varphi(g_\gamma)\\
&= \sum_{r(d)<r} \dot h_0(d)\varphi(g_{\gamma_d^-}) - \sum_{r(d)<r} \dot h_0(d)\varphi(g_{\gamma_d^+})\\
&= \sum_{r(d)<r} \dot h_0(d)\big(\varphi(g_{\gamma_d^-}) - \varphi(g_{\gamma_d^+})\big)\\
&= \sum_{r(d)<r} \dot h_0(d)\big(\varphi(g_d^-) - \varphi(g_d^+)\big) + \|\varphi\|_\nu O\Big(B^{-\nu r}\sum_{r(d)<r}\big|\dot h_0(d)\big|\Big)
\end{aligned}$$

since g_d^- realizes the same path $\gamma_d^- \in \Gamma_r$ as $g_{\gamma_d^-}$ and is therefore at distance at most $AB^{-\nu r}$ from $g_{\gamma_d^-}$, and since the same property holds for g_d^+ and $g_{\gamma_d^+}$. (And using the convention that $\varphi(g) = 0$ when the geodesic g is not defined.)

To complete the proof of Lemma 12, it now suffices to show that $\sum_{r(d)<r}|\dot{h}_0(d)| = \|\dot{\alpha}_0\|_\tau O(r^3)$. In the proof of Proposition 8, we showed that the number of components d with $r(d)=r$ is an $O(r)$, and therefore that the number of d with $r(d)<r$ is an $O(r^2)$. Also, the formula for $\dot{h}_0(d)$ given by Lemma 6 shows that $\dot{h}_0(d) = \|\dot{\alpha}_0\|_\tau O(r(d))$. It follows that $\sum_{r(d)<r}|\dot{h}_0(d)| = \|\dot{\alpha}_0\|_\tau O(r^3)$. This completes the proof of Lemma 12. □

Combining Lemmas 11, 12 and Complement 9, we conclude that the supremum limit and the infimum limit of $(\alpha_t(\varphi)-\alpha_0(\varphi))/t$ as t tends to 0^+ are both of the form

$$\sum_d \dot{h}_0(d)\big(\varphi(g_d^-)-\varphi(g_d^+)\big) + \|\dot{\alpha}_0\|_\tau \|\varphi\|_\nu O\big(r^{n+3}B^{-\nu r}\big).$$

Since this is true for every $r \geqslant 1$, it follows by letting r tend to ∞ that these supremum limit and infimum limit are both equal to the sum of the above series, namely that

$$\dot{\alpha}_0(\varphi) = \lim_{t\to 0^+} \frac{\alpha_t(\varphi)-\alpha_0(\varphi)}{t} = \sum_d \dot{h}_0(d)\big(\varphi(g_d^-)-\varphi(g_d^+)\big),$$

which concludes the proof of Theorem 7. □

From Theorem 7, we can draw a more global conclusion.

COROLLARY 13. – *Consider a 1-parameter family of measured geodesic laminations α_t, $t\in[0,t_0]$, which admits a tangent vector $\dot{\alpha}_0$ at $t=0$ for the piecewise linear structure of $\mathcal{ML}(S)$. Then, for every compact subset $K\subset G(\widetilde{S})$ and every $\nu>0$, there is a constant $C>0$ such that, for every Hölder continuous function $\varphi: G(\widetilde{S})\to\mathbb{R}$ of Hölder exponent ν whose support is contained in K, the derivative $\dot{\alpha}_0(\varphi)$ exists and is bounded by $C\|\varphi\|_\nu$. In addition, $\dot{\alpha}_0(\varphi)$ depends only on φ and on the combinatorial tangent vector $\dot{\alpha}_0$, and not on the particular family α_t tangent to $\dot{\alpha}_0$.*

Proof. – For t sufficiently small, the α_t are all carried by a certain train track τ. Enlarging τ a little, we can assume that λ_{0^+} and the supports λ_t of α_t are contained in a compact subset N of the interior of τ. Then, their essential support λ_{0^+} as t tends to 0^+ is also carried by τ. Let $\widetilde{\tau}$ and $\widetilde{N}$ be the preimages of τ and N in $\widetilde{S}$, and let $G(\widetilde{N})$ consist of those geodesics $g\in G(\widetilde{S})$ which are contained in $\widetilde{N}$.

By compactness of $K\cap G(\widetilde{N})$, it is possible to cover this subset by finitely many open subsets $U_1,\dots,U_n$ of $G(\widetilde{S})$ such that every geodesic of U_i crosses some generic tie k_i of $\widetilde{\tau}$. Choose a partition of unity by Hölder continuous functions $\xi_i: G(\widetilde{S})\to\mathbb{R}$ with support contained in U_i such that the sum $\sum_{i=1}^n \xi_i$ is equal to 1 on $K\cap G(\widetilde{N})$.

Consider a Hölder continuous function $\varphi: G(\widetilde{S})\to\mathbb{R}$ with Hölder exponent ν and support contained in K. Then, for t sufficiently close to 0, $\alpha_t(\varphi)=\sum_{i=1}^n \alpha_t(\xi_i\varphi)$. Note that $\|\xi_i\varphi\|_\nu \leqslant 3\|\xi_i\|_\nu\|\varphi\|_\nu$. In particular, $\xi_i\varphi$ is Hölder continuous and every geodesic of its support meets the tie k_i. We can therefore apply Theorem 7 and Proposition 8, which show that $\dot{\alpha}_0(\xi_i\varphi)$ exists and is an $O(\|\xi_i\varphi\|_\nu)=O(\|\varphi\|_\nu)$. Therefore, $\dot{\alpha}_0(\varphi)=\sum_{i=1}^n\dot{\alpha}_0(\xi_i\varphi)$ exists and is an $O(\|\varphi\|_\nu)$.

The second statement comes from the fact that, by Lemmas 3 and 6, the formula provided by Theorem 7 for the $\dot{\alpha}_0(\xi_i\varphi)$ does not depend on the particular family of geodesic laminations α_t tangent to $\dot{\alpha}_0$. □

In Corollary 13, the fact that $|\dot\alpha_0(\psi)| \leqslant C\|\varphi\|_\nu$ implies that the functional $\varphi \mapsto \dot\alpha(\varphi)$ defines a geodesic Hölder current, in the sense that we now define.

If X is a metric space, let $H(X)$ denote the space of Hölder continuous functions $\varphi : X \to \mathbb{R}$ with compact support. For every $\nu > 0$ and every compact subset K of X, let $H_\nu(X; K)$ denote the space of $\varphi \in H(X)$ which have Hölder exponent ν and whose support is contained in K, endowed with the topology defined by the norm $\| \ \|_\nu$. Note that there is a continuous inclusion map $H_\nu(X; K) \to H_{\nu'}(X; K)$ for every $\nu' \leqslant \nu$. Then, $H(X)$ is the union of all the $H_\nu(X; K)$ as K and ν range over all compact subsets of X and all positive numbers. By definition, a ***Hölder distribution*** on X is a linear functional $H(X) \to \mathbb{R}$ whose restriction to each $H_\nu(X; K)$ is continuous. Note that, when X is a differentiable manifold, a Hölder distribution is a distribution in the usual sense with some additional regularity properties.

The space $H(X)$ clearly depends only on the Hölder equivalence class of the metric of X, and therefore so does the notion of Hölder distribution. In particular, we can talk of Hölder distributions on $G(\widetilde{S})$, for the natural Hölder structure on this space.

Let a ***geodesic Hölder current*** on S be a Hölder distribution on $G(\widetilde{S})$ which is invariant under the action of $\pi_1(S)$. We will denote by $\mathcal{H}(S)$ the space of geodesic Hölder currents on S.

The functional $\varphi \mapsto \dot\alpha(\varphi)$ defined by Corollary 13 has the required regularity property, by the inequality $|\dot\alpha_0(\varphi)| \leqslant C\|\varphi\|_\nu$, and is clearly invariant under the action of $\pi_1(S)$. Therefore, to each tangent vector $\dot\alpha_0$ of $\mathcal{ML}(S)$, Corollary 13 associates a geodesic Hölder current which we will also denote by $\dot\alpha_0$.

4. Geodesic laminations with transverse Hölder distributions

The point of view of geodesic currents is technically powerful, as we will see in §§6-7. However, for measured laminations, it is perhaps less intuitive than the idea of a geodesic lamination with a transverse structure. In this section, we reorganize the analysis of §3 to associate to each tangent vector to $\mathcal{ML}(S)$ a geodesic lamination with some transverse structure.

For this, given a 1-parameter family α_t of measured geodesic laminations, we want to compute the derivative of the measures deposited by the α_t on an arbitrary transverse arc k.

When k is a tie of a train track carrying the α_t, the corresponding formula for this derivative is basically provided by Theorem 7. Extending this formula to the general case turns out to be more cumbersome than one could have expected. This stems from the fact that the formula of Theorem 7 strongly depends on the tie k_0 considered, and that two ties located on either side of the same switch give two apparently different formulas.

To overcome this difficulty (and explain the irrelevance of the discrepancy between formulas associated to different ties) we will use the following lemma. This lemma also plays a fundamental rôle in our classification theorem of [Bo4]. It expresses the fact that the restriction of a Hölder continuous function to a Hausdorff dimension 0 subset of an interval is completely determined by its jumps on the components of the complement of this subset, namely by its jumps over the gaps determined by this set.

LEMMA 14 (Gap Lemma). – *Let Λ be a subset of Hausdorff dimension 0 of the interior of an interval $[a,b] \subset \mathbb{R}$. For every component d of $[a,b]-\Lambda$, let x_d^- and x_d^+ be the infimum and supremum of d. Then, for every Hölder continuous function $\psi : \Lambda \cup [a,b] \to \mathbb{R}$,*

$$\psi(x_d^+) = \psi(b) + \sum_{d'>d} \left(\psi(x_{d'}^-) - \psi(x_{d'}^+)\right)$$

where the sum ranges over all components d' of $[a,b]-\Lambda$ which are above d for the ordering of these components induced by the order of $[a,b]$.

Proof. – The proof is elementary, and is given in detail in [Bo4]. The main idea is that, since Λ has Hausdorff dimension 0, its image under the Hölder continuous map ψ has Lebesgue measure 0 in $\mathbb{R}$. □

Note that the conclusion of the Gap Lemma 14 does not hold if we do not require the function ψ to be Hölder continuous

THEOREM 15. – *Consider a 1-parameter family of measured geodesic laminations α_t, $t \in [0,t_0]$, which admits a tangent vector $\dot\alpha_0$ at $t=0$ for the piecewise linear structure of $\mathcal{ML}(S)$. Let k be a compact oriented arc in S which is transverse to the support of all α_t as well as to the essential support λ_{0^+} of α_t as t tends to 0^+. For every component d of $k-\lambda_{0^+}$, let x_d^- and $x_d^+ \in k$ be the infimum and supremum of d for the orientation of k. Also, given a choice of base point $x_d \in d$, let $h_t(x_d)$ denote the α_t-integral of the subarc of k consisting of those points which are below x_d^+. Then, the right derivative $\dot h_0(x_d) = \dot h_0(d)$ exists and is independent of the choice of x_d for every d. In addition, for every Hölder continuous function $\psi : k \to \mathbb{R}$,*

$$\dot\alpha_0(\psi) = \dot\alpha_0(k)\psi(x_k^+) + \sum_d \dot h_0(d)\left(\psi(x_d^-) - \psi(x_d^+)\right)$$

where d ranges over all components of $k-\lambda_{0^+}$ and where x_k^+ is the positive end point of k.

Proof. – Decreasing t_0 if necessary without loss of generality, we can assume that the α_t are all carried by a train track τ. Enlarging τ a little bit, we can also require that the supports λ_t and the essential support λ_{0^+} are contained in the interior of τ.

First consider the case where k is contained in a generic tie k_0 of τ.

Lift k_0 to an arc in $\widetilde S$ which we will also denote by k_0, and let $\widetilde\tau$ be the train track preimage of τ in $\widetilde S$.

For every d, let $h_t^0(x_d)$ be the α_t-measure of the subarc consisting of those points of k_0 which are below the base point x_d. Then, $h_t(x_d) = h_t^0(x_d) - h_t^0(x_k^-)$ where x_k^- is the negative end point of k. It follows that $\dot h_0(x_d) = \dot h_0^0(x_d) - \dot h_0^0(x_k^-)$, which is independent of the choice of x_d by Lemma 5.

The leaves of $\widetilde\lambda_{0^+}$ cutting k form a compact subset in the open set of those geodesics of $G(\widetilde S)$ which transversely cut k in one point. Therefore, we can fix a Hölder continuous function $\xi_k : G(\widetilde S) \to [0,1]$ with compact support which is identically 1 on a neighborhood of those leaves of $\widetilde\lambda_{0^+}$ that cut k, and is identically 0 on those geodesics of $G(\widetilde S)$ that do not cut k in one point.

For every Hölder continuous function $\psi : k \to \mathbb{R}$, we can now consider the function $\varphi : G(\widetilde{S}) \to \mathbb{R}$ defined by the property that $\varphi(g) = \xi_k(g)\psi(g \cap k)$ if g transversely cuts k in one point, and $\varphi(g) = 0$ otherwise. We can choose n large enough so that, for every geodesic g that realizes the same length n edge path originating at k_0 as a leaf of $\widetilde{\lambda}_{0+}$, then $\xi_k(g) = 1$. Considering all length n edge paths originating at k_0, this decomposes $\alpha_t(\psi)$ as the sum of $\alpha_t(\varphi)$ and of a term $a_t \geqslant 0$ such that $\lim_{t\to 0^+} a_t/t = 0$, by Lemma 3. In particular, $\dot{\alpha}_0(\psi) = \dot{\alpha}_0(\varphi)$.

Applying Theorem 7 to this function φ, we get that

$$\dot{\alpha}_0(\psi) = \dot{\alpha}_0(\varphi) = \sum_d \dot{h}_0^0(d)\big(\varphi(g_d^-) - \varphi(g_d^+)\big)$$

where d ranges over all components of $k_0 - \widetilde{\lambda}_{0+}$, and where $\dot{h}_0^0(d)$ is the right derivative of $t \mapsto h_t^0(x_d)$ at $t = 0$.

If d is contained in k, it is also a component of $k - \widetilde{\lambda}_{0+}$ and $\varphi(g_d^\pm) = \psi(x_d^\pm)$. If d is disjoint from k, then $\varphi(g_d^\pm) = 0$. If d contains the positive end point x_k^+ of k, then $\varphi(g_d^+) = 0$ by definition, so that $\varphi(g_d^-) - \varphi(g_d^+) = \big(\psi(x_d^-) - \psi(x_d^+)\big) + \varphi(x_k^+)$. Similarly, $\varphi(g_d^-) - \varphi(g_d^+) = \big(\psi(x_d^-) - \psi(x_d^+)\big) - \psi(x_k^-)$ when d contains the negative end point x_k^-. It follows that

$$\begin{aligned}\dot{\alpha}_0(\psi) &= \dot{h}_0^0(x_k^+)\psi(x_k^+) - \dot{h}_0^0(x_k^-)\psi(x_k^-) + \sum_d \dot{h}_0^0(d)\big(\psi(x_d^-) - \psi(x_d^+)\big)\\ &= \dot{h}_0(x_k^+)\psi(x_k^+) + \sum_d \dot{h}_0(d)\big(\psi(x_d^-) - \psi(x_d^+)\big)\\ &\quad + \dot{h}_0^0(x_k^-)\Big(\psi(x_k^+) - \psi(x_k^-) + \sum_d \big(\psi(x_d^-) - \psi(x_d^+)\big)\Big)\\ &= \dot{h}_0(x_k^+)\psi(x_k^+) + \sum_d \dot{h}_0(d)\big(\psi(x_d^-) - \psi(x_d^+)\big)\end{aligned}$$

by the Gap Lemma 14, where the sums are over all components d of $k - \widetilde{\lambda}_{0+}$, this time. Noting that $h_t(x_k^t) = \alpha_t(k)$, this concludes the proof of Theorem 15 in the special case where k is contained in a tie k_0.

The next case we want to consider is that where k can be homotoped respecting λ_{0+} to an arc k' that is contained in a tie of the train track. This homotopy associates a component d' of $k' - \lambda_{0+}$ to each component d of $k - \lambda_{0+}$, and transports the map $\psi : k \to \mathbb{R}$ to a map $\psi' : k' \to \mathbb{R}$. By the Tracking Lemma 1, the α_t-measures $\alpha_t(\psi)$ and $\alpha_t(\psi')$ differ only by a quantity a_t such that $\lim_{t\to 0^+} a_t/t = 0$; and similarly for $h_t(x_d)$ and $h'_t(x_{d'})$. It follows that $\dot{h}_0(d) = \dot{h}'_0(d)$ and $\dot{\alpha}_0(\psi) = \dot{\alpha}_0(\psi')$. The result then follows from the previous case.

We now consider the general case. We can decompose the arc k into subarcs $k_1, \ldots, k_n$ with disjoint interiors such that each k_i can be homotoped respecting λ_{0+} to an arc contained in a tie, and where the indexing follows the orientation of k. If $x \in k_i - \lambda_{0+}$, let $h_t^{(i)}(x)$ be the α_t-mass of the set of those points of k_i which are below x. Clearly, $h_t(x) = h_t^{(i)}(x) + \sum_{j=1}^{i-1} \alpha_t(k_j)$. Combined with the previous case, this shows that the derivative $\dot{h}_0(d) = \dot{h}_0^{(i)}(d) + \sum_{j=1}^{i-1} \dot{\alpha}_0(k_j)$ exists for every component d of $k - \lambda_{0+}$ meeting k_i. Also,

$$\begin{aligned}\dot{\alpha}_0(\psi) &= \sum_{i=1}^{n} \dot{\alpha}_0(\psi_{|k_i}) \\ &= \sum_{i=1}^{n} \dot{\alpha}_0(k_i)\psi(x_{k_i}^+) + \sum_{i=1}^{n}\sum_{d_i} \dot{h}_0^{(i)}(d_i)\big(\psi(x_{d_i}^-) - \psi(x_{d_i}^+)\big) \\ &= \sum_{i=1}^{n} \dot{\alpha}_0(k_i)\psi(x_{k_i}^+) + \sum_{i=1}^{n}\sum_{d_i} \Big(\dot{h}_0(d_i) - \sum_{j=1}^{i-1}\dot{\alpha}_0(k_j)\Big)\big(\psi(x_{d_i}^-) - \psi(x_{d_i}^+)\big)\end{aligned}$$

where d_i ranges over all components of $k_i - \lambda_{0+}$. In general, a component d of $k - \lambda_{0+}$ corresponds to a unique component d_i of some $k_i - \lambda_{0+}$. However, finitely many components d of $k - \lambda_{0+}$ are the union of a component d_i of $k_i - \lambda_{0+}$ and of a component d_{i+1} of $k_{i+1} - \lambda_{0+}$; in this case, the terms $\dot{h}_0(d_i)\big(\psi(x_{d_i}^-) - \psi(x_{d_i}^+)\big)$ and $\dot{h}_0(d_{i+1})\big(\psi(x_{d_{i+1}}^-) - \psi(x_{d_{i+1}}^+)\big)$ add up to $\dot{h}_0(d)\big(\psi(x_d^-) - \psi(x_d^+)\big)$ since $\dot{h}_0(d_i) = \dot{h}_0(d_{i+1}) = \dot{h}_0(d)$ and $x_{d_i}^+ = x_{d_{i+1}}^-$. Also, the coefficient of each term $\dot{\alpha}_0(k_j)$ is

$$\psi(x_{k_j}^+) - \sum_{i=j+1}^{n}\sum_{d_i}\big(\psi(x_{d_i}^-) - \psi(x_{d_i}^+)\big) = \psi(x_k^+)$$

by the Gap Lemma 14. The theorem easily follows. □

COMPLEMENT 16. – *Under the hypotheses of Theorem 15 and given $\nu > 0$, there is a constant C such that $\dot{\alpha}_0(\psi) \leqslant C\|\psi\|_\nu$ for every Hölder continuous function $\psi : k \to \mathbb{R}$ of Hölder exponent ν. In addition, $\dot{\alpha}_0(\psi)$ depends only on ψ and on the tangent vector $\dot{\alpha}_0$.*

Proof. – This immediately follows from the proof of Theorem 15 and from Proposition 8. □

We can apply Theorem 15 to the particular case of measured laminations $\alpha_t = (1+t)\alpha$ for a fixed measured lamination α. We then get the following formula for the α-measure deposited by α on each transverse arc.

COROLLARY 17. – *Let k be an oriented arc transverse to the support λ_α of a measured lamination α. Then, for every Hölder continuous function $\psi : k \to \mathbb{R}$,*

$$\alpha(\psi) = \alpha(k)\psi(x_k^+) + \sum_d h(d)\big(\psi(x_d^-) - \psi(x_d^+)\big)$$

where d ranges over all components of $k - \lambda_\alpha$, where x_d^+ and x_d^- are the positive and negative end points of d, where $h(d)$ is the α-mass of the set of points of k which are below d, and where x_k^+ is the positive end point of k. □

The formula of Corollary 17 can also be deduced from the Gap Lemma 14, and from the estimates used in the proof of Proposition 8. *See* also [Bo4] for a generalization to geodesic laminations with transverse Hölder distributions. Note that, in the generic case where $k \cap \lambda_\alpha$ is a Cantor set, the hypothesis that ψ is Hölder continuous is absolutely necessary for the formula to hold, as shown by the numerous examples of non-constant continuous functions on k which are locally constant on $k - \lambda_\alpha$.

Theorem 15 and Complement 16 associate to each tangent vector of $\mathcal{ML}(S)$ a Hölder distribution on each arc k transverse to the essential support λ_{0+}. We would like to say that these data define a transverse invariant Hölder distribution for the geodesic lamination λ_{0+}. The invariance property under homotopy respecting the lamination requires a little care in the definition because of the Hölder condition.

To specify this property, consider a geodesic lamination λ on S, for a given negatively curved metric m with totally geodesic boundary on S. Let k and k' be two differentiable arcs transverse to λ which can be homotoped from one to the other by a continuous homotopy respecting λ. In general, we cannot assume that this homotopy is differentiable. However, this homotopy establishes a one-to-one correspondence between $k \cap \lambda$ and $k' \cap \lambda$, as well as between the components of $k - \lambda$ and the components of $k' - \lambda$. An easy geometric estimate shows that the length of a component of $k - \lambda$ is bounded above and below by constants times the length of the corresponding component of $k' - \lambda$. (The constants depend on the curvature of the metric m, on the diameter of $k \cup k'$, and on the minimum angle between k, k', and λ). Since $k \cap \lambda$ has Hausdorff dimension 0, the distance in k between two of its points is equal to the length of the components of $k' - \lambda$ separating them. It follows that the correspondence $k \cap \lambda \to k' \cap \lambda$ is Lipschitz (= Hölder continuous of Hölder exponent 1), as well as its inverse. Therefore, we can choose the homotopy so that the homeomorphism $\theta : k \to k'$ it provides is Hölder continuous, as well as its inverse; in this case, we will say that θ is ***Hölder bicontinuous***. If this holds, note that θ enables us to identify Hölder distributions on k and Hölder distributions on k'.

Given a geodesic lamination λ, a ***transverse (invariant) Hölder distribution*** for λ is a Hölder distribution defined on each differentiable arc k transverse to λ, and such that every Hölder bicontinuous homotopy sending k to another arc k' while respecting λ sends the Hölder distribution defined on k to the Hölder distribution defined on k'.

With these definitions, it is immediate that Theorem 15 and Complement 16 associate to each tangent vector to $\mathcal{ML}(S)$ a geodesic lamination endowed with a transverse Hölder distribution.

Corollary 13 and Theorem 15 suggest a relationship between, geodesic laminations with transverse Hölder distributions in S on one hand, and geodesic Hölder currents whose support in $G(\widetilde{S})$ is a geodesic lamination of $\widetilde{S}$ on the other hand. And indeed, these two notions are shown to be equivalent in [Bo4].

5. Transverse Hölder distributions and tangent vectors to the space of measured geodesic laminations

We showed that every tangent vector to $\mathcal{ML}(S)$ can be interpreted as a geodesic lamination with a transverse Hölder distribution. In this section, we characterize which geodesic laminations with transverse Hölder distributions are associated in this way to tangent vectors of $\mathcal{ML}(S)$ at $\alpha \in \mathcal{ML}(S)$.

First, we state the following classification theorem, proved in [Bo4]. If α is a transverse Hölder distribution for the geodesic lamination λ and if k is an arc transverse to λ, we define $\alpha(k)$ to be the α-integral of the constant function 1 on k. In particular, for every edge e of a train track carrying λ, we can define $\alpha(e)$ to be $\alpha(k_e)$ for any tie k_e of e.

THEOREM 18 [Bo4]. – *If the geodesic lamination λ is carried by the train track τ, a transverse Hölder distribution α for λ is uniquely determined by the numbers $\alpha(e)$ it associates to the edges of τ. In particular, the space $\mathcal{H}(\lambda)$ of transverse Hölder distributions for λ is a finite dimensional vector space. In addition, the dimension of $\mathcal{H}(\lambda)$ is equal to $-\chi(\lambda)+n_o(\lambda)$, where $\chi(\lambda)\leqslant 0$ is the Euler characteristic of λ and where $n_o(\lambda)$ is the number of orientable components of λ.* □

The Euler characteristic $\chi(\lambda)$ can be defined as the alternating sum of the Čech cohomology groups of λ, considered as a subset of S. A more practical definition, if λ is m-geodesic for some negatively curved metric m, is that $\chi(\lambda)$ is the Euler characteristic of the ε-neighborhood of λ for ε sufficiently small. In the 'generic' case where λ is maximal, namely when the complement $S-\lambda$ consists of finitely many triangles with all vertices at infinity, a counting argument easily shows that the dimension of $\mathcal{H}(\lambda)$ is equal to $-3\chi(S)$.

THEOREM 19. – *Let $\alpha_0\in\mathcal{ML}(S)$ and let λ be a geodesic lamination which contains the geodesic lamination λ_0 underlying α_0. Then, a transverse Hölder distribution α for λ represents a tangent vector of $\mathcal{ML}(S)$ at α_0 if and only if $\alpha(k)\geqslant 0$ for every arc k transverse to λ and disjoint from λ_0.*

Proof. – The condition is clearly necessary. Indeed, for such an arc k and if α is tangent to a path $t\mapsto\alpha_t\in\mathcal{ML}(S)$ starting at α_0, then $\alpha(k)$ is the derivative of $\alpha_t(k)\geqslant 0$ and $\alpha_0(k)=0$.

Conversely, assume that $\alpha(k)\geqslant 0$ for every arc k transverse to λ and disjoint from λ_0.

Let τ be a train track carrying λ. The general theory of train tracks (*see* for instance [Th1][PeH]) provides, for t sufficiently small, a measured geodesic lamination α_t carried by τ such that $\alpha_t(e)=\alpha_0(e)+t\alpha(e)$ for every edge e of τ. In addition, the geodesic laminations underlying these α_t are contained in a compact subset of the interior of τ.

This path $t\mapsto\alpha_t$ in $\mathcal{ML}(S)$ is piecewise linear, and we want to show that its tangent vector $\dot\alpha_0$ is represented by α. The core of the proof is contained in the following lemma.

As usual, we lift the situation to the universal covering $\widetilde{S}$, and we let tildes $\tilde{}$ denote preimages in $\widetilde{S}$. If γ is an edge path in $\widetilde{\tau}$, there is an arc k_γ contained in a tie of $\widetilde{\tau}$ such that the leaves of $\widetilde{\lambda}$ that realize γ are exactly those that cut k_γ. Since this number is clearly independent on the choice of k_γ, we set $\alpha(\gamma)=\alpha(k_\gamma)$.

Note that the condition that $\alpha(k)\geqslant 0$ for every arc disjoint from λ_0 implies that $\alpha(\gamma)\geqslant 0$ for every edge path γ such that $\alpha_0(\gamma)=0$.

LEMMA 20. – *For every edge path γ of $\widetilde{\tau}$, $\alpha_t(\gamma)=\alpha_0(\gamma)+t\alpha(\gamma)$ for t sufficient small.*

Proof. – Note that this property cannot hold without the hypothesis that $\alpha(k)\geqslant 0$ for every arc k disjoint from λ_0, since $\alpha_t(\gamma)$ has to be non-negative.

We will prove the lemma by induction on the length n of the edge path $\gamma_n=\langle e_1,\dots,e_n\rangle$ of $\widetilde{\tau}$.

The property holds for $n=1$ by definition of α_t. Assume as induction hypothesis that it holds for every edge path of length at most $n-1$.

As in the proof of Lemma 2, select a 'left' side and a 'right' side for γ_n. Let $G^{\mathrm{l}}_n \subset G(\widetilde{S})$ consist of those geodesics which realize some path $\langle e'_i, e_{i+1}, \dots, e_n\rangle$ with $e'_i \neq e_i$ branching in on the left side of γ_n. Similarly define G^{r}_n consisting of geodesics branching in on the right side of γ_n. Also, if s is the switch separating e_{n-1} from e_n, let A^{l}_- (resp. A^{r}_-, A^{l}_+, A^{r}_+) consist of those geodesics of $G(\widetilde{S})$ which realize an edge path consisting of one of the edges entering s on the same side as e_{n-1} (resp. e_{n-1}, e_n, e_n) and on the left (resp. right, left, right) side of γ_n. Finally, let $H^{\mathrm{l}}_n = G^{\mathrm{l}}_{n-1} \cup A^{\mathrm{l}}_- - A^{\mathrm{l}}_+$ and $H^{\mathrm{r}}_n = G^{\mathrm{r}}_{n-1} \cup A^{\mathrm{r}}_- - A^{\mathrm{r}}_+$; for instance, H^{l}_n consists of those geodesics which cross s after branching in on the left side of γ_n and which do not branch out on the left side of e_n. Then, as in the proof of Lemma 2,

$$\begin{aligned}
\alpha_t(H^{\mathrm{l}}_n) &= \max\{\alpha_t(G^{\mathrm{l}}_{n-1}) + \alpha_t(A^{\mathrm{l}}_-) - \alpha_t(A^{\mathrm{l}}_+), 0\}\\
\alpha_t(H^{\mathrm{r}}_n) &= \max\{\alpha_t(G^{\mathrm{r}}_{n-1}) + \alpha_t(A^{\mathrm{r}}_-) - \alpha_t(A^{\mathrm{r}}_+), 0\}\\
\alpha_t(G^{\mathrm{l}}_n) &= \min\{\alpha_t(e_n), \alpha_t(H^{\mathrm{l}}_n)\}\\
\alpha_t(G^{\mathrm{r}}_n) &= \min\{\alpha_t(e_n), \alpha_t(H^{\mathrm{r}}_n)\}\\
\alpha_t(\gamma_n) &= \alpha_t(e_n) - \alpha_t(G^{\mathrm{l}}_n) - \alpha_t(G^{+}_n).
\end{aligned}$$

Note that H^{l}_n is a disjoint union of G_γ associated to edge paths γ. Therefore, we can talk of $\alpha(H^{\mathrm{l}}_n)$, defined as the union of the corresponding $\alpha(\gamma)$. The same applies to H^{r}_n, G^{l}_n, G^{r}_n, A^{l}_-, A^{r}_-, A^{l}_+, A^{r}_+.

Our first goal is to prove that $\alpha_t(H^{\mathrm{l}}_n) = \alpha_0(H^{\mathrm{l}}_n) + t\alpha(H^{\mathrm{l}}_n)$ for t sufficiently small. For this, note that G^{l}_{n-1}, A^{l}_- and A^{l}_+ are disjoint unions of G_γ associated to edge paths γ of length at most $n-1$. By induction hypothesis, is follows that

$$\alpha_t(H^{\mathrm{l}}_n) = \max\{\alpha_0(G^{\mathrm{l}}_{n-1}) + \alpha_0(A^{\mathrm{l}}_-) - \alpha_0(A^{\mathrm{l}}_+) + t(\alpha(G^{\mathrm{l}}_{n-1}) + \alpha(A^{\mathrm{l}}_-) - \alpha(A^{\mathrm{l}}_+)), 0\}.$$

The intersection points of geodesics of $\widetilde{\lambda} \cap (G^{\mathrm{l}}_{n-1} \cup A^{\mathrm{l}}_-)$ form an interval of $s \cap \widetilde{\lambda}$ adjacent to the left end of the tie s. The same property holds for the geodesics of $\widetilde{\lambda} \cap A^{\mathrm{l}}_+$. It follows that, either $\widetilde{\lambda} \cap (G^{\mathrm{l}}_{n-1} \cup A^{\mathrm{l}}_-)$ is contained in $\widetilde{\lambda} \cap A^{\mathrm{l}}_+$, or $\widetilde{\lambda} \cap A^{\mathrm{l}}_+$ is contained in $\widetilde{\lambda} \cap (G^{\mathrm{l}}_{n-1} \cup A^{\mathrm{l}}_-)$. We now distinguish cases.

If $\widetilde{\lambda} \cap (G^{\mathrm{l}}_{n-1} \cup A^{\mathrm{l}}_-)$ is contained in $\widetilde{\lambda} \cap A^{\mathrm{l}}_+$, then $\alpha_0(H^{\mathrm{l}}_n) = \alpha(H^{\mathrm{l}}_n) = 0$. Also, $\alpha_0(A^{\mathrm{l}}_+) - \alpha_0(G^{\mathrm{l}}_{n-1}) - \alpha_0(A^{\mathrm{l}}_-) \geqslant 0$. In addition, $\widetilde{\lambda} \cap A^{\mathrm{l}}_+ - \widetilde{\lambda} \cap (G^{\mathrm{l}}_{n-1} \cup A^{\mathrm{l}}_-)$ is the disjoint union of finitely many $\widetilde{\lambda} \cap G_\gamma$. It follows that, if $\alpha_0(A^{\mathrm{l}}_+) - \alpha_0(G^{\mathrm{l}}_{n-1}) - \alpha_0(A^{\mathrm{l}}_-) = 0$, then $\alpha(A^{\mathrm{l}}_+) - \alpha(G^{\mathrm{l}}_{n-1}) - \alpha(A^{\mathrm{l}}_-) \geqslant 0$ since $\alpha(\gamma) \geqslant 0$ for every γ with $\alpha_0(\gamma) = 0$. As a consequence,

$$\begin{aligned}
\alpha_t(H^{\mathrm{l}}_n) &= \max\{\alpha_0(G^{\mathrm{l}}_{n-1}) + \alpha_0(A^{\mathrm{l}}_-) - \alpha_0(A^{\mathrm{l}}_+) + t(\alpha(G^{\mathrm{l}}_{n-1}) + \alpha(A^{\mathrm{l}}_-) - \alpha(A^{\mathrm{l}}_+)), 0\}\\
&= 0\\
&= \alpha_0(H^{\mathrm{l}}_n) + t\alpha(H^{\mathrm{l}}_n)
\end{aligned}$$

for t sufficiently small.

The other case is when $\widetilde{\lambda} \cap A^{\mathrm{l}}_+$ is contained in $\widetilde{\lambda} \cap (G^{\mathrm{l}}_{n-1} \cup A^{\mathrm{l}}_-)$. Then,

$$\begin{aligned}
\alpha_0(H^{\mathrm{l}}_n) &= \alpha_0(G^{\mathrm{l}}_{n-1}) + \alpha_0(A^{\mathrm{l}}_-) - \alpha_0(A^{\mathrm{l}}_+)\\
\alpha(H^{\mathrm{l}}_n) &= \alpha(G^{\mathrm{l}}_{n-1}) + \alpha(A^{\mathrm{l}}_-) - \alpha(A^{\mathrm{l}}_+).
\end{aligned}$$

In addition, $\widetilde{\lambda}\cap(G^{\mathrm{l}}_{n-1}\cup A^{\mathrm{l}}_{-})-\widetilde{\lambda}\cap A^{\mathrm{l}}_{+}$ is the disjoint union of finitely many $\widetilde{\lambda}\cap G_\gamma$. It follows that, if $\alpha_0(G^{\mathrm{l}}_{n-1})+\alpha_0(A^{\mathrm{l}}_{-})-\alpha_0(A^{\mathrm{l}}_{+})=0$, then $\alpha(G^{\mathrm{l}}_{n-1})+\alpha(A^{\mathrm{l}}_{-})-\alpha(A^{\mathrm{l}}_{+})\geqslant 0$ by hypothesis on α. As a consequence,

$$\begin{aligned}\alpha_t(H^{\mathrm{l}}_n)&=\max\{\alpha_0(G^{\mathrm{l}}_{n-1})+\alpha_0(A^{\mathrm{l}}_{-})-\alpha_0(A^{\mathrm{l}}_{+})+t(\alpha(G^{\mathrm{l}}_{n-1})+\alpha(A^{\mathrm{l}}_{-})-\alpha(A^{\mathrm{l}}_{+})),0\}\\&=\alpha_0(G^{\mathrm{l}}_{n-1})+\alpha_0(A^{\mathrm{l}}_{-})-\alpha_0(A^{\mathrm{l}}_{+})+t(\alpha(G^{\mathrm{l}}_{n-1})+\alpha(A^{\mathrm{l}}_{-})-\alpha(A^{\mathrm{l}}_{+}))\\&=\alpha_0(H^{\mathrm{l}}_n)+t\alpha(H^{\mathrm{l}}_n)\end{aligned}$$

for t sufficiently small.

This proves that, in all cases, $\alpha_t(H^{\mathrm{l}}_n)=\alpha_0(H^{\mathrm{l}}_n)+t\alpha(H^{\mathrm{l}}_n)$ for t sufficiently small.

Similarly, if $E_n\subset G(\widetilde{S})$ consists of those geodesics which realize the edge path $\langle e_n\rangle$, either $\widetilde{\lambda}\cap E_n$ is contained in $\widetilde{\lambda}\cap H^{\mathrm{l}}_n$, or $\widetilde{\lambda}\cap H^{\mathrm{l}}_n$ is contained in $\widetilde{\lambda}\cap E_n$. In both cases, the same kind of argument as above, using the fact that $\alpha(\gamma)\geqslant 0$ for every γ with $\alpha_0(\gamma)=0$, shows that

$$\alpha_t(G^{\mathrm{l}}_n)=\min\{\alpha_0(e_n)+t\alpha(e_n),\alpha_0(H^{\mathrm{l}}_n)+t\alpha(H^{\mathrm{l}}_n)\}=\alpha_0(G^{\mathrm{l}}_n)+t\alpha(G^{\mathrm{l}}_n)$$

for t sufficiently small.

Replacing 'left' by 'right' everywhere also shows that $\alpha_t(G^{\mathrm{r}}_n)=\alpha_0(G^{\mathrm{r}}_n)+t\alpha(G^{\mathrm{r}}_n)$ for t sufficiently small.

Finally, for every geodesic lamination $\widetilde{\mu}$ carried by $\widetilde{\tau}$, $\widetilde{\mu}\cap E_n$ is the disjoint union of $\widetilde{\mu}\cap G_{\gamma_n}$, $\widetilde{\mu}\cap G^{\mathrm{l}}_n$ and $\widetilde{\mu}\cap G^{\mathrm{r}}_n$. We conclude that,

$$\begin{aligned}\alpha_t(\gamma_n)&=\alpha_t(e_n)-\alpha_t(G^{\mathrm{l}}_n)-\alpha_t(G^{+}_n)\\&=\alpha_0(e_n)+t\alpha(e_n)-\alpha_0(G^{\mathrm{l}}_n)-t\alpha(G^{\mathrm{l}}_n)-\alpha_0(G^{+}_n)-t\alpha(G^{+}_n)\\&=\alpha_0(\gamma_n)+t\alpha(\gamma_n)\end{aligned}$$

for t sufficiently small.

This concludes the proof of Lemma 20. □

We can now determine the essential support λ_{0+} of the α_t as t tends to 0^+. By Lemma 3, a geodesic $g\in G(\widetilde{S})$ belongs to $\widetilde{\lambda}_{0+}$ if and only if it is carried by $\widetilde{\tau}$ and if $\alpha_0(\gamma)>0$ or $\dot{\alpha}_0(\gamma)>0$ for every edge path γ realized by g. Lemma 20 implies that $\dot{\alpha}_0(\gamma)=\underset{\sim}{\alpha}(\gamma)$ for every edge path γ. Therefore, for every edge path γ realized by a geodesic $g\in\widetilde{\lambda}_{0+}$, G_γ meets the support of α_0 or the support of $\underset{\sim}{\alpha}$. In particular, since these G_γ form a basis of neighborhoods for $g\in\widetilde{\lambda}_{0+}$ and since $\widetilde{\lambda}$ is closed, every $g\in\widetilde{\lambda}_{0+}$ belongs to $\widetilde{\lambda}$. As a consequence, the essential support λ_{0+} is contained in λ, and the tangent vector $\dot{\alpha}_0$ determines a transverse Hölder distribution for λ.

By Theorem 18, a transverse Hölder distribution for λ is determined by the weights it defines on the edges of τ. By construction of the α_t, $\dot{\alpha}_0(e)=\alpha(e)$ for every edge of τ. Therefore, α is exactly the transverse Hölder distribution associated to the tangent vector $\dot{\alpha}_0$ of $\mathcal{ML}(S)$ at α_0.

This concludes the proof of Theorem 19. □

The criterion provided by Theorem 19 can be made a little more practical as follows. A geodesic lamination can be uniquely decomposed as the union of finitely many (closed disjoint) ***minimal sublaminations***, in which all leaves are dense, and of finitely many infinite isolated leaves g for which each end is asymptotic to one of these minimal sublaminations (*see* [Th1], [CEG, §4.2]). If g is an isolated leaf of λ and if α is a transverse Hölder distribution for λ, let the α-***mass*** $\alpha(g)$ of g be the number $\alpha(k_g)$ where k_g is any arc transverse to λ such that $k_g \cap \lambda = k_g \cap g$ consists of exactly one point.

The support of a measured lamination has no infinite isolated leaves, and therefore is the union of finitely many minimal sublaminations.

THEOREM 21. – *Let the geodesic lamination λ contain the geodesic lamination underlying $\alpha_0 \in \mathcal{ML}(S)$. Then, the transverse Hölder distribution α for λ corresponds to a tangent vector of $\mathcal{ML}(S)$ at α_0 if and only if the following three conditions hold:*

(i) *every infinite isolated geodesic of λ has non-negative α-mass;*

(ii) *every infinite isolated leaf of λ which is asymptotic to a minimal sublamination of λ that is not contained in the support of α_0 has α-mass 0;*

(iii) *the restriction of α to each minimal sublamination of λ that is not contained in the support of α_0 is a transverse measure (this restriction makes sense because of Condition* (ii)*).*

Proof. – First, let us show that Conditions (i), (ii) and (iii) are necessary for α to correspond to a tangent vector of $\mathcal{ML}(S)$ at α_0.

Condition (i) immediately follows from Theorem 19 and the definitions.

Now, consider an infinite isolated geodesic g which is asymptotic to a minimal sublamination λ_1 of λ which is not in the support of α_0. Let k be an arc transverse to λ which meets λ_1 and no other minimal sublamination of λ. Then, g hits k in infinitely many isolated points. Choose small arcs around n points of $g \cap k$. The complement of these small arcs consists of $n+1$ arcs k' such that $\alpha_0(k') = 0$, and therefore such that $\alpha(k') \geqslant 0$ by Theorem 19. In particular, $\alpha(k)$ is the sum of $n\alpha(g) \geqslant 0$ and of a non-negative number. Since this is true for every n, we conclude that $\alpha(g) = 0$. This proves that Condition (ii) is necessary.

Finally, let λ_1 be a minimal sublamination of λ which is not in the support of α_0. In [Bo4, Proposition 18], we show that a transverse Hölder distribution β for λ_1 is a transverse measure if and only if $\beta(k) \geqslant 0$ for every transverse arc k (this is relatively elementary). By Theorem 19, the restriction of α to λ_1 satisfies this condition, and is therefore a transverse measure. This proves that Condition (iii) is necessary.

Conversely, assume that Conditions (i), (ii) and (iii) are satisfied. If k is an arc transverse to λ and disjoint from the support of α_0, we can split it into finitely many arcs k_i such that, either k_i intersects λ in finitely many points located on infinite isolated leaves, or k_i meets exactly one minimal sublamination λ_i and possibly some infinite isolated leaves asymptotic to λ_i. In the first case, $\alpha(k_i) \geqslant 0$ by Condition (i). In the second case, $\alpha(k_i) \geqslant 0$ by Conditions (ii) and (iii). We conclude that $\alpha(k) \geqslant 0$ for every arc k transverse to λ and disjoint from the support of α_0. By Theorem 19, this proves that α corresponds to a tangent vector of $\mathcal{ML}(S)$ at α_0. □

THEOREM 22. – *Let the geodesic lamination λ contain the geodesic lamination underlying $\alpha_0 \in \mathcal{ML}(S)$. Then, the transverse Hölder distributions α for λ corresponding to tangent* *vectors of $\mathcal{ML}(S)$ at α_0 form a closed convex cone in the vector space $\mathcal{H}(\lambda)$ of all transverse Hölder distributions for λ, and this cone is bounded by finitely many hyperplanes.*

Proof. – Let τ be a train track carrying λ. By cutting τ open along some arcs carried by τ, originating from switch points and disjoint from λ, we can arrange that the following two conditions are met:

(*a*) if a tie k of τ meets a minimal sublamination λ_1, every other leaf of λ that meets k is an isolated leaf that is asymptotic to λ_1;

(*b*) for every infinite isolated leaf g, there is a tie k of τ such that $k \cap \lambda = k \cap g$ consists of exactly one point.

By Theorem 18, the transverse Hölder distributions for λ are determined by the numbers $\alpha(e)$ they associate to the edges e of τ.

By (*b*), Conditions (i) and (ii) of Theorem 21 are equivalent to the property that some of these $\alpha(e)$ are non-negative or are equal to 0. For a minimal sublamination λ_1 that is not in the support of α_0, a result of A. Katok [Ka] (*see* also [Pa1], [PeH], [Bo4, §4]) asserts that there are finitely many transverse measures for λ_1 such that any other transverse measure can be uniquely written as a linear combination of these transverse measures with non-negative coefficients; this also follows from Theorem 18. Therefore, Condition (iii) of Theorem 21 for λ_1 can be expressed by the property that the set of those $\alpha(e)$ with e meeting λ_1 is contained in a certain linear simplex. This clearly concludes the proof. □

The representation of tangent vectors to $\mathcal{ML}(S)$ by geodesic laminations with transverse Hölder distributions also provides a nice interpretation of the faces of the piecewise linear structure of $\mathcal{ML}(S)$. Recall that two tangent vectors at the same point of a piecewise linear n-manifold belong to the same face if, when we consider the two tangent vectors of $\mathbb{R}^n$ associated to these two vectors by a local chart, the differential of every change of chart is linear on the positive cone generated by these two vectors. In the case of $\mathcal{ML}(S)$, recall that the piecewise linear structure is defined by the maps $f_k : \mathcal{ML}(S) \to \mathbb{R}^+$ where k ranges over all generic arcs of S and where $f_k(\alpha_0)$ is the total mass of the measure deposited by α_0 on k. Therefore, two tangent vectors α and β at $\alpha_0 \in \mathcal{ML}(S)$ belong to the same face if and only if, for every a, $b \geqslant 0$, there is a third tangent vector γ such that $df_k(\gamma) = a\, df_k(\alpha) + b\, df_k(\beta)$ for every generic arc k. Note that, if we interpret the tangent vector α as a geodesic lamination with transverse Hölder distribution, the image $df_k(\alpha)$ of α under the differential of f_k is just the integral $\alpha(k)$ of the constant function 1 under the Hölder distribution deposited by α on k.

PROPOSITION 23. – *Let α and β be two tangent vectors at $\alpha_0 \in \mathcal{ML}(S)$, considered as geodesic laminations with transverse Hölder distributions. Then, α and β belong to the same face of the piecewise linear structure of $\mathcal{ML}(S)$ if and only if their supports λ_α and λ_β are sublaminations of a same geodesic lamination, namely if and only if no geodesic of λ_α transversely crosses a geodesic of λ_β and conversely.*

Proof. – If λ_α and λ_β are sublaminations of a geodesic lamination λ, then α and β are transverse Hölder distributions for λ. If a and b are non-negative numbers, then $\gamma = a\alpha + b\beta$ is also a transverse Hölder distribution for λ which, by Theorem 22, is associated to a tangent vector at α_0. Since $\gamma(k) = a\alpha(k) + b\beta(k)$ for every generic arc k, this proves that α and β belong to the same face.

Conversely, assume that there is a geodesic of λ_α which crosses a geodesic of λ_β. Suppose that α and β belong to the same face. Then, for a, $b > 0$, there is a tangent vector γ such that $\gamma(k) = a\alpha(k) + b\beta(k)$ for every generic arc k. The support of γ in $G(\widetilde{S})$ contains at least the symmetric difference of the supports $\widetilde{\lambda}_\alpha$ and $\widetilde{\lambda}_\beta$. In particular, by assumption on λ_α and λ_β, there are two geodesics of the support of γ which cross each other, contradicting the fact that this support is a geodesic lamination of $\widetilde{S}$. Therefore, α and β cannot belong to the same face. □

To each face in the tangent space of $\mathcal{ML}(S)$ at α_0, Proposition 23 associates a geodesic lamination λ. This λ contains the geodesic lamination underlying α_0, and is the essential support associated by Lemma 3 to a tangent vector in the interior of this face. It is consequently important to know which geodesic laminations can be obtained in this way. By Proposition 4, this is equivalent to asking which geodesic laminations are the Hausdorff limit as t tends to 0^+ of the geodesic laminations underlying the elements of a piecewise linear path $t \mapsto \alpha_t$ starting at α_0.

An easy necessary condition is that each connected component λ_1 of a geodesic lamination λ obtained in this way must be ***chain recurrent.*** This means that, for every $\varepsilon > 0$, any two points of λ_1 can be connected by a chain of arcs contained in λ and small jumps such that the tangent vector of each arc at its terminal point is within ε of the tangent vector of the next arc at its initial point. In practice, it is easy to express this condition in terms of how the infinite isolated leaves of λ connect its minimal sublaminations, taking into account the orientations of those minimal sublaminations which are orientable.

Using train track approximations, it is not very difficult to prove that this chain recurrent condition is actually sufficient (*see* [Th4], [PeH]). Namely, for every geodesic lamination λ containing the geodesic lamination underlying α_0 and whose connected components are all chain recurrent, we can construct a piecewise linear path $t \mapsto \alpha_t$ starting at α_0 and whose Hausdorff limit as t tends to 0^+ is equal to λ. In other words, a geodesic lamination is associated to a face of $\mathcal{ML}(S)$ at α_0 if and only if it contains the geodesic lamination underlying α_0 and its connected components are all transversely chain recurrent.

It is also probably worth mentioning the following corollary of Theorem 21 and Proposition 23: There are tangent vectors to $\mathcal{ML}(S)$ which are contained in no face of maximal dimension $3|\chi(S)|$. This is another indication of the complexity of the piecewise linear structure of $\mathcal{ML}(S)$.

6. Length of Hölder geodesic currents

Consider a metric m of negative curvature on the surface S for which the boundary ∂S is totally geodesic. For a free homotopy class of closed curves γ on S, the ***length of*** γ ***with respect to*** m is the length $l_m(\gamma)$ of the (unique) multiple of closed m-geodesic contained in γ (with the convention that $l_m(\gamma) = 0$ if the curves of γ are homotopic to 0). It turns out that this length function extends to a linear continuous function $l_m : \mathcal{C}(S) \to \mathbb{R}^+$. In particular, there is a unique continuous function $l_m : \mathcal{ML}(S) \to \mathbb{R}^+$ such that, when α is a closed geodesic λ_α endowed with the transverse Dirac transverse measure of weight

$a > 0$, $\lambda_m(\alpha)$ is equal to a times the length of the closed m-geodesic homotopic to λ_α (*see* [Th1, §9.3], [Bo1, §4.2], and below).

Given a path $t \mapsto \alpha_t$ in $\mathcal{ML}(S)$ which admits a tangent vector $\dot{\alpha}_0$ at $t = 0$, we would like to compute the right derivative $\frac{\partial}{\partial t} l_m(\alpha_t)_{|t=0}$. Our interpretation of $\dot{\alpha}_0$ as a geodesic lamination with transverse Hölder distribution provides an immediate solution to this problem.

Indeed, let us recall how the length $l_m(\alpha)$ of a geodesic current $\alpha \in \mathcal{C}(S)$ is defined: Consider the projective tangent bundle $PT(S)$ of S, consisting of pairs (x, d) where $x \in S$ and d is a line passing through the origin in the tangent plane of S at x. If we endow S with the metric m, every m-geodesic can be lifted to $PT(S)$ by considering the tangent line at each of its points. Let $PT_0(S) \subset PT(S)$ be the union of the lifts of those geodesics of S which do not transversely hit the boundary ($PT_0(S) = PT(S)$ when $\partial S = \emptyset$). These lifts of geodesics foliate $PT_0(S)$, in the sense that every point of $PT_0(S)$ has a neighborhood homeomorphic to some space $D \times [0, 1]$ where each arc $* \times [0, 1]$ is contained in a lift of geodesic. This foliation $\mathcal{F}$ is the ***geodesic foliation*** of $PT_0(S)$. Locally, a point of $PT_0(S)$ is characterized by the leaf of $\mathcal{F}$ containing it, and by where it sits on that leaf. Consequently, given $\alpha \in \mathcal{C}(S)$, we can consider on $PT_0(S)$ the measure which is locally the product of α and of the length measure deposited by m on the leaves of $\mathcal{F}$. Then, $l_m(\alpha)$ is defined as the total mass of this measure.

More precisely, choose a continuous partition of unity $\xi_i : PT_0(S) \to \mathbb{R}$, $i = 1, \ldots, n$, such that $\sum_{i=1}^n \xi_i = 1$ and such that the support of each ξ_i is contained in the interior of a flow box B_i for $\mathcal{F}$ in $PT_0(S)$; namely, there is for each i a homeomorphism $\sigma_i : D_i \times [0, 1] \to B_i$ for some space D_i, such that each $\sigma_i(g \times [0, 1])$ is contained in a leaf of $\mathcal{F}$. Lifting the situation to the universal covering $\widetilde{S}$ and assuming the B_i sufficiently small, we can identify D_i to a subset of $G(\widetilde{S})$, well-defined modulo the action of $\pi_1(S)$ on $G(\widetilde{S})$. Since $\alpha \in \mathcal{C}(S)$ is invariant under the action of $\pi_1(S)$, it follows that α induces a measure α on D_i. Then,

$$l_m(\alpha) = \sum_{i=1}^{n} \int_{D_i} \int_0^1 \xi_i(\sigma_i(g, t))\, dm(t)\, d\alpha(g) = \sum_{i=1}^{n} \alpha(\varphi_i)$$

where $\varphi_i : G(\widetilde{S}) \to \mathbb{R}$ is the continuous map with compact support defined by $\varphi_i(g) = \int_0^1 \xi_i(\sigma_i(g, t))\, dm(t)$ when $g \in D_i \subset G(\widetilde{S})$ and $\varphi_i(g) = 0$ otherwise.

It turns out that the last term of this formula makes sense when α is only a Hölder distribution, provided that we choose the σ_i and ξ_i Hölder continuous (which we can always assume). Indeed, under this regularity hypothesis, it follows from the definition of the Hölder structure of $G(\widetilde{S})$ that the maps φ_i are Hölder continuous. Therefore, given $\alpha \in \mathcal{H}(S)$, we can define

$$l_m(\alpha) = \sum_{i=1}^{n} \alpha(\varphi_i).$$

By linearity of the formula and by invariance of α under the action of $\pi_1(S)$, this is clearly independent of the choice of the ξ_i, σ_i and lifts of B_i to $PT(\widetilde{S})$. Also, this $l_m(\alpha)$ is a continuous function of α by definition of the topology of $\mathcal{H}(S)$. This proves:

THEOREM 24. – *Given a metric m of negative curvature and with totally geodesic boundary on S, there is a continuous linear function $l_m : \mathcal{H}(S) \to \mathbb{R}$ such that, when $\alpha \in \mathcal{C}(S) \subset \mathcal{H}(S)$ corresponds to a closed geodesic λ endowed with a weight a, $l_m(\alpha)$ is equal to a times the length of this m-geodesic.* □

In particular, by continuity and linearity of l_m:

COROLLARY 25. – *If $t \mapsto \alpha_t$ is a 1-parameter family of measured geodesic laminations which admits a right derivative $\dot{\alpha}_0 \in \mathcal{H}(S)$ at $t = 0$, then*

$$\frac{\partial}{\partial t^+} l_m(\alpha_t)_{|t=0} = l_m(\dot{\alpha}_0).$$

In particular, the length function $l_m : \mathcal{ML}(S) \to \mathbb{R}^+$ has a differential at each point of $\mathcal{ML}(S)$, and this differential is linear on each face of the piecewise linear structure of $\mathcal{ML}(S)$. □

In [Bo5], we give an explicit formula, based on the Thurston symplectic form on $\mathcal{H}(\lambda)$, which expresses this variation $l_m(\dot{\alpha}_0)$ in terms of the shear coordinates associated to m and to any maximal geodesic lamination λ containing the support of $\dot{\alpha}_0$.

7. Lengths of realizations of measured geodesic laminations

In this section, we consider an extension of §6 to higher dimensions (in practice, dimension 3 for applications). This also extends §6 to metrics with cusps on surfaces. For simplicity, we will restrict attention to hyperbolic metrics, namely complete Riemannian metrics of constant curvature -1.

Consider a hyperbolic manifold M endowed with an isomorphism $\pi_1(M) \cong \pi_1(S)$. Note that M will not be compact if its dimension is higher than 2. Let $f_0 : S \to M$ realize the isomorphism $\pi_1(M) \cong \pi_1(S)$. Then, for every closed curve γ in S which is not homotopic to 0, let $l_M(\gamma)$ be the infimum of the lengths of those curves in M which are homotopic to $f_0(\gamma)$. Thurston proved that there is a (unique) continuous map $l_M : \mathcal{ML}(S) \to \mathbb{R}^+$ such that $l_M(\alpha) = a l_M(\lambda_\alpha)$ when α is a closed geodesic λ_α endowed with the Dirac transverse measure of weight $a > 0$; *see* [Th1], [Bo1, §5], and compare Proposition 28 below. This map $l_M : \mathcal{ML}(S) \to \mathbb{R}^+$ played an important rôle in his analysis of the structure of open hyperbolic 3-manifolds [Th1] (*see* also [Bo1]). We want to compute the differential of this map.

To do so, we will follow Thurston and use the notion of realization of a geodesic lamination. The geodesic lamination λ on S is ***realized*** by the map $f : S \to M$ if f is homotopic to f_0 and if f sends each leaf of λ to a geodesic of M. The geodesic lamination λ is ***realizable*** if it can be realized by some map $f : S \to M$. When λ is the geodesic lamination underlying some $\alpha \in \mathcal{ML}(S)$, it can be shown that λ is realizable if and only if $l_M(\alpha_1) > 0$ for every connected component α_1 of α; *see* [Th1, §8], [CEG], [Bo1, §5].

As usual, let tildes $\tilde{}$ represent lifts and/or preimages to universal coverings. In particular, $G(\widetilde{M}) \cong (\widetilde{M}_\infty \times \widetilde{M}_\infty - \Delta)/\mathbb{Z}_2$ will denote the set of geodesics of the universal covering $\widetilde{M}$ of M. Choosing a base point in $\widetilde{M}$ identifies the sphere at infinity to the sphere of unit tangent vectors at this base point, and therefore endows $\widetilde{M}_\infty$ with an 'angle' metric. By an easy estimate of hyperbolic geometry, the Hölder equivalence class of this metric is independent of the choice of base point. This defines a natural Hölder structure on $G(\widetilde{M}) \cong (\widetilde{M}_\infty \times \widetilde{M}_\infty - \Delta)/\mathbb{Z}_2$. Beware that, unlike in the case of S, the dynamics of the action of $\pi_1(M)$ on $G(\widetilde{M})$ for this Hölder structure depend on the hyperbolic metric of M, if we do not have any further hypothesis on this metric.

If A is a subset of $G(\widetilde{S})$, we will say that a geodesic $g \in G(\widetilde{S})$ is ε-***tracked*** by A if, for every $x \in g \subset \widetilde{S}$, there is a geodesic $h \in A$ and a point $y \in h$ such that the direction of g at x and the direction of h at y are within ε from each other in the projective tangent bundle of $\widetilde{S}$, for the metric induced on this bundle by the base metric m on S. The set of those geodesics of $\widetilde{S}$ which are ε-tracked by A will be denoted by $G(\widetilde{S}; A, \varepsilon)$.

LEMMA 26. – *Let $f : S \to M$ realize the geodesic lamination λ, and let $\widetilde{f} : \widetilde{S} \to \widetilde{M}$ lift f. Then, there is an $\varepsilon > 0$ such that, for every geodesic g which is ε-tracked by $\widetilde{\lambda}$, there is a unique geodesic g^* of $\widetilde{M}$ which stays at bounded distance from $\widetilde{f}(g)$. In addition, the map $r : G(\widetilde{S}; \widetilde{\lambda}, \varepsilon) \to G(\widetilde{M})$ defined by $r(g) = g^*$ is proper and, for every compact subset K of $G(\widetilde{S})$, the restriction of r is Hölder bicontinuous from $K \cap G(\widetilde{S}; \widetilde{\lambda}, \varepsilon)$ to its image in $G(\widetilde{M})$.*

Proof. – Choose an $\varepsilon > 0$, which we will later adapt to our needs.

Since $\widetilde{f}$ is proper and commutes with the actions of $\pi_1(S)$, there is a constant C such that the images of two points which are at least C apart in $\widetilde{S}$ are at least 1 apart in $\widetilde{M}$.

Let $g \in G(\widetilde{S})$ be ε-tracked by $\widetilde{\lambda}$. It is possible to decompose g as a union of intervals I_n, $n \in \mathbb{Z}$, of length $3C$ such that each I_n overlaps with I_{n+1} over a length of C, and such that each I_n is at Hausdorff distance at most ε_1 from an interval I'_n contained in a geodesic of $\widetilde{\lambda}$, where ε_1 depends on ε and tends to 0 as ε tends to 0.

By uniform continuity of f, $\widetilde{f}(I_n)$ is at Hausdorff distance at most ε_2 from $\widetilde{f}(I'_n)$ in $\widetilde{M}$, where ε_2 is a constant which tends to 0 as ε tends to 0. Since I_n overlaps with I_{n+1}, there are intervals $I^+_n \subset I'_n$ and $I^-_{n+1} \subset I'_{n+1}$ whose images under $\widetilde{f}$ are at Hausdorff distance at most $2\varepsilon_2$ from each other. In addition, by choice of the constant C, the length of the arcs $\widetilde{f}(I^-_n)$, $\widetilde{f}(I^+_n)$ and $\widetilde{f}(I'_n - (I^-_n \cup I^+_n))$ is at least $1 - \varepsilon_2$.

Since f realizes λ, the $\widetilde{f}(I'_n)$ are geodesic in $\widetilde{M}$. We can therefore connect the central parts $\widetilde{f}(I'_n - (I^-_n \cup I^+_n))$ by geodesic arcs to form a piecewise geodesic curve g' in $\widetilde{M}$ which is at Hausdorff distance at most $2\varepsilon_2$ from $\widetilde{f}(g)$, and which is made up of geodesic arcs of length at least $1 - 2\varepsilon_2$ meeting at angles at least $\pi - \delta$, where δ is a constant tending to 0 as ε tends to 0. A major property of negatively curved manifolds is that, for δ sufficiently small, such a curve g' stays at uniformly bounded distance from a geodesic g^* of $\widetilde{M}$ (*see* for instance [CEG, Theorem 4.2.10]); in addition, this geodesic is unique since no two distinct geodesics of $\widetilde{M}$ stay at bounded distance from each other. Therefore, for ε sufficiently small, the image $\widetilde{f}(g)$ of any geodesic g which is ε-tracked by $\widetilde{\lambda}$ stays at uniformly bounded distance from a unique geodesic g^* of $\widetilde{M}$.

By construction, the Hausdorff distance between $g^* = r(g)$ and $\widetilde{f}(g)$ is uniformly bounded. It follows that $r : G(\widetilde{S}; \widetilde{\lambda}, \varepsilon) \to G(\widetilde{M})$ is proper.

Given a constant D, the angle between two geodesic rays of $\widetilde{M}$ issued from a base point $\widetilde{x}_0 \in \widetilde{M}$ is bounded above and below by two functions of type AB^{-d}, where d is the distance after which these two geodesic rays stay at least D apart, and where $A > 0$ and $B > 1$ are constants depending only on D. It follows that, if h and h' are two geodesic rays originating from points at distance at most $D/2$ from $\widetilde{x}_0$, the angles under which their end points in $\widetilde{M}_\infty$ are seen from $\widetilde{x}_0$ are also bounded above and below by two functions of this type. Since a similar estimate holds in $\widetilde{S}$ and since the Hausdorff distance between $g^* = r(g)$ and $\widetilde{f}(g)$ is uniformly bounded, it immediately follows that the map $r : g \mapsto g^*$ is Hölder bicontinuous on every $K \cap G(\widetilde{S}; \widetilde{\lambda}, \varepsilon)$ with K compact in $G(\widetilde{S})$. □

We will say that the geodesic Hölder current $\alpha \in \mathcal{H}(S)$ is ε-***tracked*** by the geodesic lamination λ if every geodesic of the support of α in $G(\widetilde{S})$ is ε-tracked by $\widetilde{\lambda}$. Let $\mathcal{H}(S; \lambda, \varepsilon)$ denote the set of $\alpha \in \mathcal{H}(S)$ which are ε-tracked by λ.

Let $\mathcal{H}(M)$ denote the set of $\pi_1(M)$-invariant Hölder distributions on $G(\widetilde{M})$.

Consider the geodesic current $\alpha \in \mathcal{C}(S) \subset \mathcal{H}(S)$ associated to a closed geodesic γ of S and to a weight $a > 0$. Namely, α is the Dirac measure of weight a defined by the discrete closed subset of $G(\widetilde{S})$ consisting of all possible lifts of γ. Then, α defines an element α^* of $\mathcal{H}(M)$ as follows: If $f(\gamma)$ is homotopic to a closed geodesic γ^* of M, α^* is the Dirac measure of weight a defined by the discrete closed subset of $G(\widetilde{M})$ consisting of all possible lifts of γ^*; otherwise, $f(\gamma)$ can be homotoped to arbitrarily short curves, and $\alpha^* = 0$.

LEMMA 27. – *Under the hypotheses and conclusions of Lemma 26, there is a continuous map $\rho : \mathcal{H}(S; \lambda, \varepsilon) \to \mathcal{H}(M)$ such that, when α is defined by a weighted closed geodesic which is ε-tracked by λ, $\rho(\alpha)$ is the element α^* associated to α as above. In addition, there is a compact subset M_0 of M such that every geodesic of the support of some $\rho(\alpha)$ with $\alpha \in \mathcal{H}(S; \lambda, \varepsilon)$ is contained in the preimage of M_0 in M.*

Proof. – Let $r : G(\widetilde{S}; \widetilde{\lambda}, \varepsilon) \to G(\widetilde{M})$ be the map $g \mapsto g^*$ defined by Lemma 26. Consider $\alpha \in \mathcal{H}(S; \lambda, \varepsilon)$. If $\varphi : G(\widetilde{M}) \to \mathbb{R}$ is Hölder continuous with compact support, the composition $\varphi \circ r$ is also Hölder continuous with compact support on $G(\widetilde{S}; \widetilde{\lambda}, \varepsilon)$, by Lemma 26. We can then define $\rho(\alpha)(\varphi) = \alpha(\varphi \circ r)$ since the support of α is contained in $G(\widetilde{S}; \widetilde{\lambda}, \varepsilon)$. The Hölder distribution $\rho(\alpha)$ so defined is $\pi_1(M)$-invariant, and clearly depends continuously on α.

The fact that $\rho(\alpha) = \alpha^*$ when α is associated to a weighted closed geodesic ε-tracked by λ is immediate from the definitions.

For every $g \in G(\widetilde{S}; \widetilde{\lambda}, \varepsilon)$, the geodesic $r(g) \in G(\widetilde{M})$ stays at uniformly bounded distance from $\widetilde{f}(g)$. Its projection to M therefore stays in a compact neighborhood M_0 of $f(S)$. Since the support of each $\rho(\alpha)$ is contained in $r\big(G(\widetilde{S}; \widetilde{\lambda}, \varepsilon)\big)$, this concludes the proof. □

Given a compact subset M_0 of M, let $\mathcal{H}(M_0)$ denote the set of those $\pi_1(M)$-invariant Hölder distributions α on $G(\widetilde{M})$ such that every geodesic of the support of α is contained in the preimage of M_0. Then, we can define a length function l_M on $\mathcal{H}(M_0)$ by a formula analogous to the one used in §6.

Namely, consider the projective tangent bundle $PT(M)$, endowed with its geodesic foliation $\mathcal{F}$. Cover a neighborhood of the preimage of M_0 in $PT(M)$ by finitely many flow boxes B_i, $i = 1, \dots, n$; namely, there is for each i a Hölder bicontinuous homeomorphism $\sigma_i : D_i \times [0,1] \to B_i$ for some space D_i, such that each $\sigma_i(g \times [0,1])$ is contained in a leaf of $\mathcal{F}$. For each i, choose a Hölder continuous function $\xi_i : PT(M) \to \mathbb{R}$ with support contained in the interior of B_i, in such a way that $\sum_{i=1}^n \xi_i = 1$ on the preimage of M_0 in $PT(M)$. Lift each flow box B_i to the universal covering $\widetilde{M}$ so that, assuming these flow boxes small enough, D_i becomes identified to a subset of $G(\widetilde{M})$.

Then, for $\alpha \in \mathcal{H}(M_0)$, define

$$l_M(\alpha) = \sum_{i=1}^n \int_{D_i} \int_0^1 \xi_i(\sigma_i(g,t))\, dm(t)\, d\alpha(g) = \sum_{i=1}^n \alpha(\varphi_i)$$

where m is the metric of $\widetilde{M}$ and where $\varphi_i : G(\widetilde{M}) \to \mathbb{R}$ is the Hölder continuous map with compact support defined by $\varphi_i(g) = \int_0^1 \xi_i(\sigma_i(g,t))\, dm(t)$ when $g \in D_i \subset G(\widetilde{M})$ and $\varphi_i(g) = 0$ otherwise.

The map $l_M : \mathcal{H}(M_0) \to \mathbb{R}$ so defined is clearly continuous.

If γ is a closed geodesic of M which is contained in M_0, the set of all possible lifts of γ to $\widetilde{M}$ forms a discrete closed subset of $G(\widetilde{M})$. Then, the Dirac measure of weight 1 defined by this discrete closed subset provides an element α of $\mathcal{H}(M_0)$. For this α, it is immediate that $l_M(\alpha)$ is exactly the length of γ.

Combining this with Lemmas 26 and 27, we get:

PROPOSITION 28. – *Let the geodesic lamination λ be realized by some map $f : S \to M$. Then, there is an $\varepsilon > 0$ and a continuous linear function $l_M : \mathcal{H}(S;\lambda,\varepsilon) \to \mathbb{R}$ such that $l_M(\alpha) = a l_M(\lambda_\alpha)$ when $\alpha \in \mathcal{C}(S) \cap \mathcal{H}(S;\lambda,\varepsilon)$ is a closed geodesic λ_α endowed with the Dirac transverse measure of weight $a > 0$.* □

The measured geodesic laminations corresponding to simple closed geodesics endowed with Dirac transverse measures of positive weights are dense in the space $\mathcal{ML}(S)$ (*see* [Th1, §8]). It follows that the function l_M of Proposition 28 coincides with Thurston's function $l_M : \mathcal{ML}(S) \to \mathbb{R}^+$ on $\mathcal{ML}(S) \cap \mathcal{H}(S;\lambda,\varepsilon)$.

A geodesic lamination λ in S is ***realizable in*** M if it can be realized by some map $f : S \to M$ homotopic to the given map $f_0 : S \to M$. It can be shown that λ is realizable if and only if each of its minimal sublaminations is realizable; *see* [Th1, §8], [CEG, §4.2]. When M has dimension at most 3, the analysis of [Th1] (*see* also [Bo1]) shows that there are only finitely many non-realizable minimal geodesic laminations; the non-realizable closed geodesics of S are exactly those whose image under f_0 is homotopic to a cusp of M; the other non-realizable minimal laminations are the ending laminations of geometrically infinite ends of M.

We will say that a path $t \mapsto \alpha_t \in \mathcal{ML}(S)$, $t \in [0, t_0]$, has a ***strong tangent vector*** at 0^+ if it has a tangent vector $\dot\alpha_0$ for the piecewise linear structure of $\mathcal{ML}(S)$ and if, as t tends to 0^+, the essential support λ_{0^+} of the α_t is equal to the Hausdorff limit of the geodesic laminations λ_t underlying the α_t. As seen in §2, the essential support is always contained in this Hausdorff limit, if it exists, but is not necessarily equal to it. However, by Proposition 4, any piecewise linear path $t \mapsto \alpha_t$ has a strong tangent vector at 0^+.

THEOREM 29. – *Let $t \mapsto \alpha_t \in \mathcal{ML}(S)$, $t \in [0, t_0]$, be a 1-parameter family of measured laminations admitting a strong tangent vector $\dot\alpha_0$ at $t = 0$. Assume that the geodesic lamination λ_0 underlying α_0 is realizable in M, and decompose λ_{0+} as the union of its realizable components $\lambda_{0+}^{\mathrm{r}}$ and of its non-realizable components $\lambda_{0+}^{\mathrm{nr}}$. Then, the function $t \mapsto l_M(\alpha_t)$ admits a right derivative at 0 and, if $\dot\alpha_0^{\mathrm{r}}$ is the restriction of the Hölder distribution $\dot\alpha_0$ to $\lambda_{0+}^{\mathrm{r}}$, this derivative is equal to the length $l_M(\dot\alpha_0^{\mathrm{r}})$ defined by Proposition 28.*

Proof. – As indicated above, a geodesic lamination is realizable if and only if each of its minimal sublaminations is realizable [Th1, §8][CEG, §4.2]. By Theorem 21, λ_{0+} is the union of λ_0, of some infinite isolated leaves, and of minimal sublaminations which are not in the closure of any isolated infinite leaf. Since λ_0 is realizable, it follows that $\lambda_{0+}^{\mathrm{nr}}$ consists only of minimal sublaminations which are connected components of λ_{0+}, and that λ_0 is contained in $\lambda_{0+}^{\mathrm{r}}$.

By hypothesis, the geodesic lamination λ_t underlying α_t converges to λ_{0+} for the Hausdorff metric. Therefore, for $\varepsilon > 0$ sufficiently small, α_t naturally splits as the sum of two measured geodesic laminations α_t^{r} and α_t^{nr} which are respectively ε-tracked by $\lambda_{0+}^{\mathrm{r}}$ and $\lambda_{0+}^{\mathrm{nr}}$, for t sufficiently close to 0 (depending on ε).

Applying Proposition 28 to the realizable geodesic lamination $\lambda_{0+}^{\mathrm{r}}$, we get for $\varepsilon > 0$ small enough a linear continuous map $l_M : \mathcal{H}(S; \lambda_{0+}^{\mathrm{r}}, \varepsilon) \to \mathbb{R}$ such that $l_M(\alpha)$ coincides with Thurston's length for every measured geodesic lamination $\alpha \in \mathcal{ML}(S) \cap \mathcal{H}(S; \lambda_{0+}^{\mathrm{r}}, \varepsilon)$. Then, by linearity and continuity of l_M for the topology of $\mathcal{H}(S; \lambda_{0+}^{\mathrm{r}}, \varepsilon)$,

$$\begin{aligned}\frac{\partial}{\partial t^+} l_M(\alpha_t^{\mathrm{r}})_{|t=0} &= \lim_{t\to 0^+} \frac{l_M(\alpha_t^{\mathrm{r}}) - l_M(\alpha_0)}{t} = \lim_{t\to 0^+} l_M\left(\frac{\alpha_t^{\mathrm{r}} - \alpha_0}{t}\right)\\ &= l_M\left(\lim_{t\to 0^+} \frac{\alpha_t^{\mathrm{r}} - \alpha_0}{t}\right) = l_M(\dot\alpha_0^{\mathrm{r}}).\end{aligned}$$

Since the components $\lambda_{0+}^{\mathrm{nr}}$ are not realizable, the shortening process of [Bo1, §5] proves the following: For every $\eta > 0$, there is a map $f : S \to M$ homotopic to f_0 and a train track τ carrying $\lambda_{0+}^{\mathrm{nr}}$ such that, for every measured geodesic lamination α carried by τ, the length of $f(\alpha)$ (suitably defined) is less than $\eta l_m(\alpha)$ where m is any fixed metric of negative curvature on S; in particular, $l_M(\alpha) \leqslant \eta l_m(\alpha)$. Note that α_t^{nr} is carried by this τ for t sufficiently close to 0. Also, if τ_0 is a fixed train track carrying $\lambda_{0+}^{\mathrm{nr}}$ and disjoint from $\lambda_{0+}^{\mathrm{r}}$, the length $l_m(\alpha_t^{\mathrm{nr}})$ is bounded by a constant (depending on the m-length of the edges of τ_0) times $\|\alpha_t^{\mathrm{nr}}\|_{\tau_0}$. Since λ_0 is disjoint from $\lambda_{0+}^{\mathrm{nr}}$, $\|\alpha_t^{\mathrm{nr}}\|_{\tau_0} = O(t)$ and it follows that $l_m(\alpha_t^{\mathrm{nr}}) = O(t)$. This proves that, for every $\eta > 0$, $l_M(\alpha_t^{\mathrm{nr}}) \leqslant \eta O(t)$ for t sufficiently close to 0, where the constant hidden in the symbol $O(\ )$ is independent of η. As a consequence, the right derivative of $t \mapsto l_M(\alpha_t^{\mathrm{nr}})$ at $t = 0$ is equal to 0.

Since $l_M(\alpha_t) = l_M(\alpha_t^{\mathrm{r}}) + l_M(\alpha_t^{\mathrm{nr}})$, this concludes the proof that the right derivative of $t \mapsto l_M(\alpha_t)$ at $t = 0$ is equal to $l_M(\dot\alpha_0^{\mathrm{r}})$. □

Since every tangent vector of $\mathcal{ML}(S)$ is tangent to a piecewise linear path, and since every piecewise linear path has a strong tangent vector everywhere, this proves:

COROLLARY 30. – *The length function* $l_M : \mathcal{ML}(S) \longrightarrow \mathbb{R}^+$ *has a differential at each point of* $\mathcal{ML}(S)$ *whose underlying geodesic lamination is realizable, and this differential is linear on each face of the piecewise linear structure of* $\mathcal{ML}(S)$. □

With a little more work the hypothesis that the path $t \mapsto \alpha_t$ has a strong tangent vector at $t = 0$ can be removed from Theorem 29. However, this stronger version does not seem to be of sufficient interest to justify giving the proof here.

On the other hand, simple examples show that the conclusions of Theorem 29 and Corollary 30 fail at measured geodesic laminations whose geodesic laminations are non-realizable.

The results of Proposition 28 and Theorem 29 are particularly interesting if we realize geodesic laminations by pleated surfaces. A ***pleated surface*** is a map $f : S \longrightarrow M$ which realizes some geodesic lamination λ and which is totally geodesic on the complement $S - \lambda$; *see* [Th1], [CEG] for details. It turns out that the path metric m obtained by pulling back the metric of M by such an f is hyperbolic. If we compare the definitions of the functions l_m and l_M in §6 and Proposition 28, then we immediately see:

LEMMA 31. – *Let* $f : S \longrightarrow M$ *be a pleated surface realizing the geodesic lamination* λ, *and let* m *be the metric obtained by pulling back the metric of* M *by* f. *Then, for every transverse Hölder distribution* α *for* λ, $l_M(\alpha) = l_m(\alpha)$. □

As a corollary:

PROPOSITION 32. – *Under the hypotheses of Theorem 29, let* $f : S \longrightarrow M$ *be a pleated surface realizing the geodesic lamination* λ_{0+}, *and let* m *be the metric obtained by pulling back the metric of* M *by* f. *Then, the two functions* $t \mapsto l_M(\alpha_t)$ *and* $t \mapsto l_m(\alpha_t)$ *have the same value and the same derivative at 0.* □

In particular, up to first order, the length $l_M(\alpha_t)$ depends only on the pull back metric m, and not on the bending between the totally geodesic pieces of the pleated surface.

REFERENCES

[BiS] J. S. BIRMAN, C. Series, *Geodesics with bounded intersection numbers on surfaces are sparsely distributed* (*Topology*, Vol. 24, 1985, pp. 217-225).

[Bo1] F. BONAHON, *Bouts des variétés hyperboliques de dimension* 3 (*Ann. of Math.*, Vol. 124, 1986, pp. 71-158).

[Bo2] F. BONAHON, *The geometry of Teichmüller space via geodesic currents* (*Invent. Math.*, Vol. 92, 1988, pp. 139-162).

[Bo3] F. BONAHON, *Ensembles limites et applications* (45 minute address), Proceedings of the International Congress of Mathematicians, Kyoto, Japan, 1990 Springer-Verlag, Berlin Heidelberg, New York, 1991, pp. 599-608.

[Bo4] F. BONAHON, *Transverse Hölder distributions for geodesic laminations*, to appear in Topology.

[Bo5] F. BONAHON, *Shearing hyperbolic surfaces, bending pleated surfaces, and Thurston's, symplectic form* (*Ann. Fac. Sci.* Toulouse, Vol. 5, 1996, pp. 233-297).

[Bo6] F. BONAHON, *Variations of the boundary geometry of convex cores of hyperbolic 3-manifolds*, preprint, 1996.

[Bo7] F. BONAHON *A Schläfli-type formula for the volume of convex cores of hyperbolic 3-manifolds*, in preparation.

[CEG] R. D. CANARY, D. B. A. EPSTEIN and P. GREEN, *Notes on notes of Thurston* (*Analytical and Geometrical aspects of Hyperbolic space*, D. B. A. Epstein ed., LMS *Lecture Notes Series* Vol. 111 , 1987 Cambridge University Press, pp. 3-92).

[CaB] A. CASSON and S. A. BLEILER, *Automorphisms of surfaces after Thurston and Nielsen,* 1988 Cambridge University Press.

[EpM] D. B. A. EPSTEIN and A. MARDEN, *Convex hulls in hyperbolic spaces, a theorem of Sullivan, and measured pleated surfaces* (*Analytical and geometric aspects of hyperbolic space*, D. B. A. Epstein ed., 1986, Cambridge University Press, pp. 113-253).

[FLP] A. FATHI, F. LAUDENBACH and V. POENARU, *Travaux de Thurston sur les surfaces* (*Astérisque* 66-67, Société Mathématique de France, 1979).

[Fl] W. J. FLOYD, *Group completions and limit sets of Kleinian groups* (*Invent. Math.*, Vol. 57, 1980, pp. 205-218).

[Gh] E. GHYS, *Le cercle à l'infini des surfaces à courbure négative*, (45 minute address) (*Proceedings of the International Congress of Mathematicians*, Kyoto, Japan, 1990, Springer-Verlag, Berlin, Heidelberg, New York, 1991, pp. 501-509).

[Gr] M. GROMOV, *Hyperbolic groups*, (Essays in group theory, S. M. Gersten ed., MSRI Publications, Vol. 8, Springer-Verlag, Berlin, Heidelberg, New York, 1987, pp. 75-263).

[Ka] A. KATOK, *Invariant measures of flows on oriented surfaces*, (*Soviet Math. Dokl.*, Vol. 14, 1973, pp. 1104-1108).

[Ke] S. P. KERCKHOFF, *The Nielsen realization problem* (*Ann. of Math.*, Vol. 117, 1983, pp. 235-265).

[Le] G. LEVITT, *Foliations and laminations on hyperbolic surfaces* (*Topology*, Vol. 22, 1983, pp. 119-135).

[Mo] H. M. MORSE, *A one-to-one representation of geodesics on a surface of negative curvature* (*Amer. J. Math.*, Vol. 43, 1921, pp. 33-51).

[Pa] A. PAPADOPOULOS, *Geometric intersection functions and Hamiltonian flows on the space of measured foliations of a surface* (*Pacific J. Math.*, Vol. 124, 1986, pp. 375-402).

[PeH] R. C. PENNER and J. L. HARER, *Combinatorics of train tracks* (*Ann. Math. Studies*, Vol. 125, Princeton Univ. Press, 1992).

[Pl] J. PLANTE, *Foliations with measure preserving holonomy* (*Ann. Math.*, Vol. 127, 1975, pp. 327-362).

[Th1] W. P. THURSTON, *The topology and geometry of 3-manifolds*, *Lecture Notes*, Princeton University, 1976-1979).

[Th2] W. P. THURSTON, *Earthquakes in two-dimensional hyperbolic geometry* (*Low-dimensional Topology and Kleinian groups*, D. B. A. Epstein ed., 1986 Cambridge University press, pp. 91-112).

[Th3] W. P. THURSTON, *On the geometry and dynamics of diffeomorphisms of surfaces* (*Bull. Amer. Math. Soc.*, Vol. 19 , 1988, pp. 417-431).

[Th4] W. P. THURSTON, *Minimal stretch maps between hyperbolic surfaces*, unpublished preprint, *ca* 1986.

(Manuscript received March 10, 1994;
accepted October 1, 1996.)

F. BONAHON
Department of Mathematics,
University of Southern California,
Los Angeles CA 90089-1113, U.S.A.
E-mail FBonahon@math.USC.edu