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THE COHOMOLOGY OF PERIOD DOMAINS FOR REDUCTIVE GROUPS OVER FINITE FIELDS

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ABSTRACT. – The goal of this paper is to give an explicit formula for the ℓ-adic cohomology of period domains over finite fields for arbitrary reductive groups. The result is a generalisation of the computation in [6] which treats the case of the general linear group GL_n. © 2001 Éditions scientifiques et médicales Elsevier SAS

RÉSUMÉ. – Dans cet article, nous donnons une formule explicite pour la cohomologie des domaines de périodes sur un corps fini dans le cas d'un groupe réductif quelconque. Le résultat est une généralisation du calcul fait en [6], qui traite le cas du groupe linéaire GL_n. © 2001 Éditions scientifiques et médicales Elsevier SAS

1. Introduction

Let $k = \mathbb{F}_q$ be a finite field, and let G be a reductive algebraic group defined over k of semisimple k-rank $d = k\text{-rk}_{\mathrm{ss}}(G)$. Fix a pair $S \subset B$ consisting of a maximal k-split torus S and a Borel subgroup B defined over k. The centralizer $Z_G(S)$ of S in G is a maximal torus T, since G is quasisplit. We denote by R the roots, by R^+ the positive roots and by

$$\Delta = \{\alpha_1, \ldots, \alpha_d\}$$

the basis of simple roots of G with respect to $S \subset B$. Let k' be a finite field extension of k over which G splits, and let $\overline{k}$ be an algebraic closure of k. We denote by $\Gamma = \mathrm{Gal}(k'/k)$, respectively $\Gamma_k = \mathrm{Gal}(\overline{k}/k)$ the associated Galois groups. If $\lambda \in X_*(T)_{\mathbb{Q}}$ is a rational cocharacter we will denote by

$$P(\lambda) = \bigl\{g \in G;\ \lim_{t \to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists in } G\bigr\}$$

the associated parabolic subgroup, and by

$$U(\lambda) = \bigl\{g \in G;\ \lim_{t \to 0} \lambda(t) g \lambda(t)^{-1} = 1\bigr\}$$

its unipotent radical.

Fix a conjugacy class

$$\{\mu\} \subset X_*(G)$$

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of one-parameter subgroups (1-PS) of G, where μ denotes a representative lying in $X_*(T)$. Let

$$E=\{x\in\overline{k};\ \sigma(x)=x,\ \text{for all}\ \sigma\in\mathrm{Stab}_{\Gamma_k}(\{\mu\})\}$$

be the Shimura field of $\{\mu\}$, an intermediate field of k'/k. According to a lemma of Kottwitz [3], Lemma 1.1.3, we can suppose that $\mu\in\{\mu\}$ is defined over E. Hence the conjugacy class $\{\mu\}$ defines a flag variety

$$\mathscr{F}:=\mathscr{F}(G,\{\mu\}):=G/P(\mu)$$

over $\overline{k}$ that is defined over E. Notice that the geometric points of $\mathscr{F}$ coincide with the set

$$\{\mu\}/\sim,$$

where $\lambda_1,\lambda_2\in\{\mu\}$ are equivalent, written $\lambda_1\sim\lambda_2$, if there exists a point $g\in\mathrm{U}(\lambda_1)$ with $\mathrm{Int}(g)\circ\lambda_1=\lambda_2$. Here

$$\begin{aligned}\mathrm{Int}(g):G&\longrightarrow G,\\ h&\longmapsto ghg^{-1}\end{aligned}$$

is the inner automorphism of G, which is induced by g. Finally, we set $\Gamma_E:=\mathrm{Gal}(k'/E)$.

In the further text we identify a variety with the set of its closed points. Let $x\in\mathscr{F}$ be a point which is represented by a 1-PS λ. It is well-known that λ induces for every G-module V over $\overline{k}$ a descending $\mathbb{Z}$-filtration $\mathcal{F}^\bullet_\lambda(V)$ on V. In fact, let $V=\bigoplus V_\lambda(i)$ be the associated $\mathbb{Z}$-grading. Then $\mathcal{F}^\bullet_\lambda(V)$ is given by

$$\mathcal{F}^i_\lambda(V)=\bigoplus_{j\geqslant i}V_\lambda(j),\quad i\in\mathbb{Z}.$$

As this filtration depends only on x, we denote this filtration by $\mathcal{F}^\bullet_x(V)$. Considering the adjoint action of G on its Lie algebra $\mathrm{Lie}\,G$, we get in particular a filtration $\mathcal{F}^\bullet_x:=\mathcal{F}^\bullet_x(\mathrm{Lie}\,G)$ on $\mathrm{Lie}\,G$. We will say that x is semistable if the filtered vector space $(\mathrm{Lie}\,G,\mathcal{F}^\bullet_x)$ is semistable. For the latter definition of semistability see [8–10] or [6], Definition 1.13. Following [10], the semistable points of $\mathscr{F}$ form an open subvariety

$$\mathscr{F}^{\mathrm{ss}}:=\mathscr{F}(G,\{\mu\})^{\mathrm{ss}},$$

which is called the period domain with respect to G and $\{\mu\}$. It is defined over E and is supplied with an action of $G(k)$. In his paper [12], Totaro has shown that there exists a relationship to the concept of semistability in Geometric Invariant Theory introduced by Mumford [5]. We shall explain this relationship in Section 2.

Choose an invariant inner positive definite product on G; i.e. we have for all maximal tori T in G a non-degenerate positive definite pairing ( ) on $X_*(T)_{\mathbb{Q}}$, such that the natural maps

$$X_*(T)_{\mathbb{Q}}\longrightarrow X_*(T^g)_{\mathbb{Q}},$$

induced by conjugating with $g\in G(k)$ and

$$X_*(T)_{\mathbb{Q}}\longrightarrow X_*(T^\sigma)_{\mathbb{Q}}$$

induced by conjugating with $\sigma\in\Gamma_k$ are isometries for all $g\in G(k)$, $\sigma\in\Gamma_k$. Here $T^g=gTg^{-1}$ is the conjugate torus, respectively $T^\sigma=\sigma\cdot T$ is the image of T under the morphism $\sigma:G\to G$

induced by σ. The inner product, together with the natural pairing

$$\langle\ \rangle : X_*(T)_{\mathbb{Q}} \times X^*(T)_{\mathbb{Q}} \longrightarrow \mathbb{Q},$$

induces identifications

$$\begin{aligned} X_*(T)_{\mathbb{Q}} &\longrightarrow X^*(T)_{\mathbb{Q}}, \\ \lambda &\longmapsto \lambda^*, \end{aligned}$$

respectively

$$\begin{aligned} X^*(T)_{\mathbb{Q}} &\longrightarrow X_*(T)_{\mathbb{Q}}, \\ \chi &\longmapsto \chi^*, \end{aligned}$$

for all maximal tori T in G. We call λ^* the dual character of λ and χ^* the dual cocharacter of χ.

Before we can state the main result of this paper, we have to introduce a few more notations. Let $W = N(T)/T$ be the Weyl group of G, and let W_μ be the stabilizer of μ with respect to the action of W on $X_*(T)$. We denote by W^μ the set of Kostant-representatives with respect to W/W_μ. Consider the action of Γ_E on W. Since μ is defined over E, this action preserves W^μ. Denote the corresponding set of orbits by W^μ/Γ_E and its elements by $[w]$, where w is in W^μ. Clearly the length of an element in W only depends on its orbit. So the symbol $l([w])$ makes sense. For any orbit $[w]$ we set

$$\mathrm{ind}_{[w]} := \mathrm{Ind}^{\Gamma_E}_{\mathrm{Stab}_{\Gamma_E}(w)} \mathbb{Q}_\ell.$$

This induced representation is clearly independent of the specified representative. Let

$$\{\omega_{\alpha_1}, \ldots, \omega_{\alpha_d}\} \subset X_*(S)_{\mathbb{Q}}$$

be the dual basis of Δ, i.e. we have

$$\langle \omega_{\alpha_i}, \alpha_j \rangle = \delta_{ij}, \quad \forall\, i,\, j,\ \ 1 \leqslant i, j \leqslant d.$$

For every subset $I \subset \Delta$ we define

$$\Omega_I := \big\{ [w] \in W^\mu/\Gamma_E ; \langle w\mu, \omega^*_\alpha \rangle > 0,\ \text{for all } \alpha \notin I \big\}.$$

We get the following inclusion relation

$$I \subset J \Rightarrow \Omega_I \subset \Omega_J.$$

In the further text we denote for $[w] \in W^\mu/\Gamma_E$ by $I_{[w]}$ the smallest subset of Δ such that $[w]$ is contained in $\Omega_{I_{[w]}}$. Obviously, we have

$$I_{[w]} \subset I \Leftrightarrow [w] \in \Omega_I. \tag{1}$$

For a parabolic subgroup $P \subset G$ defined over k we consider the trivial representation of $P(k)$ on $\mathbb{Q}_\ell$. We denote by

$$i_P^G = \mathrm{Ind}^{G(k)}_{P(k)}(\mathbb{Q}_\ell)$$

the resulting induced representation of $G(k)$. Further we set

$$v_P^G = i_P^G / \sum_{P \subsetneq Q} i_Q^G.$$

In the case $P = B$, we get the Steinberg representation [11]. Finally, for any subset $I \subset \Delta$ we set

$$P_I := \bigcap_{I \subset \Delta - \{\alpha\}} P(\omega_\alpha).$$

This parabolic subgroup is defined over k since the ω_α are. Thus we can state the following theorem, which calculates the ℓ-adic cohomology with compact support of the period domain $\mathscr{F}^{\mathrm{ss}}$ as representation of the product $G(k) \times \Gamma_E$.

THEOREM 1.1. – *We have*

$$\mathrm{H}_\mathrm{c}^*(\mathscr{F}^{\mathrm{ss}}, \mathbb{Q}_\ell) = \bigoplus_{[w] \in W^\mu / \Gamma_E} v_{P_{I_{[w]}}}^G \otimes \mathrm{ind}_{[w]}\bigl(-l([w])\bigr)\bigl[-2l([w]) - \#(\Delta - I_{[w]})\bigr].$$

Here the symbol (n), $n \in \mathbb{N}$, means the nth Tate twist and $[-n]$, $n \in \mathbb{N}$, symbolizes that the corresponding module is shifted into degree n of the graded cohomology ring.

As in the case of the GL_{d+1} (*cf.* [6], Korollar 4.5) we can state the following result about the vanishing of some cohomology groups of these period domains. The proof of this corollary is similar to the GL_{d+1}-case.

COROLLARY 1.2. – *We have*

$$\mathrm{H}_\mathrm{c}^i\bigl(\mathscr{F}^{\mathrm{ss}}, \mathbb{Q}_\ell\bigr) = 0, \quad 0 \leqslant i \leqslant d-1,$$

and

$$\mathrm{H}_\mathrm{c}^d\bigl(\mathscr{F}^{\mathrm{ss}}, \mathbb{Q}_\ell\bigr) = v_B^G.$$

Theorem 1.1 has been conjectured by Kottwitz and Rapoport, who had calculated previously the Euler–Poincaré characteristic with compact support of these period domains in the Grothendieck group of $G(k) \times \Gamma_E$ representations (*cf.* [10]). The formula for the Euler–Poincaré characteristic is accordingly

$$\chi_\mathrm{c}(\mathscr{F}_g^{\mathrm{ss}}, \mathbb{Q}_\ell) = \sum_{[w] \in W^\mu / \Gamma_E} (-1)^{d - \# I_{[w]}} v_{P_{I_w}}^G \otimes \mathrm{ind}_{[w]}\bigl(-l([w])\bigr).$$

In the split case the formula of the theorem becomes

$$\mathrm{H}_\mathrm{c}^*(\mathscr{F}^{\mathrm{ss}}, \mathbb{Q}_\ell) = \bigoplus_{w \in W^\mu} v_{P_{I_w}}^G \bigl(-l(w)\bigr)\bigl[-2l(w) - \#(\Delta - I_w)\bigr],$$

which has been already calculated for $G = \mathrm{GL}_{d+1}$ in a slightly different way in [6].

2. The relationship of period domains to GIT

In this section we want to explain the relationship between period domains and Geometric Invariant Theory. For details we refer to the papers [12], respectively [10]. We mention, that

Totaro has described in his article [12] the theory of period domains in the case of local fields. But as the reader verifies easily, all the proofs and ideas work also in the case of finite fields.

Let

$$M := P(\mu)/U(\mu)$$

be the Levi-quotient of $P(\mu)$ with center Z_M. Then μ defines an element of $X_*(Z_M)$. Let T_M be a maximal torus in M. Then we have $Z_M \subset T_M$ and T_M is the isomorphic image of a maximal torus in G. So we get an invariant inner product on M. Consider the dual character $\mu^* \in X^*(T_M)_{\mathbb{Q}}$. As μ belongs to $X_*(Z_M)$, the dual character μ^* is contained in

$$X^*(M_{\mathrm{ab}})_{\mathbb{Q}} \cong \operatorname{Hom}\bigl(P(\mu), \mathbb{G}_m\bigr) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The inverse character $-\mu^*$ induces a homogeneous line bundle

$$\mathcal{L} := \mathcal{L}_{-\mu^*}$$

on $\mathcal{F}$. The reason for the sign is that this line bundle is ample.

Let $\lambda : \mathbb{G}_m \to G$ be a 1-PS of G. For any point $x \in \mathcal{F}$ we can consider the slope $\mu^{\mathcal{L}}(x, \lambda)$ of λ in x relative to the line bundle $\mathcal{L}$ (*cf.* [5], Definition 2.2). Now we are able to state the following theorem of Totaro (*cf.* [12], Theorem 3).

THEOREM 2.1 (Totaro). – *Let x be a point of $\mathcal{F}$. Then x is semistable if and only if $\mu^{\mathcal{L}}(x,\lambda) \geqslant 0$ for all 1-PS λ of G_{der} which are defined over k. Here G_{der} is the derived group of G.*

In order to investigate the GIT-semistability of points on varieties, it is useful to consider the spherical building of the given group. Let $B(G)_k$ be the k-rational spherical building of our fixed group G. Recall the definition of $B(G)_k$ (*cf.* [2]). For a maximal k-split torus S of G we consider first of all the space of rays

$$\bigl(X_*(S)_{\mathbb{R}} - \{0\}\bigr)/\mathbb{R}_{>0} := \bigl\{\mathbb{R}_{>0}\lambda;\ \lambda \in X_*(S)_{\mathbb{R}} - \{0\}\bigr\}$$

in $X_*(S)_{\mathbb{R}}$ starting in the origin. This space is homeomorphic to the $(r-1)$-sphere $\mathbb{S}^{r-1}$, where r is the k-rank of G. We can associate to every ray $\mathbb{R}_{>0}\lambda \in (X_*(S)_{\mathbb{R}} - \{0\})/\mathbb{R}_{>0}$ a well-defined parabolic subgroup $P(\mathbb{R}_{>0}\lambda)$ (*cf.* [2]), which is compatible with the old definition of $P(\lambda)$ with respect to a rational 1-PS $\lambda \in X_*(S)_{\mathbb{Q}}$. We also have a natural action of the k-rational points of $G(k)$ on the disjoint union $\coprod_{S\ k\text{-split}} (X_*(S)_{\mathbb{R}} - \{0\})/\mathbb{R}_{>0}$. We will say that two rays $\mathbb{R}_{>0}\lambda_1$, $\mathbb{R}_{>0}\lambda_2$ are equivalent, $\mathbb{R}_{>0}\lambda_1 \sim \mathbb{R}_{>0}\lambda_2$, if there exists an element $g \in P(\mathbb{R}_{>0}\lambda_1)(k)$ which transforms the one ray into the other. Finally, we set

$$B(G)_k := \Bigl(\coprod_{S\ k\text{-split}} \bigl(X_*(S)_{\mathbb{R}} - \{0\}\bigr)/\mathbb{R}_{>0}\Bigr)\Big/\sim$$

and supply this set with the induced topology. Again we can associate to every point $x \in B(G)_k$ a well-defined k-rational parabolic subgroup $P(x)$ of G. If S is any maximal k-split torus of G then we have a closed embedding

$$B(S)_k \hookrightarrow B(G)_k.$$

The space $B(S)_k$ is called the apartment belonging to S.

Assume for the remainder of this section that our group G is semisimple. In this case the space $B(G)_k$ is homeomorphic to the geometric realization of the combinatorial building (*cf.* [2], 6.1). Thus we have a simplicial structure on $B(G)_k$ which is defined as follows. For a k-rational parabolic subgroup $P \subset G$ we let

$$D(P) := \{x \in B(G)_k;\ P(x) \supset P\}$$

be the facette corresponding to P. If P is a minimal parabolic subgroup, i.e. a Borel subgroup as G is quasi-split, then we call $D(P)$ a chamber of $B(G)_k$. If in contrast P is a proper maximal subgroup, then $D(P)$ is called a vertex.

Consider the k-rational cocharacters ω_α, $\alpha \in \Delta$, introduced in the previous section. These cocharacters correspond to the vertices of the chamber $D_0 := D(B)$, since the $P(\omega_\alpha)$, $\alpha \in \Delta$, are the maximal k-rational parabolic subgroups that contain B. For any other chamber $D = D(P)$ in $B(G)_k$, there exists a $g \in G(k)$, such that the conjugated elements $\mathrm{Int}(g) \circ \omega_\alpha$, $\alpha \in \Delta$, correspond to the vertices of D. The element g is of course unique up to multiplication by an element of $B(k)$ from the right. Therefore we choose for the rest of this paper for every chamber D an element g_D with the above property. The element g_{D_0} should be of course the obvious one. With this choice, we define for every chamber D in $B(G)_k$ the simplex

$$\widetilde{D} := \left\{ \sum_{\alpha \in \Delta} r_\alpha \lambda_\alpha;\ 0 \leqslant r_\alpha \leqslant 1,\ \sum_{\alpha \in \Delta} r_\alpha = 1 \right\} \subset X_*\big(g_D S g_D^{-1}\big)_{\mathbb{R}},$$

which is the convex hull of the fixed set of representatives $\lambda_\alpha := g_D \omega_\alpha g_D^{-1} \in X_*(g_D S g_D^{-1})_{\mathbb{R}}$, $\alpha \in \Delta$. The topological spaces D and $\widetilde{D}$ are obviously homeomorphic. For the standard chamber D_0 we have in particular the description

$$\widetilde{D}_0 := \left\{ \sum_{\alpha \in \Delta} r_\alpha \omega_\alpha;\ 0 \leqslant r_\alpha \leqslant 1,\ \sum_{\alpha \in \Delta} r_\alpha = 1 \right\}.$$

We can extend $\mu^{\mathcal{L}}(x,\cdot)$ in a well-known way to a function on $X_*(T)_{\mathbb{R}}$ for every maximal torus T in G. Notice that the slope function $\mu^{\mathcal{L}}(x,\cdot)$ is not defined on D but on $\widetilde{D}$. In spite of this fact we will say that $\mu^{\mathcal{L}}(x,\cdot)$ is affine on D if it is affine on $\widetilde{D}$, i.e. if following equality holds:

$$\mu^{\mathcal{L}}\left(x, \sum_{\alpha \in \Delta} r_\alpha \lambda_\alpha\right) = \sum_{\alpha \in \Delta} r_\alpha \mu^{\mathcal{L}}(x, \lambda_\alpha) \quad \text{for all } \sum_{\alpha \in \Delta} r_a \lambda_\alpha \in \widetilde{D}.$$

It follows from [5], Proposition 2.7, that the definition of being affine does not depend on the chosen representatives $g_D \in G(k)$.

In the case of the special linear group we can calculate the slope of a point explicitly. If $\mathcal{F}$ and $\mathcal{F}'$ are two filtrations on a finite-dimensional vector space V we set

$$(\mathcal{F}, \mathcal{F}') = \sum_{\alpha, \beta \in \mathbb{Z}} \alpha\beta \dim \mathrm{gr}^{\alpha}_{\mathcal{F}}\big(\mathrm{gr}^{\beta}_{\mathcal{F}'}(V)\big).$$

LEMMA 2.2. – *Let* $G = \mathrm{SL}(V)$.
(i) *Let* $x \in \mathscr{F}$ *and* $\lambda \in X_*(G)$ *with corresponding filtration* $\mathcal{F}_\lambda$ *on* $V_{\bar{k}} = V \otimes_k \bar{k}$. *Then*

$$\mu^{\mathcal{L}}(x, \lambda) = -\big(\mathcal{F}_x(V_{\bar{k}}), \mathcal{F}_\lambda\big).$$

(ii) *Let $T \subset G$ be a maximal torus and $\lambda,\ \lambda' \in X_*(T)$. Then*

$$(\lambda, \lambda') = (\mathcal{F}_\lambda, \mathcal{F}_{\lambda'}).$$

Proof. – (i) If the point x is fixed by λ, then our statement is just a result of Totaro (*cf.* [12], Lemma 6 and part (ii) of this lemma). In general, let $x_0 := \lim_{t\to 0} \lambda(t)x \in \mathscr{F}$. Then we know that $\mu^{\mathcal{L}}(x,\lambda) = \mu^{\mathcal{L}}(x_0,\lambda)$ (*cf.* [5], Definition 2.2, Property (iv)). On the other hand let $\mathcal{F}_x^\bullet(V)$, respectively $\mathcal{F}_{x_0}^\bullet(V)$ be the corresponding filtrations on V. Then we claim that

$$\mathrm{gr}^\alpha_{\mathcal{F}_\lambda}\big(\mathcal{F}_x^\beta(V)\big) \cong \mathrm{gr}^\alpha_{\mathcal{F}_\lambda}\big(\mathcal{F}_{x_0}^\beta(V)\big) \quad \text{for all } \alpha,\ \beta \in \mathbb{Z},$$

proving our assertion. Indeed, let $W \subset V$ be any subspace. For every $\alpha \in \mathbb{Z}$ we set

$$W_\alpha := \mathrm{im}\big(\mathrm{gr}^\alpha_{\mathcal{F}_\lambda}(W) \hookrightarrow V_\lambda(\alpha)\big),$$

where $V = \bigoplus_\alpha V_\lambda(\alpha)$ is the grading of V, which is induced by λ. Then we get

$$\lim_{t\to 0} \lambda(t)\cdot W = \sum_\alpha W_\alpha,$$

considered as points of the corresponding Grassmanian variety. But then

$$\mathrm{gr}^\alpha_{\mathcal{F}_\lambda}(W) \cong W_\alpha = \mathrm{gr}^\alpha_{\mathcal{F}_\lambda}\big(\lim_{t\to 0}\lambda(t)\cdot W\big) \quad \text{for all } \alpha \in \mathbb{Z},$$

and the claim follows.

(ii) Choose a basis of V such that T is the diagonal torus of SL_{d+1}. Then we may identify λ, respectively λ', with $d+1$-tuples $\lambda = (\lambda_1,\ldots,\lambda_{d+1})$, respectively $\lambda' = (\lambda'_1,\ldots,\lambda'_{d+1}) \in \mathbb{Z}^{d+1}$. Obviously, we have $\mathrm{gr}^\alpha_{\mathcal{F}_\lambda}(V) = V_\lambda(\alpha)$, respectively $\mathrm{gr}^\alpha_{\mathcal{F}_{\lambda'}}(V) = V_{\lambda'}(\alpha)$, and

$$\mathrm{gr}^\alpha_{\mathcal{F}_\lambda}\big(\mathrm{gr}^\beta_{\mathcal{F}_{\lambda'}}(V)\big) = V_\lambda(\alpha) \cap V_{\lambda'}(\beta) \quad \text{for all } \alpha, \beta \in \mathbb{Z}.$$

But then

$$\begin{aligned}(\mathcal{F}_\lambda, \mathcal{F}_{\lambda'}) &= \sum_{\alpha,\beta} \alpha\beta \dim\big(V_\lambda(\alpha)\cap V_{\lambda'}(\beta)\big) = \sum_{\alpha,\beta}\alpha\beta\,\#\{i\,;\ \lambda_i = \alpha,\ \lambda'_i = \beta\}\\ &= \sum_{i=1}^{d+1} \lambda_i\lambda'_i = (\lambda,\lambda'). \qquad \square\end{aligned}$$

The idea of the next proposition is due to Burt Totaro which is a decisive point in proving the acyclicity of the fundamental complex in Theorem 3.2.

PROPOSITION 2.3. – *Let $x \in \mathscr{F}$ be any point. The slope function $\mu^{\mathcal{L}}(x,\cdot)$ is affine on each chamber of $B(G)_k$.*

Proof. – We may assume that our group is k-simple. Choose a faithful representation

$$i : G \hookrightarrow \mathrm{SL}_n =: G'$$

which is defined over k. Set $\mu' := i \circ \mu \in X_*(G')$. We get a closed immersion

$$i : \mathscr{F}(G,\{\mu\}) \hookrightarrow \mathscr{F}(G',\{\mu'\}) =: \mathscr{F}'$$

of the corresponding flag varieties, under which μ is mapped to μ'. We assume that we have an invariant inner product on SL_n which restricts to our fixed one on G. This is not really a restriction since two such inner products on a k-simple group differ only by a positive scalar (*cf.* [12], Lemma 7). The line bundle $\mathcal{L}' := \mathcal{L}_{-\mu'}$ on $\mathscr{F}'$, defined in a similar way as $\mathcal{L}$, restricts then via the pullback to $\mathcal{L}$. Because of the equality $\mu^{\mathcal{L}}(x,\lambda) = \mu^{\mathcal{L}'}(i(x), i\circ\lambda)$ ([5], property (iii) following Definition 2.2) we can restrict ourselves to the case $G = \mathrm{SL}(V)$. Let $\lambda \in \{\mu\}$ be a 1-PS representing x. Let $S \subset G$ be a maximal k-split torus, such that the corresponding apartment contains both D, a given chamber with representatives $\lambda_\alpha \in X_*(S)_{\mathbb{Q}}$, $\alpha \in \Delta$, of its vertices and λ. Using the previous lemma we get

$$\mu^{\mathcal{L}}\Bigl(x, \sum_{\alpha\in\Delta} r_\alpha\lambda_\alpha\Bigr) = -\bigl(\mathcal{F}_x(V_{\overline{k}}), \mathcal{F}_{\sum_{\alpha\in\Delta} r_\alpha\lambda_\alpha}\bigr) = -\Bigl(\lambda, \sum_{\alpha\in\Delta} r_\alpha\lambda_\alpha\Bigr) = -\sum_{\alpha\in\Delta} r_\alpha(\lambda,\lambda_\alpha)$$
$$= -\sum_{\alpha\in\Delta} r_\alpha\bigl(\mathcal{F}_x(V), \mathcal{F}_\lambda\bigr) = \sum_{\alpha\in\Delta} r_\alpha\mu^{\mathcal{L}}(x,\lambda_\alpha). \qquad \square$$

I want to stress that the previous corollary fails for arbitrary varieties. In general, the slope function is only convex (*cf.* [5], Corollary 2.13).

COROLLARY 2.4. – *Let x be a point in $\mathscr{F}$. Then x is not semistable $\Leftrightarrow$ there exist an element $g \in G(k)$ and an $\alpha \in \Delta$ such that $\mu^{\mathcal{L}}(x, \mathrm{Int}(g)\circ\omega_\alpha) < 0$.*

Proof. – The direction "$\Leftarrow$" is clear. So let λ be a k-rational 1-PS with $\mu^{\mathcal{L}}(x,\lambda) < 0$. Let $g \in G(k)$ such that $\mathrm{Int}(g^{-1})\circ\lambda$ lies in the simplex $\widetilde{D}_0$ spanned by the rational 1-PS ω_α^*, $\alpha \in \Delta$. Thus we can write λ in the shape $\lambda = \sum_\alpha r_\alpha \mathrm{Int}(g)\circ\omega_\alpha$, with $0 \leqslant r_\alpha \leqslant 1$. The statement follows now immediately from Proposition 2.3. $\square$

3. The fundamental complex

Let G be again an arbitrary reductive group. In this section we will construct an acyclic complex of étale sheaves on the closed complement

$$Y := \mathscr{F} \setminus \mathscr{F}^{\mathrm{ss}}$$

of the period domain $\mathscr{F}^{\mathrm{ss}}$, which is defined over E as well. This complex yields a method to calculate the cohomology of $\mathscr{F}^{\mathrm{ss}}$.

For any subset $I \subset \Delta$ we set

$$Y_I := \bigl\{x \in \mathscr{F}\,;\ \mu^{\mathcal{L}}(x,\omega_\alpha) < 0 \text{ for all } \alpha \notin I\bigr\}.$$

It is a consequence of Corollary 2.4 that we can write Y as the union

$$Y = \bigcup_{\alpha\in\Delta}\ \bigcup_{g\in G(k)} gY_{\Delta-\{\alpha\}}.$$

LEMMA 3.1. – (a) *The set Y_I induces a closed subvariety of Y, which is defined over E.*
(b) *The natural action of G on $\mathscr{F}$ restricts to an action of P_I on Y_I for every $I \subset \Delta$.*

Proof. – It is enough to show the statement in the extreme case $I = \Delta - \{\alpha\}$. Choosing an G-linearized embedding $Y \hookrightarrow \mathbb{P}(V)$ into some projective space defined over E (*cf.* [5], Proposition 1.7), we may restrict ourselves to show that the set $\{x \in \mathbb{P}(V);\ \mu(x,\lambda) \leqslant 0\}$ is closed for every $\lambda \in X(G)_k$. Let $V = \bigoplus_{i\in\mathbb{Z}} V_\lambda(i)$ be the grading induced by λ. Then the above set is just the closed subspace $\mathbb{P}(\bigoplus_{i\geqslant 0} V_\lambda(i))$, and the first assertion follows. The second statement results immediately from the fact that (*cf.* [5, Proposition 2.7])

$$\mu^{\mathcal{L}}(px,\omega_\alpha) = \mu^{\mathcal{L}}(x,\omega_\alpha) \quad \text{for all } p \in P(\omega_\alpha). \qquad \square$$

Let g be an element of $G(k)$ and $I \subset \Delta$. We denote by

$$\phi_{g,I} : gY_I \hookrightarrow Y$$

the corresponding closed embedding. Let F be an étale sheaf on Y and let $I \subset J$ be two subsets of Δ with $\#(J \setminus I) = 1$. Let further $g \in (G/P_I)(k)$, $h \in (G/P_J)(k)$ two elements, such that g is mapped to h under the canonical projection $(G/P_I)(k) \to (G/P_J)(k)$. In this case we define

$$p_{I,J}^{g,h} : (\phi_{h,J})_*(\phi_{h,J})^* F \longrightarrow (\phi_{g,I})_*(\phi_{g,I})^* F$$

to be the natural morphism of étale sheaves on Y which is induced by the closed embedding $gY_I \hookrightarrow hY_J$. If g is not mapped to h then we set $p_{I,J}^{g,h} = 0$. Finally, we define

$$\begin{aligned} p_{I,J} = \bigoplus_{(g,h)\in (G/P_I)(k)\times(G/P_J)(k)} & p_{I,J}^{g,h} : \\ \bigoplus_{h\in(G/P_J)(k)} (\phi_{h,J})_*(\phi_{h,J})^* F \longrightarrow & \bigoplus_{g\in(G/P_I)(k)} (\phi_{g,I})_*(\phi_{g,I})^* F. \end{aligned}$$

For two arbitrary subsets $I,\ J \subset \Delta$ with $\# J - \# I = 1$ we set

$$d_{I,J} = \begin{cases} (-1)^i p_{I,J}\colon & J = I \cup \{\alpha_i\}, \\ 0\colon & I \not\subset J. \end{cases}$$

We get a complex of étale sheaves on Y:

$$0 \to F \to \bigoplus_{\substack{I\subset\Delta \\ \#(\Delta-I)=1}} \bigoplus_{g\in(G/P_I)(k)} (\phi_{g,I})_*(\phi_{g,I})^* F \to \bigoplus_{\substack{I\subset\Delta \\ \#(\Delta-I)=2}} \bigoplus_{g\in(G/P_I)(k)} (\phi_{g,I})_*(\phi_{g,I})^* F$$

$$(*) \quad \cdots \to \bigoplus_{\substack{I\subset\Delta \\ \#(\Delta-I)=d-1}} \bigoplus_{g\in(G/P_I)(k)} (\phi_{g,I})_*(\phi_{g,I})^* F \to \bigoplus_{g\in(G/B)(k)} (\phi_{g,\varnothing})_*(\phi_{g,\varnothing})^* F \to 0.$$

One essential step in order to calculate the cohomology of our period domain is the following result.

THEOREM 3.2. – *The above complex is acyclic.*

Proof. – Let $x \in Y(k^{\mathrm{sep}})$ be a geometric point. Localizing in x yields a chain complex which is precisely the chain complex that computes the homology with coefficient group F_x of a subcomplex of the combinatorial Tits complex to $G(k)$. Strictly speaking this subcomplex corresponds to the following subset of the set of vertices of the Tits building:

$$\bigl\{gP(\omega_\alpha)g^{-1};\ g \in G(k),\ \alpha \in \Delta \text{ such that } \mu^{\mathcal{L}}\bigl(x, \mathrm{Int}(g)\circ\omega_\alpha\bigr) < 0\bigr\}.$$

We will show that this combinatorial subcomplex is contractible. Let T_x be its canonical geometric realization in the spherical building $B(G)_k$. Then T_x is already contained in

$B(G_{\mathrm{der}})_k \subset B(G)_k$. The next two lemmas will show that the topological space T_x is contractible. □

LEMMA 3.3. – *Let*

$$C_x := \left\{ \lambda \in B(G_{\mathrm{der}})_k;\ \frac{\mu^{\mathcal{L}}(x,\lambda)}{\sqrt{(\lambda,\lambda)}} < 0 \right\}.$$

This set is convex. The intersection of C_x with each chamber in $B(G_{\mathrm{der}})_k$ is convex. (*For the definition of convex we refer to* [5].)

Proof. – In the case that the group G_{der} is split this is just [5], Corollary 2.16. But the proof for the general case goes through in the same way. □

Notice that we get an inclusion $T_x \hookrightarrow C_x$ because the slope-function is affine on every chamber of $B(G_{\mathrm{der}})_k$.

LEMMA 3.4. – *The inclusion $T_x \hookrightarrow C_x$ is a deformation retract.*

Proof. – Let $D = g_D D_0,\ g \in G(k)$ be a chamber in the spherical building $B(G)_k$ with $D \cap C_x \neq \emptyset$. Following Lemma 3.3 this intersection is a convex set, where $D \cap T_x$ lies in the boundary of this space. Thus we can construct a deformation retract between $D \cap C_x$ and $D \cap T_x$ as follows. Denote by $\widetilde{D \cap C_x}$ the preimage of $D \cap C_x$ under the canonical homeomorphism $\widetilde{D} \to D$. Put

$$\Lambda := \{ \alpha \in \Delta \colon \operatorname{Int}(g_D) \circ \omega_\alpha \in \widetilde{D \cap T_x} \}.$$

Let

$$\phi_D : (D \cap C_x) \times [0,1] \longrightarrow C_x$$

be the map which is induced by the map

$$\phi_{\widetilde{D}} : (\widetilde{D \cap C_x}) \times [0,1] \longrightarrow \widetilde{D \cap C_x},$$

defined by

$$\begin{aligned} &\phi_{\widetilde{D}}\left(\sum_{\alpha\in\Lambda} r_\alpha \operatorname{Int}(g_D) \circ \omega_\alpha + \sum_{\alpha\notin\Lambda} r_\alpha \operatorname{Int}(g_D) \circ \omega_\alpha, t \right) \\ &\quad := \sum_{\alpha\in\Lambda} r_\alpha \operatorname{Int}(g_D) \circ \omega_\alpha + \sum_{\alpha\notin\Lambda} t\, r_\alpha \operatorname{Int}(g) \circ \omega_\alpha. \end{aligned}$$

This is a continuous map and one checks easily that the collection of these maps paste together to a continuous map

$$\phi : C_x \times [0,1] \longrightarrow C_x$$

which induces a deformation retraction from T_x to C_x. □

4. The proof of Theorem 1.1

This last part of the paper deals with the evaluation of the complex (*) in the case of the ℓ-adic sheaf $F = \mathbb{Q}_\ell$.

PROPOSITION 4.1. – *We have the following description of the closed varieties Y_I in terms of the Bruhat cells of G with respect to $P(\mu)$.*

$$Y_I = \bigcup_{\substack{w\in W^\mu \\ [w]\in\Omega_I}} P(\omega_\alpha)wP(\mu)/P(\mu) = \bigcup_{\substack{w\in W^\mu \\ [w]\in\Omega_I}} BwP(\mu)/P(\mu).$$

Proof. – It is enough to show the assertion in the case $I = \Delta - \{\alpha\}$ for an element $\alpha \in \Delta$, since the sets Ω_I and Y_I are compatible with forming intersections relative to the sets $I \subset \Delta$, i.e.

$$\Omega_{I\cap J} = \Omega_I \cap \Omega_J,$$

respectively

$$Y_{I\cap J} = Y_I \cap Y_J \quad \text{for all } I, J \subset \Delta.$$

Let p be an element of $P(\omega_\alpha)$. We have the equality

$$\mu^{\mathcal{L}}(px, \omega_\alpha) = \mu^{\mathcal{L}}(x, \omega_\alpha) \quad \text{for all } x \in \mathscr{F}$$

(*cf.* [5], Proposition 2.7). The proposition follows now immediately from the equalities

$$\mu^{\mathcal{L}}(pw\bar\mu, \omega_\alpha) = \mu^{\mathcal{L}}(w\bar\mu, \omega_\alpha) = -(w\mu, \omega_\alpha^*),$$

where $\bar\mu$ denotes the point of $\mathscr{F}$, which is induced by the 1-PS μ. □

The above cell decomposition for the varieties Y_I allows us to calculate the cohomology of them. The proof is the same as in the case of GL_{d+1} (*cf.* [6], Proposition 7.1) and will be omitted.

PROPOSITION 4.2. – *We have*

(2) $$\mathrm{H}^*_{\text{ét}}(Y_I, \mathbb{Q}_\ell) = \bigoplus_{[w]\in\Omega_I} \mathrm{ind}_{[w]}(-l[w])\big[-2l([w])\big].$$

In the following we denote for an orbit $[w] \in W^\mu/\Gamma_E$ and a subset $I \subset \Delta$ the contribution of $[w]$ with respect to the direct sum (2) by $\mathrm{H}(Y_I, [w])$, i.e.

(3) $$\mathrm{H}(Y_I, [w]) = \begin{cases} \mathrm{ind}_{[w]}(-l([w]))\big[-2l([w])\big]: & [w] \in \Omega_I, \\ 0: & [w] \notin \Omega_I. \end{cases}$$

Thus we have

(4) $$\mathrm{H}^*_{\text{ét}}(Y_I, \mathbb{Q}_\ell) = \bigoplus_{[w]\in W^\mu/\Gamma_E} \mathrm{H}(Y_I, [w]).$$

Let $I \subset J$ be two subsets of Δ. We consider the homomorphism

$$\phi_{I,J} : \mathrm{H}^*_{\text{ét}}(Y_J) \longrightarrow \mathrm{H}^*_{\text{ét}}(Y_I),$$

given by the closed embedding $Y_I \hookrightarrow Y_J$. The construction of Proposition 4.2 induces a grading of $\phi_{I,J}$,

$$\phi_{I,J} = \bigoplus_{([w],[w'])\in(W^\mu/\Gamma_E)^2} \phi_{[w],[w']} : \bigoplus_{[w]\in W^\mu/\Gamma_E} \mathrm{H}(Y_J, [w]) \longrightarrow \bigoplus_{[w']\in W^\mu/\Gamma_E} \mathrm{H}(Y_I, [w'])$$

with

$$\phi_{[w],[w']} = \begin{cases} \mathrm{id}\colon & [w]=[w'], \\ 0\colon & [w]\neq [w']. \end{cases} \tag{5}$$

We need a generalization of a result of Lehrer, respectively Björner. We will construct a complex in analogy to the sequence (*). Let $I \subset J \subset \Delta$ be two subsets with $\#(J \setminus I) = 1$. We get a homomorphism

$$p_{I,J} : i^G_{P_J} \longrightarrow i^G_{P_I},$$

which comes from the projection $(G/P_I)(k) \longrightarrow (G/P_J)(k)$. For two arbitrary subsets $I,\ J \subset \Delta$ with $\# J - \# I = 1$, we define

$$d_{I,J} = \begin{cases} (-1)^i p_{I,J}, & J = I \cup \{\alpha_i\}, \\ 0, & I \not\subset J. \end{cases}$$

Thus we get for every $I_0 \subset \Delta$ a $\mathbb{Z}$-indexed complex

$$\begin{aligned} K^\bullet_{I_0} : 0 \longrightarrow i^G_G \longrightarrow \bigoplus_{\substack{I_0 \subset I \subset \Delta \\ \#(\Delta - I) = 1}} i^G_{P_I} \longrightarrow \bigoplus_{\substack{I_0 \subset I \subset \Delta \\ \#(\Delta - I) = 2}} i^G_{P_I} \\ \longrightarrow \cdots \longrightarrow \bigoplus_{\substack{I_0 \subset I \subset \Delta \\ \#(\Delta - I) = \#(\Delta - I_0) - 1}} i^G_{P_I} \longrightarrow i^G_{P_{I_0}}, \end{aligned}$$

where the differentials are induced by the above $d_{I,J}$. The component i^G_G is in degree -1.

PROPOSITION 4.3. – *The complex $K^\bullet_{I_0}$ is acyclic.*

Proof. – In the split case this is precisely the result of Lehrer [4], respectively Björner [1]. Since the group Γ is finite taking the fix-vectors in the category of $\mathbb{Q}_\ell$-representations yields an exact functor. But the above complex is just the resulting fix-complex of the analogous complex relative to G considered as a split group defined over k'. □

We mention the following well-known lemma (*cf.* [6], Lemma 7.4).

LEMMA 4.4. – *Every extension of the Γ_E-module $\mathbb{Q}_\ell(m)$ by $\mathbb{Q}_\ell(n)$ with $m \neq n$ splits.*

The acyclic complex (*) yields the following theorem.

THEOREM 4.5. – *The spectral sequence*

$$\mathrm{E}_1^{p,q} = \mathrm{H}^q_{\text{ét}}\Biggl(Y, \bigoplus_{\substack{I \subset \Delta \\ \#(\Delta - I) = p+1}} \bigoplus_{g \in (G/P_I)(k)} (\phi_{g,I})_* (\phi_{g,I})^* \mathbb{Q}_\ell\Biggr) \Longrightarrow \mathrm{H}^{p+q}_{\text{ét}}(Y, \mathbb{Q}_\ell)$$

resulting from (*)*, degenerates in the* E_2*-term and we get for the* ℓ*-adic cohomology of* Y:

$$\begin{aligned} \mathrm{H}^*_{\text{ét}}(Y, \mathbb{Q}_\ell) = & \bigoplus_{\substack{w \in W^\mu/\Gamma_E \\ \#(\Delta - I_{[w]}) = 1}} \Bigl(i^G_{P_{I_{[w]}}} \otimes \mathrm{ind}_{[w]}\bigl(-l(w)\bigr)\bigl[-2l([w])\bigr]\Bigr) \oplus \\ & \bigoplus_{\substack{w \in W^\mu/\Gamma_E \\ \#(\Delta - I_{[w]}) > 1}} \Biggl(\Bigl(i^G_G \otimes \mathrm{ind}_{[w]}\bigl(-l([w])\bigr)\bigl[-2l([w])\bigr]\Bigr) \oplus \\ & \Bigl(v^G_{P_{I_{[w]}}} \otimes \mathrm{ind}_{[w]}\bigl(-l([w])\bigr)\bigl[-2l([w]) - \#(\Delta - I_{[w]}) + 1\bigr]\Bigr)\Biggr). \end{aligned}$$

Proof. – We have

$$\begin{aligned}\mathrm{E}_1^{p,q} &= \mathrm{H}^q_{\text{ét}}\Big(Y, \bigoplus_{\substack{I\subset\Delta\\ \#(\Delta-I)=p+1}} \bigoplus_{g\in(G/P_I)(k)} (\phi_{g,I})_*(\phi_{g,I})^*\mathbb{Q}_\ell\Big)\\ &= \bigoplus_{\substack{I\subset\Delta\\ \#(\Delta-I)=p+1}} \bigoplus_{g\in(G/P_I)(k)} \mathrm{H}^q_{\text{ét}}(Y_I, (\phi_{g,I})^*\mathbb{Q}_\ell) = \bigoplus_{\substack{I\subset\Delta\\ \#(\Delta-I)=p+1}} \bigoplus_{(G/P_I)(k)} \mathrm{H}^q_{\text{ét}}(Y_I, \mathbb{Q}_\ell).\end{aligned}$$

The application of (4) and (5) yields a decomposition

$$\mathrm{E}_1 = \bigoplus_{[w]\in W^\mu/\Gamma_E} \mathrm{E}_{1,[w]}$$

into subcomplexes with

$$\mathrm{E}_{1,[w]}^{p,q} = \begin{cases} \bigoplus\limits_{\substack{I\subset\Delta\\ \#(\Delta-I)=p+1}} \bigoplus\limits_{(G/P_I)(k)} \mathrm{H}(Y_I,[w]), & q = 2l([w]),\\ 0, & q \neq 2l([w]).\end{cases}$$

Thus $\mathrm{E}_{1,[w]}$ is the subcomplex

$$\begin{aligned}\mathrm{E}_{1,[w]}: \bigoplus_{\substack{I\subset\Delta\\ \#(\Delta-I)=1}} \bigoplus_{(G/P_I)(k)} \mathrm{H}(Y_I,[w]) &\longrightarrow \bigoplus_{\substack{I\subset\Delta\\ \#(\Delta-I)=2}} \bigoplus_{(G/P_I)(k)} \mathrm{H}(Y_I,[w])\\ &\longrightarrow \cdots \longrightarrow \bigoplus_{(G/B)(k)} \mathrm{H}(Y_\emptyset,[w]).\end{aligned}$$

In view of (1) and (3) we have

$$\mathrm{H}(Y_I,[w]) = \begin{cases} \mathrm{ind}_{[w]}(-l([w]))[-2l([w])]\colon & I_{[w]}\subset I,\\ 0\colon & I_{[w]}\not\subset I.\end{cases}$$

So $\mathrm{E}_{1,[w]}$ simplifies to

$$\begin{aligned}\Big(\bigoplus_{\substack{I_{[w]}\subset I\\ \#(\Delta-I)=1}} i_{P_I}^G\otimes \mathrm{ind}_{[w]}\big(-l([w])\big) &\longrightarrow \bigoplus_{\substack{I_{[w]}\subset I\\ \#(\Delta-I)=2}} i_{P_I}^G\otimes \mathrm{ind}_{[w]}\big(-l([w])\big)\\ &\longrightarrow\cdots\longrightarrow i_{P_{I_{[w]}}}^G\otimes\mathrm{ind}_{[w]}\big(-l([w])\big)\Big)\big[-2l([w])\big],\end{aligned}$$

and we get an exact sequence of complexes:

$$\begin{aligned}0\longrightarrow i_G^G\otimes\mathrm{ind}_{[w]}\big(-l([w])\big)\big[-2l([w])+1\big]\longrightarrow K^\bullet_{I_{[w]}}\otimes\mathrm{ind}_{[w]}\big(-l([w])\big)\big[-2l([w])\big]\\ \longrightarrow \mathrm{E}_{1,[w]}\longrightarrow 0.\end{aligned}$$

This yields the following three cases for $\mathrm{E}_{2,[w]}$:

$$\begin{aligned} I_{[w]}=\Delta\colon\ \mathrm{E}_{2,[w]}^{p,q} &= 0, \quad p\geqslant 0,\ q\geqslant 0,\\ \#(\Delta-I_{[w]})=1\colon\ \mathrm{E}_{2,[w]}^{0,2l([w])} &= i^G_{P_{I_{[w]}}}\otimes\mathrm{ind}_{[w]}\big(-l([w])\big),\\ \mathrm{E}_{2,[w]}^{p,q} &= 0,\quad (p,q)\neq\big(0,2l([w])\big),\end{aligned}$$

$$\begin{aligned}
\#(\Delta - I_{[w]}) > 1\colon\ \mathrm{E}_{2,[w]}^{0,2l([w])} &= i_G^G \otimes \mathrm{ind}_{[w]}\big(-l([w])\big),\\
\mathrm{E}_{2,[w]}^{j,2l([w])} &= 0, \quad j = 1, \ldots, \#(\Delta - I_{[w]}) - 2,\\
\mathrm{E}_{2,[w]}^{j,2l([w]))} &= v_{P_{I_{[w]}}}^G \otimes \mathrm{ind}_{[w]}\big(-l([w])\big), \quad j = \#(\Delta - I_{[w]}) - 1,\\
\mathrm{E}_{2,[w]}^{p,q} &= 0, \quad q \neq 2l([w]) \text{ or } p > \#(\Delta - I_{[w]}) - 1.
\end{aligned}$$

The Galois modules $\mathrm{E}_2^{p,q} \neq (0)$ possess the Tate twist $-q/2$. As every homomorphism of Galois modules of different Tate twists is trivial, the E_2-term coincides with the E_∞-term. Thus, for all $n \geqslant 0$,

$$\mathrm{gr}^p\big(\mathrm{H}_{\text{ét}}^n(Y)\big) = \mathrm{E}_\infty^{p,n-p} = \mathrm{E}_2^{p,n-p} = \bigoplus_{[w]\in W^\mu/\Gamma_E} \mathrm{E}_{2,[w]}^{p,n-p}$$

$$= \begin{cases} \displaystyle\bigoplus_{\substack{[w]\in W^\mu/\Gamma_E\\ \#(\Delta - I_{[w]})=1\\ 2l([w])=n}} i_{P_{I_{[w]}}}^G \otimes \mathrm{ind}_{[w]}\big(-l([w])\big) \oplus \bigoplus_{\substack{[w]\in W^\mu/\Gamma_E\\ \#(\Delta - I_{[w]})>1\\ 2l([w])=n}} i_G^G \otimes \mathrm{ind}_{[w]}\big(-l([w])\big)\colon & p = 0,\\[2ex] \displaystyle\bigoplus_{\substack{[w]\in W^\mu/\Gamma_E\\ 2l([w])+\#(\Delta - I_{[w]})-1=n}} v_{P_{I_{[w]}}}^G \otimes \mathrm{ind}_{[w]}\big(-l([w])\big)\colon & p > 0. \end{cases}$$

Following Lemma 4.4 extensions of $\mathbb{Q}_\ell(m)$ by $\mathbb{Q}_\ell(n)$ with $m \neq n$ are trivial. This yields an isomorphism

$$\begin{aligned}
\mathrm{H}_{\text{ét}}^n(Y, \mathbb{Q}_\ell) &\cong \bigoplus_{p\in\mathbb{N}} \mathrm{gr}^p\big(\mathrm{H}_{\text{ét}}^n(Y, \mathbb{Q}_\ell)\big)\\
&= \bigoplus_{\substack{[w]\in W^\mu/\Gamma_E\\ \#(\Delta/\Gamma - I_{[w]})=1\\ 2l([w])=n}} i_{P_{I_{[w]}}}^G \otimes \mathrm{ind}_{[w]}\big(-l([w])\big) \oplus \bigoplus_{\substack{[w]\in W^\mu/\Gamma_E\\ \#(\Delta/\Gamma - I_{[w]})>1\\ 2l([w])=n}} i_G^G \otimes \mathrm{ind}_{[w]}\big(-l([w])\big) \oplus\\
&\quad \bigoplus_{\substack{[w]\in W^\mu/\Gamma_E\\ 2l([w])+\#(\Delta/\Gamma - I_{[w]})-1=n}} v_{P_{I_{[w]}}}^G \otimes \mathrm{ind}_{[w]}\big(-l([w])\big).
\end{aligned}$$

The claim follows. □

Proof of Theorem 1.1. – The proof is the same as in the case of $G = \mathrm{GL}_{d+1}$ (*cf.* [6]). □

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