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LOCALLY COMPACT QUANTUM GROUPS

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ABSTRACT. – In this paper we propose a simple definition of a locally compact quantum group in reduced form. By the word 'reduced' we mean that we suppose the Haar weight to be faithful. So in fact we define and study an arbitrary locally compact quantum group, represented on the L^2-space of its Haar weight. For this locally compact quantum group we construct the antipode with polar decomposition. We construct the associated multiplicative unitary and prove that it is manageable in the sense of Woronowicz. We define the modular element and prove the uniqueness of the Haar weights. Following [15] we construct the reduced dual, which will again be a reduced locally compact quantum group. Finally we prove that the second dual is canonically isomorphic to the original reduced locally compact quantum group, extending the Pontryagin duality theorem. © 2000 Éditions scientifiques et médicales Elsevier SAS

RÉSUMÉ. – Dans cet article nous proposons une définition simple des groupes quantiques localement compacts et réduits. L'adjectif 'réduit' exprime l'hypothèse de fidélité du poids de Haar. Nous définissons et étudions des groupes quantiques localement compacts arbitraires représentés sur l'espace L^2 du poids de Haar. Nous construisons l'antipode de ce groupe quantique localement compact, ainsi que sa décomposition polaire. Nous construisons l'unitaire multiplicatif associé et nous démontrons qu'il est maniable au sens de Woronowicz. Nous définissons l'élément modulaire et démontrons l'unicité des poids de Haar. En nous inspirant de [15] nous construisons le dual réduit, qui est de nouveau un groupe quantique localement compact et réduit. Finalement, nous démontrons que le groupe quantique bidual est isomorphe au groupe quantique de départ, ce qui constitue une généralisation du théorème de dualité de Pontryagin. © 2000 Éditions scientifiques et médicales Elsevier SAS

Introduction

Historically, the first aim in constructing axiomatizations of 'quantized' locally compact groups was the extension of the Pontryagin duality to non-abelian groups. Because in general the dual of a non-abelian group will not be a group any more, the quest was for a larger category which included both groups and group duals. After pioneering work by Tannaka, Krein, Kac and Takesaki, among others, this problem was completely solved independently by M. Enock and J.-M. Schwartz (*see* [15] for a survey) and by Kac and Vainerman [56,55] in the seventies. The object they defined is called a Kac algebra.

In [73] S.L. Woronowicz constructed a C^*-algebra with comultiplication — it was the quantum $SU(2)$ — which had so many group-like properties that it was justified to consider it as being a 'quantum group'. But in this example the antipode had become an unbounded operator in the C^*-algebra, and not any more a $*$-antiautomorphism as in the case of Kac algebras. Also the algebraic examples of Drinfel'd and Jimbo showed that the antipode need not to be involutive in

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an arbitrary quantum group [13]. So it became clear that the category of Kac algebras was too small to include all quantum groups, and that it should be enlarged.

The first success in this direction was obtained by Woronowicz who succeeded to define the compact quantum groups [72,67] in a simple way and who proved, most importantly, the existence and uniqueness of a Haar state. It was also Woronowicz who made clear that, following the common paradigm that C*-algebras are quantized locally compact spaces, the C*-algebra framework would be the most natural to formulate a theory of locally compact quantum groups [74].

The next success provided us with another approach. In [6], S. Baaj and G. Skandalis made a study of multiplicative unitaries, which can be considered as an abstract study of the Kac–Takesaki operator of a locally compact group. With an irreducible and regular multiplicative unitary they associate two C*-algebras, which are each others dual, with a comultiplication and a densely defined antipode. In this way they obtain both the compact quantum groups and, in a certain sense, the Kac algebras. More precisely, their theory includes the C*-Kac algebras, which are reformulations of Kac algebras to the C*-algebra framework [16,58].

Shortly afterwards S. Baaj discovered that the multiplicative unitary associated with the quantum E(2) is not regular [3,2]. Hence more and more people got convinced that the notion of Baaj and Skandalis was too narrow to include all the quantum groups. It was Woronowicz [68] who proposed an alternative condition on the multiplicative unitaries which he called manageability, and it seems that all the multiplicative unitaries coming from quantum groups will satisfy this condition. Nevertheless, manageability is not a weaker notion than regularity. In fact, manageability is quite strong, but well adapted to quantum groups.

Still one wanted to give a more intrinsic definition of a locally compact quantum group, with a C*-algebra (or von Neumann algebra) with comultiplication as a starting point. An essential idea in this direction was put forward by Kirchberg in [19], who proposed to allow the antipode of a Kac algebra to be deformed by a 'scaling group', which should be a one-parameter group of automorphisms of the underlying von Neumann algebra. Then T. Masuda and Y. Nakagami formulated the definition of a Woronowicz algebra in [33], generalizing Kac algebras by introducing this scaling group. They were able to construct the dual within the same category, and their theory included the known examples, the Kac algebras and the compact quantum groups in a certain sense. However, there is an objection to their theory and that is the complexity of the axioms: a Woronowicz algebra is a quintuple consisting of a von Neumann algebra, a comultiplication, a Haar weight, a unitary antipode and a scaling group, satisfying a lot of relations. So a lot of nice properties that one would like to have as theorems are included as axioms. One of these axioms, which is also an axiom of Kac algebras, is the 'strong left invariance'. This gives a link between the antipode and the Haar weight (although its name suggests a link between the comultiplication and the Haar weight). Classically however, in Hopf algebras and locally compact groups, the axioms of the antipode do not refer to invariant functionals or measures, and we think this is more natural. In our theory we do not include the existence of the antipode as an axiom, but we construct it and prove the strong left invariance. We also give a characterization of the antipode solely in terms of the comultiplication and are hence close to the classical situation. Such a characterization is also given in [54], by A. Van Daele and the second author.

Another objection was already given by Masuda and Nakagami themselves: remembering the paradigm mentioned above it would be better to use the C*-algebra framework to define locally compact quantum groups. In this sense a C*-version of the theory of Masuda and Nakagami was announced by Masuda, Nakagami and Woronowicz in some lectures at the Fields Institute and at the University of Warsaw in 1995 [34]. They proposed the same complex definition, but this programme is still not achieved.

In this paper we will give a simple definition of a reduced locally compact quantum group in the C*-algebra framework. We will call a pair (A,Δ) a reduced locally compact quantum group when Δ is a comultiplication on the C*-algebra A, when (A,Δ) satisfies a certain density condition and when there exist a left invariant weight and a right invariant weight on (A,Δ) which are faithful and 'approximately KMS'. This last property is weaker than the usual KMS-property but sufficient for us.

First notice that we add the predicate 'reduced' because we assume the Haar weights to be faithful. This is of minor importance because the theories of Kac algebras, multiplicative unitaries or Woronowicz algebras are all 'reduced' theories as well. Moreover, a lot of our theory will be applicable to the non-reduced case as well, about which we will tell more later on. In fact the difference between the reduced and universal case (*see* [23]), or between the C*-algebra and von Neumann algebra case, only lies in the appearance of the same object, represented in different ways, or closed with respect to different topologies. Next, one should observe that this definition is in fact even simpler than the definition of a Kac algebra, because we do not assume any kind of antipode, nor any kind of 'strong left invariance'.

What will we be able to prove from this definition? First, we will construct the associated multiplicative unitary. The proof of its unitarity, which is a major problem in [15] as well as in [33], uses an inversion formula, which makes the proof easier. We will also show that the von Neumann algebras generated by the left and right Haar weights are isomorphic, which will be an important technical tool for us. Then we will be able to construct the antipode with its polar decomposition, and we will prove the strong left invariance. We will also prove the manageability of our multiplicative unitary. Next we will prove a certain commutation relation between the left and right Haar weight and hence get hold of the modular function, which will be the Radon–Nikodym derivative of the right Haar weight with respect to the left Haar weight. Then we will prove the uniqueness of left and right invariant weights — up to a positive scalar of course. Finally we construct the dual reduced locally compact quantum group, and we will prove that the bidual is canonically isomorphic to the original reduced locally compact quantum group. This generalizes the Pontryagin duality theorem for abelian locally compact groups.

So we think that all the relevant properties of a locally compact quantum group can be proven from our definition. Because we also believe that an existence proof for a Haar weight is still far away, our definition seems to be the simplest one can hope for at the moment.

Of course there are a lot of sources of inspiration for the present work. On the algebraic level a lot of theory has been developed by A. Van Daele. He generalizes the notion of Hopf algebras to multiplier Hopf algebras in [62] and then imposes the existence of an invariant functional in [59]. Formally, the starting point of Van Daele looks very much like ours then and a lot of the nice properties of his theory were a motivation for our theory. In [31], the first author and A. Van Daele constructed a C*-algebraic quantum group out of an algebraic quantum group in the sense of [59], and this is a source of techniques adapted to the C*-framework. Of course the theory of weights is important to us and we rely on the work of J. Verding [65] and the first author [24]. While constructing the dual we used a lot of the ideas of Baaj and Skandalis [6], Masuda and Nakagami [33] and most importantly Woronowicz [68] and Enock and Schwartz [15]. The importance of Woronowicz' work for us cannot be overestimated: he learned us to use C*-algebras to tackle the quantum groups [74] and he provided us with many examples [73,71,38,70,69,64]. He also provided a lot of techniques in his work (e.g. [70]). While proving unicity results we were inspired a lot by Enock and Schwartz [15].

A short overview of the main results of this paper appeared in [27].

Notations and conventions

For any subset X of a Banach space E, we denote the linear span by $\langle X\rangle$, its closed linear span by $[X]$.

If I is set, $\mathrm{F}(I)$ will denote the set of finite subsets of I. We turn it into a directed set by inclusion.

All tensor products in this paper are minimal ones. This implies that the tensor product functionals separate points of the tensor product (and also of its multiplier algebra). The completed tensor products will be denoted by $\otimes$, the algebraic ones by $\odot$. For the tensor product of von Neumann algebras, we use the notation $\bar{\otimes}$. The flip operator on the tensor product of an algebra with itself will be denoted by χ.

The multiplier algebra of a C*-algebra A will be denoted by $\mathrm{M}(A)$.

Consider two C*-algebras A and B and a linear map $\rho: A \to \mathrm{M}(B)$. We call ρ strict if it is norm bounded and strictly continuous on bounded sets. If ρ is strict, ρ has a unique linear extension $\bar{\rho}: \mathrm{M}(A) \to \mathrm{M}(B)$ which is strictly continuous on bounded sets (*see* Proposition 7.2 of [25]). The resulting $\bar{\rho}$ is norm bounded and has the same norm as ρ. For $a \in \mathrm{M}(A)$, we put $\rho(a) = \bar{\rho}(a)$.

Given two strict linear mappings $\rho: A \to \mathrm{M}(B)$ and $\eta: B \to \mathrm{M}(C)$, we define a new strict linear map $\eta\,\rho: A \to \mathrm{M}(C)$ by $\eta\,\rho = \bar{\eta} \circ \rho$. The two basic examples of strict linear mappings are:

- continuous linear functionals on a C*-algebra;
- non-degenerate *-homomorphisms. Recall that a *-homomorphism $\pi: A \to \mathrm{M}(B)$ is called non-degenerate $\Leftrightarrow B = [\,\pi(a)\,b \mid a \in A, b \in B\,]$.

All strict linear mappings in this paper will arise as the tensor product of continuous functionals and/or non-degenerate *-homomorphisms.

For $\omega \in A^*$ and $a \in \mathrm{M}(A)$, we define new elements $a\,\omega$ and $\omega\,a$ belonging to A^* such that $(a\,\omega)(x) = \omega(x\,a)$ and $(\omega\,a)(x) = \omega(a\,x)$ for $x \in A$.

We also define a functional $\overline{\omega} \in A^*$ such that $\overline{\omega}(x) = \overline{\omega(x^*)}$ for all $x \in A$. (Sometimes, $\overline{\omega}$ will denote the closure of a densely defined bounded functional, but it will be clear from the context what is precisely meant by $\overline{\omega}$.)

If A and B are C*-algebras, then the tensor product $\mathrm{M}(A) \otimes \mathrm{M}(B)$ is naturally embedded in $\mathrm{M}(A \otimes B)$.

We will make extensive use of the leg numbering notation. Let us give an example to illustrate it. Consider three C*-algebras A, B and C. Then there exists a unique non-degenerate *-homomorphism $\theta: A \otimes C \to \mathrm{M}(A \otimes B \otimes C)$ such that $\theta(a \otimes c) = a \otimes 1 \otimes c$ for all $a \in A$ and $c \in C$. For any element $x \in \mathrm{M}(A \otimes C)$, we define $x_{13} = \theta(x) \in \mathrm{M}(A \otimes B \otimes C)$. It will be clear from the context which C*-algebra B is under consideration.

If we have another C*-algebra D and a non-degenerate *-homomorphism $\Delta: D \to \mathrm{M}(A \otimes C)$, we define the non-degenerate *-homomorphism $\Delta_{13}: D \to \mathrm{M}(A \otimes B \otimes C)$ such that $\Delta_{13}(d) = \Delta(d)_{13}$ for all $d \in D$.

In this paper, we will also use the notion of a Hilbert C*-module over a C*-algebra A. For an excellent treatment of Hilbert C*-modules, we refer to [32].

If E and F are Hilbert C*-modules over the same C*-algebra, $\mathcal{L}(E,F)$ denotes the set of adjointable operators from E into F. When A is a C*-algebra and H is a Hilbert space, $A \otimes H$ will denote the Hilbert space over A, which is a Hilbert C*-module over A.

For the notion of elements affiliated to a C*-algebra A, we refer to [4], [70] and [32] (these affiliated elements are a generalization of closed densely defined operators in a Hilbert space). For these affiliated elements, there exist notions of self adjointness, positivity and a functional calculus similar to the notions for closed operators in a Hilbert space. We collected some extra results concerning the functional calculus in [26]. Self adjointness will be considered as a part of the definition of positivity. If δ is a positive element affiliated to a C*-algebra A, δ is called strictly positive if and only if it has dense range. For such an element δ, functional calculus allows us to define for every $z \in \mathbb{C}$ the power δ^z, which is again affiliated to A (*see* Definition 7.5 of [26]).

Let H be a Hilbert space. The space of bounded operators on H will be denoted by $\mathrm{B}(H)$, the space of compact operators on H by $\mathrm{B}_0(H)$. Notice that $\mathrm{M}(\mathrm{B}_0(H)) = \mathrm{B}(H)$.

Let A and B be C*-algebras and π a non-degenerate representation of A on H. Consider also $\omega \in \mathrm{B}_0(H)^*$.

For $a \in \mathrm{M}(A)$, we will use the notation $\omega(a) := \omega(\pi(a)) \in \mathbb{C}$. For $x \in \mathrm{M}(A \otimes B)$, we use the notation $(\omega \otimes \iota)(x) := (\omega \otimes \iota)((\pi \otimes \iota)(x)) \in \mathrm{M}(B)$.

Consider a Banach space E and denote the set of all isometric vector space isomorphisms on E by $\mathrm{Isom}(E)$. Consider a mapping $\alpha : \mathbb{R} \to \mathrm{Isom}(E)$ such that:

(1) $\alpha_s \alpha_t = \alpha_{s+t}$ for all $t \in \mathbb{R}$;

(2) we have for all $x \in E$ that the function $\mathbb{R} \to E : t \mapsto \alpha_t(x)$ is norm continuous.

Then we call α a norm continuous one-parameter representation on E.

There is a standard way to define for every $z \in \mathbb{C}$ a closed densely defined linear operator α_z in E. Denote with $S(z)$ the strip $\{y \in \mathbb{C} \mid \mathrm{Im}\, y \in [0, \mathrm{Im}\, z]\}$ and with $S(z)^\circ$ its interior.

- The domain of α_z is by definition the set of elements $x \in E$ such that there exists a function f from $S(z)$ into E satisfying:
  (1) f is continuous on $S(z)$;
  (2) f is analytic on $S(z)^0$;
  (3) we have that $\alpha_t(x) = f(t)$ for every $t \in \mathbb{R}$.
- Consider x in the domain of α_z and f the unique function from $S(z)$ into E such that:
  (1) f is continuous on $S(z)$;
  (2) f is analytic on $S(z)^0$;
  (3) we have that $\alpha_t(x) = f(t)$ for every $t \in \mathbb{R}$.
  Then we have by definition that $\alpha_z(x) = f(z)$.

If A is a C*-algebra, a norm continuous one-parameter group α on A is a norm continuous one-parameter representation on A such that α_t is a *-automorphism on A for every $t \in \mathbb{R}$. It is then easy to prove that the mapping $\mathbb{R} \to \mathrm{M}(A) : t \mapsto \alpha_t(a)$ is strictly continuous whenever $a \in M(a)$.

The mapping α_z is closable for the strict topology in $\mathrm{M}(A)$ and we define the strict closure of α_z in $\mathrm{M}(A)$ by $\bar{\alpha}_z$. For $a \in D(\bar{\alpha}_z)$, we put $\alpha_z(a) := \bar{\alpha}_z(a)$.

Using the strict topology on $\mathrm{M}(A)$, $\bar{\alpha}_z$ can be constructed from the mapping $\mathbb{R} \to \mathrm{Aut}(\mathrm{M}(A)) : t \mapsto \bar{\alpha}_t$ in a similar way as α_z is constructed from α. (*See* [25] or [12], where they used the results in [9] to prove more general results.)

If M is a von Neumann algebra, then a strongly continuous one-parameter group (and its extension to the complex plane) is defined in a similar way as a norm continuous one-parameter group but you have to replace the norm topology by the strong topology.

1. Weight theory on C*-algebras

In this section, we will collect some necessary information and conventions about weights. All weights in this paper will be assumed to be non-zero, densely defined and lower semi-continuous. These weights will be called proper weights.

1.1. Weights on C^*-algebras

In this first section, we give some information about weights. The standard reference for lower semi-continuous weights is [11]. A substantial number of results are collected in [24]. We start off with some standard notions concerning lower semi-continuous weights.

DEFINITION 1.1. – Consider a C^*-algebra A and a function $\varphi : A^+ \to [0,\infty]$ such that:
(1) $\varphi(x+y) = \varphi(x) + \varphi(y)$ for all $x,\ y \in A^+$;
(2) $\varphi(rx) = r\varphi(x)$ for all $r \in \mathbb{R}^+$ and $x \in A^+$.
Then we call φ a weight on A.

Let φ be a weight on a C^*-algebra A. We will use the following standard notations:
- $\mathcal{M}_\varphi^+ = \{a \in A^+ \mid \varphi(a) < \infty\}$;
- $\mathcal{N}_\varphi = \{a \in A \mid \varphi(a^*a) < \infty\}$;
- $\mathcal{M}_\varphi = \operatorname{span} \mathcal{M}_\varphi^+ = \mathcal{N}_\varphi^* \mathcal{N}_\varphi$,

where $\mathcal{N}_\varphi^*\mathcal{N}_\varphi = \operatorname{span}\{y^*x \mid x,\ y \in \mathcal{N}_\varphi\}$.

Condition 1 in Definition 1.1 implies that $\varphi(x) \leqslant \varphi(y)$ for all $x, y \in A^+$ such that $x \leqslant y$. Therefore $\mathcal{M}_\varphi^+$ is a hereditary cone in A^+ and $\mathcal{N}_\varphi$ is a left ideal in $\mathrm{M}(A)$. So we see that $\mathcal{M}_\varphi \subseteq \mathcal{N}_\varphi$. We also have that $\mathcal{M}_\varphi$ is a sub*-algebra of A and $\mathcal{M}_\varphi^+ = \mathcal{M}_\varphi \cap A^+$.

It is not so difficult to see that there exists a unique linear map $\psi : \mathcal{M}_\varphi \to \mathbb{C}$ such that $\psi(x) = \varphi(x)$ for all $x \in \mathcal{M}_\varphi^+$. For every $x \in \mathcal{M}_\varphi$, we put $\varphi(x) = \psi(x)$.

Also the following terminology is standard:
- we say that φ is densely defined $\Leftrightarrow$ $\mathcal{M}_\varphi^+$ is dense in A^+ $\Leftrightarrow$ $\mathcal{M}_\varphi$ is dense in A $\Leftrightarrow$ $\mathcal{N}_\varphi$ is dense in A;
- the weight φ is called faithful $\Leftrightarrow$ $[\forall a \in A^+ \colon \varphi(a) = 0 \Rightarrow a = 0]$.

Quite often we will need help from the modular theory (*see* e.g. [46]) and so we will work also with weights on von Neumann algebras. We will use the abbreviation n.f.s. weight for a normal, faithful and semifinite weight on a von Neumann algebra.

The role of the L^2-space of a measure is taken over by the GNS-construction for a weight:

DEFINITION 1.2. – Consider a weight φ on a C^*-algebra A. A GNS-construction for φ is by definition a triple $(H_\varphi, \pi_\varphi, \Lambda_\varphi)$ such that:
- H_φ is a Hilbert space;
- Λ_φ is a linear map from $\mathcal{N}_\varphi$ into H_φ such that:
  (1) $\Lambda_\varphi(\mathcal{N}_\varphi)$ is dense in H_φ;
  (2) we have for every $a, b \in \mathcal{N}_\varphi$, that $\langle \Lambda_\varphi(a), \Lambda_\varphi(b)\rangle = \varphi(b^*a)$;
- π_φ is a representation of A on H_φ such that $\pi_\varphi(a)\,\Lambda_\varphi(b) = \Lambda_\varphi(ab)$ for every $a \in A$ and $b \in \mathcal{N}_\varphi$.

It is not difficult to construct such a GNS-construction for any weight (*cf.* the GNS-construction for a positive functional) and it is unique up to a unitary transformation.

When we use one of the notations H_φ, π_φ or Λ_φ without further comment, we implicitly fixed a GNS-construction for φ.

The following sets play a central role in the theory of lower semi-continuous weights.

DEFINITION 1.3. – Consider a weight φ on a C^*-algebra A. Then we define the sets

$$\mathcal{F}_\varphi = \{\omega \in A_+^* \mid \omega(x) \leqslant \varphi(x) \text{ for } x \in A^+\}$$

and

$$\mathcal{G}_\varphi = \{\alpha\,\omega \mid \omega \in \mathcal{F}_\varphi,\ \alpha \in]0,1[\} \subseteq \mathcal{F}_\varphi.$$

On $\mathcal{F}_\varphi$, we use the order inherited from the natural order on A^*_+. The advantage of $\mathcal{G}_\varphi$ over $\mathcal{F}_\varphi$ lies in the fact that $\mathcal{G}_\varphi$ is a directed subset of $\mathcal{F}_\varphi$: for every $\omega_1, \omega_2 \in \mathcal{G}_\varphi$, there exists an element $\omega \in \mathcal{G}_\varphi$ such that $\omega_1, \omega_2 \leqslant \omega$. This implies that $\mathcal{G}_\varphi$ can be used as the index set of a net.

A proof of this fact can be found in [39] or [65]. We also included a proof in Section 3 of [24].

Notation 1.4. – Consider a weight φ on a C*-algebra A and a GNS-construction $(H_\varphi, \pi_\varphi, \Lambda_\varphi)$ of φ. Let $\omega \in \mathcal{F}_\varphi$.

- We define T_ω as the element in $\mathrm{B}(H_\varphi) \cap \pi_\varphi(A)'$ with $0 \leqslant T_\omega \leqslant 1$ such that

$$\langle T_\omega \Lambda_\varphi(a), \Lambda_\varphi(b)\rangle = \omega(b^*a) \quad \text{for } a, b \in \mathcal{N}_\varphi.$$

- There exists a unique element $\xi_\omega \in H_\varphi$ such that $T_\omega^{\frac{1}{2}} \Lambda_\varphi(a) = \pi_\varphi(a)\xi_\omega$ for $a \in \mathcal{N}_\varphi$.

In order to make weights manageable, we have to impose a continuity condition on them. It turns out that the usual lower semi-continuity is a useful continuity condition.

DEFINITION 1.5. – Consider a weight φ on a C*-algebra A. Then φ is lower semi-continuous

$\Leftrightarrow$ We have for every $\lambda \in \mathbb{R}^+$ that the set $\{a \in A^+ \mid \varphi(a) \leqslant \lambda\}$ is closed.

$\Leftrightarrow$ If $(x_i)_{i\in I}$ is a net in A^+ and $x \in A^+$ such that $(x_i)_{i \in I} \to x$, then $\varphi(x) \leqslant \liminf(\varphi(x_i))_{i \in I}$.

Notice that the last condition resembles the result in the classical lemma of Fatou. It implies also easily the next dominated convergence property:

Consider $x \in A^+$ and $(x_i)_{i \in I}$ a net in A^+ such that $x_i \leqslant x$ for $i \in I$ and $(x_i)_{i\in I} \to x$. Then the net $(\varphi(x_i))_{i \in I}$ converges to $\varphi(x)$.

The most important result concerning lower semi-continuous weights is the following one (proven in [11] by F. Combes).

THEOREM 1.6. – *Consider a lower semi-continuous weight φ on a C*-algebra A. Then we have for every $x \in A^+$ that*

$$\varphi(x) = \sup\{\omega(x) \mid \omega \in \mathcal{F}_\varphi\}.$$

By writing any element of $\mathcal{M}_\varphi$ as a sum of elements in $\mathcal{M}_\varphi^+$, we get immediately that the net $(\omega(x))_{\omega \in \mathcal{G}_\varphi}$ converges to $\varphi(x)$ for every $x \in \mathcal{M}_\varphi$.

Using this theorem, it is not hard to prove the following properties about a GNS-construction for a lower semi-continuous weight.

PROPOSITION 1.7. – *Consider a lower semi-continuous weight φ on a C*-algebra A and a GNS-construction $(H_\varphi, \pi_\varphi, \Lambda_\varphi)$ for φ. Then:*

- *the mapping $\Lambda_\varphi : \mathcal{N}_\varphi \to H_\varphi$ is closed;*
- *the *-homomorphism $\pi_\varphi : A \to \mathrm{B}(H_\varphi)$ is non-degenerate;*
- *the net $(T_\omega)_{\omega \in \mathcal{G}_\varphi}$ converges strongly to 1.*

From now on, we will only work with proper weights, i.e. weights which are non-zero, densely defined and lower semi-continuous.

1.2. Extensions of lower semi-continuous weights to the multiplier algebra

Consider a C*-algebra A. Recall that every $\omega \in A^*$ has a unique extension $\bar{\omega}$ to $\mathrm{M}(A)$ which is strictly continuous and we put $\omega(x) = \bar{\omega}(x)$ for every $x \in \mathrm{M}(A)$.

This implies immediately that any proper weight has a natural extension to a weight on $\mathrm{M}(A)$.

DEFINITION 1.8. – Consider a proper weight φ on a C*-algebra A. Then we define the weight $\overline{\varphi}$ on $\mathrm{M}(A)$ such that

$$\overline{\varphi}(x) = \sup\{\omega(x) \mid \omega \in \mathcal{F}_\varphi\}$$

for every $x \in \mathrm{M}(A)^+$. Then $\overline{\varphi}$ is an extension of φ and we put $\varphi(x) = \overline{\varphi}(x)$ for all $x \in \mathrm{M}(A)^+$.

We will use the following notations: $\overline{\mathcal{M}}_\varphi^+ = \mathcal{M}_{\overline{\varphi}}^+$, $\overline{\mathcal{M}}_\varphi = \mathcal{M}_{\overline{\varphi}}$ and $\overline{\mathcal{N}}_\varphi = \mathcal{N}_{\overline{\varphi}}$.

For any $x \in \overline{\mathcal{M}}_\varphi$, we put $\varphi(x) = \overline{\varphi}(x)$. It is then clear that the net $(\omega(x))_{\omega\in\mathcal{G}_\varphi}$ converges to $\varphi(x)$.

The GNS-construction for a proper weight has a natural extension to a GNS-construction for its extension to the multiplier algebra (*see* e.g. Definition 2.5 and Proposition 2.6 of [24]).

PROPOSITION 1.9. – *Consider a proper weight φ on a C*-algebra A and a GNS-construction $(H_\varphi, \pi_\varphi, \Lambda_\varphi)$ for φ. Then the mapping $\Lambda_\varphi : \mathcal{N}_\varphi \to H_\varphi$ is closable for the strict topology on $\mathrm{M}(A)$ and the norm topology on H_φ. We denote this closure by $\bar{\Lambda}_\varphi$. Then $(H_\varphi, \bar{\Lambda}_\varphi, \bar{\pi}_\varphi)$ is a GNS-construction for $\overline{\varphi}$.*

In particular, we have that $D(\bar{\Lambda}_\varphi) = \overline{\mathcal{N}}_\varphi$ and we put $\Lambda_\varphi(a) = \bar{\Lambda}_\varphi(a)$ for every $a \in \overline{\mathcal{N}}_\varphi$.

Consider $a \in \overline{\mathcal{N}}_\varphi$. Then there exists a net $(a_i)_{i\in I}$ in $\mathcal{N}_\varphi$ such that:
- $\|a_i\| \leqslant \|a\|$ for $i \in I$;
- $(a_i)_{i\in I}$ converges strictly to a;
- $(\Lambda_\varphi(a_i))_{i\in I}$ converges to $\Lambda_\varphi(a)$.

This follows immediately by taking an approximate unit in A and multiplying each element of the approximate unit by a from the right.

Let ω be a functional in $\mathcal{F}_\varphi$. Then it is easy to check that, using the definitions of Notation 1.4, the following holds:
- $\langle T_\omega \Lambda_\varphi(a), \Lambda_\varphi(b)\rangle = \omega(b^*a)$ for $a, b \in \overline{\mathcal{N}}_\varphi$;
- $T_\omega^{\frac{1}{2}} \Lambda_\varphi(a) = \pi_\varphi(a)\xi_\omega$ for $a \in \overline{\mathcal{N}}_\varphi$.

1.3. KMS weights on a C*-algebra

Although a C*-algebra is generally non-commutative, we would like to have some control over the non-commutativity under the weight. Therefore we will introduce the class of KMS weights. For full details, we refer to [24].

DEFINITION 1.10. – Consider a C*-algebra A and a weight φ on A. We say that φ is a KMS weight on $A \Leftrightarrow \varphi$ is a proper weight on A and there exists a norm continuous one-parameter group σ on A satisfying the following properties:

(1) φ is invariant under σ: $\varphi\sigma_t = \varphi$ for every $t \in \mathbb{R}$;

(2) We have for every $a \in D(\sigma_{\frac{i}{2}})$ that $\varphi(a^*a) = \varphi(\sigma_{\frac{i}{2}}(a)\sigma_{\frac{i}{2}}(a)^*)$.

The one-parameter group σ is called a modular group for φ.

If the weight φ is faithful, then the one-parameter group σ is uniquely determined and is called the modular group of φ.

This is not the usual definition of a KMS weight on a C*-algebra (*see* [10]), but we prove in [24] that this definition is equivalent with the usual one. More precisely,

PROPOSITION 1.11. – *Consider a proper weight φ on a C*-algebra A with GNS-construction $(H_\varphi, \pi_\varphi, \Lambda_\varphi)$. Let σ be a norm continuous one-parameter group on A such that $\varphi\sigma_t = \varphi$ for all $t \in \mathbb{R}$. Then the following conditions are equivalent.*

(1) *We have that $\varphi(a^*a) = \varphi(\sigma_{\frac{i}{2}}(a)\,\sigma_{\frac{i}{2}}(a)^*)$ for all $a \in D(\sigma_{\frac{i}{2}})$.*

(2) *There exists a non-degenerate *-antihomomorphism* $\theta : A \to \mathrm{B}(H_\varphi)$ *such that we have for all* $x \in \mathcal{N}_\varphi$ *and* $a \in D(\sigma_{\frac{i}{2}})$ *that* xa *belongs to* $\mathcal{N}_\varphi$ *and* $\Lambda_\varphi(xa) = \theta(\sigma_{\frac{i}{2}}(a))\,\Lambda_\varphi(x)$.
(3) *For all* $a, b \in \mathcal{N}_\varphi \cap \mathcal{N}_\varphi^*$, *there exists a function* $f : S(i) \to \mathbb{C}$ *such that:*
- f *is continuous and bounded on* $S(i)$;
- f *is analytic on* $S(i)^\circ$;
- $f(t) = \varphi(\sigma_t(b)\,a)$ *and* $f(t+i) = \varphi(a\,\sigma_t(b))$ *for* $t \in \mathbb{R}$.

PROPOSITION 1.12. – *Let* φ *be a KMS weight on a* C^**-algebra* A, *with GNS-construction* $(H_\varphi, \pi_\varphi, \Lambda_\varphi)$. *Then the following properties hold:*
(1) *there exists a unique anti-unitary operator* J *on* H_φ *such that* $J\Lambda_\varphi(x) = \Lambda_\varphi(\sigma_{\frac{i}{2}}(x)^*)$ *for every* $x \in \mathcal{N}_\varphi \cap D(\sigma_{\frac{i}{2}})$;
(2) *let* $a \in D(\sigma_{\frac{i}{2}})$ *and* $x \in \mathcal{N}_\varphi$. *Then* xa *belongs to* $\mathcal{N}_\varphi$ *and* $\Lambda_\varphi(xa) = J\pi_\varphi(\sigma_{\frac{i}{2}}(a))^* J\,\Lambda_\varphi(x)$;
(3) *let* $a \in D(\sigma_{-i})$ *and* $x \in \mathcal{M}_\varphi$. *Then* ax *and* $x\sigma_{-i}(a)$ *belong to* $\mathcal{M}_\varphi$ *and* $\varphi(ax) = \varphi(x\sigma_{-i}(a))$;
(4) *consider* $x \in \mathcal{N}_\varphi \cap \mathcal{N}_\varphi^*$ *and* $a \in \mathcal{N}_\varphi^* \cap D(\sigma_{-i})$ *such that* $\sigma_{-i}(a) \in \mathcal{N}_\varphi$. *Then* $\varphi(ax) = \varphi(x\sigma_{-i}(a))$.

The anti-unitary operator J will be called the modular conjugation of φ in the GNS-construction $(H_\varphi, \pi_\varphi, \Lambda_\varphi)$. We also have a strictly positive operator ∇ in H_φ such that $\nabla^{it}\Lambda_\varphi(a) = \Lambda_\varphi(\sigma_t(a))$ for $t \in \mathbb{R}$ and $a \in \mathcal{N}_\varphi$. The operator ∇ will be called the modular operator of φ in the GNS-construction $(H_\varphi, \pi_\varphi, \Lambda_\varphi)$.

Although the definition of the modular conjugation and the modular operator depend on σ, they only depend on the weight φ:
There exists a densely defined closed operator T from within H_φ into H_φ such that $\Lambda_\varphi(\mathcal{N}_\varphi \cap \mathcal{N}_\varphi^*)$ is a core for T and $T\Lambda_\varphi(a) = \Lambda_\varphi(a^*)$ for $a \in \mathcal{N}_\varphi \cap \mathcal{N}_\varphi^*$.

Then $\nabla = T^*T$ and $T = J\nabla^{\frac{1}{2}} = \nabla^{-\frac{1}{2}}J$. Also, notice that $J\nabla^t J = \nabla^{-t}$ and $J\nabla^{it}J = \nabla^{it}$ for $t \in \mathbb{R}$.

The above proposition can be easily extended to elements in the multiplier algebra by using the extensions $\overline{\varphi}$ and $\bar{\sigma}$. We will not hesitate to use this extension.

If we have a proper weight η which agrees with a KMS weight φ on the intersection $\mathcal{M}_\varphi^+ \cap \mathcal{M}_\eta^+$ and such that the proper weight η is invariant under a modular group of φ, then $\varphi = \eta$. For a proof, we refer to Corollary 1.15 of [30].

PROPOSITION 1.13. – *Consider a KMS weight* φ *on a* C^**-algebra* A *with modular group* σ. *Let* η *be a proper weight on* A *such that* $\eta\sigma_t = \eta$ *for* $t \in \mathbb{R}$ *and* $\eta(x) = \varphi(x)$ *for all* $x \in \mathcal{M}_\varphi^+ \cap \mathcal{M}_\eta^+$. *Then* $\eta = \varphi$.

1.4. Absolutely continuous KMS weights

In the first part of this subsection, we fix a C^*-algebra A and a KMS weight φ on A with modular group σ. Let (H, π, Λ) be a GNS-construction for φ.

We will also consider a strictly positive element δ affiliated with A (in the C^*-algebra sense) such that there exists a strictly positive number $\lambda > 0$ such that $\sigma_t(\delta) = \lambda^t\,\delta$ for all $t \in \mathbb{R}$.

Then it is natural to look for a good definition for a weight which is formally equal to $\varphi(\delta^{\frac{1}{2}} \cdot \delta^{\frac{1}{2}})$. If $\lambda \neq 1$, the method of defining this weight in [36] is not applicable anymore. Instead we will work with an inverse GNS-construction. This was done in full detail in Section 8 of [24] and we give a short overview of the main results of it in the first part of this subsection.

We need some extra terminology. Let T be an element affiliated with A and $a \in \mathrm{M}(A)$. Then:
(1) we say that a is a left multiplier of $T \Leftrightarrow$ There exists an element $b \in \mathrm{M}(A)$ such that $a\,T(c) = b\,c$ for $c \in D(T)$. In this case, we put $a\,T = b$;

(2) we say that a is a right multiplier of $T \Leftrightarrow a\,A \subseteq D(T)$. In this case, there exists a unique element $b \in \mathrm{M}(A)$ such that $T(a\,c) = b\,c$ for $c \in A$ and we put $T\,a = b$.

It is not so difficult to show that a is a left multiplier of $T \Leftrightarrow a^*$ is a right multiplier of T^*. If this is the case, then $(a\,T)^* = T^* a^*$.

Define the following subspace of A:

$$N = \{a \in A \mid a \text{ is a left multiplier of } \delta^{\frac{1}{2}} \text{ and } a\delta^{\frac{1}{2}} \text{ belongs to } \mathcal{N}_\varphi\}.$$

Then N is a dense left ideal of A and the mapping $N \to H : a \mapsto \Lambda(a\delta^{\frac{1}{2}})$ is closable. We define Λ_δ to be the closure of the mapping $N \to H : a \mapsto \Lambda(a\delta^{\frac{1}{2}})$.

PROPOSITION 1.14. – *There exists a unique KMS weight φ_δ on A such that (H, π, Λ_δ) is a GNS-construction for φ_δ.*

It should be noted that φ_δ is faithful $\Leftrightarrow$ φ is faithful.

Define the norm continuous one-parameter group σ' on A such that $\sigma'_t(a) = \delta^{it}\,\sigma_t(a)\,\delta^{-it}$ for $t \in \mathbb{R}$ and $a \in A$. Then σ' is a modular group for φ_δ. Also notice that $\sigma'_t(\delta) = \lambda^t\,\delta$ for $t \in \mathbb{R}$.

We have moreover for every $t \in \mathbb{R}$ that $\varphi\,\sigma'_t = \lambda^t\,\varphi$ and $\varphi_\delta\,\sigma_t = \lambda^{-t}\,\varphi_\delta$.

In the first part of this subsection, we concentrated on KMS weights on C*-algebras (which was covered in [24]). The same discussion can be easily translated to the level of faithful semi-finite normal weights on von Neumann algebras by replacing the norm topology by the σ-strong* topology. A slightly different route can be followed by using the theory of left Hilbert algebras. This approach is developed in [52], where the construction is even further generalized by allowing λ to be a certain well-behaved strictly positive operator.

For the sake of completeness, we will also repeat the definitions in this framework. This time, we consider a von Neumann algebra M acting on a Hilbert space K and a faithful semi-finite normal weight φ on M with modular group σ. Let (H, π, Λ) be a GNS-construction for φ.

Again, we will also consider a strictly positive element δ affiliated with M (in the von Neumann algebra sense this time) such that there exists a strictly positive number $\lambda > 0$ such that $\sigma_t(\delta) = \lambda^t\,\delta$ for all $t \in \mathbb{R}$.

Before getting to the definition, we want to make the following remarks. Consider an element T affiliated with M (in the von Neumann algebra sense) and $a \in M$. Using unitary elements in the commutant of M, we get easily the following results.

- If $a\,T$ is bounded, then $\overline{a\,T}$ belongs to M.
- If $aK \subseteq D(T)$, then $T\,a$ belongs to M.

Now we define the following subspace of M:

$$N = \{a \in M \mid a\,\delta^{\frac{1}{2}} \text{ is bounded and } \overline{a\,\delta^{\frac{1}{2}}} \in \mathcal{N}_\varphi\}.$$

Then N is a σ-strongly* dense left ideal in M.

This time, the mapping $N \to H : a \mapsto \Lambda(\overline{a\delta^{\frac{1}{2}}})$ is σ-strong* closable and we denote the σ-strong* closure by Λ_δ.

PROPOSITION 1.15. – *There exists a unique n.f.s. weight φ_δ on M such that (H, π, Λ_δ) is a GNS-construction for φ_δ.*

Define the strongly continuous one-parameter group σ' on M such that $\sigma'_t(x) = \delta^{it}\,\sigma_t(x)\,\delta^{-it}$ for $t \in \mathbb{R}$ and $x \in A$. Then σ' is the modular group for φ_δ. Also notice that $\sigma'_t(\delta) = \lambda^t\,\delta$ for $t \in \mathbb{R}$.

We have moreover for every $t \in \mathbb{R}$ that $\varphi\,\sigma'_t = \lambda^t\,\varphi$ and $\varphi_\delta\,\sigma_t = \lambda^{-t}\,\varphi_\delta$.

In [52], the second author generalizes considerably the usual Radon–Nikodym theorem for n.f.s. weights on von Neumann algebras due to Pedersen and Takesaki (*see* [36]). We will only need the following special case (Proposition 5.5 of [52]). This theorem is one of the major advantages of n.f.s weights on von Neumann algebras over KMS weights on C^*-algebras.

THEOREM 1.16. – *Consider a von Neumann algebra M and two n.f.s. weights φ and ψ on M with modular groups σ and σ' respectively. Consider also a number $\lambda > 0$. Then the following statements are equivalent.*
(1) $\varphi\,\sigma'_t = \lambda^t\,\varphi$ *for all* $t \in \mathbb{R}$;
(2) $\psi\,\sigma_t = \lambda^{-t}\,\psi$ *for all* $t \in \mathbb{R}$;
(3) *there exists a strictly positive operator δ affiliated with M such that $\sigma_t(\delta) = \lambda^t\,\delta$ for $t \in \mathbb{R}$ and $\psi = \varphi_\delta$.*

1.5. Slicing with weights

Fix two C^*-algebras A and B together with a proper weight φ on B. An important tool in the theory of C^*-algebraic quantum groups is the slice map $\iota \otimes \varphi$. If A and B would arise from locally compact spaces, φ is implemented by a regular Borel measure μ and $\iota \otimes \varphi$ would integrate out the second variable with respect to μ.

In this section, we will define $\iota \otimes \varphi$ and mention some properties concerning this slice map. For full details, we refer to Section 3 of [30].

DEFINITION 1.17. – We will use the following notations:
- we define the set

$$\overline{\mathcal{M}}_{\iota\otimes\varphi}^{+} = \big\{x \in \mathrm{M}(A \otimes B)^+ \mid \text{ the net } \big((\iota \otimes \omega)(x)\big)_{\omega \in \mathcal{G}_\varphi} \text{ is strictly convergent in } \mathrm{M}(A)\big\};$$

- for $x \in \overline{\mathcal{M}}_{\iota\otimes\varphi}^{+}$, we define $(\iota \otimes \varphi)(x)$ to be the element in $\mathrm{M}(A)$ such that the net $((\iota \otimes \omega)(x))_{\omega \in \mathcal{G}_\varphi}$ converges strictly to $(\iota \otimes \varphi)(x)$.

Using the uniform boundedness principle, one can prove (and it is not very difficult) a slightly different characterization of $\overline{\mathcal{M}}_{\iota\otimes\varphi}^{+}$:

PROPOSITION 1.18. – *Consider $x \in \mathrm{M}(A \otimes B)^+$. Then x belongs to $\overline{\mathcal{M}}_{\iota\otimes\varphi}^{+} \Leftrightarrow$ we have for all $a \in A$ that the net $(a^*(\iota \otimes \omega)(x)a)_{\omega \in \mathcal{G}_\varphi}$ is convergent in A.*

RESULT 1.19. – *The slice $\iota \otimes \varphi$ satisfies the following algebraic properties.*
(1) *We have for $x, y \in \overline{\mathcal{M}}_{\iota\otimes\varphi}^{+}$ that $x + y \in \overline{\mathcal{M}}_{\iota\otimes\varphi}^{+}$ and $(\iota \otimes \varphi)(x + y) = (\iota \otimes \varphi)(x) + (\iota \otimes \varphi)(y)$.*
(2) *We have for $x \in \overline{\mathcal{M}}_{\iota\otimes\varphi}^{+}$ and $\lambda \in \mathbb{R}^+$ that $\lambda x \in \overline{\mathcal{M}}_{\iota\otimes\varphi}^{+}$ and $(\iota \otimes \varphi)(\lambda x) = \lambda(\iota \otimes \varphi)(x)$.*
(3) *Consider $y \in \overline{\mathcal{M}}_{\iota\otimes\varphi}^{+}$ and $x \in \mathrm{M}(A \otimes B)^+$ such that $x \leqslant y$. Then $x \in \overline{\mathcal{M}}_{\iota\otimes\varphi}^{+}$ and $(\iota \otimes \varphi)(x) \leqslant (\iota \otimes \varphi)(y)$.*
(4) *We have for $a \in \mathrm{M}(A)^+$ and $b \in \overline{\mathcal{M}}_{\varphi}^{+}$ that $a \otimes b \in \overline{\mathcal{M}}_{\iota\otimes\varphi}^{+}$ and $(\iota \otimes \varphi)(a \otimes b) = a\,\varphi(b)$.*

As for ordinary weights, this allows us to extend $\iota \otimes \varphi$ to a linear mapping defined on a subalgebra of $\mathrm{M}(A \otimes B)$:

Notation 1.20. – The following notations will be used.
- We define $\overline{\mathcal{M}}_{\iota\otimes\varphi}$ as the linear span of $\overline{\mathcal{M}}_{\iota\otimes\varphi}^{+}$ in $\mathrm{M}(A \otimes B)$. Then $\overline{\mathcal{M}}_{\iota\otimes\varphi}$ is a sub-*-algebra of $\mathrm{M}(A \otimes B)$ such that $\overline{\mathcal{M}}_{\iota\otimes\varphi}^{+} = \overline{\mathcal{M}}_{\iota\otimes\varphi} \cap \mathrm{M}(A \otimes B)^+$.

- There exists a unique linear map $F:\overline{\mathcal{M}}_{\iota\otimes\varphi}\to \mathrm{M}(A)$ such that $F(x)=(\iota\otimes\varphi)(x)$ for $x\in\overline{\mathcal{M}}_{\iota\otimes\varphi}^{+}$.
  For every $x\in\overline{\mathcal{M}}_{\iota\otimes\varphi}$, we put $(\iota\otimes\varphi)(x)=F(x)$.
- We define $\overline{\mathcal{N}}_{\iota\otimes\varphi}=\{x\in \mathrm{M}(A\otimes B)\mid x^*x\in\overline{\mathcal{M}}_{\iota\otimes\varphi}^{+}\}$. Then $\overline{\mathcal{N}}_{\iota\otimes\varphi}$ is a left ideal in $\mathrm{M}(A\otimes B)$ such that $\overline{\mathcal{M}}_{\iota\otimes\varphi}=\overline{\mathcal{N}}_{\iota\otimes\varphi}^{\,*}\overline{\mathcal{N}}_{\iota\otimes\varphi}$.

Then we have immediately the following natural properties concerning $\overline{\mathcal{M}}_{\iota\otimes\varphi}$.

LEMMA 1.21. – *The following properties hold.*
- *Let $x\in\overline{\mathcal{M}}_{\iota\otimes\varphi}$. Then the net $((\iota\otimes\omega)(x))_{\omega\in\mathcal{G}_\varphi}$ converges strictly to $(\iota\otimes\varphi)(x)$.*
- *We have for $a\in\mathrm{M}(A)$ and $b\in\overline{\mathcal{M}}_\varphi$ that $a\otimes b\in\overline{\mathcal{M}}_{\iota\otimes\varphi}$ and $(\iota\otimes\varphi)(a\otimes b)=a\,\varphi(b)$.*
- *We have for $a\in\mathrm{M}(A)$ and $b\in\overline{\mathcal{N}}_\varphi$ that $a\otimes b\in\overline{\mathcal{N}}_{\iota\otimes\varphi}$.*

The next Fubini-like proposition is an easy consequence of the definitions above.

PROPOSITION 1.22. – *Consider $x\in\overline{\mathcal{M}}_{\iota\otimes\varphi}$ and $\theta\in A^*$. Then $(\theta\otimes\iota)(x)$ belongs to $\overline{\mathcal{M}}_\varphi$ and*

$$\varphi\big((\theta\otimes\iota)(x)\big)=\theta\big((\iota\otimes\varphi)(x)\big).$$

Using Dini's theorem, it is possible to prove the following converse. It follows immediately from Lemma A.4 (a result we borrowed from [43]).

PROPOSITION 1.23. – *Consider $x\in\mathrm{M}(A\otimes B)^+$ and $a\in\mathrm{M}(A)^+$ such that $(\theta\otimes\iota)(x)$ belongs to $\overline{\mathcal{M}}_\varphi^{+}$ and $\varphi((\theta\otimes\iota)(x))=\theta(a)$ for all $\theta\in A_+^*$. Then x belongs to $\overline{\mathcal{M}}_{\iota\otimes\varphi}^{+}$ and $(\iota\otimes\varphi)(x)=a$.*

In this setting of 'C*-valued weights', the Cauchy–Schwarz inequality is generalized in the following way. A proof can be found in Proposition 3.15 of [30].

PROPOSITION 1.24. – *Let $x,y\in\overline{\mathcal{N}}_{\iota\otimes\varphi}$. Then*

$$\big((\iota\otimes\varphi)(y^*x)\big)^*\big((\iota\otimes\varphi)(y^*x)\big)\leqslant\|(\iota\otimes\varphi)(y^*y)\|(\iota\otimes\varphi)(x^*x).$$

This follows easily by approximating $(\iota\otimes\varphi)(y^*x)$ by $(\iota\otimes\omega)(x^*y)$ for $\omega\in\mathcal{G}_\varphi$ and using a Hilbert C*-module estimate in the KSGNS-construction for $\iota\otimes\omega$.

In the next part of this section, we provide a KSGNS-construction for $\iota\otimes\varphi$ on $A\otimes H_\varphi$ as a generalization of a GNS-construction for weights. From now on, we will fix a GNS-construction $(H_\varphi,\pi_\varphi,\Lambda_\varphi)$ for φ.

RESULT 1.25. – *Consider $x\in\overline{\mathcal{N}}_{\iota\otimes\varphi}$ and $v\in H_\varphi$. Then there exists a unique element $q\in\mathrm{M}(A)$ such that $\theta(q)=\langle\Lambda_\varphi((\theta\otimes\iota)(x)),v\rangle$ for $\theta\in A^*$.*

We included a proof of this fact in Subsection A.1 of Appendix A. This is also the case for the next proposition in which we introduce a KSGNS-construction for the 'C*-valued weight' $\iota\otimes\varphi$:

PROPOSITION 1.26. – *There exists a unique linear map $\Lambda:\overline{\mathcal{N}}_{\iota\otimes\varphi}\to\mathcal{L}(A,A\otimes H_\varphi)$ such that*

$$\Lambda(x)^*\big(a\otimes\Lambda_\varphi(b)\big)=(\iota\otimes\varphi)\big(x^*(a\otimes b)\big)$$

for $a\in A$, $b\in\mathcal{N}_\varphi$ and $x\in\overline{\mathcal{N}}_{\iota\otimes\varphi}$.

For $x\in\overline{\mathcal{N}}_{\iota\otimes\varphi}$, we put $(\iota\otimes\Lambda_\varphi)(x)=\Lambda(x)$. Then we have the following properties:
- *we have for all $x,y\in\overline{\mathcal{N}}_{\iota\otimes\varphi}$ that $(\iota\otimes\Lambda_\varphi)(y)^*(\iota\otimes\Lambda_\varphi)(x)=(\iota\otimes\varphi)(y^*x)$;*
- *consider $a\in\mathrm{M}(A)$ and $b\in\overline{\mathcal{N}}_\varphi$, then $(\iota\otimes\Lambda_\varphi)(a\otimes b)=a\otimes\Lambda_\varphi(b)$;*
- *we have for $x\in\mathrm{M}(A\otimes B)$ and $y\in\overline{\mathcal{N}}_{\iota\otimes\varphi}$ that $(\iota\otimes\Lambda_\varphi)(xy)=(\iota\otimes\pi_\varphi)(x)(\iota\otimes\Lambda_\varphi)(y)$.*

1.6. Partial GNS-construction for the tensor product of two lower semi-continuous weights

Throughout this section, we will fix C*-algebras A and B, a proper weight φ on A and a proper weight ψ on B. At the same time, we fix a GNS-construction $(H_\varphi, \pi_\varphi, \Lambda_\varphi)$ for φ and a GNS-construction $(H_\psi, \pi_\psi, \Lambda_\psi)$ for ψ.

In this subsection we will quickly say something about the tensor product of φ and ψ and a partial GNS-construction for this tensor product. We refer to Section 4 of [30] for full details.

DEFINITION 1.27. – We define the tensor product weight $\varphi \otimes \psi$ on $A \otimes B$ in such a way that

$$(\varphi \otimes \psi)(x) = \sup\{(\omega \otimes \theta)(x) \mid \omega \in \mathcal{F}_\varphi, \theta \in \mathcal{F}_\psi\}$$

for every $x \in (A \otimes B)^+$. Then $\varphi \otimes \psi$ is a proper weight on $A \otimes B$.

Notice that the family $\{\omega \otimes \theta \mid \omega \in \mathcal{G}_\varphi, \theta \in \mathcal{G}_\psi\}$ is upwardly directed. This will imply that the map $\varphi \otimes \psi$ above is additive.

The defining formula for $\varphi \otimes \psi$ can be easily extended to the multiplier algebra of the tensor product: we have for all $x \in \mathrm{M}(A \otimes B)^+$ that

$$(\varphi \otimes \psi)(x) = \sup\{(\omega \otimes \theta)(x) \mid \omega \in \mathcal{F}_\varphi, \theta \in \mathcal{F}_\psi\}.$$

This will imply immediately the following results:
- $\overline{\mathcal{M}}_\varphi \odot \overline{\mathcal{M}}_\psi \subseteq \overline{\mathcal{M}}_{\varphi\otimes\psi}$ and $(\varphi \otimes \psi)(a \otimes b) = \varphi(a)\,\psi(b)$ for $a \in \overline{\mathcal{M}}_\varphi$ and $b \in \overline{\mathcal{M}}_\psi$;
- $\overline{\mathcal{N}}_\varphi \odot \overline{\mathcal{N}}_\psi \subseteq \overline{\mathcal{N}}_{\varphi\otimes\psi}$.

Although the space $H_\varphi \otimes H_\psi$ is in general (possibly) too small to serve as a GNS-space for $\varphi \otimes \psi$, we still can find interesting elements which can be suitably represented in $H_\varphi \otimes H_\psi$.

First, we introduce a special subset of $\overline{\mathcal{N}}_{\varphi\otimes\psi}$ consisting of all elements which can be properly represented in $H_\varphi \otimes H_\psi$.

DEFINITION 1.28. – We define the following objects:
- we define $\overline{\mathcal{N}}(\varphi, \psi)$ as the set of elements $x \in \overline{\mathcal{N}}_{\varphi\otimes\psi}$ such that there exists $v \in H_\varphi \otimes H_\psi$ such that $\|v\|^2 = (\varphi \otimes \psi)(x^*x)$ and $\langle v, \Lambda_\varphi(a) \otimes \Lambda_\psi(b)\rangle = (\varphi \otimes \psi)((a^* \otimes b^*)\,x)$ for all $a \in \mathcal{N}_\varphi$, $b \in \mathcal{N}_\psi$;
- also define $\mathcal{N}(\varphi, \psi) = \overline{\mathcal{N}}(\varphi, \psi) \cap (A \otimes B)$;
- we define the mapping $\Lambda_\varphi \otimes \Lambda_\psi : \mathcal{N}(\varphi, \psi) \to H_\varphi \otimes H_\psi$ as follows. Let $x \in \mathcal{N}(\varphi, \psi)$. Then we define $(\Lambda_\varphi \otimes \Lambda_\psi)(x) \in H_\varphi \otimes H_\psi$ such that

$$\big\langle (\Lambda_\varphi \otimes \Lambda_\psi)(x), \Lambda_\varphi(a) \otimes \Lambda_\psi(b)\big\rangle = (\varphi \otimes \psi)\big((a^* \otimes b^*)x\big)$$

  for $a \in \mathcal{N}_\varphi$, $b \in \mathcal{N}_\psi$.

It is easy to see that $\mathcal{N}_\varphi \odot \mathcal{N}_\psi \subseteq \mathcal{N}(\varphi, \psi)$ and that $(\Lambda_\varphi \otimes \Lambda_\psi)(x) = (\Lambda_\varphi \odot \Lambda_\psi)(x)$ for $x \in \mathcal{N}_\varphi \odot \mathcal{N}_\psi$.

We collect the GNS-like properties of $(H_\varphi \otimes H_\psi, \pi_\varphi \otimes \pi_\psi, \Lambda_\varphi \otimes \Lambda_\psi)$ in the following proposition.

PROPOSITION 1.29. – *The following properties hold.*
- *The mapping $\Lambda_\varphi \otimes \Lambda_\psi : \mathcal{N}(\varphi, \psi) \to H_\varphi \otimes H_\psi$ is a linear map which is closed with respect to the strict topology on $A \otimes B$ and the norm topology on $H_\varphi \otimes H_\psi$.*
- *The mapping $\Lambda_\varphi \otimes \Lambda_\psi : \mathcal{N}(\varphi, \psi) \to H_\varphi \otimes H_\psi$ is closable with respect to the strict topology on $\mathrm{M}(A \otimes B)$ and the norm topology on $H_\varphi \otimes H_\psi$. Denote its closure by $\overline{\Lambda_\varphi \otimes \Lambda_\psi}$. Then $D(\overline{\Lambda_\varphi \otimes \Lambda_\psi}) = \overline{\mathcal{N}}(\varphi, \psi)$ and we put $(\Lambda_\varphi \otimes \Lambda_\psi)(a) = (\overline{\Lambda_\varphi \otimes \Lambda_\psi})(a)$ for $a \in \overline{\mathcal{N}}(\varphi, \psi)$.*

- $\langle(\Lambda_\varphi\otimes\Lambda_\psi)(x),(\Lambda_\varphi\otimes\Lambda_\psi)(y)\rangle=(\varphi\otimes\psi)(y^*x)$ *for* $x,y\in\overline{\mathcal{N}}(\varphi,\psi)$.
- $\overline{\mathcal{N}}(\varphi,\psi)$ *and* $\mathcal{N}(\varphi,\psi)$ *are left ideals in* $\mathrm{M}(A\otimes B)$ *and* $(\pi_\varphi\otimes\pi_\psi)(x)(\Lambda_\varphi\otimes\Lambda_\psi)(a)=(\Lambda_\varphi\otimes\Lambda_\psi)(x\,a)$ *for* $x\in\mathrm{M}(A\otimes B)$ *and* $a\in\overline{\mathcal{N}}(\varphi,\psi)$.

It is now also clear that $\overline{\mathcal{N}}_\varphi\odot\overline{\mathcal{N}}_\psi\subseteq\overline{\mathcal{N}}(\varphi,\psi)$ and that $(\Lambda_\varphi\otimes\Lambda_\psi)(x)=(\Lambda_\varphi\odot\Lambda_\psi)(x)$ for $x\in\overline{\mathcal{N}}_\varphi\odot\overline{\mathcal{N}}_\psi$.

We included a proof of the previous result in Subsection A.2 of Appendix A.

If one of the weights φ and ψ is a KMS weight, it can be shown (*see* Proposition 4.14 of [30]) that $\overline{\mathcal{N}}_{\varphi\otimes\psi}=\overline{\mathcal{N}}(\varphi,\psi)$. So $(H_\varphi\otimes H_\psi,\pi_\varphi\otimes\pi_\psi,\Lambda_\varphi\otimes\Lambda_\psi)$ is a GNS-construction for $\varphi\otimes\psi$ in this case.

If both of them are KMS, then $\mathcal{N}_\varphi\odot\mathcal{N}_\psi$ is a core for $\Lambda_\varphi\otimes\Lambda_\psi$ (*see* e.g. Section 7 of [24]).

Throughout this paper, we will be able to restrict ourselves to the following kind of elements in $\overline{\mathcal{N}}(\varphi,\psi)$ for which we have explicit formulas. See Subsection A.2 in Appendix A for a proof.

LEMMA 1.30. – *Consider an orthonormal basis* $(e_i)_{i\in I}$ *for* H_φ. *Let* $x\in\overline{\mathcal{N}}_{\iota\otimes\psi}$ *and* $y\in\overline{\mathcal{N}}_\varphi$. *Then* $x\,(y\otimes 1)$ *belongs to* $\overline{\mathcal{N}}(\varphi,\psi)$,

$$\sum_{i\in I}\big\|\Lambda_\psi\big((\omega_{\Lambda_\varphi(y),e_i}\otimes\iota)(x)\big)\big\|^2=(\varphi\otimes\psi)\big((y^*\otimes 1)\,x^*x\,(y\otimes 1)\big)$$
$$=\varphi\big(y^*(\iota\otimes\psi)(x^*x)y\big)<\infty$$

and

$$(\Lambda_\varphi\otimes\Lambda_\psi)\big(x\,(y\otimes 1)\big)=\sum_{i\in I}e_i\otimes\Lambda_\psi\big((\omega_{\Lambda_\varphi(y),e_i}\otimes\iota)(x)\big).$$

1.7. W^*-lifts of lower semi-continuous weights

In this subsection, we fix a proper weight φ on a C^*-algebra A. Let $(H_\varphi,\pi_\varphi,\Lambda_\varphi)$ be a GNS-construction for φ. We will discuss the canonical extension of φ to a normal weight on $\pi_\varphi(A)''$ and produce a useful and natural GNS-construction for it.

By Notation 1.4, we have for all $\omega\in\mathcal{F}_\varphi$ that $\omega(a)=\langle\pi_\varphi(a)\,\xi_\omega,\xi_\omega\rangle$ for $a\in A$. So there exists a unique element $\widetilde{\omega}\in(\pi_\varphi(A)'')^+_*$ such that $\widetilde{\omega}\,\pi_\varphi=\omega$, i.e. $\widetilde{\omega}(x)=\langle x\,\xi_\omega,\xi_\omega\rangle$ for $x\in\pi_\varphi(A)''$.

Because $\mathcal{G}_\varphi$ is upwardly directed, the same is true for the set $\{\widetilde{\omega}\mid\omega\in\mathcal{G}_\varphi\}$. This will imply that we really get a weight in the next definition.

DEFINITION 1.31. – We define the function $\widetilde{\varphi}:(\pi_\varphi(A)'')^+\to[0,\infty]$ such that $\widetilde{\varphi}(x)=\sup\{\widetilde{\omega}(x)\mid\omega\in\mathcal{F}_\varphi\}$ for $x\in(\pi_\varphi(A)'')^+$. Then $\widetilde{\varphi}$ is a normal semi-finite weight on $\pi_\varphi(A)''$ such that $\widetilde{\varphi}\,\pi_\varphi=\varphi$.

Notice that the last statement follows from Theorem 1.6. We call $\widetilde{\varphi}$ the W^*-lift of φ in the GNS-construction $(H_\varphi,\pi_\varphi,\Lambda_\varphi)$.

Although this definition is conceptually appealing it is not that practical in some applications. It will turn out that it is more useful to get a GNS-construction for $\widetilde{\varphi}$ by closing Λ_φ with respect to the σ-strong*-topology. Using a result of U. Haagerup (*see* [18]), S. Baaj provided in [5] the necessary ammunition to prove the next proposition (we included a full exposition in Section 2 of [30]).

PROPOSITION 1.32. – *There exists a unique linear map* $\widetilde{\Lambda}_\varphi:\mathcal{N}_{\widetilde{\varphi}}\to H_\varphi$ *such that*:

- $(H_\varphi,\iota,\widetilde{\Lambda}_\varphi)$ *is a GNS-construction for* $\widetilde{\varphi}$;
- $\widetilde{\Lambda}_\varphi(\pi_\varphi(a))=\Lambda_\varphi(a)$ *for all* $a\in\mathcal{N}_\varphi$.

We have moreover the following property. For every $x\in\mathcal{N}_{\widetilde{\varphi}}$, *there exists a net* $(a_i)_{i\in I}$ *in* $\mathcal{N}_\varphi$ *such that*:

(1) $\|a_i\| \leqslant \|x\|$ *for every* $i \in I$;
(2) $(\pi_\varphi(a_i))_{i\in I}$ *converges strongly* to* x;
(3) $(\Lambda_\varphi(a_i))_{i\in I}$ *converges to* $\widetilde{\Lambda}_\varphi(x)$.

We will call $(H_\varphi, \iota, \widetilde{\Lambda}_\varphi)$ the W*-lift of $(H_\varphi, \pi_\varphi, \Lambda_\varphi)$. It is not difficult to see that (*see* Proposition 2.18 of [30]):

- $\widetilde{\varphi}(\pi_\varphi(x)) = \varphi(x)$ for all $x \in \mathrm{M}(A)^+$;
- $\widetilde{\Lambda}_\varphi(\pi_\varphi(x)) = \Lambda_\varphi(x)$ for all $x \in \overline{\mathcal{N}}_\varphi$.

We will use the following terminology (which obviously comes from left Hilbert algebra theory).

DEFINITION 1.33. – Consider a vector $v \in H_\varphi$. Then we say that v is right bounded with respect to $(H_\varphi, \pi_\varphi, \Lambda_\varphi) \Leftrightarrow$ There exists a number $M \geqslant 0$ such that $\|\pi_\varphi(x)v\| \leqslant M\|\Lambda_\varphi(x)\|$ for all $x \in \mathcal{N}_\varphi$.

It is clear that the set of right bounded elements is a subspace of H_φ.

DEFINITION 1.34. – We say that φ is approximately KMS $\Leftrightarrow$ The subspace of right bounded elements is dense in H_φ.

A KMS weight is always approximately KMS.

Using the theory of n.f.s. weights, the following result follows easily.

PROPOSITION 1.35. – *The weight* φ *is approximately KMS* $\Leftrightarrow$ $\widetilde{\varphi}$ *is faithful.*

Proof. –

$\Rightarrow$ Choose $y \in \pi_\varphi(A)''$ such that $\widetilde{\varphi}(y^*y) = 0$.

Take a right bounded element v. Then there exists clearly a bounded operator $T \in \mathrm{B}(H_\varphi)$ such that $T\Lambda_\varphi(a) = \pi_\varphi(a)\,v$ for all $a \in \mathcal{N}_\varphi$. Using the second part of Proposition 1.32, it is easy to see that $T\widetilde{\Lambda}_\varphi(x) = x\,v$ for all $x \in \mathcal{N}_{\widetilde{\varphi}}$. In particular, $yv = T\widetilde{\Lambda}_\varphi(y) = 0$.

Because the set of right bounded elements is dense in H_φ (by assumption), this implies that $y = 0$.

$\Leftarrow$ So $\widetilde{\varphi}$ is now a n.f.s. weight on $\pi_\varphi(A)''$ and we can use the standard technique to produce right bounded elements. Call $\widetilde{\sigma}$ the modular group of $\widetilde{\varphi}$, $(H_\varphi, \iota, \widetilde{\Lambda}_\varphi)$ the W*-lift of $(H_\varphi, \pi_\varphi, \Lambda_\varphi)$ and J the modular conjugation of $\widetilde{\varphi}$ in the GNS-construction $(H_\varphi, \iota, \widetilde{\Lambda}_\varphi)$.

For every $a \in \mathcal{N}_{\widetilde{\varphi}}$ and $n \in \mathbb{N}$, we define (the integral is defined in the strong topology)

$$a_n = \int \exp(-n^2t^2)\widetilde{\sigma}_t(a)\,\mathrm{d}t,$$

then $a_n \in \mathcal{N}_{\widetilde{\varphi}} \cap D(\widetilde{\sigma}_{\frac{i}{2}})$. So we have for every $x \in \mathcal{N}_{\widetilde{\varphi}}$ that

$$x\,\widetilde{\Lambda}_\varphi(a_n) = \widetilde{\Lambda}_\varphi(x\,a_n) = J\,\widetilde{\sigma}_{\frac{i}{2}}(a_n)^*\,J\widetilde{\Lambda}_\varphi(x).$$

This implies that $\widetilde{\Lambda}_\varphi(a_n)$ is a right bounded element.

But it is also clear that $\langle \widetilde{\Lambda}_\varphi(a_n) \mid a \in \mathcal{N}_{\widetilde{\varphi}},\ n \in \mathbb{N}\rangle$ is dense in H_φ. So φ is approximately KMS. □

The second part of this proof can be immediately translated to show that φ is approximately KMS if φ is KMS.

If φ is approximately KMS, this proposition implies the existence of a closed operator T in H_φ such that $\widetilde{\Lambda}_\varphi(\mathcal{N}_{\widetilde{\varphi}} \cap \mathcal{N}_{\widetilde{\varphi}}^*)$ is a core for T and $T\widetilde{\Lambda}_\varphi(x) = \widetilde{\Lambda}_\varphi(x^*)$ for all $x \in \mathcal{N}_{\widetilde{\varphi}} \cap \mathcal{N}_{\widetilde{\varphi}}^*$.

So we get also that $\Lambda_\varphi(\overline{\mathcal{N}}_\varphi \cap \overline{\mathcal{N}}_\varphi^*) \subseteq D(T)$ and $T\Lambda_\varphi(x) = \Lambda_\varphi(x^*)$ for all $x \in \overline{\mathcal{N}}_\varphi \cap \overline{\mathcal{N}}_\varphi^*$.

We will also need the following result concerning the tensor product of approximate KMS weights.

PROPOSITION 1.36. – *Consider* C*-*algebras A and B and*
- *an approximate KMS weight φ on A with GNS-construction $(H_\varphi, \pi_\varphi, \Lambda_\varphi)$;*
- *an approximate KMS weight ψ on B with GNS-construction $(H_\psi, \pi_\psi, \Lambda_\psi)$.*

Then the weight $\varphi \otimes \psi$ is approximately KMS and $(H_\varphi \otimes H_\psi, \pi_\varphi \otimes \pi_\psi, \Lambda_\varphi \otimes \Lambda_\psi)$ is a GNS-construction for $\varphi \otimes \psi$.

Proof. – Denote the respective W*-lifts of φ and ψ in their respective GNS-constructions by $\widetilde{\varphi}$ and $\widetilde{\psi}$. We also denote the W*-lifts of $(H_\varphi, \pi_\varphi, \Lambda_\varphi)$ and $(H_\psi, \pi_\psi, \Lambda_\psi)$ by $(H_\varphi, \iota, \widetilde{\Lambda}_\varphi)$ and $(H_\psi, \iota, \widetilde{\Lambda}_\psi)$ respectively. Define $\widetilde{\theta} = \widetilde{\varphi} \overline{\otimes} \widetilde{\psi}$, the von Neumann tensor product of $\widetilde{\varphi}$ and $\widetilde{\psi}$ (*see* Theorem 8.2 of [46]). By Proposition 8.3 of [46], we know that

$$\widetilde{\theta} = \sup_{\omega \in \mathcal{G}_\varphi,\, \mu \in \mathcal{G}_\psi} \widetilde{\omega} \overline{\otimes} \widetilde{\mu}.$$

Then we have immediately that $\widetilde{\theta}(\pi_\varphi \otimes \pi_\psi) = \varphi \otimes \psi$. Let $(H_\varphi \otimes H_\psi, \iota, \Lambda_{\widetilde{\theta}})$ be the canonical GNS-construction for $\widetilde{\theta}$ (*see* Proposition 8.1 of [46]). We know that for $a \in \mathcal{N}_{\widetilde{\varphi}}$ and $b \in \mathcal{N}_{\widetilde{\psi}}$: $\Lambda_{\widetilde{\theta}}(a \otimes b) = \widetilde{\Lambda}_\varphi(a) \otimes \widetilde{\Lambda}_\psi(b)$. But now we have for all $x \in \overline{\mathcal{N}}_{\varphi \otimes \psi}$ that $(\pi_\varphi \otimes \pi_\psi)(x) \in \mathcal{N}_{\widetilde{\theta}}$ and for all $a \in \mathcal{N}_\varphi$ and $b \in \mathcal{N}_\psi$

$$\begin{aligned}\langle \Lambda_{\widetilde{\theta}}((\pi_\varphi \otimes \pi_\psi)(x)), \Lambda_\varphi(a) \otimes \Lambda_\psi(b) \rangle &= \langle \Lambda_{\widetilde{\theta}}((\pi_\varphi \otimes \pi_\psi)(x)), \Lambda_{\widetilde{\theta}}(\pi_\varphi(a) \otimes \pi_\psi(b)) \rangle \\ &= \widetilde{\theta}\big((\pi_\varphi \otimes \pi_\psi)((a^* \otimes b^*)x)\big) \\ &= (\varphi \otimes \psi)\big((a^* \otimes b^*)x\big).\end{aligned}$$

It is also clear that $\|\Lambda_{\widetilde{\theta}}((\pi_\varphi \otimes \pi_\psi)(x))\|^2 = (\varphi \otimes \psi)(x^*x)$. From this we can conclude that $x \in \overline{\mathcal{N}}(\varphi, \psi)$ and

$$(\Lambda_\varphi \otimes \Lambda_\psi)(x) = \Lambda_{\widetilde{\theta}}\big((\pi_\varphi \otimes \pi_\psi)(x)\big).$$

Then we get immediately that $(H_\varphi \otimes H_\psi, \pi_\varphi \otimes \pi_\psi, \Lambda_\varphi \otimes \Lambda_\psi)$ is a GNS-construction for $\varphi \otimes \psi$. Let $\widetilde{\varphi \otimes \psi}$ be its W*-lift with lifted GNS-construction $(H_\varphi \otimes H_\psi, \iota, \widetilde{\Lambda}_{\varphi \otimes \psi})$. It then follows from the construction of $\widetilde{\Lambda}_{\varphi \otimes \psi}$ and the strong-norm closedness of $\Lambda_{\widetilde{\theta}}$ that for all $a \in \mathcal{N}_{\widetilde{\varphi \otimes \psi}}$ we have $a \in \mathcal{N}_{\widetilde{\theta}}$, and

$$\widetilde{\Lambda}_{\varphi \otimes \psi}(a) = \Lambda_{\widetilde{\theta}}(a).$$

From this we may conclude that $\widetilde{\varphi \otimes \psi}$ is faithful, and so $\varphi \otimes \psi$ is approximately KMS. □

2. The multiplicative partial isometries

Consider a C*-algebra A with comultiplication Δ and a left invariant weight on A. It is then customary to define the multiplicative partial isometry. Up to now, it was a non-trivial matter to prove that this multiplicative partial isometry is unitary. What has been lacking in order to achieve this, was a concrete formula for the inverse. In this section, we will first introduce some terminology and then prove a formula for the inverse. We will give some important applications of this formula in the next section.

2.1. Left invariance of weights

First we introduce some terminology concerning bi-C*-algebras and left invariant weights on them. The used terminology is by no means a standard one but will be used throughout this paper.

DEFINITION 2.1. – Consider a C*-algebra A and a non-degenerate *-homomorphism $\Delta : A \to \mathrm{M}(A \otimes A)$ such that $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$. Then we call (A, Δ) a bi-C*-algebra.

We define the non-degenerate *-homomorphism $\Delta^{(2)} : A \to \mathrm{M}(A \otimes A \otimes A)$ as $\Delta^{(2)} = (\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$.

Now we define a form of left invariance for weights on bi-C*-algebras. We will use a very weak form of left invariance but in the cases of interest, this weak form of left invariance will imply a much stronger one.

DEFINITION 2.2. – Consider a bi-C*-algebra (A, Δ) and a proper weight φ on A. Then:
- we call φ left invariant $\Leftrightarrow$ we have for all $a \in \mathcal{M}_\varphi^+$ and $\omega \in A_+^*$ that $\varphi((\omega \otimes \iota)\Delta(a)) = \omega(1)\,\varphi(a)$;
- we call φ right invariant $\Leftrightarrow$ we have for all $a \in \mathcal{M}_\varphi^+$ and $\omega \in A_+^*$ that $\varphi((\iota \otimes \omega)\Delta(a)) = \omega(1)\,\varphi(a)$.

Notice that we use the extensions of φ to $\mathrm{M}(A)^+$ in the previous definition because we only know that $(\omega \otimes \iota)\Delta(a) \in \mathrm{M}(A)^+$. In a proper framework for quantum groups, $(\omega \otimes \iota)\Delta(a)$ will belong to A^+ but we will not assume this immediately.

If φ is left invariant, it is easy to see that $\varphi((\omega \otimes \iota)\Delta(a)) = \varphi(a)\,\omega(1)$ for all $a \in \overline{\mathcal{M}}_\varphi^+$ (approximate a strictly from below by elements in $\mathcal{M}_\varphi^+$).

RESULT 2.3. – *Consider a bi-*C*-algebra* (A, Δ) *and a left invariant proper weight* φ *on* A *with GNS-construction* (H, π, Λ). *Then*:
1. *we have for all* $a \in \overline{\mathcal{M}}_\varphi$ *and* $\omega \in A^*$ *that* $(\omega \otimes \iota)\Delta(a) \in \overline{\mathcal{M}}_\varphi$ *and* $\varphi((\omega \otimes \iota)\Delta(a)) = \omega(1)\,\varphi(a)$;
2. *we have for all* $a \in \overline{\mathcal{N}}_\varphi$ *and* $\omega \in A^*$ *that* $(\omega \otimes \iota)\Delta(a) \in \overline{\mathcal{N}}_\varphi$ *and* $\|\Lambda((\omega \otimes \iota)\Delta(a))\| \leqslant \|\omega\|\,\|\Lambda(a)\|$.

The first statement is immediate. For the second one, use for instance Lemma 3.10 of [30].

We also want to work with the slice map $\iota \otimes \varphi$. Therefore we state the following result. We introduced the necessary terminology in Section 1.5.

RESULT 2.4. – *Consider a bi-*C*-algebra* (A, Δ) *and a left invariant proper weight* φ *on* A *with GNS-construction* (H, π, Λ). *Then*:
1. *consider* $a \in \overline{\mathcal{M}}_\varphi$; *then* $\Delta(a) \in \overline{\mathcal{M}}_{\iota \otimes \varphi}$ *and* $(\iota \otimes \varphi)\Delta(a) = \varphi(a)\,1$;
2. *consider* $a \in \overline{\mathcal{N}}_\varphi$; *then* $\Delta(a)$ *belongs to* $\overline{\mathcal{N}}_{\iota \otimes \varphi}$ *and* $(\iota \otimes \Lambda_\varphi)(\Delta(a))^*(\iota \otimes \Lambda_\varphi)(\Delta(a)) = \varphi(a^* a)1$.

The first statement follows from Proposition 1.23. The second one is an immediate consequence of the first one.

Although all properties are stated in terms of left invariant weights, there are of course also similar properties for right invariant weights.

Let (A, Δ) be a bi-C*-algebra and ψ a right invariant proper weight on A. Consider $a, b \in \overline{\mathcal{N}}_\psi$. Then the previous result implies for all $x \in \mathrm{M}(A \otimes A)$ that $\Delta(a^*)x(b \otimes 1) \in \overline{\mathcal{M}}_{\psi \otimes \iota}$ and Proposition 1.24 implies that

$$
\begin{aligned}
(2.1)\qquad & (\psi \otimes \iota)\big(\Delta(a^*)x(b \otimes 1)\big)^*(\psi \otimes \iota)\big(\Delta(a^*)x(b \otimes 1)\big) \\
& \leqslant \psi(a^* a)(\psi \otimes \iota)\big((b^* \otimes 1)x^* x(b \otimes 1)\big).
\end{aligned}
$$

This is a simple observation which will be used throughout the paper.

RESULT 2.5. – *Consider a bi-*C*-algebra (A,Δ) such that $(\omega\otimes\iota)\Delta(x)\in A$ for all $x\in A$ and $\omega\in A^*$. Let ψ be a right invariant proper weight on (A,Δ) and $a,b\in\mathcal{N}_\psi$. Then $(\psi\otimes\iota)(\Delta(a^*)(b\otimes 1))$ and $(\psi\otimes\iota)((a^*\otimes 1)\Delta(b))$ belong to A.*

Proof. – We have for $\omega\in\mathcal{G}_\psi$ that

$$\begin{aligned}\big\|(\omega\otimes\iota)\big(\Delta(a^*)(b\otimes 1)\big)\big\|^2 &\leqslant \big\|(\omega\otimes\iota)(\Delta(a^*a))\big\|\,\omega(b^*b)\\ &\leqslant \big\|(\psi\otimes\iota)(\Delta(a^*a))\big\|\,\omega(b^*b)=\psi(a^*a)\,\omega(b^*b).\end{aligned}$$

Hence we get for all $\omega,\ \mu\in\mathcal{G}_\psi$ with $\mu\leqslant\omega$ that

$$\big\|(\omega\otimes\iota)\big(\Delta(a^*)(b\otimes 1)\big)-(\mu\otimes\iota)\big(\Delta(a^*)(b\otimes 1)\big)\big\|^2\leqslant\psi(a^*a)(\omega-\mu)(b^*b).$$

This implies that the net $\big((\omega\otimes\iota)(\Delta(a^*)(b\otimes 1))\big)_{\omega\in\mathcal{G}_\psi}$ is Cauchy and therefore norm convergent. So we conclude that this net converges in norm to $(\psi\otimes\iota)(\Delta(a^*)(b\otimes 1))$.

Since, by assumption, all elements $(\omega\otimes\iota)(\Delta(a^*)(b\otimes 1))$ $(\omega\in\mathcal{G}_\psi)$ belong to A, we get that $(\psi\otimes\iota)(\Delta(a^*)(b\otimes 1))$ belongs to A. □

In the next result, we use inequality (2.1) once more and combine it with the left invariance of φ. We get a basic result which will be used throughout the paper.

RESULT 2.6. – *Consider a bi-*C*-algebra (A,Δ), a left invariant proper weight φ on A and a right invariant proper weight ψ on A. Let $a,b\in\overline{\mathcal{N}}_\psi$ and $c\in\overline{\mathcal{N}}_\varphi$. Then $(\psi\otimes\iota)(\Delta(a^*c)(b\otimes 1))$ belongs to $\overline{\mathcal{N}}_\varphi$ and*

$$\big\|\Lambda_\varphi\big((\psi\otimes\iota)(\Delta(a^*c)(b\otimes 1))\big)\big\|\leqslant\|\Lambda_\psi(a)\|\,\|\Lambda_\psi(b)\|\,\|\Lambda_\varphi(c)\|.$$

This implies easily the following technical result.

RESULT 2.7. – *Consider a bi-*C*-algebra (A,Δ), a left invariant proper weight φ on A and a right invariant proper weight ψ on A. Let $a,b\in\overline{\mathcal{N}}_\psi$ and $c\in\overline{\mathcal{N}}_\varphi$.*

Suppose that θ is a non-degenerate representation of A on a Hilbert space K and consider $v\in K$ together with an orthonormal basis $(e_i)_{i\in I}$ for K. Then

$$\begin{aligned}&\sum_{i\in I}\big\|\Lambda_\varphi\big((\psi\otimes\iota)(\Delta(a^*c)\big((\iota\otimes\omega_{v,e_i})(\Delta(b))\otimes 1\big))\big)\big\|^2\\ &\qquad\leqslant\|\Lambda_\psi(a)\|^2\,\|\Lambda_\psi(b)\|^2\,\|\Lambda_\varphi(c)\|^2\,\|v\|^2<\infty.\end{aligned}$$

Proof. – By the previous result, we know that

$$\begin{aligned}&\sum_{i\in I}\big\|\Lambda_\varphi\big((\psi\otimes\iota)(\Delta(a^*c)((\iota\otimes\omega_{v,e_i})(\Delta(b))\otimes 1))\big)\big\|^2\\ &\quad\leqslant\sum_{i\in I}\|\Lambda_\psi(a)\|^2\,\big\|\Lambda_\psi((\iota\otimes\omega_{v,e_i})\Delta(b))\big\|^2\,\|\Lambda_\varphi(c)\|^2\\ &\quad=\sum_{i\in I}\|\Lambda_\psi(a)\|^2\,\|\Lambda_\varphi(c)\|^2\psi\big((\iota\otimes\omega_{v,e_i})(\Delta(b))^*(\iota\otimes\omega_{v,e_i})(\Delta(b))\big).\end{aligned}$$

Lemma A.6 implies that

$$\sum_{i\in I}\|\Lambda_\psi(a)\|^2\,\|\Lambda_\varphi(c)\|^2\,\psi\big((\iota\otimes\omega_{v,e_i})(\Delta(b))^*(\iota\otimes\omega_{v,e_i})(\Delta(b))\big)$$

$$= \|\Lambda_\psi(a)\|^2 \, \|\Lambda_\varphi(c)\|^2 \, \psi\big((\iota \otimes \omega_{v,v})(\Delta(b^*b))\big)$$
$$= \|\Lambda_\psi(a)\|^2 \, \|\Lambda_\varphi(c)\|^2 \, \|v\|^2 \, \psi(b^*b),$$

where we used the right invariance of ψ in the last equality. Now the result follows. □

2.2. The multiplicative partial isometry

In the next part of this section, we will introduce (as usual) the multiplicative partial isometry and prove an inversion formula which is the cornerstone of this paper. This inversion formula will imply the unitarity of the partial isometry and the existence of the antipode together with its polar decomposition under rather weak assumptions.

For the rest of this subsection, we will fix a bi-C*-algebra (A, Δ) together with a left invariant proper weight φ on it. Let (H, π, Λ) denote a GNS-construction for φ.

At the same time, we fix a proper weight η on A with GNS-construction (K, θ, Γ). (Notice that we do not assume any form of left or right invariance.)

In Definition 1.27, we defined the tensor product $\eta \otimes \varphi$ of the two proper weights. Although the Hilbert space $K \otimes H$ is in general probably too small to serve as a GNS-space for $\eta \otimes \varphi$, it is however possible to define a partial GNS-construction for $\eta \otimes \varphi$. We discussed this in Section 1.6 where we introduced the notations $\overline{\mathcal{N}}(\eta, \varphi)$ and $\Gamma \otimes \Lambda$ (Notation 1.28). We want to stress that we will always be working with elements as in Lemma 1.30.

Using Lemma 1.30 and the left invariance of φ, it is immediate that we have for every $a \in \overline{\mathcal{N}}_\eta$ and $b \in \overline{\mathcal{N}}_\varphi$ that $\Delta(b)(a \otimes 1) \in \overline{\mathcal{N}}(\eta, \varphi)$ and that $(\eta \otimes \varphi)((a^* \otimes 1)\Delta(b^*b)(a \otimes 1)) = \eta(a^*a)\,\varphi(b^*b)$.

By polarization, we get that

$$\big\langle (\Gamma \otimes \Lambda)(\Delta(b)(a \otimes 1)), (\Gamma \otimes \Lambda)(\Delta(d)(c \otimes 1)) \big\rangle = \langle \Gamma(a), \Gamma(c) \rangle \, \langle \Lambda(b), \Lambda(d) \rangle$$

for all $a, c \in \overline{\mathcal{N}}_\eta$ and $b, d \in \overline{\mathcal{N}}_\varphi$. This justifies the following definition.

Notation 2.8. – We define the isometry $U : K \otimes H \to K \otimes H$ such that

$$U\big(\Gamma(a) \otimes \Lambda(b)\big) = (\Gamma \otimes \Lambda)\big(\Delta(b)(a \otimes 1)\big)$$

for $a \in \mathcal{N}_\eta$ and $b \in \mathcal{N}_\varphi$.

Now U^* could be called a multiplicative partial isometry associated to (A, Δ). Using the fact that $\mathcal{N}_\eta$ and $\mathcal{N}_\varphi$ are bounded strict cores for Γ and Λ respectively (*see* the remarks after Proposition 1.9) and the strict closedness of $\overline{\Gamma \otimes \Lambda}$ (first statement of Proposition 1.29), we get easily that

$$U\big(\Gamma(a) \otimes \Lambda(b)\big) = (\Gamma \otimes \Lambda)\big(\Delta(b)(a \otimes 1)\big)$$

for $a \in \overline{\mathcal{N}}_\eta$ and $b \in \overline{\mathcal{N}}_\varphi$.

We will also need the following formula for U.

RESULT 2.9. – *Consider an orthonormal basis $(e_i)_{i \in I}$ for K, $a \in \overline{\mathcal{N}}_\varphi$ and $v \in K$. Then we have that $\sum_{i \in I} \| \Lambda((\omega_{v,e_i} \otimes \iota)\Delta(a)) \|^2 = \|v\|^2 \, \varphi(a^*a) < \infty$ and*

$$U\big(v \otimes \Lambda(a)\big) = \sum_{i \in I} e_i \otimes \Lambda\big((\omega_{v,e_i} \otimes \iota)\Delta(a)\big).$$

Proof. – We have for every $w \in K$ that

$$\sum_{i\in I} \|\Lambda\big((\omega_{w,e_i}\otimes 1)\Delta(a)\big)\|^2 = \sum_{i\in I} \varphi\big([(\omega_{w,e_i}\otimes 1)\Delta(a)]^*[(\omega_{w,e_i}\otimes 1)\Delta(a)]\big).$$

So Lemma A.6 implies that

$$\sum_{i\in I} \|\Lambda\big((\omega_{w,e_i}\otimes 1)\Delta(a)\big)\|^2 = \varphi\big((\omega_{w,w}\otimes 1)\Delta(a^*a)\big) = \|w\|^2\,\varphi(a^*a),$$

where we used the left invariance in the last equality.

Also, notice that we get that both expressions in the statement of the lemma depend continuously on v. So it is enough to prove the equality for a dense set of elements v in K.

Therefore choose $b \in \mathcal{N}_\eta$. Because $\Delta(a)$ belongs to $\overline{\mathcal{N}}_{\iota\otimes\varphi}$, we can apply Lemma 1.30. This lemma gives us immediately that

$$U\big(\Gamma(b)\otimes\Lambda(a)\big) = (\Gamma\otimes\Lambda)\big(\Delta(a)(b\otimes 1)\big) = \sum_{i\in I} e_i\otimes\Lambda\big((\omega_{\Gamma(b),e_i}\otimes\iota)\Delta(a)\big). \qquad \square$$

For the sake of completeness, we also include the standard formulas for the slice of U with a functional (left and right).

RESULT 2.10. – *The following properties hold.*

- *We have for all* $a,b \in \overline{\mathcal{N}}_\varphi$ *that* $(\iota\otimes\omega_{\Lambda(a),\Lambda(b)})(U) = \theta((\iota\otimes\varphi)((1\otimes b^*)\Delta(a)))$.
- *We have for all* $\omega \in \mathrm{B}_0(H_\eta)^*$ *and* $a \in \overline{\mathcal{N}}_\varphi$ *that* $(\omega\otimes\iota)(U)\,\Lambda(a) = \Lambda((\omega\otimes\iota)\Delta(a))$.

Proof. – • Choose $c,d \in \mathcal{N}_\eta$. Then

$$\begin{aligned}
\big\langle(\iota\otimes\omega_{\Lambda(a),\Lambda(b)})(U)\Gamma(c),\Gamma(d)\big\rangle &= \big\langle U(\Gamma(c)\otimes\Lambda(a)),\Gamma(d)\otimes\Lambda(b)\big\rangle\\
&= \big\langle(\Gamma\otimes\Lambda)(\Delta(a)(c\otimes 1)),(\Gamma\otimes\Lambda)(d\otimes b)\big\rangle\\
&= (\eta\otimes\varphi)\big((d^*\otimes b^*)\Delta(a)(c\otimes 1)\big)\\
&= \eta\big(d^*(\iota\otimes\varphi)((1\otimes b^*)\Delta(a))c\big)\\
&= \big\langle\theta\big((\iota\otimes\varphi)((1\otimes b^*)\Delta(a))\big)\Gamma(c),\Gamma(d)\big\rangle
\end{aligned}$$

and the stated formula follows.

• Take $c,d \in \mathcal{N}_\eta$. Choose $b \in \mathcal{N}_\varphi$. Then

$$\begin{aligned}
\big\langle(\omega_{\Gamma(c),\Gamma(d)}\otimes\iota)(U)\Lambda(a),\Lambda(b)\big\rangle &= \big\langle U(\Gamma(c)\otimes\Lambda(a)),\Gamma(d)\otimes\Lambda(b)\big\rangle\\
&= \big\langle(\Gamma\otimes\Lambda)(\Delta(a)(c\otimes 1)),(\Gamma\otimes\Lambda)(d\otimes b)\big\rangle\\
&= (\eta\otimes\varphi)\big((d^*\otimes b^*)\Delta(a)(c\otimes 1)\big)\\
&= \varphi\big(b^*(\eta\otimes\iota)((d^*\otimes 1)\Delta(a)(c\otimes 1))\big)\\
&= \big\langle\Lambda\big((\omega_{\Gamma(c),\Gamma(d)}\otimes\iota)\Delta(a)\big),\Lambda(b)\big\rangle.
\end{aligned}$$

So we get that $(\omega_{\Gamma(c),\Gamma(d)}\otimes\iota)(U)\Lambda(a) = \Lambda((\omega_{\Gamma(c),\Gamma(d)}\otimes\iota)\Delta(a))$ for all $c,d \in \mathcal{N}_\eta$.

Because $\mathrm{B}_0(H_\eta)^* = [\omega_{\Gamma(c),\Gamma(d)} \mid c,d \in \mathcal{N}_\eta]$, the result follows by Result 2.3. $\square$

As mentioned in the beginning of this section, a crucial step in this paper is a formula for the inverse of the isometry U. We will provide one in the following results.

In order to deal with the unboundedness of the weight φ, we will use the following simple lemma to make things bounded. For the definition of $\mathcal{F}_\varphi$, T_ω and ξ_ω used in the next lemma, we refer to Notation 1.4.

LEMMA 2.11. – *Consider $\omega \in \mathcal{F}_\varphi$, $u, v \in K$, $w \in H$ and $a \in \overline{\mathcal{N}}_\varphi$. Then*

$$\big\langle \big(1 \otimes T_\omega^{\frac{1}{2}}\big) U\big(u \otimes \Lambda(a)\big), v \otimes w \big\rangle = (\omega_{u,v} \otimes \omega_{\xi_\omega, w})\Delta(a).$$

Proof. – It is clear that we only need to prove the lemma for elements u which are dense in K. Therefore choose $b \in \mathcal{N}_\eta$. Also take $\rho \in \mathcal{F}_\eta$.

We have for all $x \in \overline{\mathcal{N}}(\eta, \varphi)$ that

$$\big\|(\theta \otimes \pi)(x)(\xi_\rho \otimes \xi_\omega)\big\|^2 = (\rho \otimes \omega)(x^* x) \leqslant (\eta \otimes \varphi)(x^* x) = \big\|(\Gamma \otimes \Lambda)(x)\big\|^2.$$

So we can define a bounded operator $F: K \otimes H \to K \otimes H$ such that $F(\Gamma \otimes \Lambda)(x) = (\theta \otimes \pi)(x)(\xi_\rho \otimes \xi_\omega)$ for all $x \in \overline{\mathcal{N}}(\eta, \varphi)$. It is then clear that

$$F\big(\Gamma(p) \otimes \Lambda(q)\big) = \big(T_\rho^{\frac{1}{2}} \otimes T_\omega^{\frac{1}{2}}\big)\big(\Gamma(p) \otimes \Lambda(q)\big)$$

for $p \in \mathcal{N}_\eta$, $q \in \mathcal{N}_\varphi$. Thus $F = T_\rho^{\frac{1}{2}} \otimes T_\omega^{\frac{1}{2}}$.

Hence,

$$\begin{aligned}\big(T_\rho^{\frac{1}{2}} \otimes T_\omega^{\frac{1}{2}}\big) U\big(\Gamma(b) \otimes \Lambda(a)\big) &= F(\Gamma \otimes \Lambda)\big(\Delta(a)(b \otimes 1)\big)\\ &= (\theta \otimes \pi)\big(\Delta(a)(b \otimes 1)\big)(\xi_\rho \otimes \xi_\omega).\end{aligned}$$

Consequently,

$$\begin{aligned}\big\langle \big(T_\rho^{\frac{1}{2}} \otimes T_\omega^{\frac{1}{2}}\big) U(\Gamma(b) \otimes \Lambda(a)), v \otimes w \big\rangle &= \big\langle (\theta \otimes \pi)(\Delta(a)(b \otimes 1))(\xi_\rho \otimes \xi_\omega), v \otimes w \big\rangle\\ &= \big\langle (\theta \otimes \pi)(\Delta(a))(\theta(b)\xi_\rho \otimes \xi_\omega), v \otimes w \big\rangle\\ &= \big\langle (\theta \otimes \pi)(\Delta(a))\big(T_\rho^{\frac{1}{2}}\Gamma(b) \otimes \xi_\omega\big), v \otimes w \big\rangle.\end{aligned}$$

Because $(T_\rho^{\frac{1}{2}})_{\rho \in \mathcal{G}_\eta}$ converges strongly to 1, this implies that

$$\big\langle \big(1 \otimes T_\omega^{\frac{1}{2}}\big) U(\Gamma(b) \otimes \Lambda(a)), v \otimes w \big\rangle = \big\langle (\theta \otimes \pi)(\Delta(a))(\Gamma(b) \otimes \xi_\omega), v \otimes w \big\rangle. \qquad \square$$

Now, we can easily prove the crucial result.

PROPOSITION 2.12. – *Consider a right invariant proper weight ψ on (A, Δ), $a, b \in \overline{\mathcal{N}}_\psi$, $c \in \overline{\mathcal{N}}_\varphi$, $v \in K$ and an orthonormal basis $(e_i)_{i \in I}$ for K. Then*

$$\sum_{i \in I} \big\|\Lambda\big((\psi \otimes \iota)(\Delta(a^* c)((\iota \otimes \omega_{v, e_i})(\Delta(b)) \otimes 1))\big)\big\|^2 < \infty$$

and

$$U\Big(\sum_{i \in I} e_i \otimes \Lambda\big((\psi \otimes \iota)(\Delta(a^* c)((\iota \otimes \omega_{v, e_i})(\Delta(b)) \otimes 1))\big)\Big) = v \otimes \Lambda\big((\psi \otimes \iota)(\Delta(a^* c)(b \otimes 1))\big).$$

Proof. – The first inequality was already proven in Result 2.7. Let us now prove the last equality. Choose $\omega \in \mathcal{F}_\varphi$, $u \in K$ and $w \in H$.

Using Lemma 2.11, we get that

$$\begin{aligned}
&\Big\langle (1 \otimes T_\omega^{\frac{1}{2}}) U\Big(\sum_{i \in I} e_i \otimes \Lambda((\psi \otimes \iota)(\Delta(a^*c)((\iota \otimes \omega_{v,e_i})(\Delta(b)) \otimes 1)))\Big), u \otimes w\Big\rangle \\
&= \sum_{i \in I} \big\langle (1 \otimes T_\omega^{\frac{1}{2}}) U\big(e_i \otimes \Lambda((\psi \otimes \iota)(\Delta(a^*c)((\iota \otimes \omega_{v,e_i})(\Delta(b)) \otimes 1)))\big), u \otimes w\big\rangle \\
&= \sum_{i \in I} (\omega_{e_i,u} \otimes \omega_{\xi_\omega,w})\Delta\big((\psi \otimes \iota)(\Delta(a^*c)((\iota \otimes \omega_{v,e_i})(\Delta(b)) \otimes 1))\big) \\
&= \sum_{i \in I} \psi\big((\iota \otimes (\omega_{e_i,u} \otimes \omega_{\xi_\omega,w})\Delta)(\Delta(a^*c))(\iota \otimes \omega_{v,e_i})(\Delta(b))\big) \\
&= \sum_{i \in I} \psi\big((\iota \otimes \omega_{e_i,u} \otimes \omega_{\xi_\omega,w})((\Delta \otimes \iota)\Delta(a^*c))(\iota \otimes \omega_{v,e_i})(\Delta(b))\big) \\
&= \sum_{i \in I} \psi\big((\iota \otimes \omega_{e_i,u})\big(\Delta((\iota \otimes \omega_{\xi_\omega,w})\Delta(a^*c))\big)(\iota \otimes \omega_{v,e_i})(\Delta(b))\big).
\end{aligned}$$

But Lemma A.6 implies that the last sum is equal to

$$\psi\big((\iota \otimes \omega_{v,u})(\Delta\big((\iota \otimes \omega_{\xi_\omega,w})\Delta(a^*c)\big)\,\Delta(b))\big).$$

Using now the right invariance of ψ, we get that

$$\begin{aligned}
&\Big\langle (1 \otimes T_\omega^{\frac{1}{2}}) U\Big(\sum_{i \in I} e_i \otimes \Lambda((\psi \otimes \iota)(\Delta(a^*c)((\iota \otimes \omega_{v,e_i})(\Delta(b)) \otimes 1)))\Big), u \otimes w\Big\rangle \\
&= \psi\big((\iota \otimes \omega_{v,u})\Delta((\iota \otimes \omega_{\xi_\omega,w})(\Delta(a^*c))\,b)\big) \\
&= \langle v,u\rangle\, \psi\big((\iota \otimes \omega_{\xi_\omega,w})(\Delta(a^*c))\,b\big) \\
&= \langle v,u\rangle\, \omega_{\xi_\omega,w}\big((\psi \otimes \iota)(\Delta(a^*c)(b \otimes 1))\big) \\
&= \langle v,u\rangle\, \big\langle \pi\big((\psi \otimes \iota)(\Delta(a^*c)(b \otimes 1))\big)\xi_\omega, w\big\rangle \\
&= \langle v,u\rangle\, \big\langle T_\omega^{\frac{1}{2}} \Lambda\big((\psi \otimes \iota)(\Delta(a^*c)(b \otimes 1))\big), w\big\rangle.
\end{aligned}$$

Because $(T_\omega^{\frac{1}{2}})_{\omega \in \mathcal{G}_\varphi}$ converges strongly to 1, we get for all $u \in K$ and $w \in H$ that

$$\begin{aligned}
&\Big\langle U\Big(\sum_{i \in I} e_i \otimes \Lambda((\psi \otimes \iota)(\Delta(a^*c)((\iota \otimes \omega_{v,e_i})(\Delta(b)) \otimes 1)))\Big), u \otimes w\Big\rangle \\
&= \langle v,u\rangle\, \big\langle \Lambda\big((\psi \otimes \iota)(\Delta(a^*c)(b \otimes 1))\big), w\big\rangle. \qquad \square
\end{aligned}$$

Let us put this inversion formula in a form which is more transparent.

PROPOSITION 2.13. – *Consider a right invariant proper weight ψ on (A, Δ), $a, b \in \overline{\mathcal{N}}_\psi$, $c \in \overline{\mathcal{N}}_\varphi$, $d \in \mathcal{N}_\eta$ and put $x = (\psi \otimes \iota \otimes \iota)(\Delta_{13}(a^*c)\Delta_{12}(b))$. Then x belongs to $\overline{\mathcal{N}}_{\iota \otimes \varphi}$, $x(d \otimes 1)$ belongs to $\overline{\mathcal{N}}(\eta, \varphi)$ and*

$$U(\Gamma \otimes \Lambda)\big(x(d \otimes 1)\big) = \Gamma(d) \otimes \Lambda\big((\psi \otimes \iota)(\Delta(a^*c)(b \otimes 1))\big).$$

Proof. – Using inequality (2.1), we have that

$$\begin{aligned} x^*x &= (\psi\otimes\iota\otimes\iota)\big(\Delta_{13}(a^*c)\Delta_{12}(b)\big)^*(\psi\otimes\iota\otimes\iota)\big(\Delta_{13}(a^*c)\Delta_{12}(b)\big) \\ &\leqslant \big\|(\psi\otimes\iota\otimes\iota)\big(\Delta_{13}(a^*a)\big)\big\|\,(\psi\otimes\iota\otimes\iota)\big(\Delta_{12}(b^*)\Delta_{13}(c^*c)\Delta_{12}(b)\big) \\ &= \psi(a^*a)(\psi\otimes\iota\otimes\iota)\big(\Delta_{12}(b^*)\Delta_{13}(c^*c)\Delta_{12}(b)\big). \end{aligned}$$

For every $\omega\in\mathcal{G}_\varphi$ we have

$$\begin{aligned} &(\iota\otimes\omega)\big((\psi\otimes\iota\otimes\iota)(\Delta_{12}(b^*)\Delta_{13}(c^*c)\Delta_{12}(b))\big) \\ &\quad= (\psi\otimes\iota)\big(\Delta(b^*)((\iota\otimes\omega)(\Delta(c^*c))\otimes 1)\Delta(b)\big) \\ &\quad= (\Lambda_\psi\otimes\iota)(\Delta(b))^*\big(\pi_\psi\big((\iota\otimes\omega)(\Delta(c^*c))\big)\otimes 1\big)(\Lambda_\psi\otimes\iota)(\Delta(b)). \end{aligned}$$

Therefore the left invariance of φ implies that the net

$$\big((\iota\otimes\omega)\big((\psi\otimes\iota\otimes\iota)(\Delta_{12}(b^*)\Delta_{13}(c^*c)\Delta_{12}(b))\big)\big)_{\omega\in\mathcal{G}_\varphi}$$

converges strictly to $\varphi(c^*c)\,(\psi\otimes\iota)(\Delta(b^*b))$. Hence we get that

$$(\psi\otimes\iota\otimes\iota)\big(\Delta_{12}(b^*)\Delta_{13}(c^*c)\Delta_{12}(b)\big)\in\overline{\mathcal{M}}^+_{\iota\otimes\varphi}.$$

By Result 1.19 and the inequality above we have $x^*x\in\overline{\mathcal{M}}^+_{\iota\otimes\varphi}$. Therefore x belongs to $\overline{\mathcal{N}}_{\iota\otimes\varphi}$ and we can apply Lemma 1.30. This lemma says that $x(d\otimes 1)$ belongs to $\overline{\mathcal{N}}(\eta,\varphi)$ and

$$\begin{aligned} (\Gamma\otimes\Lambda)\big(x(d\otimes 1)\big) &= \sum_{i\in I} e_i\otimes\Lambda\big((\omega_{\Gamma(d),e_i}\otimes\iota)(x)\big) \\ &= \sum_{i\in I} e_i\otimes\Lambda\big((\psi\otimes\iota)(\Delta(a^*c)((\iota\otimes\omega_{\Gamma(d),e_i})(\Delta(b))\otimes 1))\big). \end{aligned}$$

Therefore the previous proposition implies that

$$U(\Gamma\otimes\Lambda)\big(x(d\otimes 1)\big) = \Gamma(d)\otimes\Lambda\big((\psi\otimes\iota)(\Delta(a^*c)(b\otimes 1))\big). \qquad\square$$

3. Bi-C*-algebras possessing a left and a right invariant weight

This section revolves around the applications of Propositions 2.12 and 2.13. They are threefold:

- the C*-algebras and von Neumann algebras in the GNS-spaces are independent of the left or right invariant weight;
- unitarity of the multiplicative partial isometries;
- polar decomposition of the antipode.

3.1. Uniqueness of the reduced C*- and W*-algebras

Consider a bi-C*-algebra (A,Δ) and proper weights φ and ψ such that both are left or right invariant. Let $(H_\varphi,\pi_\varphi,\Lambda_\varphi)$ be a GNS-construction for φ and $(H_\psi,\pi_\psi,\Lambda_\psi)$ a GNS-construction for ψ. We will prove that under some fairly weak conditions the von Neumann algebras $\pi_\varphi(A)''$ and $\pi_\psi(A)''$ are $*$-isomorphic (and similarly for the C*-algebras). This will be done by establishing a weak absolute continuity property between φ and ψ.

We start off with the lemma which makes it all possible.

LEMMA 3.1. – *Consider a bi-C*-algebra (A,Δ). Let φ be a left invariant proper weight on (A,Δ) and ψ a right invariant proper weight on (A,Δ). Let $b,\, d \in \overline{\mathcal{N}}_\psi$, $c \in \overline{\mathcal{N}}_\varphi$ and put $a = (\psi \otimes \iota)(\Delta(b^*c)(d \otimes 1)) \in \overline{\mathcal{N}}_\varphi$.*

Suppose that θ is a non-degenerate representation of A on a Hilbert space H. Let $v \in H$. Then there exists a family of elements $(a_i)_{i\in I}$ in the set $\{(\psi\otimes\iota)(\Delta(b^*c)(e\otimes 1)) \mid e \in \overline{\mathcal{N}}_\psi\} \subseteq \overline{\mathcal{N}}_\varphi$ such that $\sum_{i\in I} \varphi(a_i^* a_i) < \infty$ and $(\omega_{v,v} \otimes a\varphi a^*)\Delta(x) = \sum_{i\in I} \varphi(a_i^* x a_i)$ for all $x \in A^+$.

Proof. – Define $\omega \in A_+^*$ by $\omega(x) = \langle \theta(x)v, v\rangle$ for $x \in A$. Then we have a natural GNS-construction $(H_\omega, \pi_\omega, \Lambda_\omega)$ for ω:
- $H_\omega = [\theta(a)v \mid a \in A]$;
- $\pi_\omega(a)w = \theta(a)w$ for all $a \in A$ and $w \in H_\omega$;
- $\Lambda_\omega(a) = \theta(a)v$ for all $a \in A$.

Take an orthonormal basis $(e_i)_{i\in I}$ for H_ω and define, for $i \in I$,

$$a_i = (\psi \otimes \iota)(\Delta(b^*c)((\iota \otimes \omega_{v,e_i})(\Delta(d)) \otimes 1)) \in \overline{\mathcal{N}}_\varphi.$$

By Result 2.7, we already know that $\sum_{i\in I} \varphi(a_i^* a_i) < \infty$.

Now we want to apply the results of Section 2.2 (the η in that section will be our ω). As in Notation 2.8, we define an isometry $U : H_\omega \otimes H_\varphi \to H_\omega \otimes H_\varphi$ such that $U(\Lambda_w(p) \otimes \Lambda_\varphi(q)) = (\Lambda_\omega \otimes \Lambda_\varphi)(\Delta(q)(p \otimes 1))$ for $p \in A$ and $q \in \mathcal{N}_\varphi$.

Then Proposition 2.12 implies that $U(\sum_{i\in I} e_i \otimes \Lambda_\varphi(a_i)) = v \otimes \Lambda_\varphi(a)$.

By Proposition 1.29, it is not difficult to see that $U(1 \otimes \pi_\varphi(x)) = (\pi_\omega \otimes \pi_\varphi)(\Delta(x))U$ for $x \in A$. This implies for $x \in A^+$ that

$$\begin{aligned}
(\omega_{v,v} \otimes a\varphi a^*)\Delta(x) &= \big\langle (\pi_\omega \otimes \pi_\varphi)(\Delta(x))(v \otimes \Lambda_\varphi(a)), (v \otimes \Lambda_\varphi(a))\big\rangle \\
&= \left\langle (\pi_\omega \otimes \pi_\varphi)(\Delta(x))U\left(\sum_{i\in I} e_i \otimes \Lambda_\varphi(a_i)\right), U\left(\sum_{i\in I} e_i \otimes \Lambda_\varphi(a_i)\right)\right\rangle \\
&= \left\langle U(1 \otimes \pi_\varphi(x))\left(\sum_{i\in I} e_i \otimes \Lambda_\varphi(a_i)\right), U\left(\sum_{i\in I} e_i \otimes \Lambda_\varphi(a_i)\right)\right\rangle \\
&= \left\langle U\left(\sum_{i\in I} e_i \otimes \Lambda_\varphi(xa_i)\right), U\left(\sum_{i\in I} e_i \otimes \Lambda_\varphi(a_i)\right)\right\rangle \\
&= \left\langle \sum_{i\in I} e_i \otimes \Lambda_\varphi(xa_i), \sum_{i\in I} e_i \otimes \Lambda_\varphi(a_i)\right\rangle \\
&= \sum_{i\in I} \big\langle \Lambda_\varphi(xa_i), \Lambda_\varphi(a_i)\big\rangle = \sum_{i\in I} \varphi(a_i^* x a_i). \qquad \square
\end{aligned}$$

In a next step, we want to use this last result to prove a weak form of absolute continuity between left and right invariant weights. In order to do so, we will need a weak extra assumption on (A,Δ).

Terminology 3.2. – Consider a bi-C*-algebra (A,Δ). Then we say that (A,Δ) satisfies condition [D] if and only if there exist a left invariant proper weight φ and a right invariant proper weight ψ on (A,Δ) such that

$$A = \big[(\psi \otimes \iota)(\Delta(a^*)(b \otimes 1)) \mid a, b \in \mathcal{N}_\psi\big] = \big[(\iota \otimes \varphi)(\Delta(a^*)(1 \otimes b)) \mid a, b \in \mathcal{N}_\varphi\big].$$

Terminology 3.3. – Consider a bi-C*-algebra (A,Δ). Then we say that (A,Δ) satisfies the KMS condition if and only if there exist a left invariant approximate KMS weight φ and a right invariant approximate KMS weight ψ on (A,Δ).

We will give two natural conditions which imply condition [D].

RESULT 3.4. – *Consider a bi-*C*-*-algebra* (A,Δ) *such that*:
- $A=[(\iota\otimes\omega)\Delta(a)\mid a\in A,\ \omega\in A^*]=[(\omega\otimes\iota)\Delta(a)\mid a\in A,\ \omega\in A^*]$;
- *there exist a faithful left invariant proper weight* φ *and a faithful right invariant proper weight* ψ *on* (A,Δ).

Then (A,Δ) *satisfies condition* [D].

Proof. – We will prove that $[(\iota\otimes\varphi)(\Delta(a^*)(1\otimes b))\mid a,b\in\mathcal{N}_\varphi]=A$.

From Result 2.5, we know that $(\iota\otimes\varphi)(\Delta(a^*)(1\otimes b))$ belongs to A for all $a,b\in\mathcal{N}_\varphi$. To prove density, we will use Hahn–Banach.

Therefore choose $\theta\in A^*$ such that $\theta((\iota\otimes\varphi)(\Delta(a^*)(1\otimes b)))=0$ for $a,b\in\mathcal{N}_\varphi$. Then we have for every $a,b\in\mathcal{N}_\varphi$ that $\varphi((\theta\otimes\iota)(\Delta(a^*))b)=0$.

Hence the faithfulness of φ implies that $(\theta\otimes\iota)\Delta(a^*)=0$ for all $a\in\mathcal{N}_\varphi$. So we get for every $\omega\in A^*$ and $a\in\mathcal{N}_\varphi$ that

$$\theta\big((\iota\otimes\omega)\Delta(a^*)\big)=\omega\big((\theta\otimes\iota)\Delta(a^*)\big)=0.$$

Because we have that $A=[(\iota\otimes\omega)\Delta(a^*)\mid a\in\mathcal{N}_\varphi,\ \omega\in A^*]$ by assumption, we get that $\theta=0$.

So we have proven that $[(\iota\otimes\varphi)(\Delta(a^*)(1\otimes b))\mid a,b\in\mathcal{N}_\varphi]=A$. The case for ψ is dealt with in the same way. □

RESULT 3.5. – *Consider a bi-*C*-*-algebra* (A,Δ) *such that*:
- $\Delta(A)(1\otimes A)$ *and* $\Delta(A)(A\otimes 1)$ *are dense in* $A\otimes A$;
- *there exist a left invariant proper weight* φ *and a right invariant proper weight* ψ *on* (A,Δ).

Then (A,Δ) *satisfies condition* [D].

Proof. – We will prove that $[(\iota\otimes\varphi)(\Delta(a^*)(1\otimes b))\mid a,b\in\mathcal{N}_\varphi]=A$.

From Result 2.5, we know that $(\iota\otimes\varphi)(\Delta(a^*)(1\otimes b))$ belongs to A for all $a,b\in\mathcal{N}_\varphi$. To prove density, we will use Hahn–Banach again.

Suppose that $[(\iota\otimes\varphi)(\Delta(a^*)(1\otimes b))\mid a,b\in\mathcal{N}_\varphi]$ is not equal to A. By Hahn–Banach, there exists $\theta\in A^*$ such that $\theta\neq 0$ and $\theta((\iota\otimes\varphi)(\Delta(a^*)(1\otimes b)))=0$ for $a,b\in\mathcal{N}_\varphi$. Then we have for every $a,b\in\mathcal{N}_\varphi$ that $\varphi((\theta\otimes\iota)(\Delta(a^*))b)=0$. So we have in particular for every $a\in\mathcal{N}_\varphi$ that $\varphi((\theta\otimes\iota)(\Delta(a^*))(\overline{\theta}\otimes\iota)(\Delta(a)))=0$.

Hence we get for all $a\in\mathcal{N}_\varphi$ and $x\in A$ that

$$\begin{aligned}&\varphi\big((\theta\otimes\iota)(\Delta(a^*)(1\otimes x^*))(\overline{\theta}\otimes\iota)((1\otimes x)\Delta(a))\big)\\&\quad=\varphi\big((\theta\otimes\iota)(\Delta(a^*))x^*x(\overline{\theta}\otimes\iota)(\Delta(a))\big)\\&\quad\leqslant\|x\|^2\varphi\big((\theta\otimes\iota)(\Delta(a^*))(\overline{\theta}\otimes\iota)(\Delta(a))\big)=0.\end{aligned}$$

So we get that $\varphi((\theta\otimes\iota)(\Delta(a^*)(1\otimes x^*))(\overline{\theta}\otimes\iota)((1\otimes x)\Delta(a)))=0$ for all $a\in\mathcal{N}_\varphi$ and $x\in A$.

Define $N=\{y\in A\mid\varphi(y^*y)=0\}$. Because φ is lower semi-continuous, N is closed in A. But the above discussion implies that N contains $\langle(\overline{\theta}\otimes\iota)((1\otimes x)\Delta(a))\mid x\in A,\ a\in\mathcal{N}_\varphi\rangle$. But since $(1\otimes A)\Delta(A)$ is by assumption dense in $A\otimes A$ and $\overline{\theta}$ is not zero,

$$\langle(\overline{\theta}\otimes\iota)((1\otimes x)\Delta(a))\mid x\in A,\ a\in\mathcal{N}_\varphi\rangle$$

is dense in A. Consequently, $N=A$ and $\varphi=0$. A contradiction.

So we have proven that $[(\iota\otimes\varphi)(\Delta(a^*)(1\otimes b)) \mid a,b\in\mathcal{N}_\varphi]=A$. The case for ψ is dealt with in the same way. □

If a bi-C*-algebra satisfies condition [D], it is now easy to prove that we have a weak absolute continuity between left and right invariant weights.

PROPOSITION 3.6. – *Consider a bi-*C**-algebra* (A,Δ) *which satisfies condition* [D]. *Let* φ *and* ψ *be proper weights on* A *such that* φ *is left or right invariant and such that* ψ *is left or right invariant. Then there exists a family of elements* $(a_i)_{i\in I}$ *in* $\mathcal{N}_\varphi$ *such that* $\psi(x)=\sum_{i\in I}\varphi(a_i^*xa_i)$ *for* $x\in\mathcal{M}_\psi^+$.

Proof. – Suppose first that φ is left invariant and that ψ is right invariant. Let (H,π,Λ) be a GNS-construction for ψ. We know that there exists a family of vectors $(v_k)_{k\in K}$ in H such that $\psi(x)=\sum_{k\in K}\omega_{v_k,v_k}(x)$ for $x\in A^+$ (combine Definition 1.31 and the results in [18]). Now, we want to apply Lemma 3.1. In this case, A will act on H via π.

By assumption, there exists a right invariant proper weight η on (A,Δ) such that

$$A=\big[(\eta\otimes\iota)\big(\Delta(b^*)(d\otimes 1)\big) \mid b,d\in\mathcal{N}_\eta\big].$$

By referring to inequality (2.1), we see that $A=[(\eta\otimes\iota)(\Delta(b^*c)(d\otimes 1)) \mid b,d\in\mathcal{N}_\eta,\ c\in\mathcal{N}_\varphi]$. Put $N=\{(\eta\otimes\iota)(\Delta(b^*c)(d\otimes 1)) \mid b,d\in\mathcal{N}_\eta,\ c\in\mathcal{N}_\varphi\}$. Then there exists an element $a\in N$ such that $\varphi(a^*a)=1$: Suppose that there does not exist an element $p\in N$ such that $\varphi(p^*p)\neq 0$. Then Cauchy–Schwarz implies for every p, $q\in N$ that $\varphi(q^*p)=0$. So we get that $\varphi(p^*p)=0$ for $p\in\langle N\rangle$.

Now $\{y\in A \mid \varphi(y^*y)=0\}$ is a closed subset of A which contains $\langle N\rangle$. Because $\langle N\rangle$ is dense in A, we see that $\{y\in A \mid \varphi(y^*y)=0\}=A$. A contradiction with the fact that $\varphi\neq 0$.

Now take $b,d\in\mathcal{N}_\eta$ and $c\in\mathcal{N}_\varphi$ such that $a=(\eta\otimes\iota)(\Delta(b^*c)(d\otimes 1))$.

Let $k\in K$. By Lemma 3.1, we know that there exist a set $J(k)$ and a family of elements $(a(k,j))_{j\in J(k)}$ in the set $\{(\eta\otimes\iota)(\Delta(b^*c)(e\otimes 1)) \mid e\in\overline{\mathcal{N}}_\eta\}\subseteq\overline{\mathcal{N}}_\varphi$ such that

$$(\omega_{v_k,v_k}\otimes a\varphi a^*)\Delta(x)=\sum_{j\in J(k)}\varphi\big(a(k,j)^*xa(k,j)\big)$$

for $x\in A^+$.

Referring to inequality (2.1), we see that

$$\begin{aligned}A&=\big[(\eta\otimes\iota)\big(\Delta(f^*g)(e\otimes 1)\big) \mid f\in\mathcal{N}_\eta,\ e\in\mathcal{N}_\eta,\ g\in\mathcal{N}_\varphi\big]\\&=\big[(\eta\otimes\iota)\big(\Delta(f^*g)(e\otimes 1)\big) \mid f\in\mathcal{N}_\eta,\ e\in\overline{\mathcal{N}}_\eta,\ g\in\mathcal{N}_\varphi\big],\end{aligned}$$

so we get that $a(k,j)\in A$, hence $a(k,j)\in\mathcal{N}_\varphi$ for all $j\in J(k)$.

Using the right invariance of ψ, we get now for $x\in\mathcal{M}_\psi^+$ that

$$\begin{aligned}\psi(x)&=\psi(x)(a\varphi a^*)(1)=\psi\big((\iota\otimes a\varphi a^*)\Delta(x)\big)\\&=\sum_{k\in K}(\omega_{v_k,v_k}\otimes a\varphi a^*)\Delta(x)=\sum_{k\in K}\sum_{j\in J(k)}\varphi\big(a(k,j)^*xa(k,j)\big).\end{aligned}$$

So we have proven the proposition in the case that φ is left invariant and ψ is right invariant. If φ is right invariant and ψ is left invariant, the proposition is proven in the same way.

If φ and ψ are both left invariant, we use the right invariant weight η as an intermediator. The case where φ and ψ are both right invariant is dealt with in the same way. □

The same result can also be proven under other conditions. One of them is the KMS condition. The proof is almost the same as the previous one. The only difference is that we have to find an element $a \in N$ such that $\varphi(a^*a) = 1$ in another way. But this will follow easily from Proposition 3.15.

So we get the following proposition:

PROPOSITION 3.7. – *Consider a bi-*C^**-algebra* (A, Δ) *which satisfies the KMS condition. Let* φ *and* ψ *be proper weights on* A *such that* φ *is left or right invariant and such that* ψ *is left or right invariant. Then there exists a family of elements* $(a_i)_{i \in I}$ *in* $\overline{\mathcal{N}}_\varphi$ *such that* $\psi(x) = \sum_{i \in I} \varphi(a_i^* x a_i)$ *for* $x \in \mathcal{M}_\psi^+$.

Suppose that $(\omega \otimes \iota)\Delta(a)$ and $(\iota \otimes \omega)\Delta(a)$ both belong to A for all $\omega \in A^*$ and $a \in A$. Appealing to the proof of Proposition 3.6 and Result 2.5, we see that the elements $(a_i)_{i \in I}$ in the proposition above can then be chosen in such a way that they belong to $\mathcal{N}_\varphi$.

Now it is extremely easy to prove the main result of this section.

THEOREM 3.8. – *Consider a bi-*C^**-algebra* (A, Δ) *which satisfies condition* [D] *or the KMS condition. Let* φ *and* ψ *be proper weights on* A *such that* φ *is left or right invariant and such that* ψ *is left or right invariant. Then*:
- *there exists a unique *-isomorphism* $\pi : \pi_\varphi(A) \to \pi_\psi(A)$ *such that* $\pi(\pi_\varphi(a)) = \pi_\psi(a)$ *for* $a \in A$;
- *there exists a unique *-isomorphism* $\theta : \pi_\varphi(A)'' \to \pi_\psi(A)''$ *such that* $\theta(\pi_\varphi(a)) = \pi_\psi(a)$ *for* $a \in A$.

Proof. – By Propositions 3.6 and 3.7, there exists a family of elements $(a_i)_{i \in I}$ in $\overline{\mathcal{N}}_\varphi$ such that $\psi(x) = \sum_{i \in I} \varphi(a_i^* x a_i)$ for $x \in \mathcal{M}_\psi^+$. Then we get for all $x \in \mathcal{N}_\psi$ that $\sum_{i \in I} \|\Lambda_\varphi(x a_i)\|^2 = \|\Lambda_\psi(x)\|^2 < \infty$. Let $(e_i)_{i \in I}$ be the orthonormal basis for $\ell^2(I)$.

So we can define the isometry $U : H_\psi \to \ell^2(I) \otimes H_\varphi$ such that $U\Lambda_\psi(x) = \sum_{i \in I} e_i \otimes \Lambda_\varphi(x a_i)$ for $x \in \mathcal{N}_\psi$.

It is then easy to see that $(1 \otimes \pi_\varphi(a))U = U\pi_\psi(a)$ for $a \in A$. So we get for every $a \in A$ that $\pi_\psi(a) = U^*(1 \otimes \pi_\varphi(a))U$. This implies that $\pi_\psi(A) = U^*(1 \otimes \pi_\varphi(A))U$ and $\pi_\psi(A)'' = U^*(1 \otimes \pi_\varphi(A)'')U$.

Now, define linear mappings $\pi : \pi_\varphi(A) \to \pi_\psi(A)$, $y \mapsto U^*(1 \otimes y)U$ and $\theta : \pi_\varphi(A)'' \to \pi_\psi(A)''$, $y \mapsto U^*(1 \otimes y)U$. So we have for every $a \in A$ that $\pi(\pi_\varphi(a)) = \theta(\pi_\varphi(a)) = \pi_\psi(a)$. This implies that π and θ are *-homomorphisms.

We get in a similar way *-homomorphisms $\pi_0 : \pi_\psi(A) \to \pi_\varphi(A)$ and $\theta_0 : \pi_\psi(A)'' \to \pi_\varphi(A)''$ such that $\pi_0(\pi_\psi(a)) = \theta_0(\pi_\psi(a)) = \pi_\varphi(a)$ for $a \in A$.

It is then clear that $\pi_0 \circ \pi = \iota$, $\pi \circ \pi_0 = \iota$, $\theta_0 \circ \theta = \iota$ and $\theta \circ \theta_0 = \iota$. Hence π and θ are *-isomorphisms. □

This result will be crucial to prove the uniqueness of left invariant weights in certain cases. Given two left invariant weights, we can extend both of them to normal weights in their respective GNS-spaces. In order to apply Radon–Nikodym, we need to get these normal weights represented on the same von Neumann algebra. This will be possible due to the previous result.

We will also use the same principle to get hold of the 'modular function' of quantum groups. In this case, we will have to work with a left and a right invariant weight (*see* Section 7).

3.2. Liftings of weights and one-parameter groups to the reduced level

In this subsection, we will fix a bi-C^*-algebra (A, Δ) which satisfies condition [D] or the KMS condition. We also fix a left or right invariant proper weight φ on (A, Δ) together with a GNS-construction (H, π, Λ) for φ. The next discussion will mainly serve to fix some terminology.

Theorem 3.8 will play a crucial role in the next results.

PROPOSITION 3.9. – *Consider a proper weight* ψ *on* (A,Δ) *which is left or right invariant. For every* $\omega\in\mathcal{F}_\psi$*, there exists a unique* $\widetilde{\omega}\in(\pi(A)'')^+_*$ *such that* $\widetilde{\omega}\,\pi=\omega$.

We define the mapping $\widetilde{\psi}:(\pi(A)'')^+\to[0,\infty]$ *such that*

$$\widetilde{\psi}(x)=\sup\{\widetilde{\omega}(x)\mid\omega\in\mathcal{F}_\psi\}$$

for all $x\in(\pi(A)'')^+$. *Then* $\widetilde{\psi}$ *is a semi-finite normal weight on* $\pi(A)''$ *such that* $\widetilde{\psi}\,\pi=\psi$. *We call* $\widetilde{\psi}$ *the* W*-*lift of* ψ *in the GNS-construction* (H,π,Λ).

By Definition 1.31, we can define such a weight in the GNS-construction for ψ. By using Theorem 3.8, we can transport everything to $\pi(A)''$. The same remark applies to the next proposition. In this case, we have to use Proposition 1.32.

PROPOSITION 3.10. – *Consider a proper weight* ψ *on* (A,Δ) *which is left or right invariant. Let* (K,θ,Γ) *be a GNS-construction for* ψ *and let* $\widetilde{\theta}:\pi(A)''\to\theta(A)''$ *be the *-isomorphism such that* $\widetilde{\theta}(\pi(a))=\theta(a)$ *for* $a\in A$. *Then there exists a unique linear mapping* $\widetilde{\Gamma}:\mathcal{N}_{\widetilde{\psi}}\to K$ *such that*:

- $(K,\widetilde{\theta},\widetilde{\Gamma})$ *is a GNS-construction for* $\widetilde{\psi}$;
- $\widetilde{\Gamma}(\pi(a))=\Gamma(a)$ *for* $a\in\mathcal{N}_\psi$.

We call $(K,\widetilde{\theta},\widetilde{\Gamma})$ *the* W*-*lift of* (K,θ,Γ) *in the GNS-construction* (H,π,Λ).

We have moreover the following property. Consider $x\in\mathcal{N}_{\widetilde{\psi}}$. *Then there exists a net* $(a_i)_{i\in I}$ *in* $\mathcal{N}_\psi$ *such that*:

(1) $\|a_i\|\leqslant\|x\|$ *for* $i\in I$;
(2) $(\pi(a_i))_{i\in I}$ *converges to* x *in the strong*-topology*;
(3) $(\Gamma(a_i))_{i\in I}$ *converges to* $\widetilde{\Gamma}(x)$.

Let us also quickly look into the liftings of one-parameter groups. The following result is (easily) proven in Proposition 2.19 of [30]. We can then again use Theorem 3.8 to transport everything to $\pi(A)''$.

PROPOSITION 3.11. – *Consider a norm continuous one-parameter group* τ *on* A *such that there exists a left or right invariant proper weight* ψ *on* A *such that* ψ *is relatively invariant under* τ, *i.e. such that there exists a number* $\lambda>0$ *satisfying* $\psi\,\tau_t=\lambda^t\,\psi$ *for all* $t\in\mathbb{R}$.

Then there exists a unique strongly continuous one-parameter group $\widetilde{\tau}$ *on* $\pi(A)''$ *such that* $\widetilde{\tau}_t\circ\pi=\pi\circ\tau_t$ *for* $t\in\mathbb{R}$. *We call* $\widetilde{\tau}$ *the* W*-*lift of* τ *in* (H,π,Λ).

Approximate KMS weights are lifted to n.f.s. weights. This follows from Proposition 1.35 and Theorem 3.8. See also the remark after Proposition 3.9.

PROPOSITION 3.12. – *Consider a left or right invariant approximate KMS weight* ψ *on* (A,Δ). *Let* $\widetilde{\psi}$ *be the* W*-*lift of* ψ. *Then* $\widetilde{\psi}$ *is a normal semi-finite faithful weight.*

Suppose that ψ happens to be a KMS weight with modular group σ. Then we can define the W*-lift $\widetilde{\sigma}$ of σ such that $\widetilde{\sigma}_t\pi=\pi\sigma_t$ for $t\in\mathbb{R}$. Then $\widetilde{\sigma}$ is the modular group of $\widetilde{\psi}$. (*See* e.g. Proposition 2.22 of [30].)

We will need that the left/right invariance on the C*-algebra level is also lifted to the left/right invariance on the W*-level. For this, the last part of Proposition 3.10 is crucial.

PROPOSITION 3.13. – *Suppose that there exists a normal *-homomorphism*

$$\widetilde{\Delta}:\pi(A)''\to\pi(A)''\,\overline{\otimes}\,\pi(A)''$$

such that $\widetilde{\Delta}\pi = (\pi \otimes \pi)\Delta$. Consider a left invariant weight ψ on (A, Δ). Then the W*-*lift* $\widetilde{\psi}$ *is left invariant, i.e.*

$$\widetilde{\psi}\big((\omega \,\overline{\otimes}\, \iota)\widetilde{\Delta}(x)\big) = \widetilde{\psi}(x)\,\omega(1)$$

for all $x \in \mathcal{M}^+_{\widetilde{\psi}}$ and $\omega \in (\pi(A)'')^+_*$.

Proof. – We have for $x \in \pi(\mathcal{N}_\psi)$ and $\omega \in (\pi(A)'')^+_*$ that $\widetilde{\psi}((\omega \,\overline{\otimes}\, \iota)\widetilde{\Delta}(x^*x)) = \omega(1)\,\widetilde{\psi}(x^*x)$: there exists $a \in \mathcal{N}_\psi$ such that $\pi(a) = x$; then

$$(\omega \,\overline{\otimes}\, \iota)\big(\widetilde{\Delta}(x^*x)\big) = (\omega \,\overline{\otimes}\, \iota)\big((\pi \otimes \pi)\Delta(a^*a)\big) = \pi\big((\omega\pi \otimes \iota)\Delta(a^*a)\big), \tag{3.1}$$

so that the left invariance of ψ implies that

$$\begin{aligned}\widetilde{\psi}\big((\omega \,\overline{\otimes}\, \iota)(\widetilde{\Delta}(x^*x))\big) &= \psi\big((\omega\pi \otimes \iota)\Delta(a^*a)\big) = (\omega\pi)(1)\,\psi(a^*a)\\ &= \omega(1)\,\widetilde{\psi}\big(\pi(a^*a)\big) = \omega(1)\,\widetilde{\psi}(x^*x).\end{aligned}$$

Fix $\omega \in (\pi(A)'')^+_*$. Define the sesquilinear mapping $T: \mathcal{N}_{\widetilde{\psi}} \times \mathcal{N}_{\widetilde{\psi}} \to \pi(A)''$ such that $T(x, y) = (\omega \,\overline{\otimes}\, \iota)\widetilde{\Delta}(y^*x)$ for all $x, y \in \mathcal{N}_{\widetilde{\psi}}$.

Take a GNS-construction $(H_{\widetilde{\psi}}, \pi_{\widetilde{\psi}}, \Lambda_{\widetilde{\psi}})$ for $\widetilde{\psi}$. Now we apply Lemma A.1 with $V = \mathcal{N}_{\widetilde{\psi}}$, $W = \pi(\mathcal{N}_\psi)$, $X = H_{\widetilde{\psi}}$, $\Lambda = \Lambda_{\widetilde{\psi}}$ and $K = \omega(1)$. This is possible thanks to the last part of Proposition 3.10 and Eq. (3.1). So Lemma A.1 implies for every $x \in \mathcal{N}_{\widetilde{\psi}}$ that

$$\widetilde{\psi}\big((\omega \,\overline{\otimes}\, \iota)(\widetilde{\Delta}(x^*x))\big) = \widetilde{\psi}\big(T(x, x)\big) = \omega(1)\,\|\Lambda_{\widetilde{\psi}}(x)\|^2. \qquad \square$$

We have of course an analogous result and proof for the right invariant case.

3.3. Multiplicative unitaries

In this subsection, we fix a bi-C*-algebra (A, Δ). Let φ be a left invariant proper weight on (A, Δ) with GNS-construction (H, π, Λ). We will prove under some fairly weak conditions that the multiplicative partial isometries (*see* the remark after Notation 2.8) are unitary.

We will also assume the existence of a right invariant proper weight ψ on (A, Δ) such that ψ is approximately KMS. Let $(H_\psi, \pi_\psi, \Lambda_\psi)$ be a GNS-construction for ψ. Using Definition 1.31 and Proposition 1.32, we define the following objects:

- $\widetilde{\psi} =$ the W*-lift of ψ in the GNS-construction $(H_\psi, \pi_\psi, \Lambda_\psi)$;
- $(H_\psi, \iota, \widetilde{\Lambda}_\psi) =$ the W*-lift of $(H_\psi, \pi_\psi, \Lambda_\psi)$.

Since ψ is approximately KMS, $\widetilde{\psi}$ is a n.f.s. weight on $\pi_\psi(A)''$. We will denote its modular group by $\sigma^{\widetilde{\psi}}$.

For technical reasons, we will need to use a Tomita *-algebra:

$$\mathcal{T}_{\widetilde{\psi}} = \big\{x \in \mathcal{N}_{\widetilde{\psi}} \cap \mathcal{N}^*_{\widetilde{\psi}} \,\big|\, x \text{ is analytic with respect to } \sigma^{\widetilde{\psi}} \text{ and } \sigma^{\widetilde{\psi}}_z(x) \in \mathcal{N}_{\widetilde{\psi}} \cap \mathcal{N}^*_{\widetilde{\psi}} \text{ for } z \in \mathbb{C}\big\}.$$

Using the right invariant version of Notation 2.8, we define the isometry $U: H_\psi \otimes H \to H_\psi \otimes H$ such that $U(\Lambda_\psi(p) \otimes \Lambda(q)) = (\Lambda_\psi \otimes \Lambda)(\Delta(p)(1 \otimes q))$ for $p \in \mathcal{N}_\psi$ and $q \in \mathcal{N}_\varphi$.

The right invariant version of result 2.10 implies for every $a, b \in \mathcal{N}_\psi$ that

$$(\omega_{\Lambda_\psi(a), \Lambda_\psi(b)} \otimes \iota)(U^*) = \pi\big((\psi \otimes \iota)(\Delta(b^*)(a \otimes 1))\big). \tag{3.2}$$

This implies that $(\omega_{v,w} \otimes \iota)(U^*)$ belongs to $M(\pi(A))$ for all $v, w \in H_\psi$.

LEMMA 3.14. – *Consider $a \in \mathcal{N}_{\widetilde{\psi}}$, $b \in \mathcal{T}_{\widetilde{\psi}}$, $c \in \mathcal{N}_{\psi}$. Then*

$$\pi\big((\omega_{\widetilde{\Lambda}_\psi(a),\widetilde{\Lambda}_\psi(b)} \otimes \iota)\Delta(c^*)\big) = \big(\omega_{\widetilde{\Lambda}_\psi(a\,\sigma^{\widetilde{\psi}}_{-i}(b^*)),\Lambda_\psi(c)} \otimes \iota\big)(U^*).$$

Proof. – Choose $\omega \in \mathrm{B}_0(H)^*$. Using Eq. (3.2) above, we get for all $p \in \mathcal{N}_\psi$ that

$$\begin{aligned}\omega\big((\omega_{\Lambda_\psi(p),\Lambda_\psi(c)} \otimes \iota)(U^*)\big) &= \psi\big((\iota\otimes\omega)(\Delta(c^*))\,p\big) = \big\langle \Lambda_\psi(p), \Lambda_\psi((\iota\otimes\overline{\omega})\Delta(c))\big\rangle \\ &= \big\langle \widetilde{\Lambda}_\psi(\pi(p)), \widetilde{\Lambda}_\psi\big(\pi((\iota\otimes\overline{\omega})\Delta(c))\big)\big\rangle.\end{aligned}$$

Therefore, Proposition 1.32 implies for every $y \in \mathcal{N}_{\widetilde{\psi}}$ that

$$\omega\big((\omega_{\widetilde{\Lambda}_\psi(y),\Lambda_\psi(c)} \otimes \iota)(U^*)\big) = \big\langle \widetilde{\Lambda}_\psi(y), \widetilde{\Lambda}_\psi\big(\pi((\iota\otimes\overline{\omega})\Delta(c))\big)\big\rangle = \widetilde{\psi}\big(\pi((\iota\otimes\omega)\Delta(c^*))y\big).$$

So we get that

$$\begin{aligned}\omega\big((\omega_{\widetilde{\Lambda}_\psi(a\,\sigma^{\widetilde{\psi}}_{-i}(b^*)),\Lambda_\psi(c)} \otimes \iota)(U^*)\big) &= \widetilde{\psi}\big(\pi((\iota\otimes\omega)\Delta(c^*))a\,\sigma^{\widetilde{\psi}}_{-i}(b^*)\big) \\ &= \widetilde{\psi}\big(b^*\pi((\iota\otimes\omega)\Delta(c^*))a\big) \\ &= \omega_{\widetilde{\Lambda}_\psi(a),\widetilde{\Lambda}_\psi(b)}\big(\pi((\iota\otimes\omega)\Delta(c^*))\big) \\ &= \omega\big(\pi((\omega_{\widetilde{\Lambda}_\psi(a),\widetilde{\Lambda}_\psi(b)} \otimes \iota)\Delta(c^*))\big). \qquad \square\end{aligned}$$

PROPOSITION 3.15. – *We have that*

$$\begin{aligned}H &= \big[\Lambda\big((\psi\otimes\iota)(\Delta(b^*c)(a\otimes 1))\big) \mid a,b \in \mathcal{N}_\psi, c \in \mathcal{N}_\varphi\big] \\ &= \big[\Lambda\big((\omega_{v,w}\otimes\iota)\Delta(c)\big) \mid v,w \in H_\psi, c \in \mathcal{N}_\varphi\big].\end{aligned}$$

Proof. – We define φ_r to be the proper weight on $\pi(A)$ such that $\varphi_r\pi = \varphi_r$ and Λ_r to be the linear map from $\mathcal{N}_{\varphi_r}$ to H such that $\Lambda_r(\pi(a)) = \Lambda(a)$ for $a \in \mathcal{N}_\varphi$. Then (H, ι, Λ_r) is a GNS-construction for φ_r (φ_r is just the restriction of $\widetilde{\varphi}$ in Definition 1.31). It is then not difficult to check that $\bar{\varphi}_r\bar{\pi} = \bar{\varphi}$ and that $\Lambda_r(\pi(a)) = \Lambda(a)$ for all $a \in \overline{\mathcal{N}}_\varphi$.

Define the closed subspace K of H as

$$\begin{aligned}K &= \big[\Lambda\big((\psi\otimes\iota)(\Delta(b^*c)(a\otimes 1))\big) \mid a,b \in \mathcal{N}_\psi,\ c \in \mathcal{N}_\varphi\big] \\ &= \big[\Lambda\big((\psi\otimes\iota)(\Delta(b^*c)(a\otimes 1))\big) \mid b \in \mathcal{N}_\psi,\ a \in \overline{\mathcal{N}}_\psi,\ c \in \mathcal{N}_\varphi\big], \qquad (3.3)\end{aligned}$$

where we used result 2.6 to get the last equality.

We first show quickly that

$$K = \big[\Lambda\big((\omega_{v,w}\otimes\iota)\Delta(c)\big) \mid v,w \in H_\psi,\ c \in \mathcal{N}_\varphi\big].$$

From Result 2.6 and Eq. (3.2) in the discussion before the previous lemma, we know for all $a,b \in \mathcal{N}_\psi$ and $c \in \mathcal{N}_\varphi$ that $(\omega_{\Lambda_\psi(a),\Lambda_\psi(c^*b)} \otimes \iota)(U^*) = \pi((\psi\otimes\iota)(\Delta(b^*c)(a\otimes 1)))$ belongs to $\overline{\mathcal{N}}_{\varphi_r}$ and

$$\big\|\Lambda_r\big((\omega_{\Lambda_\psi(a),\Lambda_\psi(c^*b)} \otimes \iota)(U^*)\big)\big\| \leqslant \|\Lambda_\psi(a)\|\,\|\Lambda_\psi(b)\|\,\|\Lambda(c)\|.$$

Hence the closedness of $\bar{\Lambda}_r$ (*see* Proposition 1.9) implies for every $v \in H_\psi$, every $b \in \mathcal{N}_\psi$ and every $c \in \mathcal{N}_\varphi$ that $(\omega_{v,\Lambda_\psi(c^*b)} \otimes \iota)(U^*)$ belongs to $\overline{\mathcal{N}}_{\varphi_r}$ and

$$\big\|\Lambda_r\big((\omega_{v,\Lambda_\psi(c^*b)} \otimes \iota)(U^*)\big)\big\| \leqslant \|v\|\,\|\Lambda_\psi(b)\|\,\|\Lambda(c)\|. \qquad (3.4)$$

We know that $\mathcal{T}_{\widetilde{\psi}}$ is strongly* dense in $\pi_\psi(A)''$ and that $\widetilde{\Lambda}_\psi(\mathcal{T}_{\widetilde{\psi}})$ is dense in H_ψ. Combining this with the fact that $\sigma^{\widetilde{\psi}}_{-i}(\mathcal{T}_{\widetilde{\psi}}) = \mathcal{T}_{\widetilde{\psi}}$, we get that $H_\psi = [\widetilde{\Lambda}_\psi(x\,\sigma^{\widetilde{\psi}}_{-i}(y^*)) \mid x, y \in \mathcal{T}_{\widetilde{\psi}}]$. Hence inequality (3.4) implies that

$$\begin{aligned}K &= \big[\Lambda_r\big(\pi\big((\psi\otimes\iota)(\Delta(b^*c)(a\otimes 1))\big)\big) \mid a,b\in\mathcal{N}_\psi,\ c\in\mathcal{N}_\varphi\big]\\ &= \big[\Lambda_r\big((\omega_{\Lambda_\psi(a),\Lambda_\psi(c^*b)}\otimes\iota)(U^*)\big) \mid a,b\in\mathcal{N}_\psi,\ c\in\mathcal{N}_\varphi\big]\\ &= \big[\Lambda_r\big((\omega_{\widetilde{\Lambda}_\psi(x\,\sigma^{\widetilde{\psi}}_{-i}(y^*)),\Lambda_\psi(c^*b)}\otimes\iota)(U^*)\big) \mid x,y\in\mathcal{T}_{\widetilde{\psi}},\ b\in\mathcal{N}_\psi,\ c\in\mathcal{N}_\varphi\big].\end{aligned}$$

Using Lemma 3.14, this gives

$$K = \big[\Lambda\big((\omega_{\widetilde{\Lambda}_\psi(x),\widetilde{\Lambda}_\psi(y)}\otimes\iota)\Delta(b^*c)\big) \mid x,y\in\mathcal{T}_{\widetilde{\psi}},\ b\in\mathcal{N}_\psi,\ c\in\mathcal{N}_\varphi\big].$$

Since $\widetilde{\Lambda}_\psi(\mathcal{T}_{\widetilde{\psi}})$ is dense in H_ψ and $\mathcal{N}_\psi$ is dense in A, Result 2.3 now implies that

$$K = \big[\Lambda\big((\omega_{v,w}\otimes\iota)\Delta(c)\big) \mid v,w\in H_\psi,\ c\in\mathcal{N}_\varphi\big]. \tag{3.5}$$

Now, we can finish the proof rather smoothly. Using Notation 2.8, we define an isometry $V : H_\psi\otimes H \to H_\psi\otimes H$ such that $V(\Lambda_\psi(p)\otimes\Lambda(q)) = (\Lambda_\psi\otimes\Lambda)(\Delta(q)(p\otimes 1))$ for $p\in\mathcal{N}_\psi$ and $q\in\mathcal{N}_\varphi$. Now,

- the expression for K in Eq. (3.5) and Result 2.9 imply immediately that $V(H_\psi\otimes H)\subseteq H_\psi\otimes K$;
- on the other hand, the expression for K in Eq. (3.3) and Proposition 2.12 imply that $H_\psi\otimes K\subseteq V(H_\psi\otimes K)$.

So we see that $V(H_\psi\otimes H) = V(H_\psi\otimes K)$, hence the injectivity of V implies that $H_\psi\otimes H = H_\psi\otimes K$. Because $H_\psi\neq 0$, this implies that $K = H$. □

If we combine this result with Proposition 2.12, we get immediately the following crucial result.

THEOREM 3.16. – *Let η be a proper weight on A with GNS-construction $(H_\eta,\pi_\eta,\Lambda_\eta)$. Define the isometry $V : H_\eta\otimes H\to H_\eta\otimes H$ such that $V(\Lambda_\eta(a)\otimes\Lambda(b)) = (\Lambda_\eta\otimes\Lambda)(\Delta(b)(a\otimes 1))$ for $a\in\mathcal{N}_\eta$ and $b\in\mathcal{N}_\varphi$. Then V is a unitary element in $\mathrm{B}(H_\eta\otimes H)$.*

Using Proposition 1.29, it is easy to check that $V(1\otimes\pi(a)) = (\pi_\eta\otimes\pi)(\Delta(a))V$ for $a\in A$. Hence $(\pi_\eta\otimes\pi)(\Delta(a)) = V(1\otimes\pi(a))V^*$ for $a\in A$.

It should be mentioned that V^* is considered as a multiplicative unitary (rather than V).

With this result in hand, it is now easy to lift the comultiplication Δ to the reduced level.

PROPOSITION 3.17. – *Define the unitary $W\in\mathrm{B}(H\otimes H)$ such that*

$$W^*\big(\Lambda(a)\otimes\Lambda(b)\big) = (\Lambda\otimes\Lambda)\big(\Delta(b)(a\otimes 1)\big)$$

for $a,b\in\mathcal{N}_\varphi$. Then:

- *there exists a unique injective normal unital *-homomorphism $\widetilde{\Delta} : \pi(A)''\to\pi(A)''\,\overline{\otimes}\,\pi(A)''$ such that $\widetilde{\Delta}\,\pi = (\pi\otimes\pi)\,\Delta$. We have moreover that $\widetilde{\Delta}(x) = W^*(1\otimes x)W$ for all $x\in\pi(A)''$;*
- *there exists a unique injective non-degenerate *-homomorphism*

$$\bar{\Delta} : \pi(A)\to M\big(\pi(A)\otimes\pi(A)\big)$$

such that $\bar{\Delta}\,\pi = (\pi\otimes\pi)\,\Delta$. We have moreover that $\bar{\Delta}(x) = W^*(1\otimes x)W$ for all $x\in\pi(A)$.

As usual, the coassociativity of Δ implies the multiplicativity of the operator W.

PROPOSITION 3.18. – *The operator W satisfies the pentagonal equation*: $W_{12}W_{13}W_{23} = W_{23}W_{12}$.

Proof. – Take $\omega, \theta \in \mathrm{B}_0(H)^*$. By Result 2.10, we have for $a \in \mathcal{N}_\varphi$ that

$$(\omega\otimes\iota)(W^*)(\theta\otimes\iota)(W^*)\Lambda(a) = (\omega\otimes\iota)(W^*)\Lambda\big((\theta\otimes\iota)\Delta(a)\big) \\ = \Lambda\big((\omega\otimes\iota)\Delta((\theta\otimes\iota)\Delta(a))\big) = \Lambda\big(((\theta\pi\otimes\omega\pi)\Delta\otimes\iota)\Delta(a)\big).$$

Since $(\theta\pi\otimes\omega\pi)\Delta(x) = (\theta\otimes\omega)(W^*(1\otimes\pi(x))W)$ for all $x\in A$, this implies that

$$(\theta\otimes\omega\otimes\iota)(W_{23}^*W_{13}^*)\Lambda(a) = (\omega\otimes\iota)(W^*)(\theta\otimes\iota)(W^*)\Lambda(a) = (\theta\otimes\omega\otimes\iota)(W_{12}^*W_{23}^*W_{12})\Lambda(a).$$

So we conclude that $W_{23}^*W_{13}^* = W_{12}^*W_{23}^*W_{12}$. □

3.4. The left regular corepresentation

In this section, we define the left regular corepresentation of a bi-C*-algebra. We will prove the associated inversion formula like in Proposition 2.13 and formulate the corresponding unitarity result.

The left regular corepresentation will be crucial in pulling down objects existing on the reduced level to the C*-algebra A itself (which will be dealt with in a subsequent paper).

We will define the left regular corepresentation within the framework of Hilbert C*-modules but notice that there is a direct link to multiplier algebras.

For the rest of this subsection, we will fix a bi-C*-algebra (A,Δ), a left invariant proper weight φ on (A,Δ) together with a GNS-construction (H,π,Λ) for φ.

Since

$$\big\langle(\iota\otimes\Lambda)(\Delta(b))\,a,(\iota\otimes\Lambda)(\Delta(d))\,c\big\rangle = c^*a\,\langle\Lambda(b),\Lambda(d)\rangle$$

for all $a,c\in A$ and $b,d\in\mathcal{N}_\varphi$, the following notation is justified.

Notation 3.19. – We define the isometric A-linear mapping $U: A\otimes H\to A\otimes H$ such that

$$U(a\otimes\Lambda(b)) = (\iota\otimes\Lambda)(\Delta(b))\,a$$

for $a\in A$ and $b\in\mathcal{N}_\varphi$.

Using the last statement of Proposition 1.26, it is not difficult to see that $U(1\otimes\pi(x)) = (\iota\otimes\pi)(\Delta(x))U$ for $x\in A$.

We want to prove a version of Proposition 2.13 in this framework. We will do this by reducing it to the framework developed in Subsection 2.2.

In order to reduce this case to one we already dealt with, we will need some extra notation. Fix $\omega\in A_+^*$. We will use Lemma 1.30. We have for all $z\in\overline{\mathcal{N}}_{\iota\otimes\varphi}$ and $a\in A$ that $z(a\otimes 1)\in\overline{\mathcal{N}}(\omega,\varphi)$ and

$$\big\langle(\Lambda_\omega\otimes\Lambda)(z(a\otimes 1)),(\Lambda_\omega\otimes\Lambda)(z(a\otimes 1))\big\rangle = (\omega\otimes\varphi)\big((a^*\otimes 1)\,z^*z\,(a\otimes 1)\big) \\ = \omega\big(a^*\,(\iota\otimes\varphi)(z^*z)\,a\big) \\ = \omega\big(\big\langle(\iota\otimes\Lambda)(z)\,a,(\iota\otimes\Lambda)(z)\,a\big\rangle\big).$$

By a polarization we obtain, for all $x\in\langle z(a\otimes 1)\mid z\in\overline{\mathcal{N}}_{\iota\otimes\varphi},\,a\in A\rangle$,

$$\|(\Lambda_\omega\otimes\Lambda)(x)\| \leqslant \|\omega\|^{\frac12}\,\|(\iota\otimes\Lambda)(x)\|.$$

So there exists a bounded linear map $P_\omega : A \otimes H \to H_\omega \otimes H$ such that $P_\omega((\iota \otimes \Lambda)(z)\, a) = (\Lambda_\omega \otimes \Lambda)(z(a \otimes 1))$ for $z \in \overline{\mathcal{N}}_{\iota\otimes\varphi}$ and $a \in A$. It is not difficult to see that

$$P_\omega(p \otimes \Lambda(q)) = \Lambda_\omega(p) \otimes \Lambda(q) \quad \text{for } p \in A,\ q \in \mathcal{N}_\varphi. \tag{3.6}$$

Because

$$\big\langle P_\omega(\iota \odot \Lambda)(y), P_\omega(\iota \odot \Lambda)(z)\big\rangle = \omega\big(\big\langle(\iota \odot \Lambda)(y), (\iota \odot \Lambda)(z)\big\rangle\big)$$

for $y, z \in A \odot \mathcal{N}_\varphi$, we have immediately that $\langle P_\omega v, P_\omega w\rangle = \omega(\langle v, w\rangle)$ for $v, w \in A \otimes H$.

Let $v \in A \otimes H$. Then the last equality implies immediately that:

$$v = 0 \Leftrightarrow P_\omega v = 0 \quad \text{for all } \omega \in A^*_+. \tag{3.7}$$

Now we have enough extra terminology to prove the desired result.

PROPOSITION 3.20. – *Consider a right invariant proper weight ψ on (A, Δ), $a, c \in \mathcal{N}_\psi$, $b \in \mathcal{N}_\varphi$, $y \in A$ and put $x = (\psi \otimes \iota \otimes \iota)(\Delta_{13}(a^*b)\Delta_{12}(c))$. Then x belongs to $\overline{\mathcal{N}}_{\iota\otimes\varphi}$ and*

$$U(\iota \otimes \Lambda)(x)\, y = y \otimes \Lambda\big((\psi \otimes \iota)(\Delta(a^*b)(c \otimes 1))\big).$$

Proof. – By Proposition 2.13, we know that $x \in \overline{\mathcal{N}}_{\iota\otimes\varphi}$. Choose $\omega \in A^*_+$. Then we define the isometry $U_\omega : H_\omega \otimes H \to H_\omega \otimes H$ such that $U_\omega(\Lambda_\omega(p) \otimes \Lambda(q)) = (\Lambda_\omega \otimes \Lambda)(\Delta(q)(p \otimes 1))$ for $p \in A$ and $q \in \mathcal{N}_\varphi$. Using Eq. (3.6) among the remarks before this proposition, it is then clear that $U_\omega P_\omega = P_\omega U$.

Using Proposition 2.13, we now see that

$$\begin{aligned} P_\omega U(\iota \otimes \Lambda)(x)\, y = U_\omega P_\omega(\iota \otimes \Lambda)(x)\, y &= U_\omega(\Lambda_\omega \otimes \Lambda)\big(x(y \otimes 1)\big) \\ &= \Lambda_\omega(y) \otimes \Lambda\big((\psi \otimes \iota)(\Delta(a^*b)(c \otimes 1))\big) \\ &= P_\omega\big(y \otimes \Lambda\big((\psi \otimes \iota)(\Delta(a^*b)(c \otimes 1))\big)\big). \end{aligned}$$

Therefore Remark 3.7 before this proposition implies that

$$U(\iota \otimes \Lambda)(x)\, y = y \otimes \Lambda\big((\psi \otimes \iota)(\Delta(a^*b)(c \otimes 1))\big). \qquad \square$$

Combining the previous result with Proposition 3.15, we immediately get the following one.

PROPOSITION 3.21. – *Suppose that there exists a right invariant approximate KMS weight on (A, Δ). Then U is a unitary element in $\mathcal{L}(A \otimes H)$.*

As usual, U^* will be called a left regular corepresentation of (A, Δ). Remember that U can also be considered as an element of $M(A \otimes \mathrm{B}_0(H))$.

Notice that $(\iota \otimes \pi)\Delta(a) = U^*(1 \otimes \pi(a))U$ for all $a \in A$ in this case.

3.5. The first step in the construction of the antipode

In this subsection, we fix a bi-C*-algebra (A, Δ) and
- a left invariant approximate KMS weight φ on (A, Δ);
- a right invariant proper weight ψ on (A, Δ).

Let (H, π, Λ) denote a GNS-construction for φ and define the closed subspace K of H by

$$K = \big[\Lambda\big((\psi \otimes \iota)(\Delta(a^*b)(c \otimes 1))\big) \mid a, c \in \mathcal{N}_\psi,\ b \in \mathcal{N}_\varphi\big].$$

Using Result 2.6, we get that

$$K = \left[\Lambda\big((\psi \otimes \iota)(\Delta(x^*)(y \otimes 1))\big) \mid x, y \in \mathcal{N}_\varphi^* \mathcal{N}_\psi\right].$$

We will construct a closed operator in H which is formally the closure of the operator $\Lambda(x) \mapsto \Lambda(S(x^*))$, where S denotes the antipode. By taking the polar decomposition of this closed operator, we get operators which potentially induce the scaling group and unitary antipode appearing in the polar decomposition of the antipode (*see* Section 5).

PROPOSITION 3.22. – *There exists a unique densely defined closed antilinear operator G in K such that*

$$\left\langle \Lambda\big((\psi \otimes \iota)(\Delta(x^*)(y \otimes 1))\big) \mid x, y \in \mathcal{N}_\varphi^* \mathcal{N}_\psi \right\rangle$$

is a core for G and

$$G\Lambda\big((\psi \otimes \iota)(\Delta(x^*)(y \otimes 1))\big) = \Lambda\big((\psi \otimes \iota)(\Delta(y^*)(x \otimes 1))\big)$$

for $x, y \in \mathcal{N}_\varphi^* \mathcal{N}_\psi$. We have moreover that G is involutive.

Proof. – By Proposition 1.36, we know that $\varphi \otimes \varphi$ is approximately KMS and that

$$(H \otimes H, \pi \otimes \pi, \Lambda \otimes \Lambda)$$

is a GNS-construction for $\varphi \otimes \varphi$ (where $\Lambda \otimes \Lambda$ is introduced in Definition 1.28).

Because $\varphi \otimes \varphi$ is approximately KMS, we get by the remarks after Proposition 1.35 the existence of a closed operator T in $H \otimes H$ such that $T(\Lambda \otimes \Lambda)(x) = (\Lambda \otimes \Lambda)(x^*)$ for $x \in \overline{\mathcal{N}}_{\varphi \otimes \varphi} \cap \overline{\mathcal{N}}_{\varphi \otimes \varphi}^{\,*}$.

Let Σ denote the flip operator on $H \otimes H$.

Define the isometry $U \in \mathrm{B}(H \otimes H)$ such that $U(\Lambda(a) \otimes \Lambda(b)) = (\Lambda \otimes \Lambda)(\Delta(b)(a \otimes 1))$ for $a, b \in \mathcal{N}_\varphi$.

Choose $v \in K$ and for every $n \in \mathbb{N}$ an element $k_n \in \mathbb{N}$ and elements $x(n,1), \ldots, x(n,k_n)$ and $y(n,1), \ldots, y(n,k_n)$ in $\mathcal{N}_\varphi^* \mathcal{N}_\psi$ such that:

(1) $\big(\sum_{i=1}^{k_n} \Lambda((\psi \otimes \iota)(\Delta(x(n,i)^*)(y(n,i) \otimes 1)))\big)_{n=1}^\infty \to 0$;

(2) $\big(\sum_{i=1}^{k_n} \Lambda((\psi \otimes \iota)(\Delta(y(n,i)^*)(x(n,i) \otimes 1)))\big)_{n=1}^\infty \to v$.

We have to prove that $v = 0$.

Choose $c, d \in \mathcal{N}_\varphi$. Proposition 2.13 implies for $n \in \mathbb{N}$ that

$$\begin{aligned} &U\left(\sum_{i=1}^{k_n} (\Lambda \otimes \Lambda)\big((1 \otimes d^*)(\psi \otimes \iota \otimes \iota)(\Delta_{13}(x(n,i)^*)\Delta_{12}(y(n,i)))(c \otimes 1)\big)\right) \\ &\quad = (\pi \otimes \pi)(\Delta(d^*))\left(\Lambda(c) \otimes \sum_{i=1}^{k_n} \Lambda\big((\psi \otimes \iota)(\Delta(x(n,i)^*)(y(n,i) \otimes 1))\big)\right). \end{aligned}$$

Therefore the convergence in 1 implies that the net

$$\left(\sum_{i=1}^{k_n} (\Lambda \otimes \Lambda)\big((1 \otimes d^*)(\psi \otimes \iota \otimes \iota)(\Delta_{13}(x(n,i)^*)\Delta_{12}(y(n,i)))(c \otimes 1)\big)\right)_{n=1}^\infty \tag{3.8}$$

converges to 0.

Using the convergence in (2), we get in the same way that the net

$$\left(\sum_{i=1}^{k_n}(\Lambda\otimes\Lambda)\big((1\otimes c^*)(\psi\otimes\iota\otimes\iota)(\Delta_{13}(y(n,i)^*)\Delta_{12}(x(n,i)))(d\otimes 1)\big)\right)_{n=1}^{\infty} \tag{3.9}$$

converges to $U^*(\pi\otimes\pi)(\Delta(c^*))(\Lambda(d)\otimes v)$.

Fix $n\in\mathbb{N}$ and $i\in k_n$ for the moment. By Proposition 2.13, we know that

$$(1\otimes d^*)(\psi\otimes\iota\otimes\iota)\big(\Delta_{13}(x(n,i)^*)\Delta_{12}(y(n,i))\big)(c\otimes 1)\in\overline{\mathcal{N}}_{\varphi\otimes\varphi}.$$

By using the flip maps on $A\otimes A$ and $H\otimes H$, we get that

$$(d^*\otimes 1)(\psi\otimes\iota\otimes\iota)\big(\Delta_{12}(x(n,i)^*)\Delta_{13}(y(n,i))\big)(1\otimes c)\in\overline{\mathcal{N}}_{\varphi\otimes\varphi}$$

and

$$\begin{aligned}(\Lambda\otimes\Lambda)\big((d^*\otimes 1)(\psi\otimes\iota\otimes\iota)(\Delta_{12}(x(n,i)^*)\Delta_{13}(y(n,i)))(1\otimes c)\big)\\ =\Sigma(\Lambda\otimes\Lambda)\big((1\otimes d^*)(\psi\otimes\iota\otimes\iota)(\Delta_{13}(x(n,i)^*)\Delta_{12}(y(n,i)))(c\otimes 1)\big).\end{aligned} \tag{3.10}$$

We have furthermore that

$$\begin{aligned}\big[(d^*\otimes 1)(\psi\otimes\iota\otimes\iota)\big(\Delta_{12}(x(n,i)^*)\Delta_{13}(y(n,i))\big)(1\otimes c)\big]^*\\ =(1\otimes c^*)(\psi\otimes\iota\otimes\iota)\big(\Delta_{13}(y(n,i)^*)\Delta_{12}(x(n,i))\big)(d\otimes 1),\end{aligned} \tag{3.11}$$

which, by Proposition 2.13, also belongs to $\overline{\mathcal{N}}_{\varphi\otimes\varphi}$.

Hence, we see that $(d^*\otimes 1)(\psi\otimes\iota\otimes\iota)(\Delta_{12}(x(n,i)^*)\Delta_{13}(y(n,i)))(1\otimes c)$ belongs to $\overline{\mathcal{N}}_{\varphi\otimes\varphi}\cap\overline{\mathcal{N}}_{\varphi\otimes\varphi}^{\,*}$.

Combining the convergence of expression (3.8) and Eq. (3.10), we see that the net

$$\left(\sum_{i=1}^{k_n}(\Lambda\otimes\Lambda)\big((d^*\otimes 1)(\psi\otimes\iota\otimes\iota)(\Delta_{12}(x(n,i)^*)\Delta_{13}(y(n,i)))(1\otimes c)\big)\right)_{n=1}^{\infty}$$

converges to 0.

From the convergence of expression (3.9) and Eq. (3.11), we conclude that the net

$$\left(T\left(\sum_{i=1}^{k_n}(\Lambda\otimes\Lambda)\big((d^*\otimes 1)(\psi\otimes\iota\otimes\iota)(\Delta_{12}(x(n,i)^*)\Delta_{13}(y(n,i)))(1\otimes c)\big)\right)\right)_{n=1}^{\infty}$$

converges to $U^*(\pi\otimes\pi)(\Delta(c^*))(\Lambda(d)\otimes v)$.

So the closedness of T implies that $U^*(\pi\otimes\pi)(\Delta(c^*))(\Lambda(d)\otimes v)=0$. By Proposition 2.13, we know that there exists an element $w\in H\otimes H$ such that $U\,w=\Lambda(d)\otimes v$. Therefore

$$0=U^*(\pi\otimes\pi)(\Delta(c^*))\,Uw=U^*U\big(1\otimes\pi(c^*)\big)\,w=\big(1\otimes\pi(c^*)\big)\,w.$$

Because $\mathcal{N}_\varphi$ is dense in A, we see that $w=0$, thus $\Lambda(d)\otimes v=0$. Hence $v=0$.

From all this, it follows that we can define a closed operator G in K as stated in the proposition. It is clear that G is involutive. □

4. The definition of a reduced C^*-algebraic quantum group

We want to give a simple definition for C^*-algebraic quantum groups in which the existence of a left invariant weight and a right invariant weight plays the central role. We will prove that from our small list of axioms all the axioms of an upcoming definition of Masuda, Nakagami and Woronowicz can be proved. In particular the existence of the antipode and its polar decomposition will be proved. The only thing that we do not prove is the invariance of the Haar weights under the scaling group. We can only obtain relative invariance, which is in a sense invariance up to a positive scalar, the scaling constant. Recently, Woronowicz and Van Daele have discovered an example in which the scaling constant is really non-trivial: the quantum $ax+b$-group.

But let us first start with our definition of a C^*-algebraic quantum group.

DEFINITION 4.1. – Consider a C^*-algebra A and a non-degenerate $*$-homomorphism $\Delta : A \to \mathrm{M}(A \otimes A)$ such that:

- $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$;
- $A = [(\omega \otimes \iota)\Delta(a) \mid \omega \in A^*,\ a \in A] = [(\iota \otimes \omega)\Delta(a) \mid \omega \in A^*,\ a \in A]$.

Assume moreover the existence of:

- a faithful left invariant approximate KMS weight φ on (A, Δ);
- a right invariant approximate KMS weight ψ on (A, Δ).

Then we call (A, Δ) a reduced C^*-algebraic quantum group.

Remark 4.2. – Notice that the weight ψ is also faithful:

The bi-C^*-algebra satisfies the KMS condition, so the results of Section 3.2 apply. Let (H, π, Λ) be a GNS-construction for φ. By Proposition 3.12, the W^*-lift $\widetilde{\psi}$ of ψ in the GNS-construction (H, π, Λ) is a faithful weight. Because φ is faithful, π is injective. Since $\psi = \widetilde{\psi}\,\pi$, we see that ψ is faithful.

Remark 4.3. – It should be noted that the last axiom in the definition (concerning the right invariant weight) is not too worrying. In practice, it should be not too difficult to establish the existence of a $*$-antiautomorphism $\theta : A \to A$ such that $\chi(\theta \otimes \theta)\Delta = \Delta\theta$ (and we will later see that such a map θ always exists). Then $\varphi\,\theta$ gives us a right invariant approximate KMS weight.

Notice also that a definition in which the existence of a right invariant approximate KMS weight is replaced by the existence of such a $*$-antiautomorphism θ is very appealing if one wants a definition comparable to the definition of a Kac algebra.

Remark 4.4. – Although we assume faithfulness of the left invariant weight, this should not be seen as a restriction. It is more a preparatory stage to the general case. First we develop the theory and construct all relevant objects for the reduced case. In the general case, one constructs first the reduced C^*-algebraic quantum group and uses the left regular corepresentation to pull down the objects from the reduced C^*-algebraic quantum group to the C^*-algebra one started from. This procedure runs smoothly in the 'universal' case and we will go into this in a subsequent paper (see [23]). But let us first make the statement about the reduction procedure more precise.

In the general case we will look at a C^*-algebra A and a non-degenerate $*$-homomorphism $\Delta : A \to \mathrm{M}(A \otimes A)$ such that:

- $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$;
- $\Delta(A)(A \otimes 1)$ and $\Delta(A)(1 \otimes A)$ are dense subsets of $A \otimes A$.

We will moreover assume the existence of:

- a left invariant approximate KMS weight φ on (A, Δ);
- a right invariant approximate KMS weight ψ on (A, Δ).

We will show later on that the density conditions above also holds for our reduced quantum groups.

In a next step, we take a GNS-construction (H,π,Λ) for φ. The reduced C*-algebra is by definition $A_r = \pi(A)$. By Proposition 3.17, there exists a unique non-degenerate *-homomorphism $\Delta_r : A_r \to \mathrm{M}(A_r \otimes A_r)$ such that $(\pi \otimes \pi)\Delta = \Delta_r \pi$. It is then clear that:

- $(\Delta_r \otimes \iota)\Delta_r = (\iota \otimes \Delta_r)\Delta_r$;
- $\Delta_r(A_r)(A_r \otimes 1)$ and $\Delta_r(A_r)(1 \otimes A_r)$ are dense subsets of $A_r \otimes A_r$.

We can also use the results of Section 3.2. So we get the W*-lifts $\widetilde{\varphi}$ and $\widetilde{\psi}$ of φ and ψ respectively. Define φ_r to be the restriction of $\widetilde{\varphi}$ to A_r^+ and ψ_r to be the restriction of $\widetilde{\psi}$ to A_r^+. These weights φ_r and ψ_r are determined by $\varphi_r\,\pi = \varphi$ and $\psi_r\,\pi = \psi$. From the results in Section 3.2, we conclude that:

- φ_r is a faithful left invariant approximate KMS weight;
- ψ_r is a faithful right invariant approximate KMS weight.

Therefore (A_r, Δ_r) is a reduced C*-algebraic quantum group.

Some properties can now be easily pulled down from the reduced level to A. For instance, we will show uniqueness of left invariant proper weights for (A_r, Δ_r) (up to a scalar of course). If we combine this with Proposition 3.9, we also get immediately uniqueness of left invariant proper weights on the level of (A, Δ).

For the rest of this paper, we will fix a reduced C*-algebraic quantum group (A, Δ). In the rest of this paper, we will gradually prove all the basic properties and construct the relevant objects. Let us recall the axioms and formulate the results which can be derived from the previous sections.

So A is a C*-algebra and $\Delta : A \to \mathrm{M}(A \otimes A)$ is a non-degenerate *-homomorphism such that:

- $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$;
- $A = [(\omega \otimes \iota)\Delta(a) \mid \omega \in A^*,\ a \in A] = [(\iota \otimes \omega)\Delta(a) \mid \omega \in A^*,\ a \in A]$.

We will also fix a faithful left invariant approximate KMS weight φ on (A, Δ). Remember that we also assumed the existence of a faithful right invariant approximate KMS weight but we will not fix one at the moment.

Recall from the proof of Result 3.4 that

$$A = \big[(\iota \otimes \varphi)\big((1 \otimes a^*)\Delta(b)\big) \mid a, b \in \mathcal{N}_\varphi\big] = \big[(\iota \otimes \varphi)\big(\Delta(a^*)(1 \otimes b)\big) \mid a, b \in \mathcal{N}_\varphi\big]. \tag{4.1}$$

A similar result holds of course also for any faithful right invariant proper weight.

Throughout the rest of this paper, we will work with a fixed GNS-construction (H, π, Λ) for φ.

Using Notation 2.8 and Theorem 3.16, we know that there exists a unitary operator $W \in \mathrm{B}(H \otimes H)$ such that

$$W^*\big(\Lambda(a) \otimes \Lambda(b)\big) = (\Lambda \otimes \Lambda)\big(\Delta(b)(a \otimes 1)\big)$$

for all $a, b \in \mathcal{N}_\varphi$.

The operator W is called a multiplicative unitary of (A, Δ). As usual, it satisfies the pentagonal equation (*see* Proposition 3.18):

$$W_{12}W_{13}W_{23} = W_{23}W_{12}.$$

Notice that $(\pi \otimes \pi)(\Delta(x)) = W^*(1 \otimes \pi(x))W$ for all $x \in A$.

Also recall the following formulas for W (*see* Result 2.10):

(1) $(\iota \otimes \omega_{\Lambda(a),\Lambda(b)})(W) = \pi((\iota \otimes \varphi)(\Delta(b^*)(1 \otimes a)))$ for all $a, b \in \mathcal{N}_\varphi$;
(2) $(\omega \otimes \iota)(W^*)\Lambda(a) = \Lambda((\omega \otimes \iota)\Delta(a))$ for all $a \in A$, $\omega \in \mathrm{B}_0(H)^*$.

Combining the first of these equalities with the density in Eq. (4.1), we see that

$$\pi(A)=[(\iota\otimes\omega)(W)\mid\omega\in \mathrm{B}_0(H)^*], \tag{4.2}$$

a familiar result.

From time to time, we need to get help from the von Neumann algebra world. We will therefore need to lift all objects to the von Neumann algebra in the GNS-space of φ.

We define the von Neumann algebra $\widetilde{A}$ on H by $\widetilde{A}=\pi(A)''$, the von Neumann algebra generated by $\pi(A)$. By Proposition 3.17, we know that there exists a unique injective normal *-homomorphism $\widetilde{\Delta}:\widetilde{A}\to\widetilde{A}\,\overline{\otimes}\,\widetilde{A}$ such that $\widetilde{\Delta}\pi=(\pi\otimes\pi)\Delta$. Clearly, $\widetilde{\Delta}(x)=W^*(1\otimes x)W$ for all $x\in\widetilde{A}$.

By using the results in Section 3.2, we can lift immediately the weight φ to $\widetilde{A}$. We will use the following notations.

- $\widetilde{\varphi}=$ the W*-lift of φ in the GNS-construction (H,π,Λ) as described in Proposition 3.9.
- $(H,\iota,\widetilde{\Lambda})=$ the W*-lift of (H,π,Λ) in the GNS-construction (H,π,Λ) as described in Proposition 3.10.

Recall that $\widetilde{\varphi}\pi=\varphi$ and that $(H,\iota,\widetilde{\Lambda})$ is a GNS-construction for $\widetilde{\varphi}$.

Proposition 3.12 tells us that $\widetilde{\varphi}$ is a normal semi-finite faithful weight on $\widetilde{A}$, we denote its modular group by $\widetilde{\sigma}$. We will reserve the following notations for the modular objects of $\widetilde{\varphi}$:

- $J=$ modular conjugation of $\widetilde{\varphi}$ in the GNS-construction $(H,\iota,\widetilde{\Lambda})$;
- $\nabla=$ modular operator of $\widetilde{\varphi}$ in the GNS-construction $(H,\iota,\widetilde{\Lambda})$.

For calculation reasons, we will need a Tomita *-algebra:

$$\mathcal{T}_{\widetilde{\varphi}}=\{x\in\mathcal{N}_{\widetilde{\varphi}}\cap\mathcal{N}_{\widetilde{\varphi}}^*\mid x \text{ is analytic with respect to } \widetilde{\sigma} \text{ and } \widetilde{\sigma}_z(x)\in\mathcal{N}_{\widetilde{\varphi}}\cap\mathcal{N}_{\widetilde{\varphi}}^* \text{ for } z\in\mathbb{C}\}.$$

From Proposition 3.13, we know that $\widetilde{\varphi}$ is left invariant:

$$\widetilde{\varphi}\big((\omega\,\overline{\otimes}\,\iota)\widetilde{\Delta}(x)\big)=\omega(1)\,\widetilde{\varphi}(x)$$

for all $x\in\mathcal{M}_{\widetilde{\varphi}}^+$ and $\omega\in\widetilde{A}_*^+$.

It is clear that the lifting properties of Section 3.2 apply in this case. All W*-liftings of weights, GNS-constructions and one-parameter groups will always be considered with respect to (H,π,Λ).

5. The antipode and its polar decomposition

In this section we will construct the antipode together with its polar decomposition. We will work as follows: given a right invariant weight ψ on (A,Δ) such that ψ is approximately KMS, we construct the antipode S, the scaling group τ and the unitary antipode R. At the end of this section, we will show that these objects do not depend on the particular choice of ψ (and the already fixed left invariant weight φ). The reader should keep in mind however that this independence is only established towards the end of this section.

Throughout this section, we will fix a right invariant weight ψ on (A,Δ) such that ψ is approximately KMS. Recall from Remark 4.2 that ψ is faithful. Take also a GNS-construction $(H_\psi,\pi_\psi,\Lambda_\psi)$ for ψ.

By Proposition 1.36, we know that $\psi\otimes\psi$ is an approximate KMS weight on $A\otimes A$ with GNS-construction $(H_\psi\otimes H_\psi,\pi_\psi\otimes\pi_\psi,\Lambda_\psi\otimes\Lambda_\psi)$. Recall that the mapping $\Lambda_\psi\otimes\Lambda_\psi$ was introduced in Definition 1.28.

Thanks to Propositions 3.22 and 3.15, we can introduce the following notation.

Notation 5.1. – We define the closed densely defined anti-linear operator G in H such that the space

$$\langle \Lambda\big((\psi\otimes\iota)(\Delta(x^*)(y\otimes 1))\big) \mid x,y\in\mathcal{N}_\varphi^*\mathcal{N}_\psi\rangle$$

is a core for G and

$$G\Lambda\big((\psi\otimes\iota)(\Delta(x^*)(y\otimes 1))\big)=\Lambda\big((\psi\otimes\iota)(\Delta(y^*)(x\otimes 1))\big)$$

for $x,y\in\mathcal{N}_\varphi^*\mathcal{N}_\psi$. We have moreover that G is involutive.

By taking the polar decomposition of G, we get the following operators in H.

Notation 5.2. –
(1) We define $N=G^*G$, so N is a strictly positive operator in H.
(2) We define the anti-unitary operator I on H such that $G=I\,N^{\frac{1}{2}}$.
Because G is involutive, we have that:

- $I=I^*$;
- $I^2=1$;
- $I\,N\,I=N^{-1}$.

5.1. A first step towards strong left invariance

The following technical lemma will be needed in the proof of Proposition 5.5

Consider a Hilbert space K and $v\in K$. Then θ_v denotes the element in $\mathrm{B}(K,\mathbb{C})$ given by $\theta_v(w)=\langle w,v\rangle$ for all $w\in K$. So $\theta_v^*(c)=c\,v$ for all $c\in\mathbb{C}$.

LEMMA 5.3. – *Let $a,c\in\mathcal{N}_\psi$. Then the following properties hold:*
(1) $\Delta^{(2)}(c)\,\Delta_{23}(a)$ *belongs to* $\overline{\mathcal{N}}_{(\psi\otimes\psi)\otimes\iota}$ *and*

$$\big\|\big((\Lambda_\psi\otimes\Lambda_\psi)\otimes\iota\big)\big(\Delta^{(2)}(c)\,\Delta_{23}(a)\big)\big\|^2=\psi(c^*c)\,\psi(a^*a);$$

(2) $\Delta(c)(1\otimes a)$ *belongs to* $\overline{\mathcal{N}}_{\psi\otimes\psi}$;
(3) *let* $v\in H_\psi$ *and put* $p=(\theta_v\otimes 1)(\Lambda_\psi\otimes\iota)(\Delta(c))$. *Then p belongs to* $\mathrm{M}(A)$ *and*

- $(\Lambda_\psi\otimes\iota)(\Delta(p\,a))=(\theta_v\otimes 1\otimes 1)((\Lambda_\psi\otimes\Lambda_\psi)\otimes\iota)(\Delta^{(2)}(c)\,\Delta_{23}(a))$;
- $\Lambda_\psi(p\,a)=(\theta_v\otimes 1)(\Lambda_\psi\otimes\Lambda_\psi)(\Delta(c)(1\otimes a))$.

Proof. – (1) Choose $\theta\in A_+^*$. Then we have by definition of $\psi\otimes\psi$ that:

$$\begin{aligned}
&(\psi\otimes\psi)\big((\iota\otimes\iota\otimes\theta)(\Delta_{23}(a^*)\,\Delta^{(2)}(c^*c)\Delta_{23}(a))\big)\\
&\quad=\sup_{\omega_1\in\mathcal{G}_\psi,\,\omega_2\in\mathcal{G}_\psi}(\omega_1\otimes\omega_2)\big((\iota\otimes\iota\otimes\theta)(\Delta_{23}(a^*)\,\Delta^{(2)}(c^*c)\Delta_{23}(a))\big)\\
&\quad=\sup_{\omega_2\in\mathcal{G}_\psi}\Big(\sup_{\omega_1\in\mathcal{G}_\psi}(\omega_1\otimes\omega_2)\big((\iota\otimes\iota\otimes\theta)(\Delta_{23}(a^*)\,\Delta^{(2)}(c^*c)\Delta_{23}(a))\big)\Big)\\
&\quad=\sup_{\omega_2\in\mathcal{G}_\psi}\Big(\sup_{\omega_1\in\mathcal{G}_\psi}(\omega_2\otimes\theta)\big(\Delta(a^*\,(\omega_1\otimes\iota)(\Delta(c^*c))\,a)\big)\Big).
\end{aligned}$$

Since $((\omega_1\otimes\iota)\Delta(c^*c))_{\omega_1\in\mathcal{G}_\varphi}$ is an increasing net which converges strictly to $\psi(c^*c)\,1$, the previous chain of equalities implies that

$$\begin{aligned}
(\psi\otimes\psi)\big((\iota\otimes\iota\otimes\theta)(\Delta_{23}(a^*)\,\Delta^{(2)}(c^*c)\Delta_{23}(a))\big)&=\psi(c^*c)\sup_{\omega_2\in\mathcal{G}_\psi}\big((\omega_2\otimes\theta)(\Delta(a^*a))\big)\\
&=\psi(c^*c)\,\psi\big((\iota\otimes\theta)\Delta(a^*a)\big)\\
&=\psi(c^*c)\,\psi(a^*a)\,\theta(1).
\end{aligned}$$

Therefore, Proposition 1.23 implies that $\Delta_{23}(a^*)\,\Delta^{(2)}(c^*c)\Delta_{23}(a)$ belongs to $\overline{\mathcal{M}}^+_{(\psi\otimes\psi)\otimes\iota}$ and

$$\big((\psi\otimes\psi)\otimes\iota\big)\big(\Delta_{23}(a^*)\,\Delta^{(2)}(c^*c)\Delta_{23}(a)\big)=\psi(c^*c)\,\psi(a^*a)1.$$

(2) We already know this by the right invariant version of the remarks before Notation 2.8.

(3) Notice that by its definition p belongs to $\mathcal{L}(A)=\mathrm{M}(A)$.

Let us proceed with the proof of the first equality. Choose $\omega_1\in\mathcal{F}_\psi$, $\omega_2\in\mathcal{F}_\psi$ and $w\in H_\psi$. Then Result 3.21 of [30] implies that

$$\begin{aligned}&\big(\theta_v T_{\omega_1}^{\frac{1}{2}}\otimes\theta_w T_{\omega_2}^{\frac{1}{2}}\otimes 1\big)\big((\Lambda_\psi\otimes\Lambda_\psi)\otimes\iota\big)\big(\Delta^{(2)}(c)\Delta_{23}(a)\big)\\&\quad=(\omega_{\xi_{\omega_1},v}\otimes\omega_{\xi_{\omega_2},w}\otimes\iota)\big(\Delta^{(2)}(c)\Delta_{23}(a)\big)\\&\quad=(\omega_{\xi_{\omega_2},w}\otimes\iota)\big(\Delta((\omega_{\xi_{\omega_1},v}\otimes\iota)(\Delta(c))\,a)\big)\\&\quad=(\omega_{\xi_{\omega_2},w}\otimes\iota)\big(\Delta((\theta_v T_{\omega_1}^{\frac{1}{2}}\otimes 1)(\Lambda_\psi\otimes\iota)(\Delta(c))\,a)\big).\end{aligned}$$

Because $(T_{\omega_1}^{\frac{1}{2}})_{\omega_1\in\mathcal{G}_\psi}$ converges strongly to 1, the above equality gives us that

$$\begin{aligned}&\big(\theta_v\otimes\theta_w T_{\omega_2}^{\frac{1}{2}}\otimes 1\big)\big((\Lambda_\psi\otimes\Lambda_\psi)\otimes\iota\big)\big(\Delta^{(2)}(c)\Delta_{23}(a)\big)\\&\quad=(\omega_{\xi_{\omega_2},w}\otimes\iota)\big(\Delta((\theta_v\otimes 1)(\Lambda_\psi\otimes\iota)(\Delta(c))\,a)\big)\\&\quad=\big(\theta_w T_{\omega_2}^{\frac{1}{2}}\otimes 1\big)(\Lambda_\psi\otimes\iota)\big(\Delta(p\,a)\big).\end{aligned}$$

Since $(T_{\omega_2}^{\frac{1}{2}})_{\omega_2\in\mathcal{G}_\psi}$ converges strongly to 1, we get that

$$(\theta_v\otimes\theta_w\otimes 1)\big((\Lambda_\psi\otimes\Lambda_\psi)\otimes\iota\big)\big(\Delta^{(2)}(c)\Delta_{23}(a)\big)=(\theta_w\otimes 1)(\Lambda_\psi\otimes\iota)(\Delta(p\,a))$$

for all $w\in H_\psi$. So we conclude that

$$(\Lambda_\psi\otimes\iota)(\Delta(p\,a))=(\theta_v\otimes 1\otimes 1)\big((\Lambda_\psi\otimes\Lambda_\psi)\otimes\iota\big)\big(\Delta^{(2)}(c)\Delta_{23}(a)\big).$$

Now we quickly prove the second equality. Take $\omega_1,\,\omega_2\in\mathcal{F}_\psi$ and $w\in H_\psi$. Then

$$\begin{aligned}&\big\langle\big(T_{\omega_1}^{\frac{1}{2}}\otimes T_{\omega_2}^{\frac{1}{2}}\big)(\Lambda_\psi\otimes\Lambda_\psi)\big(\Delta(c)(1\otimes a)\big),v\otimes w\big\rangle\\&\quad=(\omega_{\xi_{\omega_1},v}\otimes\omega_{\xi_{\omega_2},w})\big(\Delta(c)(1\otimes a)\big)\\&\quad=\omega_{\xi_{\omega_2},w}\big((\omega_{\xi_{\omega_1},v}\otimes\iota)(\Delta(c))\,a\big)\\&\quad=\omega_{\xi_{\omega_2},w}\big(\big(\theta_v T_{\omega_1}^{\frac{1}{2}}\otimes 1\big)(\Lambda_\psi\otimes\iota)(\Delta(c))\,a\big).\end{aligned}$$

As above, a two step procedure yields that

$$\big\langle(\Lambda_\psi\otimes\Lambda_\psi)(\Delta(c)(1\otimes a)),v\otimes w\big\rangle=\langle\Lambda_\psi(p\,a),w\rangle$$

for all $w\in H_\psi$. So $\Lambda_\psi(p\,a)=(\theta_v\otimes 1)(\Lambda_\psi\otimes\Lambda_\psi)(\Delta(c)(1\otimes a))$. □

Remark 5.4. – Consider $x,y\in\overline{\mathcal{N}}_{\psi\otimes\iota}$. Then we have for all $a\in\mathrm{M}(A\otimes A)$ that $(\psi\otimes\iota)(y^*a\,x)=(\Lambda_\psi\otimes\iota)(y)^*(\pi_\psi\otimes\iota)(a)(\Lambda_\psi\otimes\iota)(x)$. This implies that the mapping

$$\mathrm{M}(A\otimes A)\to\mathrm{M}(A),\quad a\mapsto(\psi\otimes\iota)(y^*a\,x)$$

is strictly continuous on bounded subsets of $\mathrm{M}(A\otimes A)$.

The next proposition is the crucial result of this section. It will allow us to define the antipode S through its polar decomposition.

PROPOSITION 5.5. – *Consider $a, b \in \mathcal{N}_\psi$. Then*

$$\pi\big((\psi \otimes \iota)(\Delta(b^*)(a \otimes 1))\big)\, G \subseteq G\pi\big((\psi \otimes \iota)(\Delta(a^*)(b \otimes 1))\big).$$

Proof. – Choose $c, d \in \mathcal{N}_\varphi^* \mathcal{N}_\psi$. Let $(e_i)_{i \in I}$ be an orthonormal basis for H_ψ. For every $i \in I$, we put $\theta_i = \theta_{e_i} \in \mathrm{B}(H_\psi, \mathbb{C})$.

Define for $i \in I$ the elements

$$v_i = (\theta_i \otimes 1)\big(\Lambda_\psi \otimes \iota\big)\big(\Delta(d)\big) \quad \text{and} \quad u_i = (\theta_i \otimes 1)(\Lambda_\psi \otimes \iota)(\Delta(c)).$$

It is then clear that both v_i and u_i belong to $\mathrm{M}(A)$.

Since $\Delta(c)$ and $\Delta(d)$ belong to $\overline{\mathcal{N}}_{\psi \otimes \iota}$, it is not very difficult to see (using the definition) that $\Delta^{(2)}(c)$ and $\Delta_{12}(d)$ belong to $\overline{\mathcal{N}}_{\psi \otimes \iota \otimes \iota}$. Using Proposition 1.22 and the right invariance of ψ, we get moreover for all $\omega, \theta \in A^*$ that

$$\begin{aligned}(\omega \otimes \theta)\big((\psi \otimes \iota \otimes \iota)(\Delta^{(2)}(c^*)\,\Delta_{12}(d))\big) &= \psi\big((\iota \otimes \omega \otimes \theta)(\Delta^{(2)}(c^*)\,\Delta_{12}(d))\big)\\ &= \psi\big((\iota \otimes \omega)(\Delta((\iota \otimes \theta)(\Delta(c^*)(d \otimes 1))))\big)\\ &= \omega(1)\,\psi\big((\iota \otimes \theta)(\Delta(c^*)(d \otimes 1))\big)\\ &= \omega(1)\,\theta\big((\psi \otimes \iota)(\Delta(c^*)(d \otimes 1))\big).\end{aligned}$$

Hence $(\psi \otimes \iota \otimes \iota)(\Delta^{(2)}(c^*)\,\Delta_{12}(d)) = 1 \otimes (\psi \otimes \iota)(\Delta(c^*)(d \otimes 1))$.

Using Proposition 3.38 of [30], we have for $J \in F(I)$

$$\begin{aligned}\sum_{i \in J} \Delta(u_i^*)(v_i \otimes 1) &= \sum_{i \in J} (\Lambda_\psi \otimes \iota \otimes \iota)(\Delta^{(2)}(c))^*(\theta_i^*\theta_i \otimes 1 \otimes 1)(\Lambda_\psi \otimes \iota \otimes \iota)(\Delta_{12}(d))\\ &= (\Lambda_\psi \otimes \iota \otimes \iota)(\Delta^{(2)}(c))^*(P_J \otimes 1 \otimes 1)(\Lambda_\psi \otimes \iota \otimes \iota)(\Delta_{12}(d)),\end{aligned}$$

where P_J denotes the orthogonal projection onto the subspace $\langle e_i \mid i \in J \rangle$. So we get that the net

$$\left(\sum_{i \in J} \Delta(u_i^*)(v_i \otimes 1)\right)_{J \in F(I)}$$

is bounded and converges strictly to

$$(\psi \otimes \iota \otimes \iota)\big(\Delta^{(2)}(c^*)\Delta_{12}(d)\big) = 1 \otimes (\psi \otimes \iota)\big(\Delta(c^*)(d \otimes 1)\big)$$

Because of the remark before this proposition, we have that

$$\begin{aligned}&(\psi \otimes \iota)\big(\Delta(a^*)(b \otimes 1)\big)(\psi \otimes \iota)\big(\Delta(c^*)(d \otimes 1)\big)\\ &\quad = (\psi \otimes \iota)\big(\Delta(a^*)(1 \otimes (\psi \otimes \iota)(\Delta(c^*)(d \otimes 1)))(b \otimes 1)\big)\\ &\quad = \sum_{i \in I} (\psi \otimes \iota)\big(\Delta(a^* u_i^*)(v_i b \otimes 1)\big)\end{aligned}$$

in the strict topology.

We claim that for all $i \in I$, $u_i, v_i \in \mathcal{N}_\varphi^*$. Choose $i \in I$. Take a sequence $x_n \in \mathcal{N}_\psi$ such that $\Lambda_\psi(x_n) \to e_i$. We have for every $n \in \mathbb{N}$ that

$$(\psi \otimes \iota)((x_n^* \otimes 1)\Delta(d)) = (\theta_{\Lambda(x_n)} \otimes 1)(\Lambda_\psi \otimes \iota)(\Delta(d)),$$

so it is clear that $(\psi \otimes \iota)((x_n^* \otimes 1)\Delta(d)) \to v_i$ in norm. By Result 2.5 we know that

$$(\psi \otimes \iota)\big((x_n^* \otimes 1)\Delta(d)\big)$$

belongs to A for every $n \in \mathbb{N}$. Hence v_i belongs to A.

Moreover, because $d \in \mathcal{N}_\varphi^* \mathcal{N}_\psi$, Result 2.6 implies that $(\psi \otimes \iota)((x_n^* \otimes 1)\Delta(d)) \in \mathcal{N}_\varphi^*$ and that the sequence $\Lambda((\psi \otimes \iota)((x_n^* \otimes 1)\Delta(d))^*)$ is Cauchy in H. Since Λ is closed, we get that $v_i \in \mathcal{N}_\varphi^*$. Analogously, $u_i \in \mathcal{N}_\varphi^*$. This proves our claim.

Now we claim that the net

$$\left(\sum_{i \in J} \Lambda\big((\psi \otimes \iota)(\Delta(a^* u_i^*)(v_i b \otimes 1))\big)\right)_{J \in F(I)}$$

is norm convergent in H. It suffices to look at the case $c = u^* v$ with $u \in \mathcal{N}_\varphi$ and $v \in \mathcal{N}_\psi$.

Put $\xi = (\Lambda_\psi \otimes \Lambda_\psi)(\Delta(d)(1 \otimes b)) \in H_\psi \otimes H_\psi$.

Let $i \in I$. From the lemma before this proposition, we may conclude that

$$(\Lambda_\psi \otimes \iota)(\Delta(u_i a)) = (\theta_i \otimes 1 \otimes 1)\big((\Lambda_\psi \otimes \Lambda_\psi) \otimes \iota\big)\big(\Delta^{(2)}(c)\Delta_{23}(a)\big)$$

and

$$\Lambda_\psi(v_i b) = (\theta_i \otimes 1)(\Lambda_\psi \otimes \Lambda_\psi)\big(\Delta(d)(1 \otimes b)\big) = (\theta_i \otimes 1)\,\xi.$$

So we get for $J \in F(I)$ that

$$\begin{aligned}
w_J &:= \sum_{i \in J} (\psi \otimes \iota)\big(\Delta(a^* u_i^*)(v_i b \otimes 1)\big) \\
&= \sum_{i \in J} (\Lambda_\psi \otimes \iota)(\Delta(u_i a))^* (\Lambda_\psi \otimes \iota)(v_i b \otimes 1) \\
&= \big((\Lambda_\psi \otimes \Lambda_\psi) \otimes \iota\big)\big(\Delta^{(2)}(c)\Delta_{23}(a)\big)^* (P_J \otimes 1 \otimes 1)(\theta_\xi^* \otimes 1) \\
&= T^*(\pi_\psi \otimes \pi_\psi \otimes \iota)(\Delta^{(2)}(u))(\theta^*_{(P_J \otimes 1)\,\xi} \otimes 1),
\end{aligned}$$

where $T = ((\Lambda_\psi \otimes \Lambda_\psi) \otimes \iota)(\Delta^{(2)}(v)\Delta_{23}(a))$. Therefore

$$\begin{aligned}
w_J^* w_J &= (\theta_{(P_J \otimes 1)\xi} \otimes 1)(\pi_\psi \otimes \pi_\psi \otimes \iota)(\Delta^{(2)}(u^*))TT^*(\pi_\psi \otimes \pi_\psi \otimes \iota)(\Delta^{(2)}(u))(\theta^*_{(P_J \otimes 1)\xi} \otimes 1) \\
&\leqslant \|T\|^2 (\theta_{(P_J \otimes 1)\,\xi} \otimes 1)(\pi_\psi \otimes \pi_\psi \otimes \iota)(\Delta^{(2)}(u^* u))(\theta^*_{(P_J \otimes 1)\,\xi} \otimes 1) \\
&= \psi(v^* v)\psi(a^* a)(\omega_{(P_J \otimes 1)\xi,(P_J \otimes 1)\xi} \otimes \iota)(\Delta^{(2)}(u^* u)).
\end{aligned}$$

So we get by left invariance of φ that

$$\left\| \sum_{i \in J} \Lambda\big((\psi \otimes \iota)(\Delta(a^* u_i^*)(v_i b \otimes 1))\big) \right\|^2 = \varphi(w_J^* w_J) \leqslant \psi(v^* v)\psi(a^* a)\varphi(u^* u)\,\|(P_J \otimes 1)\xi\|^2.$$

Because $(P_J \otimes 1)_{J \in F(I)}$ increases to 1 in the strong topology, our claim has been proved (use the Cauchy criterion).

Since the map Λ is closed with respect to the strict topology on A and the norm topology on H we get that

$$\Big[\sum_{i\in J}\Lambda\big((\psi\otimes\iota)(\Delta(a^*u_i^*)(v_ib\otimes 1))\big)\Big] \underset{J\in F(I)}{\longrightarrow} \pi\big((\psi\otimes\iota)(\Delta(a^*)(b\otimes 1))\big)\Lambda\big((\psi\otimes\iota)(\Delta(c^*)(d\otimes 1))\big).$$

By symmetry we get that

$$\Big[\sum_{i\in J}\Lambda\big((\psi\otimes\iota)(\Delta(b^*v_i^*)(u_ia\otimes 1))\big)\Big] \underset{J\in F(I)}{\longrightarrow} \pi\big((\psi\otimes\iota)(\Delta(b^*)(a\otimes 1))\big)\Lambda\big((\psi\otimes\iota)(\Delta(d^*)(c\otimes 1))\big).$$

By definition of G, we have for every $J\in F(I)$ that $\sum_{i\in J}\Lambda((\psi\otimes\iota)(\Delta(a^*u_i^*)(v_ib\otimes 1)))$ belongs to $D(G)$ and

$$G\Big(\sum_{i\in J}\Lambda\big((\psi\otimes\iota)(\Delta(a^*u_i^*)(v_ib\otimes 1))\big)\Big)=\sum_{i\in J}\Lambda\big((\psi\otimes\iota)(\Delta(b^*v_i^*)(u_ia\otimes 1))\big).$$

Because the map G is closed we obtain that

$$\pi\big((\psi\otimes\iota)(\Delta(a^*)(b\otimes 1))\big)\Lambda\big((\psi\otimes\iota)(\Delta(c^*)(d\otimes 1))\big)\in D(G)$$

and

$$G\pi\big((\psi\otimes\iota)(\Delta(a^*)(b\otimes 1))\big)\Lambda\big((\psi\otimes\iota)(\Delta(c^*)(d\otimes 1))\big) = \pi\big((\psi\otimes\iota)(\Delta(b^*)(a\otimes 1))\big)\,G\Lambda\big((\psi\otimes\iota)(\Delta(c^*)(d\otimes 1))\big).$$

Because G is closed and the linear span of such elements $\Lambda((\psi\otimes\iota)(\Delta(c^*)(d\otimes 1)))$ form by definition a core for G, we can finally conclude that

$$\pi\big((\psi\otimes\iota)(\Delta(b^*)(a\otimes 1))\big)\,G\subseteq G\,\pi\big((\psi\otimes\iota)(\Delta(a^*)(b\otimes 1))\big). \qquad \square$$

Remark 5.6. – Let K be a Hilbert space, T a closed densely defined operator in K and $x,y\in B(K)$. If $xT\subseteq Ty$, then $y^*T^*\subseteq T^*x^*$: If $xT\subseteq Ty$, then we get that $(Ty)^*\subseteq(xT)^*$ by taking the adjoint. It is true in general that $y^*T^*\subseteq(Ty)^*$. Since x is bounded, we have that $(xT)^*=T^*x^*$.

Applying this remark to the previous proposition, we infer that

$$\pi\big((\psi\otimes\iota)((a^*\otimes 1)\Delta(b))\big)\,G^*\subseteq G^*\pi\big((\psi\otimes\iota)((b^*\otimes 1)\Delta(a))\big) \tag{5.1}$$

for every $a,b\in\mathcal{N}_\psi$.

In the rest of this section, we will be able to draw some important conclusions from this proposition.

5.2. The scaling group

We will repeat some of the discussion from the beginning of Section 3.3. This time, we will however lift everything in the GNS-construction for φ.

Using Propositions 3.9 and 3.10, we define the following objects:

- $\widetilde{\psi} =$ the W*-lift of ψ in the GNS-construction (H, π, Λ);
- $(H_\psi, \widetilde{\pi}_\psi, \widetilde{\Lambda}_\psi) =$ the W*-lift of $(H_\psi, \pi_\psi, \Lambda_\psi)$ in the GNS-construction (H, π, Λ).

Since ψ is approximately KMS, $\widetilde{\psi}$ is a n.f.s. weight on $\widetilde{A}$. We will denote its modular group by $\sigma^{\widetilde{\psi}}$.

We will moreover use the following notations:

- $J_\psi =$ modular conjugation of $\widetilde{\psi}$ in the GNS-construction $(H_\psi, \widetilde{\pi}_\psi, \widetilde{\Lambda}_\psi)$;
- $\nabla_\psi =$ modular operator of $\widetilde{\psi}$ in the GNS-construction $(H_\psi, \widetilde{\pi}_\psi, \widetilde{\Lambda}_\psi)$.

For technical reasons, we will need to use a Tomita *-algebra once again:

$$\mathcal{T}_{\widetilde{\psi}} = \{x \in \mathcal{N}_{\widetilde{\psi}} \cap \mathcal{N}_{\widetilde{\psi}}^* \mid x \text{ is analytic with respect to } \sigma^{\widetilde{\psi}} \text{ and } \sigma_z^{\widetilde{\psi}}(x) \in \mathcal{N}_{\widetilde{\psi}} \cap \mathcal{N}_{\widetilde{\psi}}^* \text{ for } z \in \mathbb{C}\}.$$

Recall also that $\widetilde{\Lambda}_\psi(a) \in D(\nabla_\psi^z)$ and $\nabla_\psi^z \widetilde{\Lambda}_\psi(a) = \widetilde{\Lambda}_\psi(\sigma_{-iz}^{\widetilde{\psi}}(a))$ for all $z \in \mathbb{C}$ and $a \in \mathcal{T}_{\widetilde{\psi}}$.

Using the right invariant version of Notation 2.8, we define the unitary $U : H_\psi \otimes H \to H_\psi \otimes H$ such that $U(\Lambda_\psi(p) \otimes \Lambda(q)) = (\Lambda_\psi \otimes \Lambda)(\Delta(p)(1 \otimes q))$ for $p \in \mathcal{N}_\psi$ and $q \in \mathcal{N}_\varphi$. (Remember that U is unitary because of the obvious right invariant version of Theorem 3.16.)

The right invariant version of Result 2.10 implies for every $a, b \in \mathcal{N}_\psi$ that

$$(\omega_{\Lambda_\psi(a), \Lambda_\psi(b)} \otimes \iota)(U) = \pi\big((\psi \otimes \iota)((b^* \otimes 1)\Delta(a))\big). \tag{5.2}$$

This implies that

$$\pi(A) = [(\omega_{v,w} \otimes \iota)(U) \mid v, w \in H_\psi]. \tag{5.3}$$

LEMMA 5.7. – *Consider $a \in \mathcal{N}_{\widetilde{\psi}} \cap \mathcal{N}_{\widetilde{\psi}}^*$ and $b \in \mathcal{T}_{\widetilde{\psi}}$. Then*

$$\big(\omega_{\widetilde{\Lambda}_\psi(a), \widetilde{\Lambda}_\psi(b)} \otimes \iota\big)(U)^* = \big(\omega_{\widetilde{\Lambda}_\psi(a^*), \widetilde{\Lambda}_\psi(\sigma_{-i}^{\widetilde{\psi}}(b^*))} \otimes \iota\big)(U).$$

Proof. – Choose $c, d \in \mathcal{T}_{\widetilde{\psi}}$. Lemma 3.14 implies that

$$\big(\omega_{\Lambda_\psi(p), \widetilde{\Lambda}_\psi(c\sigma_{-i}^{\widetilde{\psi}}(d^*))} \otimes \iota\big)(U) = \big(\omega_{\widetilde{\Lambda}_\psi(d), \widetilde{\Lambda}_\psi(c)} \overline{\otimes} \iota\big)\widetilde{\Delta}(\pi(p))$$

for all $p \in \mathcal{N}_\psi$. By Proposition 3.10, this gives

$$\big(\omega_{\widetilde{\Lambda}_\psi(q), \widetilde{\Lambda}_\psi(c\sigma_{-i}^{\widetilde{\psi}}(d^*))} \otimes \iota\big)(U) = \big(\omega_{\widetilde{\Lambda}_\psi(d), \widetilde{\Lambda}_\psi(c)} \overline{\otimes} \iota\big)\widetilde{\Delta}(q)$$

for all $q \in \mathcal{N}_{\widetilde{\psi}}$. So we get for $e \in \mathcal{T}_{\widetilde{\psi}}$ that

$$\begin{aligned}\big(\omega_{\widetilde{\Lambda}_\psi(a), \widetilde{\Lambda}_\psi(b\sigma_{-i}^{\widetilde{\psi}}(e^*))} \otimes \iota\big)(U)^* &= \big(\omega_{\widetilde{\Lambda}_\psi(e), \widetilde{\Lambda}_\psi(b)} \overline{\otimes} \iota\big)(\widetilde{\Delta}(a))^* \\ &= \big(\omega_{\widetilde{\Lambda}_\psi(b), \widetilde{\Lambda}_\psi(e)} \overline{\otimes} \iota\big)\widetilde{\Delta}(a^*) = \big(\omega_{\widetilde{\Lambda}_\psi(a^*), \widetilde{\Lambda}_\psi(e\sigma_{-i}^{\widetilde{\psi}}(b^*))} \otimes \iota\big)(U).\end{aligned} \tag{5.4}$$

Take a bounded net $(u_k)_{k \in K}$ in $\mathcal{N}_{\widetilde{\psi}} \cap \mathcal{N}_{\widetilde{\psi}}^*$ such that $(u_k)_{k \in K}$ converges strongly* to 1 (such a net clearly exists). For every $k \in K$, we put

$$e_k = \frac{1}{\sqrt{\pi}} \int \exp(-t^2)\, \sigma_t^{\widetilde{\psi}}(u_k)\, dt,$$

then e_k clearly belongs to $\mathcal{T}_{\widetilde{\psi}}$. We have moreover

- $(e_k)_{k\in K}$ is bounded and converges strongly* to 1;
- $(\sigma^{\widetilde{\psi}}_{\frac{i}{2}}(e_k))_{k\in K}$ is bounded and converges strongly* to 1.

Then we have for all $k\in K$ that

$$\widetilde{\Lambda}_\psi\big(b\,\sigma^{\widetilde{\psi}}_{-i}(e_k^*)\big)=J_\psi\widetilde{\pi}_\psi\big(\sigma^{\widetilde{\psi}}_{-\frac{i}{2}}(e_k^*)\big)^*J_\psi\widetilde{\Lambda}_\psi(b)=J_\psi\widetilde{\pi}_\psi\big(\sigma^{\widetilde{\psi}}_{\frac{i}{2}}(e_k)\big)J_\psi\widetilde{\Lambda}_\psi(b).$$

So we see that $(\widetilde{\Lambda}_\psi(b\,\sigma^{\widetilde{\psi}}_{-i}(e_k^*)))_{k\in K}$ converges to $\widetilde{\Lambda}_\psi(b)$.

If we now combine these convergences with Eq. (5.4), we conclude that

$$\big(\omega_{\widetilde{\Lambda}_\psi(a),\widetilde{\Lambda}_\psi(b)}\otimes\iota\big)(U)^*=\big(\omega_{\widetilde{\Lambda}_\psi(a^*),\widetilde{\Lambda}_\psi\left(\sigma^{\widetilde{\psi}}_{-i}(b^*)\right)}\otimes\iota\big)(U).\qquad\Box$$

PROPOSITION 5.8. – *Consider $v,w\in H_\psi$. Then*

$$(\omega_{v,w}\otimes\iota)(U^*)\,G\subseteq G\,(\omega_{w,v}\otimes\iota)(U^*)\quad\textit{and}\quad(\omega_{v,w}\otimes\iota)(U)\,G^*\subseteq G^*\,(\omega_{w,v}\otimes\iota)(U).$$

Proof. – Take $u\in D(G)$. Take nets $(p_i)_{i\in I}$ and $(q_i)_{i\in I}$ in $\mathcal{N}_\psi$ such that $(\Lambda_\psi(p_i))_{i\in I}$ converges to v and $(\Lambda_\psi(q_i))_{i\in I}$ converges to w. Notice that

$$(\omega_{\Lambda_\psi(p),\Lambda_\psi(q)}\otimes\iota)(U^*)=\pi\big((\psi\otimes\iota)(\Delta(q^*)(p\otimes1))\big)$$

for $p,\,q\in\mathcal{N}_\psi$. So Proposition 5.5 implies that

$$(\omega_{\Lambda_\psi(p_i),\Lambda_\psi(q_i)}\otimes\iota)(U^*)\,G\subseteq G\,(\omega_{\Lambda_\psi(q_i),\Lambda_\psi(p_i)}\otimes\iota)(U^*)$$

for every $i\in I$. This implies for every $i\in I$ that $(\omega_{\Lambda_\psi(q_i),\Lambda_\psi(p_i)}\otimes\iota)(U^*)\,u$ belongs to $D(G)$ and

$$G\big((\omega_{\Lambda_\psi(q_i),\Lambda_\psi(p_i)}\otimes\iota)(U^*)\,u\big)=(\omega_{\Lambda_\psi(p_i),\Lambda_\psi(q_i)}\otimes\iota)(U^*)\,G\,u.$$

So we get that $((\omega_{\Lambda_\psi(q_i),\Lambda_\psi(p_i)}\otimes\iota)(U^*)\,u)_{i\in I}$ converges to $(\omega_{w,v}\otimes\iota)(U^*)\,u$ and that

$$\big(G\big((\omega_{\Lambda_\psi(q_i),\Lambda_\psi(p_i)}\otimes\iota)(U^*)\,u\big)\big)_{i\in I}$$

converges to $(\omega_{v,w}\otimes\iota)(U^*)\,G\,u$. Therefore the closedness of G implies that $(\omega_{w,v}\otimes\iota)(U^*)\,u$ belongs to $D(G)$ and $G\big((\omega_{w,v}\otimes\iota)(U^*)\,u\big)=(\omega_{v,w}\otimes\iota)(U^*)\,G\,u$.
So we have proven the first inclusion. The second follows from the first by taking the adjoint (*see* Remark 5.6). $\Box$

Before we can prove the next result we need a simple lemma which we will use several times later on.

LEMMA 5.9. – *Consider Hilbert spaces K and L, a unitary $V\in B(K\otimes L)$, strictly positive operators S_1 and S_2 in K and strictly positive operators T_1 and T_2 in L such that $(\omega_{v,w}\otimes\iota)(V)\,T_1\subseteq T_2\,(\omega_{S_1^{-1}v,S_2w}\otimes\iota)(V)$ for $v\in D(S_1^{-1})$ and $w\in D(S_2)$. Then $V(S_1\otimes T_1)=(S_2\otimes T_2)V$.*

Proof. – Choose $p_1\in D(S_1)$, $q_1\in D(T_1)$, $p_2\in D(S_2)$, $q_2\in D(T_2)$. Then

$$\begin{aligned}\langle V(p_1\otimes q_1),S_2p_2\otimes T_2q_2\rangle&=\langle(\omega_{p_1,S_2p_2}\otimes\iota)(V)\,q_1,T_2q_2\rangle\\&=\langle(\omega_{S_1^{-1}(S_1p_1),S_2p_2}\otimes\iota)(V)\,q_1,T_2q_2\rangle.\end{aligned}$$

Therefore the assumptions of this lemma imply that

$$\begin{aligned}\langle V(p_1\otimes q_1), S_2p_2\otimes T_2q_2\rangle &= \langle T_2\,(\omega_{S_1^{-1}(S_1p_1),S_2p_2}\otimes\iota)(V)\,q_1,q_2\rangle\\ &= \langle(\omega_{S_1p_1,p_2}\otimes\iota)(V)\,T_1q_1,q_2\rangle = \langle V(S_1p_1\otimes T_1q_1),p_2\otimes q_2\rangle.\end{aligned}$$

Because $D(S_2)\odot D(T_2)$ is a core for $S_2\otimes T_2$, this implies for all $p_1\in D(S_1)$, $q_1\in D(T_1)$ and $v\in D(S_2\otimes T_2)$ that

$$\langle V\,(p_1\otimes q_1),(S_2\otimes T_2)\,v\rangle = \langle V\,(S_1p_1\otimes T_1q_1),v\rangle.$$

Since $S_2\otimes T_2$ is selfadjoint, this implies for all $p_1\in D(S_1)$ and $q_1\in D(T_1)$ that $V(p_1\otimes q_1)\in D(S_2\otimes T_2)$ and $(S_2\otimes T_2)(V(p_1\otimes q_1)) = V(S_1p_1\otimes T_1q_1)$.

Because $D(S_1)\odot D(T_1)$ is a core for $S_1\otimes T_1$ and $S_2\otimes T_2$ is closed, we conclude that $V(S_1\otimes T_1)\subseteq(S_2\otimes T_2)V$. Hence $(S_1\otimes T_1)V^*\subseteq V^*(S_2\otimes T_2)$. By taking he adjoint of this inclusion, we see that $(S_2\otimes T_2)V\subseteq V(S_1\otimes T_1)$. Consequently, $(S_2\otimes T_2)V = V(S_1\otimes T_1)$. □

RESULT 5.10. – *The following commutation relation holds*: $U(\nabla_\psi\otimes N) = (\nabla_\psi\otimes N)U$.

Proof. – Choose $a,b\in\mathcal{T}_{\tilde\psi}$. Using twice Lemma 5.7 and once the previous proposition, we get that

$$\begin{aligned}\big(\omega_{\widetilde\Lambda_\psi(b),\widetilde\Lambda_\psi(a)}\otimes\iota\big)(U)\,G &= \big(\omega_{\widetilde\Lambda_\psi(b^*),\widetilde\Lambda_\psi(\sigma^{\tilde\psi}_{-i}(a^*))}\otimes\iota\big)(U)^*\,G\\ &\subseteq G\,\big(\omega_{\widetilde\Lambda_\psi(\sigma^{\tilde\psi}_{-i}(a^*)),\widetilde\Lambda_\psi(b^*)}\otimes\iota\big)(U)^*\\ &= G\,\big(\omega_{\widetilde\Lambda_\psi(\sigma^{\tilde\psi}_{i}(a)),\widetilde\Lambda_\psi(\sigma^{\tilde\psi}_{-i}(b))}\otimes\iota\big)(U).\end{aligned}$$

Using the previous proposition once more, the chain of inclusions above gives

$$\begin{aligned}\big(\omega_{\widetilde\Lambda_\psi(a),\widetilde\Lambda_\psi(b)}\otimes\iota\big)(U)N &= \big(\omega_{\widetilde\Lambda_\psi(a),\widetilde\Lambda_\psi(b)}\otimes\iota\big)(U)\,G^*G\\ &\subseteq G^*\big(\omega_{\widetilde\Lambda_\psi(b),\widetilde\Lambda_\psi(a)}\otimes\iota\big)(U)\,G\\ &\subseteq G^*G\big(\omega_{\widetilde\Lambda_\psi(\sigma^{\tilde\psi}_{i}(a)),\widetilde\Lambda_\psi(\sigma^{\tilde\psi}_{-i}(b))}\otimes\iota\big)(U)\\ &= N\big(\omega_{\nabla_\psi^{-1}\widetilde\Lambda_\psi(a),\nabla_\psi\widetilde\Lambda_\psi(b)}\otimes\iota\big)(U).\end{aligned}$$

Since $\widetilde\Lambda_\psi(\mathcal{T}_{\tilde\psi})$ is a core for both ∇_ψ^{-1} and ∇_ψ, the previous inclusion implies that

$$(\omega_{v,w}\otimes\iota)(U)N\subseteq N\big(\omega_{\nabla_\psi^{-1}v,\nabla_\psi w}\otimes\iota\big)(U)$$

for all $v\in D(\nabla_\psi^{-1})$ and $w\in D(\nabla_\psi)$ (compare with the proof of the previous proposition). Hence Lemma 5.9 implies that $U(\nabla_\psi\otimes N) = (\nabla_\psi\otimes N)U$. □

Now, we have proved the necessary commutation relation to be able to define the scaling group of our reduced C*-algebraic quantum group.

PROPOSITION 5.11. – (1) *There exists a unique strongly continuous one-parameter group* $\tilde\tau$ *of automorphisms of* $\widetilde A$ *such that* $\tilde\tau_t(x) = N^{-it}\,x\,N^{it}$ *for all* $t\in\mathbb{R}$ *and* $x\in\widetilde A$.

(2) *There exists a unique norm continuous one-parameter group* τ *of automorphisms of* A *such that* $\tilde\tau_t\pi = \pi\tau_t$ *for all* $t\in\mathbb{R}$.

(3) *We have that* $(\sigma^{\tilde\psi}_t\,\overline{\otimes}\,\tilde\tau_{-t})\widetilde\Delta = \widetilde\Delta\,\sigma^{\tilde\psi}_t$ *for all* $t\in\mathbb{R}$.

Proof. – Recall from Eq. (5.3) that

$$\pi(A) = [(\omega \otimes \iota)(U) \mid \omega \in \mathrm{B}_0(H_\psi)^*]. \tag{5.5}$$

By the commutation relation $(\nabla_\psi \otimes N)U = U(\nabla_\psi \otimes N)$, we have

$$N^{-it}(\omega \otimes \iota)(U)N^{it} = (\nabla_\psi^{-it}\omega\nabla_\psi^{it} \otimes \iota)(U)$$

for all $t \in \mathbb{R}$. Therefore Eq. (5.5) implies that:

(1) $N^{-it}\pi(A)N^{it} = \pi(A)$ for all $t \in \mathbb{R}$;

(2) $N^{-it}\widetilde{A}N^{it} = \widetilde{A}$ for all $t \in \mathbb{R}$;

(3) we have for every $a \in A$ that the function $\mathbb{R} \mapsto \pi(A)$, $t \mapsto N^{-it}\pi(a)N^{it}$ is norm continuous.

The equality in (2) implies immediately that we can define a strongly continuous one-parameter group $\widetilde{\tau}$ of automorphisms of $\widetilde{A}$ such that $\widetilde{\tau}_t(x) = N^{-it}xN^{it}$ for all $t \in \mathbb{R}$ and $x \in \widetilde{A}$.

The equality in (1) and the injectivity of π allows us to define a one-parameter group τ of automorphisms of A such that $\widetilde{\tau}_t\pi = \pi\tau_t$ for all $t \in \mathbb{R}$. The third property implies that τ is norm continuous.

It is easy to see that $(\pi_\psi \otimes \pi)(\Delta(a)) = U(\pi_\psi(a) \otimes 1)U^*$ for all $a \in A$. This implies that $(\widetilde{\pi}_\psi \,\overline{\otimes}\, \iota)(\widetilde{\Delta}(x)) = U(\widetilde{\pi}_\psi(x) \otimes 1)U^*$ for $x \in \widetilde{A}$. So we have for $x \in \widetilde{A}$ and $t \in \mathbb{R}$

$$\begin{aligned}
(\widetilde{\pi}_\psi \otimes \iota)\big((\sigma_t^{\widetilde{\psi}} \,\overline{\otimes}\, \widetilde{\tau}_{-t})\widetilde{\Delta}(x)\big) &= (\nabla_\psi^{it} \otimes N^{it})\big((\widetilde{\pi}_\psi \,\overline{\otimes}\, \iota)\widetilde{\Delta}(x)\big)(\nabla_\psi^{-it} \otimes N^{-it}) \\
&= (\nabla_\psi^{it} \otimes N^{it})U(\widetilde{\pi}_\psi(x) \otimes 1)\,U^*(\nabla_\psi^{-it} \otimes N^{-it}) \\
&= U(\nabla_\psi^{it}\,\widetilde{\pi}_\psi(x)\,\nabla_\psi^{-it} \otimes 1)\,U^* \\
&= (\widetilde{\pi}_\psi \,\overline{\otimes}\, \iota)\widetilde{\Delta}(\sigma_t^{\widetilde{\psi}}(x)).
\end{aligned}$$

Because $\widetilde{\pi}_\psi$ is faithful, $(\sigma_t^{\widetilde{\psi}} \,\overline{\otimes}\, \widetilde{\tau}_{-t})\widetilde{\Delta}(x) = \widetilde{\Delta}(\sigma_t^{\widetilde{\psi}}(x))$. □

The next commutation rule is now more or less obvious.

RESULT 5.12. – *We have* $\Delta\tau_t = (\tau_t \otimes \tau_t)\Delta$ *for all* $t \in \mathbb{R}$.

Proof. – Let $t \in \mathbb{R}$. From $\widetilde{\Delta}\,\sigma_t^{\widetilde{\psi}} = (\sigma_t^{\widetilde{\psi}} \,\overline{\otimes}\, \widetilde{\tau}_{-t})\widetilde{\Delta}$ we can conclude that

$$\begin{aligned}
(\sigma_t^{\widetilde{\psi}} \,\overline{\otimes}\, \widetilde{\Delta}\,\widetilde{\tau}_{-t})\widetilde{\Delta} &= (\iota \,\overline{\otimes}\, \widetilde{\Delta})\widetilde{\Delta}\,\sigma_t^{\widetilde{\psi}} = (\widetilde{\Delta} \,\overline{\otimes}\, \iota)\widetilde{\Delta}\,\sigma_t^{\widetilde{\psi}} = \big(\widetilde{\Delta}\,\sigma_t^{\widetilde{\psi}} \,\overline{\otimes}\, \widetilde{\tau}_{-t}\big)\,\widetilde{\Delta} \\
&= \big(\sigma_t^{\widetilde{\psi}} \,\overline{\otimes}\, \widetilde{\tau}_{-t} \,\overline{\otimes}\, \widetilde{\tau}_{-t}\big)(\widetilde{\Delta} \,\overline{\otimes}\, \iota)\widetilde{\Delta} = \big(\sigma_t^{\widetilde{\psi}} \,\overline{\otimes}\, (\widetilde{\tau}_{-t} \,\overline{\otimes}\, \widetilde{\tau}_{-t})\widetilde{\Delta}\big)\widetilde{\Delta},
\end{aligned}$$

implying that $(\iota \,\overline{\otimes}\, \widetilde{\Delta}\,\widetilde{\tau}_{-t})\widetilde{\Delta} = (\iota \,\overline{\otimes}\, (\widetilde{\tau}_{-t} \,\overline{\otimes}\, \widetilde{\tau}_{-t})\widetilde{\Delta})\widetilde{\Delta}$. By composing this equality with π (from the right of course), this gives $(\pi \otimes \pi \otimes \pi)(\iota \otimes \Delta\,\tau_{-t})\Delta = (\pi \otimes \pi \otimes \pi)(\iota \otimes (\tau_{-t} \otimes \tau_{-t})\Delta)\Delta$ so that the injectivity of π implies that $(\iota \otimes \Delta\,\tau_{-t})\Delta = (\iota \otimes (\tau_{-t} \otimes \tau_{-t})\Delta)\Delta$. Then we get that

$$\Delta\big(\tau_{-t}((\omega \otimes \iota)\Delta(a))\big) = (\tau_{-t} \otimes \tau_{-t})\Delta((\omega \otimes \iota)\Delta(a))$$

for all $\omega \in A^*$ and $a \in A$. Density gives the conclusion. □

5.3. Invariant approximately KMS weights are KMS

In a next step, we want to show that the weights ψ and φ which are assumed to be approximately KMS, are in fact KMS. We will however need some extra technical results. The following result was already an important tool in the theory of Kac algebras. This will also be the case in our framework.

RESULT 5.13. – *Consider* $x \in \tilde{A}$. *If* $\tilde{\Delta}(x) = 1 \otimes x$ *or* $\tilde{\Delta}(x) = x \otimes 1$, *then* $x \in \mathbb{C}1$.

Proof. – First we suppose that $\tilde{\Delta}(x) = 1 \otimes x$. For $n \in \mathbb{N}$, we define $x_n, y_n \in \tilde{A}$ such that (the integrals are in the strong topology)

$$x_n = \frac{n}{\sqrt{\pi}} \int \exp(-n^2t^2)\sigma_t^{\tilde{\psi}}(x)\,\mathrm{d}t \quad \text{and} \quad y_n = \frac{n}{\sqrt{\pi}} \int \exp(-n^2t^2)\tilde{\tau}_{-t}(x)\,\mathrm{d}t.$$

Then x_n is analytic with respect to $\sigma^{\tilde{\psi}}$. Because $(\sigma_t^{\tilde{\psi}} \otimes \tilde{\tau}_{-t})\tilde{\Delta} = \tilde{\Delta}\,\sigma_t^{\tilde{\psi}}$ for $t \in \mathbb{R}$ (*see* Proposition 5.11), it is moreover clear that $\tilde{\Delta}(x_n) = 1 \otimes y_n$.

Now, we take $d \in \mathcal{M}_{\tilde{\psi}}^{+}$ such that $\tilde{\psi}(d) = 1$. Because $x_n \in D(\sigma_{\frac{i}{2}}^{\tilde{\psi}})$, we know that $dx_n \in \mathcal{M}_{\tilde{\psi}}$. Take $\omega \in \tilde{A}_*$. Then we have that

$$(\iota \bar{\otimes} \omega)\tilde{\Delta}(dx_n) = (\iota \bar{\otimes} y_n\omega)\tilde{\Delta}(d).$$

By right invariance of $\tilde{\psi}$, we know that $(\iota \bar{\otimes} \omega)\tilde{\Delta}(dx_n) \in \mathcal{M}_{\tilde{\psi}}$, $(\iota \bar{\otimes} y_n\omega)\tilde{\Delta}(d) \in \mathcal{M}_{\tilde{\psi}}$ and

$$\tilde{\psi}(dx_n)\,\omega(1) = \tilde{\psi}\big((\iota \bar{\otimes} \omega)\tilde{\Delta}(dx_n)\big) = \tilde{\psi}\big((\iota \bar{\otimes} y_n\omega)\tilde{\Delta}(d)\big) = (y_n\omega)(1)\,\tilde{\psi}(d) = \omega(y_n).$$

So we get that $y_n = \tilde{\psi}(dx_n)1$.
Because $(y_n)_{n=1}^{\infty}$ converges strongly to x, the sequence $(\tilde{\psi}(dx_n))_{n=1}^{\infty}$ is Cauchy so it converges to a number $c \in \mathbb{C}$. Then $x = c\,1$.

If $\tilde{\Delta}(x) = x \otimes 1$, we can use the opposite comultiplication to get that x is a scalar. We refer to the proof of Proposition 5.15 for some more details. □

Although our definitions of left and right invariance were rather weak, it is possible to improve the invariance properties considerably. More precisely:

PROPOSITION 5.14. – *Consider* $x \in \tilde{A}^+$ *such that* $(\iota \bar{\otimes} \omega_{v,v})\tilde{\Delta}(x)$ *belongs to* $\mathcal{M}_{\tilde{\psi}}^{+}$ *for every* $v \in H$. *Then* x *belongs to* $\mathcal{M}_{\tilde{\psi}}^{+}$.

Proof. – Take $y \in \tilde{A}^+$ such that $(\iota \bar{\otimes} \omega_{v,v})\tilde{\Delta}(y) \in \mathcal{M}_{\tilde{\psi}}^{+}$ for all $v \in H$. Put $y_\omega = (\tilde{\omega} \bar{\otimes} \iota)\tilde{\Delta}(y)$ for $\omega \in \mathcal{G}_\psi$ ($\tilde{\omega}$ was defined in Proposition 3.9).

Let $v \in H$. We have for every $\omega \in \mathcal{G}_\psi$ that

$$\big\|y_\omega^{\frac{1}{2}}\,v\big\|^2 = \langle y_\omega\,v, v\rangle = \tilde{\omega}\big((\iota \bar{\otimes} \omega_{v,v})\tilde{\Delta}(y)\big) \leqslant \tilde{\psi}\big((\iota \bar{\otimes} \omega_{v,v})\tilde{\Delta}(y)\big).$$

So the net $(y_\omega^{\frac{1}{2}}\,v)_{\omega \in \mathcal{G}_\psi}$ is bounded. Therefore the uniform boundedness principle implies that the net $(y_\omega^{\frac{1}{2}})_{\omega \in \mathcal{G}_\psi}$ is bounded.

Hence we have a bounded increasing net $(y_\omega)_{\omega \in \mathcal{G}_\psi}$ in $\tilde{A}^+$. Denote its strong limit by $z \in \tilde{A}^+$.

Since $\langle y_\omega\,v, v\rangle = \tilde{\omega}((\iota \bar{\otimes} \omega_{v,v})\tilde{\Delta}(y))$ for $\omega \in \mathcal{G}_\psi$, Proposition 3.9 implies that $\langle z\,v, v\rangle = \tilde{\psi}((\iota \bar{\otimes} \omega_{v,v})\tilde{\Delta}(y))$ for all $v \in H$. Because any element in $\tilde{A}_*^+$ can be written as a norm convergent sum of vector functionals, the lower semi-continuity of $\tilde{\psi}$ now implies for every $\mu \in \tilde{A}_*^+$ that $\mu(z) = \tilde{\psi}((\iota \bar{\otimes} \mu)\tilde{\Delta}(y))$, so we get in particular that $(\iota \bar{\otimes} \mu)\tilde{\Delta}(y) \in \mathcal{M}_{\tilde{\psi}}^{+}$.

Take $\omega, \mu \in \tilde{A}_*^+$. Because $(\iota \bar{\otimes} \mu)\tilde{\Delta}(y) \in \mathcal{M}_{\tilde{\psi}}^+$, the right invariance of $\tilde{\psi}$ gives

$$(\omega \bar{\otimes} \mu)\tilde{\Delta}(z) = \tilde{\psi}\big((\iota \bar{\otimes} (\omega \bar{\otimes} \mu)\tilde{\Delta})\tilde{\Delta}(y)\big) = \tilde{\psi}\big((\iota \bar{\otimes} \omega)\tilde{\Delta}((\iota \bar{\otimes} \mu)\tilde{\Delta}(y))\big)$$
$$= \omega(1)\tilde{\psi}\big((\iota \bar{\otimes} \mu)\tilde{\Delta}(y)\big) = \omega(1)\mu(z).$$

Consequently, $\tilde{\Delta}(z) = 1 \otimes z$. Therefore Result 5.13 implies the existence of $\lambda \in \mathbb{R}^+$ such that $z = \lambda 1$ and hence

$$\tilde{\psi}\big((\iota \bar{\otimes} \omega_{v,v})\tilde{\Delta}(y)\big) = \langle z\, v, v\rangle = \lambda \|v\|^2$$

for all $v \in H$.

Now define the mapping $\theta : \tilde{A}^+ \to [0, +\infty]$ in the following way. Let $y \in \tilde{A}^+$.

- If $(\iota \bar{\otimes} \omega_{v,v})\tilde{\Delta}(y) \in \mathcal{M}_{\tilde{\psi}}^+$ for all $v \in H$, we define $\theta(y) \in \mathbb{R}^+$ such that $\tilde{\psi}((\iota \bar{\otimes} \omega_{v,v})\tilde{\Delta}(y)) = \|v\|^2\, \theta(y)$ for all $v \in H$.
- If there exists a $v \in H$ such that $(\iota \bar{\otimes} \omega_{v,v})\tilde{\Delta}(y) \notin \mathcal{M}_{\tilde{\psi}}^+$, we put $\theta(y) = +\infty$.

It is easy to see that θ is a weight. Since

$$\theta(y) = \sup\big\{\tilde{\psi}((\iota \bar{\otimes} \omega_{v,v})\tilde{\Delta}(y)) \mid v \in H, \|v\| = 1\big\} \tag{5.6}$$

for all $y \in \tilde{A}^+$, the weight θ is normal (as a supremum of σ-weakly lower semi-continuous functions).

Take $t \in \mathbb{R}$ and $y \in \tilde{A}^+$. Using Proposition 5.11, we have for all $v \in H$ that

$$\tilde{\psi}\big((\iota \bar{\otimes} \omega_{v,v})\tilde{\Delta}(\sigma_t^{\tilde{\psi}}(y))\big) = \tilde{\psi}\big(\sigma_t^{\tilde{\psi}}\big((\iota \bar{\otimes} \omega_{v,v}\,\tilde{\tau}_{-t})\tilde{\Delta}(y)\big)\big) = \tilde{\psi}\big((\iota \bar{\otimes} \omega_{N^{-it}v, N^{-it}v})\tilde{\Delta}(y)\big).$$

Therefore Eq. (5.6) implies that $\theta(\sigma_t^{\tilde{\psi}}(y)) = \theta(y)$.

Because $\tilde{\psi}$ is right invariant, the definition of θ implies immediately that θ is an extension of $\tilde{\psi}$. Combining this with the fact that θ is invariant under $\sigma^{\tilde{\psi}}$, the W*-version of Proposition 1.13 implies that $\theta = \tilde{\psi}$ and the result follows. □

The next result is true because of symmetry. For the sake of completeness, we will make this argument very precise.

PROPOSITION 5.15. – *Consider $x \in \tilde{A}^+$ such that $(\omega_{v,v} \bar{\otimes} \iota)\tilde{\Delta}(x)$ belongs to $\mathcal{M}_{\tilde{\varphi}}^+$ for every $v \in H$. Then x belongs to $\mathcal{M}_{\tilde{\varphi}}^+$.*

Proof. – By imitating the first paragraph of the proof of the previous proposition we get that $(\omega \bar{\otimes} \iota)\tilde{\Delta}(x) \in \mathcal{M}_{\tilde{\varphi}}^+$ for all $\omega \in \tilde{A}_*^+$.

Let Δ° denote the opposite comultiplication on A, i.e. $\Delta^\circ = \chi\,\Delta$. Then (A, Δ°) is of course still a reduced C*-algebraic quantum group but this time φ is right invariant and ψ is left invariant.

Let $\tilde{A}^\circ = \pi_\psi(A)''$ which is a von Neumann algebra on H_ψ. Then we can lift the opposite comultiplication to a normal *-homomorphism $\tilde{\Delta}^\circ : \tilde{A}^\circ \to \tilde{A}^\circ \bar{\otimes} \tilde{A}^\circ$ such that $\tilde{\Delta}^\circ \pi_\psi = (\pi_\psi \otimes \pi_\psi)\Delta^\circ$.

By Theorem 3.8, we know that there exists a *-isomorphism $\theta : \tilde{A}^\circ \to \tilde{A}$ such that $\theta(\pi_\psi(a)) = \pi(a)$ for all $a \in A$.

Call $\tilde{\varphi}^\circ$ the W*-lift of φ in the GNS-construction $(H_\psi, \pi_\psi, \Lambda_\psi)$. For every $\omega \in \mathcal{F}_\varphi$, let $\tilde{\omega}^\circ$ denote the element in $(\tilde{A}^\circ)_*^+$ such that $\tilde{\omega}^\circ \pi_\psi = \omega$. It is then clear that $\tilde{\omega}\,\theta = \tilde{\omega}^\circ$ So we get for every $y \in (\widetilde{A}^\circ)^+$ that

$$\widetilde{\varphi}(\theta(y)) = \sup\{\widetilde{\omega}(\theta(y)) \mid \omega \in \mathcal{F}_\varphi\} = \sup\{\widetilde{\omega}^\circ(y) \mid \omega \in \mathcal{F}_\varphi\} = \widetilde{\varphi}^\circ(y).$$

It is easy to see that $\chi(\theta \overline{\otimes} \theta)\widetilde{\Delta}^\circ = \widetilde{\Delta}\,\theta$. This implies that $(\iota \overline{\otimes} \omega)\widetilde{\Delta}^\circ(\theta^{-1}(x)) \in \mathcal{M}^{\pm}_{\widetilde{\varphi}^\circ}$ for all $\omega \in (\widetilde{A}^\circ)^*_+$. Applying the previous proposition to $\widetilde{\varphi}^\circ$ gives $\theta^{-1}(x) \in \mathcal{M}^{\pm}_{\widetilde{\varphi}^\circ}$, hence $x \in \mathcal{M}^{\pm}_{\widetilde{\varphi}}$. □

Now we can at last prove the desired result:

THEOREM 5.16. – *The weight ψ is a KMS weight on A.*

Proof. – Take $t \in \mathbb{R}$. Define $\kappa_t = \widetilde{\tau}_t\,\sigma_t^{\widetilde{\psi}}$, so κ_t is a *-automorphism on $\widetilde{A}$. Proposition 5.11 and Result 5.12 give

$$(\kappa_t \overline{\otimes} \iota)\widetilde{\Delta} = (\widetilde{\tau}_t\,\sigma_t^{\widetilde{\psi}} \overline{\otimes} \widetilde{\tau}_t\,\widetilde{\tau}_{-t})\widetilde{\Delta} = (\widetilde{\tau}_t \overline{\otimes} \widetilde{\tau}_t)\widetilde{\Delta}\,\sigma_t^{\widetilde{\psi}} = \widetilde{\Delta}\,\widetilde{\tau}_t\,\sigma_t^{\widetilde{\psi}} = \widetilde{\Delta}\,\kappa_t.$$

This allows us to use the previous result. Take $y \in \mathcal{M}^{\pm}_{\widetilde{\varphi}}$. Choose $\omega \in \widetilde{A}^+_*$. Then $(\omega \overline{\otimes} \iota)\widetilde{\Delta}(\kappa_t(y)) = (\omega\,\kappa_t \overline{\otimes} \iota)\widetilde{\Delta}(y)$. Therefore the left invariance of $\widetilde{\varphi}$ implies that $(\omega \overline{\otimes} \iota)\widetilde{\Delta}(\kappa_t(y))$ belongs to $\mathcal{M}^{\pm}_{\widetilde{\varphi}}$.

Now, the previous proposition guarantees that $\kappa_t(y)$ belongs to $\mathcal{M}^{\pm}_{\widetilde{\varphi}}$. Take $\omega \in \widetilde{A}^+_*$ such that $\omega(1) = 1$. The left invariance of $\widetilde{\varphi}$ implies that

$$\widetilde{\varphi}(\kappa_t(y)) = \widetilde{\varphi}\big((\omega \overline{\otimes} \iota)\widetilde{\Delta}(\kappa_t(y))\big) = \widetilde{\varphi}((\omega\,\kappa_t \overline{\otimes} \iota)\widetilde{\Delta}(y)) = (\omega\,\kappa_t)(1)\,\widetilde{\varphi}(y) = \widetilde{\varphi}(y).$$

The relation $(\kappa_t^{-1} \overline{\otimes} \iota)\widetilde{\Delta} = \widetilde{\Delta}\,\kappa_t^{-1}$ gives in a similar way that $\kappa_t^{-1}(\mathcal{M}^{\pm}_{\widetilde{\varphi}}) \subseteq \mathcal{M}^{\pm}_{\widetilde{\varphi}}$. So we have proven that $\widetilde{\varphi}$ is invariant under κ_t. Define the unitary operator $U_t \in \mathrm{B}(H)$ such that $U_t\widetilde{\Lambda}(a) = \widetilde{\Lambda}(\kappa_t(a))$ for all $a \in \mathcal{N}_{\widetilde{\varphi}}$.

Arguing as in Lemma 3.14 and as in the beginning of Lemma 5.7, one checks easily that

$$(\iota \otimes \omega_{\widetilde{\Lambda}(a\,\widetilde{\sigma}_{-i}(b^*)),\widetilde{\Lambda}(c)})(W) = (\iota \overline{\otimes} \omega_{\widetilde{\Lambda}(a),\widetilde{\Lambda}(b)})\widetilde{\Delta}(c^*)$$

for $c \in \mathcal{N}_{\widetilde{\varphi}}$ and $a, b \in \mathcal{T}_{\widetilde{\varphi}}$ (we introduced the Tomita *-algebra $\mathcal{T}_{\widetilde{\varphi}}$ at the end of Section 4).

This implies, for $c \in \mathcal{N}_{\widetilde{\varphi}}$ and $a, b \in \mathcal{T}_{\widetilde{\varphi}}$, that

$$\begin{aligned}
\kappa_t\big((\iota \otimes \omega_{\widetilde{\Lambda}(a\,\widetilde{\sigma}_{-i}(b^*)),\widetilde{\Lambda}(c)})(W)\big) &= \kappa_t\big((\iota \overline{\otimes} \omega_{\widetilde{\Lambda}(a),\widetilde{\Lambda}(b)})\widetilde{\Delta}(c^*)\big) = \big(\iota \overline{\otimes} \omega_{\widetilde{\Lambda}(a),\widetilde{\Lambda}(b)}\big)\widetilde{\Delta}(\kappa_t(c)^*) \\
&= \big(\iota \otimes \omega_{\widetilde{\Lambda}(a\,\widetilde{\sigma}_{-i}(b^*)),\widetilde{\Lambda}(\kappa_t(c))}\big)(W) \\
&= \big(\iota \otimes \omega_{\widetilde{\Lambda}(a\,\widetilde{\sigma}_{-i}(b^*)),U_t\widetilde{\Lambda}(c)}\big)(W).
\end{aligned}$$

So we get that

$$\kappa_t\big((\iota \otimes \omega)(W)\big) = (\iota \otimes \omega\,U_t^*)(W) \tag{5.7}$$

for all $\omega \in \mathrm{B}_0(H)^*$. Because $\pi(A) = [(\iota \otimes \omega)(W) \mid \omega \in \mathrm{B}_0(H)^*]$, this implies immediately that $\kappa_t(\pi(A)) = \pi(A)$ for all $t \in \mathbb{R}$.

Choose $c, d \in \mathcal{N}_{\widetilde{\varphi}}$ and $\omega \in \mathcal{G}_\varphi$. Then we have that

$$\langle U_t\,\widetilde{\Lambda}(c), T_\omega\widetilde{\Lambda}(d)\rangle = \langle \kappa_t(c)\,\xi_\omega, d\,\xi_\omega\rangle,$$

implying that the function $\mathbb{R} \to \mathbb{C} : t \mapsto \langle U_t \widetilde{\Lambda}(c), T_\omega \widetilde{\Lambda}(d) \rangle$ is continuous. Since $(T_\omega)_{\omega \in \mathcal{G}_\varphi}$ converges weakly to 1 and $\widetilde{\Lambda}(\mathcal{N}_{\widetilde{\varphi}})$ is dense in H, this implies that the function $\mathbb{R} \to \mathbb{C} : t \mapsto \langle U_t v, w \rangle$ is continuous for all $v, w \in H$.

So we see that the function $\mathbb{R} \to \mathrm{B}(H) : t \mapsto U_t$ is weakly continuous and therefore strongly* continuous (because both topologies agree on the set of unitaries). Combining this with Eq. (5.7), we get for every $x \in \pi(A)$ that the function $\mathbb{R} \to \pi(A) : t \mapsto \kappa_t(x)$ is norm continuous.

Since $\widetilde{\sigma}_t^\psi = \widetilde{\tau}_{-t}\, \kappa_t$ for all $t \in \mathbb{R}$ (and $\widetilde{\tau}$ satisfies similar properties as κ), we conclude that:

- $\widetilde{\sigma}_t^\psi(\pi(A)) = \pi(A)$ for $t \in \mathbb{R}$;
- we have for $x \in \pi(A)$ that the function $\mathbb{R} \to \pi(A)$, $t \mapsto \widetilde{\sigma}_t^\psi(x)$ is norm continuous.

Therefore the injectivity of π implies the existence of a norm continuous one-parameter group σ^ψ on A such that $\pi\, \sigma_t^\psi = \widetilde{\sigma}_t^\psi\, \pi$ for $t \in \mathbb{R}$. It is now easy to check that ψ is a KMS weight on A with modular group σ^ψ. □

For the rest of this section, we will stick to the notation σ^ψ for the modular group of ψ. Remember that $\pi\, \sigma_t^\psi = \widetilde{\sigma}_t^\psi\, \pi$ for $t \in \mathbb{R}$. Also notice that ∇_ψ is the modular operator of ψ in the GNS-construction $(H_\psi, \Lambda_\psi, \pi_\psi)$. Let us also use the following notation:

$$\mathcal{T}_\psi = \{ x \in \mathcal{N}_\psi \cap \mathcal{N}_\psi^* \mid x \text{ is analytic with respect to } \sigma^\psi \text{ and } \sigma_z^\psi(x) \in \mathcal{N}_\psi \cap \mathcal{N}_\psi^* \text{ for } z \in \mathbb{C} \}.$$

Proposition 5.11 implies immediately the following commutation relation.

PROPOSITION 5.17. – *We have for all $t \in \mathbb{R}$ that $(\sigma_t^\psi \otimes \tau_{-t})\Delta = \Delta\, \sigma_t^\psi$.*

Using the opposite comultiplication, we get the following proposition for free.

PROPOSITION 5.18. – *Consider a left invariant weight η on (A, Δ). If η is approximately KMS, then η is a KMS weight on A.*

Therefore we have in particular that φ is a KMS weight on A. For the rest of this paper, we will denote the modular group of φ by σ. Notice that $\pi\, \sigma_t = \widetilde{\sigma}_t\, \pi$ for $t \in \mathbb{R}$. We also have that:

- $J =$ modular conjugation of φ in the GNS-construction (H, π, Λ);
- $\nabla =$ modular operator of φ in the GNS-construction (H, π, Λ).

5.4. The antipode through its polar decomposition

Let us start off this subsection with an easy consequence of Result 5.10.

RESULT 5.19. – *For all $z \in \mathbb{C}$ and $a, b \in \mathcal{T}_\psi$ we have*

$$\pi\big((\psi \otimes \iota)((b^* \otimes 1)\Delta(a))\big) N^z \subseteq N^z \pi\big((\psi \otimes \iota)((\sigma_{iz}^\psi(b^*) \otimes 1)\Delta(\sigma_{iz}^\psi(a)))\big).$$

Proof. – From Result 5.10, we know that $U(\nabla_\psi \otimes N) = (\nabla_\psi \otimes N)U$. So we get for all $t \in \mathbb{R}$ and $a, b \in \mathcal{N}_\psi$

$$\begin{aligned} N^{-it} \pi\big((\psi \otimes \iota)((b^* \otimes 1)\Delta(a))\big) N^{it} &= N^{-it} \big(\omega_{\Lambda_\psi(a), \Lambda_\psi(b)} \otimes \iota\big)(U) N^{it} \\ &= \big(\nabla_\psi^{-it} \omega_{\Lambda_\psi(a), \Lambda_\psi(b)} \nabla_\psi^{it} \otimes \iota\big)(U) \\ &= \big(\omega_{\nabla_\psi^{-it}\Lambda_\psi(a), \nabla_\psi^{-it}\Lambda_\psi(b)} \otimes \iota\big)(U). \end{aligned}$$

The analytic function

$$\mathbb{C} \to \mathrm{B}(H), \quad z \mapsto \big(\omega_{\nabla_\psi^{-z}\Lambda_\psi(a),\nabla_\psi^{\bar z}\Lambda_\psi(b)} \otimes \iota\big)(U) = \pi\big((\psi \otimes \iota)((\sigma^\psi_{iz}(b^*) \otimes 1)\Delta(\sigma^\psi_{iz}(a)))\big)$$

attains in it the value

$$N^{-it}\pi\big((\psi \otimes \iota)((b^* \otimes 1)\Delta(a))\big)N^{it},$$

so that our conclusion follows from Proposition 9.24 of [47]. □

We can also define the unitary antipode now.

PROPOSITION 5.20. – (1) *There exists a unique *-antiautomorphism $\widetilde{R}$ of $\widetilde{A}$ such that $\widetilde{R}(x) = I x^* I$ for all $x \in \widetilde{A}$.*

(2) *There exists a unique *-antiautomorphism R of A such that $\pi R = \widetilde{R}\pi$. Moreover we have*

$$R\big((\psi \otimes \iota)((a^* \otimes 1)\Delta(b))\big) = (\psi \otimes \iota)\big(\Delta(\sigma^\psi_{-\frac{i}{2}}(a^*))(\sigma^\psi_{-\frac{i}{2}}(b) \otimes 1)\big)$$

for all $a, b \in \mathcal{T}_\psi$.

Proof. – Choose $a, b \in \mathcal{T}_\psi$. Then Proposition 5.19 implies that

$$\pi\big((\psi \otimes \iota)((b^* \otimes 1)\Delta(a))\big)N^{\frac{1}{2}} \subseteq N^{\frac{1}{2}}\pi\big((\psi \otimes \iota)((\sigma^\psi_{\frac{i}{2}}(b^*) \otimes 1)\Delta(\sigma^\psi_{\frac{i}{2}}(a)))\big).$$

Remembering that $G^* = N^{\frac{1}{2}} I$, inclusion (5.1) implies that

$$\pi\big((\psi \otimes \iota)((b^* \otimes 1)\Delta(a))\big)N^{\frac{1}{2}} \subseteq N^{\frac{1}{2}} I\pi\big((\psi \otimes \iota)((a^* \otimes 1)\Delta(b))\big)I.$$

Since $N^{\frac{1}{2}}$ is densely defined and injective, these two inclusions imply that

$$I\pi\big((\psi \otimes \iota)((a^* \otimes 1)\Delta(b))\big)I = \pi\big((\psi \otimes \iota)((\sigma^\psi_{\frac{i}{2}}(b^*) \otimes 1)\Delta(\sigma^\psi_{\frac{i}{2}}(a)))\big).$$

Hence,

$$I\pi\big((\psi \otimes \iota)((a^* \otimes 1)\Delta(b))\big)^* I = \pi\big((\psi \otimes \iota)(\Delta(\sigma^\psi_{-\frac{i}{2}}(a^*))(\sigma^\psi_{-\frac{i}{2}}(b) \otimes 1))\big).$$

Thus we have $I\pi(A)I = \pi(A)$. The rest of the proof is obvious now. □

Since $I^2 = 1$, we get immediately that $R^2 = \iota$ and $\widetilde{R}^2 = \iota$.

Combining the previous proposition with Proposition 5.11, we can now define the antipode of our reduced C*-algebraic quantum group.

DEFINITION 5.21. – We define

$$S = R\tau_{-\frac{i}{2}} \quad \text{and} \quad \widetilde{S} = \widetilde{R}\widetilde{\tau}_{-\frac{i}{2}}.$$

Let us collect the most basic properties of S in the following proposition. They follow easily from the corresponding results for $\tau_{-\frac{i}{2}}$.

Notice that $\tau_t R = R\tau_t$ for all $t \in \mathbb{R}$ since $INI = N^{-1}$.

PROPOSITION 5.22. – *The linear map S has the following properties*:

(1) *S is densely defined and has dense range*;

(2) *S is injective and $S^{-1} = R\tau_{\frac{i}{2}} = \tau_{\frac{i}{2}}R$*;

(3) *S is antimultiplicative*: *we have for all $x, y \in D(S)$ that $xy \in D(S)$ and $S(xy) = S(y)S(x)$*;

(4) *we have for all* $x \in D(S)$ *that* $S(x)^* \in D(S)$ *and* $S(S(x)^*)^* = x$;
(5) $S^2 = \tau_{-i}$.

RESULT 5.23. – *We have the following commutation relations*:
- $RS = SR$;
- $\tau_t S = S\tau_t$ *for* $t \in \mathbb{R}$.

We want to stress the following fact. The definitions of S, τ and R (and their von Neumann algebraic counterparts) depend on the choice of φ and ψ and we should keep this in mind. But we will show later on that S, τ and R actually do not depend on the choice of φ and ψ.

In the next proposition we will prove a formula that one could call the strong right invariance of the weight ψ. The strong left invariance of the weight φ will be tackled in the next subsection.

PROPOSITION 5.24. – *For all* $a, b \in \mathcal{N}_\psi$, *we have* $(\psi \otimes \iota)((a^* \otimes 1)\Delta(b)) \in D(S)$ *and*

$$S\big((\psi \otimes \iota)((a^* \otimes 1)\Delta(b))\big) = (\psi \otimes \iota)\big(\Delta(a^*)(b \otimes 1)\big).$$

We have moreover that $\langle (\psi \otimes \iota)((a^* \otimes 1)\Delta(b)) \mid a, b \in \mathcal{N}_\psi \rangle$ *is a core for* S.

Proof. – Choose $a, b \in \mathcal{N}_\psi$. Remembering that $G^* = N^{\frac{1}{2}} I$, inclusion (5.1) implies that

$$\pi\big((\psi \otimes \iota)((a^* \otimes 1)\Delta(b))\big) N^{\frac{1}{2}} \subseteq N^{\frac{1}{2}} I \pi\big((\psi \otimes \iota)((b^* \otimes 1)\Delta(a))\big) I.$$

Because $\tilde{\tau}_t$ is implemented by N^{-it}, this implies that $\pi((\psi \otimes \iota)((a^* \otimes 1)\Delta(b)))$ belongs to $D(\tilde{\tau}_{-\frac{i}{2}})$ and

$$\begin{aligned}\tilde{\tau}_{-\frac{i}{2}}\big(\pi\big((\psi \otimes \iota)((a^* \otimes 1)\Delta(b))\big)\big) &= I\pi\big((\psi \otimes \iota)((b^* \otimes 1)\Delta(a))\big)I \\ &= \pi\big(R\big((\psi \otimes \iota)(\Delta(a^*)(b \otimes 1))\big)\big).\end{aligned}$$

Because this last element belongs to $\pi(A)$, we get (*see* e.g. Proposition 1.24 of [25]) that

$$(\psi \otimes \iota)\big((a^* \otimes 1)\Delta(b)\big) \in D(\tau_{-\frac{i}{2}})$$

and

$$\tau_{-\frac{i}{2}}\big((\psi \otimes \iota)((a^* \otimes 1)\Delta(b))\big) = R\big((\psi \otimes \iota)(\Delta(a^*)(b \otimes 1))\big).$$

Since $S = R\tau_{-\frac{i}{2}}$, the first statement of the proposition follows.

The second one follows by observing that the stated subspace of A is dense, invariant under τ_t by the commutation rule $(\sigma_t^\psi \otimes \tau_{-t})\Delta = \Delta\sigma_t^\psi$, and is a subspace of $D(\tau_{-\frac{i}{2}})$ (*see* e.g. Corollary 1.22 of [25]). □

We will finish this section by proving yet another familiar formula. We first need a lemma, which formally implies the well known algebraic formula $\chi(S \otimes S)\Delta = \Delta S$.

LEMMA 5.25. – *Consider* $\omega, \theta \in A^*$ *such that* ωS *and* θS *are bounded and let* $\overline{\omega S}$ *and* $\overline{\theta S}$ *denote their respective continuous extensions to* A. *Then we have for all* $x \in D(S)$ *that* $(\theta \otimes \omega)\Delta(S(x)) = (\overline{\omega S} \otimes \overline{\theta S})\Delta(x)$.

Proof. – Choose $a, b \in \mathcal{N}_\psi$ and put $y = (\psi \otimes \iota)((a^* \otimes 1)\Delta(b))$.

Then we have that

$$\begin{aligned}(\overline{\omega S} \otimes \overline{\theta S})\Delta(y) &= (\overline{\omega S} \otimes \overline{\theta S})\Delta\big((\psi \otimes \iota)((a^* \otimes 1)\Delta(b))\big) \\ &= \overline{\omega S}\big((\iota \otimes \overline{\theta S})((\psi \otimes \iota \otimes \iota)((a^* \otimes 1 \otimes 1)\Delta^{(2)}(b)))\big) \\ &= \overline{\omega S}\big((\psi \otimes \iota)((a^* \otimes 1)\Delta((\iota \otimes \overline{\theta S})\Delta(b)))\big).\end{aligned}$$

The last equality is justified by Proposition 1.22. Bring first $\overline{\omega S}$ in together with $\overline{\theta S}$. Then pull $\overline{\omega S}$ back out.

By Proposition 5.24, we get that $(\psi\otimes\iota)((a^*\otimes 1)\Delta((\iota\otimes\overline{\theta S})\Delta(b)))$ belongs to $D(S)$ and

$$S\big((\psi\otimes\iota)((a^*\otimes 1)\Delta((\iota\otimes\overline{\theta S})\Delta(b)))\big)=(\psi\otimes\iota)\big(\Delta(a^*)((\iota\otimes\overline{\theta S})(\Delta(b))\otimes 1)\big).$$

Therefore,

$$\begin{aligned}(\overline{\omega S}\otimes\overline{\theta S})\Delta(y)&=\omega\big((\psi\otimes\iota)(\Delta(a^*)((\iota\otimes\overline{\theta S})(\Delta(b))\otimes 1))\big)\\&=\psi\big((\iota\otimes\omega)(\Delta(a^*))(\iota\otimes\overline{\theta S})(\Delta(b))\big)\\&=\overline{\theta S}\big((\psi\otimes\iota)(((\iota\otimes\omega)(\Delta(a^*))\otimes 1)\Delta(b))\big)\\&=\theta\big((\psi\otimes\iota)(\Delta((\iota\otimes\omega)\Delta(a^*))(b\otimes 1))\big),\end{aligned}$$

where we used Proposition 5.24 once again.

On the other hand,

$$\begin{aligned}(\theta\otimes\omega)\Delta(S(y))&=(\theta\otimes\omega)\Delta\big(S((\psi\otimes\iota)((a^*\otimes 1)\Delta(b)))\big)\\&=(\theta\otimes\omega)\Delta\big((\psi\otimes\iota)(\Delta(a^*)(b\otimes 1))\big)\\&=(\theta\otimes\omega)\big((\psi\otimes\iota\otimes\iota)(\Delta^{(2)}(a^*)(b\otimes 1\otimes 1))\big)\\&=\theta\big((\psi\otimes\iota)(\Delta((\iota\otimes\omega)\Delta(a^*))(b\otimes 1))\big).\end{aligned}$$

So we see that $(\overline{\omega S}\otimes\overline{\theta S})\Delta(y)=(\theta\otimes\omega)\Delta(S(y))$.

Because such elements y form a core for S, the lemma follows. □

PROPOSITION 5.26. – *We have that* $\chi(R\otimes R)\Delta=\Delta R$.

Proof. – Let $\omega,\mu\in A^*$ be analytic with respect to τ. By this we mean that the function $\mathbb{R}\to A^*:t\mapsto\omega\tau_t$ has an extension to an analytic function from $\mathbb{C}$ to A^*. From this, it follows that $\omega\tau_z$ is bounded for every $z\in\mathbb{C}$ and that the function $\mathbb{C}\to A^*:z\mapsto\overline{\omega\tau_z}$ is analytic.
Then we clearly have that ωRS is bounded with closure $\overline{\omega RS}=\overline{\omega\tau_{-\frac{i}{2}}}$. The same is valid for μ.

Now the function $\mathbb{C}\to(A\otimes A)^*:z\mapsto\overline{\mu\tau_z}\otimes\overline{\omega\tau_z}$ is analytic as a function from $\mathbb{C}$ to $(A\otimes A)^*$. Because the embedding of $(A\otimes A)^*$ in $\mathrm{M}(A\otimes A)^*$ is isometric, this function is analytic, when we consider it as a function from $\mathbb{C}$ to $\mathrm{M}(A\otimes A)^*$.

Take $a\in A$ such that a is analytic with respect to τ. Then we have that $\mathbb{C}\to\mathbb{C}$: $z\mapsto(\overline{\mu\tau_z}\otimes\overline{\omega\tau_z})\Delta(a)$ and $\mathbb{C}\to\mathbb{C}:z\mapsto(\mu\otimes\omega)\Delta(\tau_z(a))$ are both analytic and coincide on $\mathbb{R}$ because of Result 5.12. So they are equal.

Then we have by the previous lemma

$$\begin{aligned}(\omega\otimes\mu)\big((R\otimes R)\Delta(S(a))\big)&=(\overline{\mu RS}\otimes\overline{\omega RS})\Delta(a)=(\overline{\mu\tau_{-\frac{i}{2}}}\otimes\overline{\omega\tau_{-\frac{i}{2}}})\Delta(a)\\&=(\mu\otimes\omega)\Delta(\tau_{-\frac{i}{2}}(a))=(\mu\otimes\omega)\Delta\big(R(S(a))\big).\end{aligned}$$

Because S has a dense range we get

$$(\omega\otimes\mu)\big((R\otimes R)\Delta(x)\big)=(\mu\otimes\omega)\Delta(R(x))\tag{5.8}$$

for all $x\in A$.

If now $\rho\in A^*$ and $n\in\mathbb{N}$, we define $\rho(n)\in A^*$ such that

$$\rho(n)(a)=\frac{n}{\sqrt{\pi}}\int\exp(-n^2t^2)\rho(\tau_t(a))\,\mathrm{d}t$$

for all $a \in A$. It is clear that $\rho(n)$ is weak* analytic with respect to τ, and the standard application of the uniform boundedness theorem gives that $\rho(n)$ is analytic with respect to τ.

Now, take $\rho_1, \rho_2 \in A^*$. By Eq. (5.8), we have for all $n \in \mathbb{N}$ and $x \in A$ that

$$\big(\rho_1(n) \otimes \rho_2(n)\big)\big(\chi(R \otimes R)\Delta(x)\big) = \big(\rho_1(n) \otimes \rho_2(n)\big)\Delta(x).$$

We have, for all $y \in \mathrm{M}(A \otimes A)$,

$$\big(\rho_1(n) \otimes \rho_2(n)\big)(y) \to (\rho_1 \otimes \rho_2)(y) \quad \text{as } n \to \infty.$$

So we conclude that $(\rho_1 \otimes \rho_2)(\chi(R \otimes R)\Delta(x)) = (\rho_1 \otimes \rho_2)\Delta(x)$ for all $x \in A$. □

5.5. Strong left invariance

We will prove the strong left invariance of the weight φ (in fact, of every left invariant proper weight). Before we can do so, we need some technical results.

First we need some extra terminology.

DEFINITION 5.27. – Consider a C*-algebra B and an index set I. Then we define the following sets:

1) $\mathrm{MC}_I(B) = \{x \text{ an } I\text{-tuple in } \mathrm{M}(B) \mid (x_i^* x_i)_{i \in I} \text{ is strictly summable in } \mathrm{M}(B)\}$.
2) $\mathrm{MR}_I(B) = \{x \text{ an } I\text{-tuple in } \mathrm{M}(B) \mid (x_i x_i^*)_{i \in I} \text{ is strictly summable in } \mathrm{M}(B)\}$.

Elements of $\mathrm{MC}_I(B)$ can be thought of as infinite columns, elements of $\mathrm{MR}_I(B)$ as infinite rows.

Notice that the *-operation gives you a bijection between $\mathrm{MC}_I(B)$ and $\mathrm{MR}_I(B)$.

LEMMA 5.28. – *Consider a* C*-*algebra* B *and an index set* I. *Let* $x, y \in \mathrm{MC}_I(B)$. *Then* $(x_i^* y_i)_{i \in I}$ *is strictly summable and the net* $(\sum_{i \in J} x_i^* y_i)_{J \in F(I)}$ *is bounded.*

Proof. – Define $c, d \in \mathrm{M}(B)^+$ such that $c = \sum_{i \in I} x_i^* x_i$ and $d = \sum_{i \in I} y_i^* y_i$.

Choose $b \in A$. Take a finite subset J of I.

Look at the Hilbert C*-module E over $\mathrm{M}(A)$ given by $E = \bigoplus_{i \in J} \mathrm{M}(A)$ and such that the inner product is defined in such a way that $\langle v, w \rangle = \sum_{i \in J} w_i^* v_i$ for $v, w \in E$.

Then we have by the Cauchy–Schwarz inequality for Hilbert C*-modules that

$$\begin{aligned}
\Big(\sum_{i \in J} x_i^* y_i b\Big)^* \Big(\sum_{i \in J} x_i^* y_i b\Big) &= \big\langle (y_i b)_{i \in J}, (x_i)_{i \in J}\big\rangle^* \big\langle (y_i b)_{i \in J}, (x_i)_{i \in J}\big\rangle \\
&\leqslant \big\|\big\langle (x_i)_{i \in J}, (x_i)_{i \in J}\big\rangle\big\| \big\langle (y_i b)_{i \in J}, (y_i b)_{i \in J}\big\rangle \\
&= \Big\|\sum_{i \in J} x_i^* x_i\Big\| \sum_{i \in J} b^* y_i^* y_i b \leqslant \|c\| \sum_{i \in J} b^* y_i^* y_i b,
\end{aligned}$$

implying that

$$\Big\|\sum_{i \in J} x_i^* y_i b\Big\|^2 \leqslant \|c\| \Big\|\sum_{i \in J} b^* y_i^* y_i b\Big\| \leqslant \|c\| \, \|b^* d b\|. \tag{5.9}$$

So we get for finite subsets J, K of I such that $J \subseteq K$ that

$$\Big\|\sum_{i \in K} x_i^* y_i b - \sum_{i \in J} x_i^* y_i b\Big\|^2 \leqslant \|c\| \Big\|\sum_{i \in K} b^* y_i^* y_i b - \sum_{i \in J} b^* y_i^* y_i b\Big\|.$$

By assumption, we have that $(\sum_{i\in J} b^* y_i^* y_i b)_{J\in F(I)}$ is convergent so the above inequality implies that the net $\big(\sum_{i\in J} x_i^* y_i b\big)_{J\in F(I)}$ is Cauchy and hence convergent in B.

We have then of course also that the net $(\sum_{i\in J} y_i^* x_i b^*)_{J\in F(I)}$ is convergent in B. So by using the *-operation, we see that $(\sum_{i\in J} b\, x_i^* y_i)_{J\in F(I)}$ is convergent in B. Notice that the boundedness statement follows from inequality (5.9). □

This lemma implies easily the following algebraic properties.

RESULT 5.29. – *Consider a* C^**-algebra* B *and an index set* I*. Then* $\mathrm{MR}_I(B)$ *and* $\mathrm{MC}_I(B)$ *are vector spaces for the componentwise addition and scalar multiplication.*

Restating Lemma 5.28 in terms of rows and columns, we get the following result:

RESULT 5.30. – *Consider a* C^**-algebra* B *and an index set* I*. Let* $x\in \mathrm{MR}_I(B)$ *and* $y\in \mathrm{MC}_I(B)$*. Then* $(x_i\, y_i)_{i\in I}$ *is strictly summable and the net* $\big(\sum_{i\in J} x_i\, y_i\big)_{J\in F(I)}$ *is bounded.*

We will also need the following simple result.

RESULT 5.31. – *Consider* C^**-algebras* B *and* C*, a non-degenerate* *-*homomorphism* θ *from* B *into* $\mathrm{M}(C)$ *and an index set* I*.*

(1) *Let* $x\in \mathrm{MR}_I(B)$*. Then* $(\theta(x_i))_{i\in I}$ *belongs to* $\mathrm{MR}_I(C)$*.*

(2) *Let* $y\in \mathrm{MC}_I(B)$*. Then* $(\theta(y_i))_{i\in I}$ *belongs to* $\mathrm{MC}_I(C)$*.*

Remark 5.32. – Take a bounded net $(u_j)_{j\in J}$ in A such that $(u_j)_{j\in J}$ converges strictly to 1. For $j\in J$, we define the element $e_j\in A$ by $e_j=\frac{1}{\sqrt{\pi}}\int \exp(-t^2)\tau_t(e_j)\,\mathrm{d}t$. Then $(e_j)_{j\in J}$ is a net in $D(\tau_{-\frac{i}{2}})$ such that:

- $(e_j)_{j\in J}$ is bounded and converges strictly to 1;
- $(\tau_{-\frac{i}{2}}(e_j))_{j\in J}$ is bounded and converges strictly to 1.

So $(e_j)_{j\in J}$ is a net in $D(S)$ such that $(S(e_j))_{j\in J}$ converges strictly to 1. Due to the closedness of S, this implies easily the following property.

Consider $a,b\in A$ such that we have for every $x\in D(S)$ that $ax\in D(S)$ and $S(ax)=S(x)b$. Then a belongs to $D(S)$ and $S(a)=b$.

Also notice that this, together with the closedness of S, implies that S is closed with respect to the strict topology on A.

We now get the following interesting result. The original idea of looking at the elements in A of the form in the next proposition is due to A. Van Daele and was further developed by A. Van Daele and the second author in [54].

PROPOSITION 5.33. – *Consider* $a,b\in A$ *such that there exist an index set* I*,* $p\in \mathrm{MR}_I(A)$ *and* $q\in \mathrm{MC}_I(A)$ *such that*

$$1\otimes a=\sum_{i\in I}(p_i\otimes 1)\Delta(q_i)\quad \textit{and}\quad 1\otimes b=\sum_{i\in I}\Delta(p_i)(q_i\otimes 1).$$

Then $a\in D(S)$ *and* $S(a)=b$*.*

Notice that by results 5.30 & 5.31, the families $((p_i\otimes 1)\Delta(q_i))_{i\in I}$ and $(\Delta(p_i)(q_i\otimes 1))_{i\in I}$ are strictly summable.

Proof. – Choose $c,d\in \mathcal{N}_\psi$. Because of Remark 5.4, we have that the net

$$\Big(\sum_{i\in J}(\psi\otimes\iota)\big((c^*p_i\otimes 1)\Delta(q_i d)\big)\Big)_{J\in F(I)}$$

converges strictly to the element $a(\psi\otimes\iota)((c^*\otimes 1)\Delta(d))\in A$. By Proposition 5.24, all the elements of the net belong to $D(S)$ and

$$S\Big(\sum_{i\in J}(\psi\otimes\iota)\big((c^*p_i\otimes 1)\Delta(q_i d)\big)\Big)=\sum_{i\in J}(\psi\otimes\iota)\big(\Delta(c^*p_i)(q_i d\otimes 1)\big)$$

for $J\in F(I)$. This net converges strictly to $(\psi\otimes\iota)(\Delta(c^*)(d\otimes 1))\,b\in A$. Because S is closed with respect to the strict topology on A we get

$$a(\psi\otimes\iota)\big((c^*\otimes 1)\Delta(d)\big)\in D(S)$$

and

$$S\big(a(\psi\otimes\iota)((c^*\otimes 1)\Delta(d))\big)=(\psi\otimes\iota)\big(\Delta(c^*)(d\otimes 1)\big)b.$$

Because of Proposition 5.24 and the closedness of S we get for all $x\in D(S)$ that $ax\in D(S)$ and $S(ax)=S(x)b$. The previous remark gives our proposition. □

COROLLARY 5.34. – *Consider $a,b\in A$ such that there exist an index set I, $p\in \mathrm{MR}_I(A)$ and $q\in \mathrm{MC}_I(A)$ such that*

$$a\otimes 1=\sum_{i\in I}\Delta(p_i)(1\otimes q_i)\quad\textit{and}\quad b\otimes 1=\sum_{i\in I}(1\otimes p_i)\Delta(q_i).$$

Then $a\in D(S)$ and $S(a)=b$.

Proof. – Apply $\chi(R\otimes R)$ to both of the given equations and use Proposition 5.26. Then the observation that $(R(p_i))_{i\in I}\in \mathrm{MC}_I(A)$ and $(R(q_i))_{i\in I}\in \mathrm{MR}_I(A)$ (remember that R is antimultiplicative) and the previous proposition imply that $R(a)\in D(S)$ and $S(R(a))=R(b)$. Since $SR=RS$, this gives that $a\in D(S)$ and $S(a)=b$. □

We will formulate the next corollary in a more general context which will be needed later on. In particular, this gives us the strong left invariance of φ with respect to S.

COROLLARY 5.35. – *Consider a left invariant proper weight η on (A,Δ). Then we have for all $a,b\in\mathcal{N}_\eta$ that*

$$(\iota\otimes\eta)\big(\Delta(a^*)(1\otimes b)\big)\in D(S)\quad\textit{and}\quad S\big((\iota\otimes\eta)(\Delta(a^*)(1\otimes b))\big)=(\iota\otimes\eta)\big((1\otimes a^*)\Delta(b)\big).$$

Proof. – Let $(H_\eta,\pi_\eta,\Lambda_\eta)$ be a GNS-construction for η. Take an orthonormal basis $(e_i)_{i\in I}$ for H_η.

Take $i\in I$ and define the operator $\theta_i\in \mathrm{B}(H_\eta,\mathbb{C})$ given by $\theta_i(v)=\langle v,e_i\rangle$ for $v\in H_\eta$. Now put

$$p_i=\big[(1\otimes\theta_i)\big(\iota\otimes\Lambda_\eta\big)\big(\Delta(a)\big)\big]^*\quad\text{and}\quad q_i=(1\otimes\theta_i)\big(\iota\otimes\Lambda_\eta\big)\big(\Delta(b)\big).$$

Then we have

$$\sum_{i\in J}p_ip_i^*=\sum_{i\in J}(\iota\otimes\Lambda_\eta)(\Delta(a))^*(1\otimes\theta_i^*\theta_i)(\iota\otimes\Lambda_\eta)(\Delta(a))$$

for all $J\in F(I)$ and this net converges strictly to

$$(\iota\otimes\Lambda_\eta)(\Delta(a))^*(\iota\otimes\Lambda_\eta)(\Delta(a)).$$

So $(p_i)_{i\in I} \in \mathrm{MR}_I(A)$. Analogously $(q_i)_{i\in I} \in \mathrm{MC}_I(A)$. With the same kind of argument as in the proof of Proposition 5.5, we get

$$\sum_{i\in I} \Delta(p_i)(1\otimes q_i) = (\iota\otimes\iota\otimes\Lambda_\eta)(\Delta^{(2)}(a))^*(\iota\otimes\iota\otimes\Lambda_\eta)(\Delta_{23}(b))$$
$$= (\iota\otimes\iota\otimes\eta)\big(\Delta^{(2)}(a^*)\Delta_{23}(b)\big) = (\iota\otimes\eta)\big(\Delta(a^*)(1\otimes b)\big)\otimes 1$$

and

$$\sum_{i\in I}(1\otimes p_i)\Delta(q_i) = (\iota\otimes\eta)\big((1\otimes a^*)\Delta(b)\big)\otimes 1$$

analogously. Now the result follows from the previous corollary. □

Again, we will use Notation 2.8 and Theorem 3.16 in the formulation of the next lemma.

LEMMA 5.36. – *Consider a left invariant proper weight η on (A,Δ) and let $(H_\eta,\pi_\eta,\Lambda_\eta)$ be a GNS-construction for η. Define the unitary element $V\in B(H\otimes H_\eta)$ such that $V(\Lambda\otimes\Lambda_\eta)(\Delta(b)(a\otimes 1)) = \Lambda(a)\otimes\Lambda_\eta(b)$ for $a\in\mathcal{N}_\varphi$ and $b\in\mathcal{N}_\eta$. Then we have for $v\in D(N^{-\frac{1}{2}})$ and $w\in D(N^{\frac{1}{2}})$ that*

$$(\omega_{v,w}\otimes\iota)(V)^* = \big(\omega_{IN^{-\frac{1}{2}}v,IN^{\frac{1}{2}}w}\otimes\iota\big)(V).$$

Proof. – Choose $a,b\in\mathcal{N}_\eta$. We have that $(\iota\otimes\omega_{\Lambda_\eta(b),\Lambda_\eta(a)})(V) = \pi((\iota\otimes\eta)(\Delta(a^*)(1\otimes b)))$. By Corollary 5.35, we know that $(\iota\otimes\eta)(\Delta(a^*)(1\otimes b))$ belongs to $D(S) = D(\tau_{-\frac{i}{2}})$. Since $\tilde{\tau}_t$ is implemented by N^{-it}, we get that

$$\big(\iota\otimes\omega_{\Lambda_\eta(b),\Lambda_\eta(a)}\big)(V)N^{\frac{1}{2}} \subseteq N^{\frac{1}{2}}\pi\big(\tau_{-\frac{i}{2}}((\iota\otimes\eta)(\Delta(a^*)(1\otimes b)))\big)$$
$$= N^{\frac{1}{2}}\pi\big(R\big(S((\iota\otimes\eta)(\Delta(a^*)(1\otimes b)))\big)\big)$$
$$= N^{\frac{1}{2}}I\pi\big((\iota\otimes\eta)((1\otimes a^*)\Delta(b))\big)^*I$$
$$= N^{\frac{1}{2}}I\pi\big((\iota\otimes\eta)(\Delta(b^*)(1\otimes a))\big)I$$
$$= N^{\frac{1}{2}}I(\iota\otimes\omega_{\Lambda_\eta(a),\Lambda_\eta(b)})(V)I.$$

Then we get for $v\in D(N^{-\frac{1}{2}})$ and $w\in D(N^{\frac{1}{2}})$ and all $a,b\in\mathcal{N}_\eta$

$$\big\langle\big(\omega_{IN^{-\frac{1}{2}}v,IN^{\frac{1}{2}}w}\otimes\iota\big)(V)\Lambda_\eta(b),\Lambda_\eta(a)\big\rangle = \big\langle(\iota\otimes\omega_{\Lambda_\eta(b),\Lambda_\eta(a)})(V)N^{\frac{1}{2}}Iv, N^{-\frac{1}{2}}Iw\big\rangle$$
$$= \big\langle N^{\frac{1}{2}}I\big(\iota\otimes\omega_{\Lambda_\eta(a),\Lambda_\eta(b)}\big)(V)v, N^{-\frac{1}{2}}Iw\big\rangle$$
$$= \big\langle w, \big(\iota\otimes\omega_{\Lambda_\eta(a),\Lambda_\eta(b)}\big)(V)v\big\rangle$$
$$= \big\langle(\omega_{w,v}\otimes\iota)(V^*)\Lambda_\eta(b),\Lambda_\eta(a)\big\rangle.$$

Because this is valid for all a, $b\in\mathcal{N}_\eta$, we get our result. □

In the next proposition, we will work with a left invariant KMS weight on A. In the first step of the proof of the uniqueness of a left invariant weight, we will show that every proper left invariant weight is automatically KMS. The basic idea of the proof of the next proposition comes from Lemma 2.5.5 in [15]. Because τ can be non-trivial, we have to be a little bit more careful and work with the unbounded operator N.

PROPOSITION 5.37. – *Consider a left invariant KMS weight η on (A,Δ) with modular group κ. Let $(H_\eta,\pi_\eta,\Lambda_\eta)$ be a GNS-construction for η and define the unitary element $V\in B(H\otimes H_\eta)$ such that $V(\Lambda\otimes\Lambda_\eta)(\Delta(b)(a\otimes 1))=\Lambda(a)\otimes\Lambda_\eta(b)$ for $a\in\mathcal{N}_\varphi$ and $b\in\mathcal{N}_\eta$. Let:*

- *$\mathcal{J}=$ the modular conjugation of η in the GNS-construction $(H_\eta,\pi_\eta,\Lambda_\eta)$;*
- *$M=$ the modular operator of η in $(H_\eta,\pi_\eta,\Lambda_\eta)$.*

Then:

(1) $(I\otimes\mathcal{J})V=V^*(I\otimes\mathcal{J})$;
(2) $(N^{-1}\otimes M)V=V(N^{-1}\otimes M)$;
(3) $(\tau_t\otimes\kappa_t)\Delta=\Delta\kappa_t$ *for* $t\in\mathbb{R}$.

Proof. – Define X to be the closed operator in H_η such that $\Lambda_\eta(\mathcal{N}_\eta\cap\mathcal{N}_\eta^*)$ is a core for X and $X\Lambda_\eta(x)=\Lambda_\eta(x^*)$ for every $x\in\mathcal{N}_\eta\cap\mathcal{N}_\eta^*$. By definition, $M=X^*X$ and $X=\mathcal{J}M^{\frac{1}{2}}$. We have moreover that $\pi_\eta(\kappa_t(x))=M^{it}\pi_\eta(x)M^{-it}$ for $t\in\mathbb{R}$ and $x\in A$.

Choose $v,w\in H$.

Take $y\in\mathcal{N}_\eta\cap\mathcal{N}_\eta^*$. By the left invariance of η, we have that $(\omega_{v,w}\otimes\iota)\Delta(y)\in\mathcal{N}_\eta\cap\mathcal{N}_\eta^*$.

Because $(\omega_{v,w}\otimes\iota)(V^*)\Lambda_\eta(y)=\Lambda_\eta((\omega_{v,w}\otimes\iota)\Delta(y))$ (*see* Result 2.10), we get that $(\omega_{v,w}\otimes\iota)(V^*)\Lambda_\eta(y)$ belongs to $D(X)$ and

$$\begin{aligned}X(\omega_{v,w}\otimes\iota)(V^*)\Lambda_\eta(y)&=X\Lambda_\eta\big((\omega_{v,w}\otimes\iota)\Delta(y)\big)=\Lambda_\eta\big((\omega_{v,w}\otimes\iota)(\Delta(y))^*\big)\\&=\Lambda_\eta\big((\omega_{w,v}\otimes\iota)\Delta(y^*)\big)=(\omega_{w,v}\otimes\iota)(V^*)\Lambda_\eta(y^*)\\&=(\omega_{w,v}\otimes\iota)(V^*)X\Lambda_\eta(y).\end{aligned}$$

Because $\Lambda_\eta(\mathcal{N}_\eta\cap\mathcal{N}_\eta^*)$ is a core for X, this implies easily that

$$(\omega_{w,v}\otimes\iota)(V^*)X\subseteq X(\omega_{v,w}\otimes\iota)(V^*). \tag{5.10}$$

By taking the adjoint of this inclusion, we get that

$$(\omega_{w,v}\otimes\iota)(V)X^*\subseteq X^*(\omega_{v,w}\otimes\iota)(V). \tag{5.11}$$

(1) Choose $v\in D(N)$ and $w\in D(N^{-1})$. Using inclusion (5.11) and Lemma 5.36, we infer that

$$\begin{aligned}(\omega_{v,w}\otimes\iota)(V)M&=(\omega_{v,w}\otimes\iota)(V)X^*X\subseteq X^*(\omega_{w,v}\otimes\iota)(V)X\\&=X^*\big(\omega_{IN^{-\frac{1}{2}}w,IN^{\frac{1}{2}}v}\otimes\iota\big)(V)^*X.\end{aligned}$$

By using inclusion (5.10) and Lemma 5.36 once more, we get that

$$\begin{aligned}(\omega_{v,w}\otimes\iota)(V)M&\subseteq X^*X\big(\omega_{IN^{\frac{1}{2}}v,IN^{-\frac{1}{2}}w}\otimes\iota\big)(V)^*\\&=M\big(\omega_{N^{\frac{1}{2}}IIN^{\frac{1}{2}}v,N^{-\frac{1}{2}}IIN^{-\frac{1}{2}}w}\otimes\iota\big)(V)=M\big(\omega_{Nv,N^{-1}w}\otimes\iota\big)(V).\end{aligned}$$

So we have proven that

$$(\omega_{v,w}\otimes\iota)(V)M\subseteq M(\omega_{Nv,N^{-1}w}\otimes\iota)(V)$$

for $v\in D(N)$ and $w\in D(N^{-1})$. Therefore Lemma 5.9 implies that

$$V(N^{-1}\otimes M)=(N^{-1}\otimes M)V.$$

(2) Choose $v\in D(N^{-\frac{1}{2}})$ and $w\in D(N^{\frac{1}{2}})$.

Using inclusion (5.10), we see that

$$\begin{aligned}\mathcal{J}(\omega_{w,v}\otimes\iota)(V^*)\mathcal{J}M^{\frac{1}{2}} &= \mathcal{J}(\omega_{w,v}\otimes\iota)(V^*)X\\ &\subseteq \mathcal{J}X(\omega_{v,w}\otimes\iota)(V^*) = M^{\frac{1}{2}}(\omega_{v,w}\otimes\iota)(V^*).\end{aligned}\tag{5.12}$$

The result proven in (1) implies that $V^*(N^{-\frac{1}{2}}\otimes M^{\frac{1}{2}}) = (N^{-\frac{1}{2}}\otimes M^{\frac{1}{2}})V^*$.

Take $p,q\in D(M^{\frac{1}{2}})$. Then

$$\begin{aligned}\langle(\omega_{v,w}\otimes\iota)(V^*)p, M^{\frac{1}{2}}q\rangle &= \langle V^*(v\otimes p), w\otimes M^{\frac{1}{2}}q\rangle\\ &= \langle V^*(v\otimes p), N^{-\frac{1}{2}}(N^{\frac{1}{2}}w)\otimes M^{\frac{1}{2}}q\rangle\\ &= \langle (N^{-\frac{1}{2}}\otimes M^{\frac{1}{2}})V^*(v\otimes p), N^{\frac{1}{2}}w\otimes q\rangle\\ &= \langle V^*(N^{-\frac{1}{2}}v\otimes M^{\frac{1}{2}}p), N^{\frac{1}{2}}w\otimes q\rangle\\ &= \langle(\omega_{N^{-\frac{1}{2}}v,N^{\frac{1}{2}}w}\otimes\iota)(V^*)M^{\frac{1}{2}}p, q\rangle.\end{aligned}$$

Because $M^{\frac{1}{2}}$ is selfadjoint, this implies for every $p\in D(M^{\frac{1}{2}})$ that $(\omega_{v,w}\otimes\iota)(V^*)p$ belongs to $D(M^{\frac{1}{2}})$ and $M^{\frac{1}{2}}(\omega_{v,w}\otimes\iota)(V^*)p = (\omega_{N^{-\frac{1}{2}}v,N^{\frac{1}{2}}w}\otimes\iota)(V^*)M^{\frac{1}{2}}p$.

In other words,

$$(\omega_{N^{-\frac{1}{2}}v,N^{\frac{1}{2}}w}\otimes\iota)(V^*)M^{\frac{1}{2}} \subseteq M^{\frac{1}{2}}(\omega_{v,w}\otimes\iota)(V^*).$$

Using Lemma 5.36, we also have that

$$\begin{aligned}(\omega_{N^{-\frac{1}{2}}v,N^{\frac{1}{2}}w}\otimes\iota)(V^*) &= (\omega_{N^{\frac{1}{2}}w,N^{-\frac{1}{2}}v}\otimes\iota)(V)^*\\ &= (\omega_{IN^{-\frac{1}{2}}N^{\frac{1}{2}}w,IN^{\frac{1}{2}}N^{-\frac{1}{2}}v}\otimes\iota)(V) = (\omega_{Iw,Iv}\otimes\iota)(V).\end{aligned}$$

Therefore,

$$(\omega_{Iw,Iv}\otimes\iota)(V)M^{\frac{1}{2}} \subseteq M^{\frac{1}{2}}(\omega_{v,w}\otimes\iota)(V^*).$$

Because $M^{\frac{1}{2}}$ has dense range, this inclusion together with inclusion (5.12) implies that

$$(\omega_{Iw,Iv}\otimes\iota)(V) = \mathcal{J}(\omega_{w,v}\otimes\iota)(V^*)\mathcal{J}.$$

This implies for all $v,w\in H$ and $p,q\in H_\eta$ that

$$\begin{aligned}\langle(I\otimes\mathcal{J})V^*(I\otimes\mathcal{J})(v\otimes p), w\otimes q\rangle &= \langle Iw\otimes\mathcal{J}q, V^*(Iv\otimes\mathcal{J}p)\rangle\\ &= \langle V(Iw\otimes\mathcal{J}q), Iv\otimes\mathcal{J}p\rangle = \langle(\omega_{Iw,Iv}\otimes\iota)(V)\mathcal{J}q,\mathcal{J}p\rangle\\ &= \langle\mathcal{J}(\omega_{w,v}\otimes\iota)(V^*)q,\mathcal{J}p\rangle = \langle p,(\omega_{w,v}\otimes\iota)(V^*)q\rangle\\ &= \langle(\omega_{v,w}\otimes\iota)(V)p,q\rangle = \langle V(v\otimes p), w\otimes q\rangle.\end{aligned}$$

Hence $(I\otimes\mathcal{J})V^*(I\otimes\mathcal{J}) = V$.

(3) It is easy to see that $(\pi\otimes\pi_\eta)(\Delta(a)) = V^*(1\otimes\pi_\eta(a))V$ for $a\in A$. Recall also that $\pi_\eta(\kappa_t(a)) = M^{it}\pi_\eta(a)M^{-it}$ and $\pi(\tau_t(a)) = N^{-it}\pi(a)N^{it}$ for all $a\in A$ and $t\in\mathbb{R}$. Theorem 3.8 and the injectivity of π imply that π_η is injective. Hence, arguing as in Proposition 5.11, the commutation $V(N^{-1}\otimes M) = (N^{-1}\otimes M)V$ implies that $(\tau_t\otimes\kappa_t)\Delta = \Delta\kappa_t$ for $t\in\mathbb{R}$. □

Let us formulate these results in the case that $\eta=\varphi$. Then $V=W$, $\mathcal{J}=J$ and $M=\nabla$.

PROPOSITION 5.38. – *We have the following commutation relations*:
(1) $(I \otimes J)W = W^*(I \otimes J)$;
(2) $(N^{-1} \otimes \nabla)W = W(N^{-1} \otimes \nabla)$;
(3) $(\tau_t \otimes \sigma_t)\Delta = \Delta\,\sigma_t$ *for* $t \in \mathbb{R}$.

PROPOSITION 5.39. – *Consider* $v \in D(N^{-\frac{1}{2}})$ *and* $w \in D(N^{\frac{1}{2}})$. *Then*

$$(\omega_{v,w} \otimes \iota)(W)^* = \big(\omega_{IN^{-\frac{1}{2}}v, IN^{\frac{1}{2}}w} \otimes \iota\big)(W).$$

With this information in hand, we can formulate a left invariant version of Proposition 5.24.

PROPOSITION 5.40. – *For all* $a, b \in \mathcal{N}_\varphi$, *we have that* $(\iota \otimes \varphi)(\Delta(a^*)(1 \otimes b)) \in D(S)$ *and*

$$S\big((\iota \otimes \varphi)(\Delta(a^*)(1 \otimes b))\big) = (\iota \otimes \varphi)\big((1 \otimes a^*)\Delta(b)\big).$$

We have moreover that the set $\langle(\iota \otimes \varphi)(\Delta(a^*)(1 \otimes b)) \mid a, b \in \mathcal{N}_\varphi\rangle$ *is a core for* S.

Proof. – The first statement was already proven in Corollary 5.35. By Proposition 5.38 (3) the considered subspace of A is invariant under τ. It is also clearly a dense subspace of $D(S)$. Now the conclusion follows. □

Remark 5.41. – Because $\tau_t((\iota \otimes \omega)\Delta(a)) = (\iota \otimes \omega\,\sigma_{-t})\Delta(\sigma_t(a))$ and because $A = [(\iota \otimes \omega)\Delta(a) \mid a \in A,\ \omega \in A^*]$ we see that τ_t does not depend on ψ. Because of the previous proposition, S does not depend on ψ either. So R does not depend on ψ.

But there is more. Because of the commutation relation $\Delta\,\sigma_t^\psi = (\sigma_t^\psi \otimes \tau_{-t})\,\Delta$, we see in an analogous way that R, S and τ do not depend on the choice of φ we made in the beginning of the story. So R, S and τ are completely determined by the pair (A, Δ).

The previous remark justifies the following terminology.

Terminology 5.42. – We will use the following more or less standard terminology.
- S is called the antipode of (A, Δ);
- τ is called the scaling group of (A, Δ);
- R is called the unitary antipode of (A, Δ).

The triple (S, R, τ) will be referred to as the antipodal triple of (A, Δ).

There is also a way to characterize S in terms of (A, Δ) without referring to left or right invariant weights. This characterization is described by Corollary 5.34 and the following result.

PROPOSITION 5.43. – *Define* C *to be the set consisting of all elements* $a \in A$ *such that there exist an element* $b \in A$, *an index set* I *and* $p \in \mathrm{MR}_I(A)$, $q \in \mathrm{MC}_I(A)$ *satisfying*

$$a \otimes 1 = \sum_{i \in I} \Delta(p_i)(1 \otimes q_i) \quad \textit{and} \quad b \otimes 1 = \sum_{i \in I} (1 \otimes p_i)\Delta(q_i).$$

Then C *is a core for* S.

Proof. – It is easy to see that C is a subspace of A. Corollary 5.34 implies that $C \subseteq D(S)$. The proof of Corollary 5.35 implies that $\langle(\iota \otimes \varphi)(\Delta(a^*)(1 \otimes b)) \mid a, b \in \mathcal{N}_\varphi\rangle$ is a subset of C. The result now follows by Proposition 5.40. □

So we see that S can be completely defined in terms of (A, Δ). Also notice that the pair R, τ is completely determined by the following properties:
- R is an involutive *-antiautomorphism on A;

- τ is a norm continuous one-parameter group on A;
- R and τ commute;
- $S = R\tau_{-\frac{i}{2}}$.

Notice that the above properties imply that $\tau_{-i} = S^2$ so that the C*-analogue of Lemma 4.4 in [17] determines τ. Then R is determined by the fact that $S = R\tau_{-\frac{i}{2}}$.

Let us illustrate how these different characterizations of the antipode can be used to prove easily a basic result about *-homomorphisms commuting with the comultiplication. First we need to extend the antipode to the multiplier algebra.

Remark 5.44. – We know that the mapping $\tau_{-\frac{i}{2}}$ is strictly closable in $\mathrm{M}(A)$ and we denote its strict closure in $\mathrm{M}(A)$ by $\bar{\tau}_{-\frac{i}{2}}$. Since $S = R\tau_{-\frac{i}{2}}$, we see that S is also strictly closable. We denote the strict closure of S by $\overline{S}$ and put $S(a) = \overline{S}(a)$ for all $a \in D(\overline{S})$. It is also clear that $\overline{S} = \overline{R}\bar{\tau}_{-\frac{i}{2}}$, where $\overline{R}$ denotes the unique *-antiautomorphism on $\mathrm{M}(A)$ which extends R.

If $a \in D(\overline{S})$ and $S(a)$ belongs to A, then a belongs to $D(S)$ (because we know it to be true for $\tau_{-\frac{i}{2}}$, *see* e.g. Proposition 1.24 of [25]). Using the net $(e_j)_{j\in J}$ of Remark 5.32, we see that for every $x \in D(\overline{S})$, there exists a net $(x_i)_{i\in I}$ in $D(S)$ such that:

- $(x_i)_{i\in I}$ is bounded and converges strictly to x;
- $(S(x_i))_{i\in I}$ is bounded and converges strictly to $S(x)$.

Arguing as in Remark 5.32, we get: consider $a, b \in \mathrm{M}(A)$ such that we have for every $x \in D(S)$ that $ax \in D(S)$ and $S(ax) = S(x)b$. Then a belongs to $D(\overline{S})$ and $S(a) = b$.

It is then clear that Proposition 5.33 and Corollary 5.34 remain true if we let a and b belong to $\mathrm{M}(A)$ and replace S by $\overline{S}$.

PROPOSITION 5.45. – *Consider two reduced* C*-*algebraic quantum groups* (A_1, Δ_1) *and* (A_2, Δ_2) *with antipodal triples* (S_1, R_1, τ_1) *and* (S_2, R_2, τ_2) *respectively. Let* $\alpha : A_1 \to \mathrm{M}(A_2)$ *be a non-degenerate *-homomorphism such that* $(\alpha \otimes \alpha)\Delta_1 = \Delta_2\,\alpha$. *Then*

$$\alpha S_1 \subseteq S_2\alpha, \quad R_2\alpha = \alpha R_1, \quad \alpha(\tau_1)_t = (\tau_2)_t\alpha \quad \textit{for all } t \in \mathbb{R},$$

where we use the usual convention that $S_2\alpha := \overline{S}_2\alpha$, $R_2\alpha := \overline{R}_2\alpha$ *and* $(\tau_2)_t\alpha := (\bar{\tau}_2)_t\alpha$.

Proof. – Define C to be the set consisting of all elements $a \in A_1$ such that there exist an element $b \in A_1$, an index set I and $p \in \mathrm{MR}_I(A_1)$, $q \in \mathrm{MC}_I(A_1)$ satisfying

$$a \otimes 1 = \sum_{i\in I} \Delta_1(p_i)(1 \otimes q_i) \quad \text{and} \quad b \otimes 1 = \sum_{i\in I} (1 \otimes p_i)\Delta_1(q_i).$$

Using Corollary 5.34 (and its extension to the multiplier algebra following the previous remark), it is not so difficult to see that the commutation relation $(\alpha \otimes \alpha)\Delta_1 = \Delta_2\alpha$ implies that $\alpha(C) \subseteq D(\overline{S}_2)$ and that $S_2(\alpha(a)) = \alpha(S_1(a))$ for $a \in C$. Since C is a core for S_1 (*see* Proposition 5.43), we get that $\alpha S_1 \subseteq S_2\alpha$. Using the fact that C is a strict 'bounded' core for $\overline{S}_1$ (*see* the previous remark), the strict closedness of $\overline{S}_2$ implies that $\bar{\alpha}\overline{S}_1 \subseteq \overline{S}_2\bar{\alpha}$.

This implies immediately that

$$\bar{\alpha}(\bar{\tau}_1)_{-i} = \bar{\alpha}\overline{S}_1^2 \subseteq \overline{S}_2^2\bar{\alpha} = (\bar{\tau}_2)_{-i}\bar{\alpha}.$$

Arguing as in Definition 4.1–Lemma 4.4 of [17], one concludes that

$$\bar{\alpha}(\bar{\tau}_1)_t = (\bar{\tau}_2)_t\bar{\alpha} \quad \text{for } t \in \mathbb{R}. \tag{5.13}$$

Choose $a \in D((\tau_1)_{-\frac{i}{2}}) = D(S_1)$. Since $\alpha(\tau_1)_{-\frac{i}{2}} \subseteq (\tau_2)_{-\frac{i}{2}}\alpha$ (which follows from commutation (5.13)), we get that $\alpha(a) \in D((\bar{\tau}_2)_{-\frac{i}{2}}) = D(\overline{S}_2)$ and

$$R_2\big(\alpha((\tau_1)_{-\frac{i}{2}}(a))\big) = R_2\big((\tau_2)_{-\frac{i}{2}}(\alpha(a))\big) = S_2(\alpha(a)) = \alpha(S_1(a)) = \alpha\big(R_1((\tau_1)_{-\frac{i}{2}}(a))\big).$$

Because $(\tau_1)_{-\frac{i}{2}}$ has dense range, this implies that $R_2(\alpha(x)) = \alpha(R_1(x))$ for all $x \in A$. □

COROLLARY 5.46. – *Consider two reduced* C*-*algebraic quantum groups* (A_1, Δ_1) *and* (A_2, Δ_2) *with antipodal triples* (S_1, R_1, τ_1) *and* (S_2, R_2, τ_2) *respectively. Let* $\alpha : A_1 \to A_2$ *be a *-isomorphism such that* $(\alpha \otimes \alpha)\Delta_1 = \Delta_2\alpha$. *Then*

$$\alpha S_1 = S_2\alpha, \quad R_2\alpha = \alpha R_1, \quad \alpha(\tau_1)_t = (\tau_2)_t\alpha \quad \textit{for all } t \in \mathbb{R}.$$

Remark 5.47. – In this case, the use of the results in [17] can be simply avoided by the following reasoning. Take a left invariant KMS weight φ on (A_2, Δ_2) with modular group σ.

The weight $\varphi\alpha$ is a left invariant KMS weight on (A_1, Δ_1) with modular group $\mathbb{R} \to \mathrm{Aut}(A_1) : t \mapsto \alpha^{-1}\,\sigma_t\,\alpha$. Therefore, Proposition 5.37 implies for every $t \in \mathbb{R}$ that

$$\begin{aligned}\big((\tau_1)_t \otimes \alpha^{-1}\sigma_t\alpha\big)\Delta_1 &= \Delta_1\alpha^{-1}\sigma_t\alpha = \big(\alpha^{-1} \otimes \alpha^{-1}\big)\Delta_2\sigma_t\alpha \\ &= \big(\alpha^{-1}(\tau_2)_t \otimes \alpha^{-1}\sigma_t\big)\Delta_2\alpha = \big(\alpha^{-1}(\tau_2)_t\alpha \otimes \alpha^{-1}\sigma_t\alpha\big)\Delta_1,\end{aligned}$$

implying that $((\tau_1)_t \otimes \iota)\Delta_1 = (\alpha^{-1}(\tau_2)_t\alpha \otimes \iota)\Delta_1$ so that $(\tau_1)_t = \alpha^{-1}(\tau_2)_t\alpha$ by the density conditions. Thus $\alpha(\tau_1)_t = (\tau_2)_t\alpha$.

6. Relative invariance properties and commutations

From now on, we will fix a particular right invariant weight ψ on (A, Δ), namely $\psi = \varphi R$. Because of Proposition 5.26, this weight ψ is indeed right invariant. Also notice that ψ is a KMS weight on (A, Δ), with modular group σ' given by $\sigma'_t = R\sigma_{-t}R$ for all $t \in \mathbb{R}$ (the minus sign appears because R is antimultiplicative).

Remember also that we have a Tomita *-algebra running around:

$$\mathcal{T}_\psi = \big\{x \in \mathcal{N}_\psi \cap \mathcal{N}_\psi^* \mid x \text{ is analytic with respect to } \sigma' \text{ and } \sigma'_z(x) \in \mathcal{N}_\psi \cap \mathcal{N}_\psi^* \text{ for } z \in \mathbb{C}\big\}.$$

We will also use the following notations:

- G = the unbounded operator defined in Notation 5.1;
- $N = G^*G$ and I is the anti-unitary on H such that $G = IN^{\frac{1}{2}}$.

Recall the following result (which is an immediate consequence of Result 5.13).

RESULT 6.1. – *Consider* $x \in \mathrm{M}(A)$. *If* $\Delta(x) = x \otimes 1$ *or* $\Delta(x) = 1 \otimes x$, *then* $x \in \mathbb{C}1$.

Notice also that Propositions 5.14 and 5.15 imply immediately the following results.

PROPOSITION 6.2. – *Consider* $x \in A^+$ *such that* $(\omega_{v,v} \otimes \iota)\Delta(x)$ *belongs to* $\mathcal{M}_\varphi^+$ *for every* $v \in H$. *Then* x *belongs to* $\mathcal{M}_\varphi^+$.

PROPOSITION 6.3. – *Consider* $x \in A^+$ *such that* $(\iota \otimes \omega_{v,v})\Delta(x)$ *belongs to* $\mathcal{M}_\psi^+$ *for every* $v \in H$. *Then* x *belongs to* $\mathcal{M}_\psi^+$.

In the following part of this section, we prove a uniqueness result for left invariant weights which will be the essential step for a general uniqueness result in the next section.

In a first step, we want to prove that any left invariant proper weight is automatically KMS. This will follow easily from the next lemma which we borrowed from [15] (Corollary 2.7.3).

LEMMA 6.4. – *Consider a projection P in $\widetilde{A}$ such that $\widetilde{\Delta}(P) \leqslant P \otimes 1$ or $\widetilde{\Delta}(P) \leqslant 1 \otimes P$. Then $P = 0$ or $P = 1$.*

Proof. – Suppose that $\widetilde{\Delta}(P) \leqslant 1 \otimes P$. Then $\widetilde{\Delta}(P)(1 \otimes P) = \widetilde{\Delta}(P)$ which in terms of W becomes $W^*(1 \otimes P)W(1 \otimes P) = W^*(1 \otimes P)W$. So we get that

$$(1 \otimes P)W(1 \otimes P) = (1 \otimes P)W. \tag{6.1}$$

Choose $v \in D(N^{-\frac{1}{2}})$, $w \in D(N^{\frac{1}{2}})$ and apply $\omega_{IN^{-\frac{1}{2}}v, IN^{\frac{1}{2}}w} \otimes \iota$ to the above equation. Hence

$$P(\omega_{IN^{-\frac{1}{2}}v, IN^{\frac{1}{2}}w} \otimes \iota)(W)P = P(\omega_{IN^{-\frac{1}{2}}v, IN^{\frac{1}{2}}w} \otimes \iota)(W). \tag{6.2}$$

Using Proposition 5.39, we see that $(\omega_{IN^{-\frac{1}{2}}v, IN^{\frac{1}{2}}w} \otimes \iota)(W)^* = (\omega_{v,w} \otimes \iota)(W)$. So if we take the adjoint of Eq. (6.2), we get that

$$P(\omega_{v,w} \otimes \iota)(W)P = (\omega_{v,w} \otimes \iota)(W)P.$$

Therefore we conclude that $(1 \otimes P)W(1 \otimes P) = W(1 \otimes P)$. Combining this with Eq. (6.1), this gives us that $W(1 \otimes P) = (1 \otimes P)W$, hence $\widetilde{\Delta}(P) = W^*(1 \otimes P)W = 1 \otimes P$. So Result 5.13 implies that $P \in \mathbb{C}1$, and hence $P = 0$ or $P = 1$.

If $\widetilde{\Delta}(P) \leqslant P \otimes 1$, then the W*-version of Proposition 5.26 implies that $\widetilde{\Delta}(\widetilde{R}(P)) \leqslant 1 \otimes \widetilde{R}(P)$ and the previous part of the proof implies that $\widetilde{R}(P) = 0$ or $\widetilde{R}(P) = 1$. Therefore $P = 0$ or $P = 1$. □

RESULT 6.5. – *Consider a left invariant proper weight η on (A, Δ). Then η is a KMS weight on A.*

Proof. – Let $\widetilde{\eta}$ denote the W*-lift of η in the GNS-construction (H, π, Λ) (as described in Proposition 3.9). By Proposition 3.13, we know that $\widetilde{\eta}$ is left invariant.

Put $N = \{x \in \widetilde{A} \mid \widetilde{\eta}(x^*x) = 0\}$. Then N is a σ-weakly closed left ideal in $\widetilde{A}$. So there exists a projection $P \in \widetilde{A}$ such that $N = \widetilde{A}P$ ($1 - P$ is of course nothing else but the support projection of $\widetilde{\eta}$).

We have in particular that $P \in N$ which by left invariance of $\widetilde{\eta}$ gives us for every $\omega \in \widetilde{A}_*^+$ that

$$\widetilde{\eta}\big([(\omega \overline{\otimes} \iota)\widetilde{\Delta}(P)]^*[(\omega \overline{\otimes} \iota)\widetilde{\Delta}(P)]\big) \leqslant \|\omega\|\, \widetilde{\eta}\big((\omega \overline{\otimes} \iota)\widetilde{\Delta}(P^*P)\big) = \|\omega\|^2\, \widetilde{\eta}(P^*P) = 0,$$

implying that $(\omega \overline{\otimes} \iota)(\widetilde{\Delta}(P))P = (\omega \overline{\otimes} \iota)(\widetilde{\Delta}(P))$. So we get that $\widetilde{\Delta}(P)(1 \otimes P) = \widetilde{\Delta}(P)$. Therefore the previous lemma implies that $P = 0$ or $P = 1$. Because $\widetilde{\eta} \neq 0$, we must have that $P = 0$. So we have proven that $\widetilde{\eta}$ faithful.

From the remarks after Proposition 3.9, it follows now that also the W*-lift of η in its own GNS-construction is faithful. So η is approximately KMS and the result follows now from Proposition 5.18. □

The next result is the essential step towards the general uniqueness result. In a corollary, we will prove uniqueness under some relative invariance condition. The spirit of the proof of Theorem 2.7.7 of [15] is still present in the proof of the next proposition but we prove a weaker result under weaker conditions. By changing strategy, we can stick to basic properties of weights (except for the ever present Tomita Takesaki theory) and avoid liftings to the von Neumann algebra where we would use Radon Nikodym.

It should be pointed out however that the proof of Theorem 2.7.7 of [15] could also be used in our approach since we will prove later on that any left invariant proper weight is automatically relatively invariant under σ (so it is in particular true for the weight η_+ in the proof of Corollary 6.7).

PROPOSITION 6.6. – *Consider left invariant proper weights η_1 and η_2 on (A, Δ) such that $\eta_1 \leqslant \eta_2$. Then there exists a number $r \in]0,1]$ such that $\eta_1(x) = r\eta_2(x)$ for $x \in \mathcal{M}_{\eta_2}^+$.*

Proof. – Take GNS-constructions (H_i, π_i, Λ_i) for η_i $(i = 1,2)$. Using Theorem 3.16, we define a unitary operator U on $H \otimes H_2$ such that

$$U\big(\Lambda(a) \otimes \Lambda_2(b)\big) = (\Lambda \otimes \Lambda_2)\big(\Delta(b)(a \otimes 1)\big)$$

for $a \in \mathcal{N}_\varphi$ and $b \in \mathcal{N}_{\eta_2}$.

By Theorem 3.8, there exists an isomorphism $\widetilde{\pi}_2 : \widetilde{A} \to \pi_2(A)''$ such that $\widetilde{\pi}_2(\pi(a)) = \pi_2(a)$ for all $a \in A$.

Using Proposition 1.29, it follows easily that $(\pi \otimes \pi_2)(\Delta(a)) = U(1 \otimes \pi_2(a))U^*$ for all $a \in A$. Therefore $(\iota \otimes \widetilde{\pi}_2)(\widetilde{\Delta}(x)) = U(1 \otimes \widetilde{\pi}_2(x))U^*$ for all $x \in \widetilde{A}$.

Result 6.5 implies that η_2 is a KMS weight. Denote by $\mathcal{J}$ the modular conjugation of η_2 in the GNS-construction (H_2, π_2, Λ_2). Tomita–Takesaki theory tells us that

$$\mathcal{J}\widetilde{\pi}_2(\widetilde{A})\mathcal{J} = \mathcal{J}\pi_2(A)''\mathcal{J} = \pi_2(A)'.$$

Since $\eta_1 \leqslant \eta_2$, there exists a bounded operator $F \in \mathrm{B}(H_2, H_1)$ such that $F\Lambda_2(a) = \Lambda_1(a)$ for all $a \in \mathcal{N}_{\eta_2}$. Put $T = F^*F$, then T is a positive operator in $\mathrm{B}(H_2)$ such that $\eta_1(b^*a) = \langle T\Lambda_2(a), \Lambda_2(b)\rangle$ for all $a, b \in \eta_2$. This equality gives us easily that $T \in \pi_2(A)'$, so $\mathcal{J}T\mathcal{J}$ belongs to $\widetilde{\pi}_2(\widetilde{A})$.

Choose $a \in \mathcal{N}_\varphi$ and $b \in \mathcal{N}_{\eta_2}$. Take an orthonormal basis $(e_i)_{i \in I}$ for H.

By Result 2.9, we know that $\sum_{i \in I} \|\Lambda_2((\omega_{\Lambda(a),e_i} \otimes \iota)(\Delta(b)))\|^2 < \infty$ and

$$U(\Lambda(a) \otimes \Lambda_2(b)) = \sum_{i \in I} e_i \otimes \Lambda_2\big((\omega_{\Lambda(a),e_i} \otimes \iota)(\Delta(b))\big).$$

Consequently,

$$\begin{aligned}
&\big\langle (1 \otimes T)U(\Lambda(a) \otimes \Lambda_2(b)), U(\Lambda(a) \otimes \Lambda_2(b))\big\rangle \\
&\quad = \sum_{i \in I} \big\langle T\Lambda_2\big((\omega_{\Lambda(a),e_i} \otimes \iota)(\Delta(b))\big), \Lambda_2\big((\omega_{\Lambda(a),e_i} \otimes \iota)(\Delta(b))\big)\big\rangle \\
&\quad = \sum_{i \in I} \eta_1\big((\omega_{\Lambda(a),e_i} \otimes \iota)(\Delta(b))^*(\omega_{\Lambda(a),e_i} \otimes \iota)(\Delta(b))\big).
\end{aligned}$$

Therefore, Lemma A.6 implies that

$$\begin{aligned}
\big\langle (1 \otimes T)U(\Lambda(a) \otimes \Lambda_2(b)), U(\Lambda(a) \otimes \Lambda_2(b))\big\rangle &= \eta_1\big((\omega_{\Lambda(a),\Lambda(a)} \otimes \iota)(\Delta(b^*b))\big) \\
&= \langle \Lambda(a), \Lambda(a)\rangle \eta_1(b^*b),
\end{aligned}$$

where we used the left invariance of η_1. So we get that

$$\begin{aligned}
&\big\langle (1 \otimes T)U(\Lambda(a) \otimes \Lambda_2(b)), U(\Lambda(a) \otimes \Lambda_2(b))\big\rangle \\
&\quad = \big\langle \Lambda(a), \Lambda(a)\big\rangle \langle T\Lambda_2(b), \Lambda_2(b)\big\rangle = \big\langle (1 \otimes T)(\Lambda(a) \otimes \Lambda_2(b)), \Lambda(a) \otimes \Lambda_2(b)\big\rangle.
\end{aligned}$$

By polarization, we get that

$$\langle (1 \otimes T)U(\Lambda(a) \otimes \Lambda_2(b)), U(\Lambda(c) \otimes \Lambda_2(d)) \rangle = \langle (1 \otimes T)(\Lambda(a) \otimes \Lambda_2(b)), \Lambda(c) \otimes \Lambda_2(d) \rangle$$

for all $a, c \in \mathcal{N}_\varphi$ and $b, d \in \mathcal{N}_{\eta_2}$.

So we have proven that $U^*(1 \otimes T)U = 1 \otimes T$. Because $U^*(I \otimes \mathcal{J}) = (I \otimes \mathcal{J})U$ (*see* Proposition 5.37), we get that

$$U(1 \otimes \mathcal{J}T\mathcal{J})U^* = 1 \otimes \mathcal{J}T\mathcal{J}.$$

Remembering that $\mathcal{J}T\mathcal{J} \in \widetilde{\pi}_2(\widetilde{A})$, we see that

$$(\iota \overline{\otimes} \widetilde{\pi}_2)\big(\widetilde{\Delta}(\widetilde{\pi}_2^{-1}(\mathcal{J}T\mathcal{J}))\big) = U(1 \otimes \mathcal{J}T\mathcal{J})U^* = 1 \otimes \mathcal{J}T\mathcal{J} = (\iota \overline{\otimes} \widetilde{\pi}_2)(1 \otimes \widetilde{\pi}_2^{-1}(\mathcal{J}T\mathcal{J})).$$

Consequently, the injectivity of $\iota \overline{\otimes} \widetilde{\pi}_2$ implies that $\widetilde{\Delta}(\widetilde{\pi}_2^{-1}(\mathcal{J}T\mathcal{J})) = 1 \otimes \widetilde{\pi}_2^{-1}(\mathcal{J}T\mathcal{J})$. Therefore, Result 5.13 implies that $\widetilde{\pi}_2^{-1}(\mathcal{J}T\mathcal{J})$ is a scalar. So we find a number $r \geqslant 0$ such that $T = r1$, thus $\eta_1(x) = r\eta_2(x)$ for all $x \in \mathcal{M}_{\eta_2}^+$. Because $\eta_1 \neq 0$, we have that $r \neq 0$ (remember that $\{x \in A^+ \mid \eta_1(x) = 0\}$ is closed). Since $\eta_2 \neq 0$ and $\eta_1 \leqslant \eta_2$, $r \leqslant 1$. □

COROLLARY 6.7. – *Consider a left invariant proper weight η on (A, Δ) such that there exists a number $\lambda > 0$ satisfying $\eta\sigma_t = \lambda^t \eta$ for all $t \in \mathbb{R}$. Then there exists a number $r > 0$ such that $\eta = r\varphi$.*

Proof. – Put $\eta_+ = \eta + \varphi$. Because $\eta\sigma_t = \lambda^t\eta$ for $t \in \mathbb{R}$, Proposition 1.14 of [30] implies that η_+ is a proper weight on A. It is also easy to check that η_+ is left invariant (notice that $\mathcal{M}_{\eta_+}^+ = \mathcal{M}_\eta^+ \cap \mathcal{M}_\varphi^+$). Because $\varphi \leqslant \eta_+$, we conclude from the previous proposition that there exists a number $s \in]0, 1]$ such that $\varphi(x) = s\eta_+(x)$ for $x \in \mathcal{M}_{\eta_+}^+$.

So we have for $x \in \mathcal{M}_\eta^+ \cap \mathcal{M}_\varphi^+$ that $\varphi(x) = s\eta(x) + s\,\varphi(x)$. Put $r = \frac{1-s}{s} \in \mathbb{R}^+$. Then we have for $x \in \mathcal{M}_\eta^+ \cap \mathcal{M}_\varphi^+$ that $\eta(x) = r\varphi(x)$. Since $\eta \neq 0$, $r \neq 0$.

This implies that η is invariant under σ and thus $\eta = r\varphi$ by Proposition 1.13. □

By using the unitary antipode R, it is clear that a similar result holds for right invariant weights.

Once we have the modular element of (A, Δ) at our disposal, it will be easy to prove that any left invariant proper weight is relatively invariant under σ. But we will already be able to use this restricted version to draw some important conclusions in the last part of this section.

Now we have gathered enough technical material to prove all the relevant relative invariance properties.

PROPOSITION 6.8. –

(1) *The automorphism groups σ, σ' and τ commute pairwise.*

(2) *We have the following commutation relations for all $t \in \mathbb{R}$:*

$$\Delta\sigma_t = (\tau_t \otimes \sigma_t)\Delta, \qquad \Delta\sigma'_t = (\sigma'_t \otimes \tau_{-t})\Delta,$$
$$\Delta\tau_t = (\tau_t \otimes \tau_t)\Delta, \qquad \Delta\tau_t = (\sigma_t \otimes \sigma'_{-t})\Delta.$$

(3) *There exists a number $\nu > 0$ such that*

$$\varphi\sigma'_t = \nu^t\varphi, \qquad \psi\sigma_t = \nu^{-t}\psi,$$
$$\psi\tau_t = \nu^{-t}\psi, \qquad \varphi\tau_t = \nu^{-t}\varphi,$$

for all $t \in \mathbb{R}$.

Proof. – Notice that the first three commutation relations in (2) were already proven in Proposition 5.17, Result 5.12 and Proposition 5.38. We will proceed to the proof of the other results.

(i) Choose $t \in \mathbb{R}$ and define $\kappa = \sigma_t \tau_{-t}$. Then

$$(\iota \otimes \kappa)\Delta = (\tau_t \tau_{-t} \otimes \sigma_t \tau_{-t})\Delta = \Delta \sigma_t \tau_{-t} = \Delta \kappa.$$

Let $a \in \mathcal{M}_\psi^+$. Then $\Delta(a) \in \overline{\mathcal{M}}_{\psi \otimes \iota}^+$ by right invariance of ψ. Hence,

$$\Delta(\kappa(a)) = (\iota \otimes \kappa)\Delta(a) \in \overline{\mathcal{M}}_{\psi \otimes \iota}^+.$$

By Proposition 6.3, we get that $\kappa(a) \in \mathcal{M}_\psi^+$ and so

$$\psi(\kappa(a))1 = (\psi \otimes \iota)\big(\Delta(\kappa(a))\big) = (\psi \otimes \iota)\big((\iota \otimes \kappa)\Delta(a)\big) = \kappa\big((\psi \otimes \iota)\Delta(a)\big) = \psi(a)1.$$

Working with κ^{-1}, we get completely analogously that $\kappa^{-1}(\mathcal{M}_\psi^+) \subseteq \mathcal{M}_\psi^+$. Hence $\psi \sigma_t \tau_{-t} = \psi \kappa = \psi$.

Then we get that $\sigma'_s \sigma_t \tau_{-t} = \sigma_t \tau_{-t} \sigma'_s$ for all $s, t \in \mathbb{R}$. Thus we have

$$\begin{aligned}(\iota \otimes \sigma_{-t}\tau_t)\Delta &= (\tau_{-t}\tau_t \otimes \sigma_{-t}\tau_t)\Delta = \Delta \sigma_{-t}\tau_t = \Delta \sigma'_{-s}\sigma_{-t}\tau_t\sigma'_s \\ &= (\sigma'_{-s} \otimes \tau_s)\Delta \sigma_{-t}\tau_t\sigma'_s = (\sigma'_{-s} \otimes \tau_s\sigma_{-t}\tau_t)\Delta\sigma'_s = (\iota \otimes \tau_s\sigma_{-t}\tau_t\tau_{-s})\Delta.\end{aligned}$$

So for all $a \in A$ and $\omega \in A^*$, we have

$$(\sigma_{-t}\tau_t)\big((\omega \otimes \iota)\Delta(a)\big) = (\tau_s\sigma_{-t}\tau_t\tau_{-s})\big((\omega \otimes \iota)\Delta(a)\big).$$

Hence,

$$\sigma_{-t}\tau_t = \tau_s\sigma_{-t}\tau_t\tau_{-s} = \tau_s\sigma_{-t}\tau_{-s}\tau_t.$$

Thus we have $\sigma_{-t} = \tau_s\sigma_{-t}\tau_{-s}$ for all $s, t \in \mathbb{R}$, so that σ and τ commute.

(ii) Fix $t \in \mathbb{R}$. If $a \in \mathcal{M}_{\varphi\tau_t}^+$ we have $\tau_t(a) \in \mathcal{M}_\varphi^+$ and thus

$$(\tau_t \otimes \tau_t)\Delta(a) = \Delta(\tau_t(a)) \in \overline{\mathcal{M}}_{\iota \otimes \varphi}^+.$$

So $(\iota \otimes \tau_t)\Delta(a) \in \overline{\mathcal{M}}_{\iota \otimes \varphi}^+$, and thus $\Delta(a) \in \overline{\mathcal{M}}_{\iota \otimes \varphi\tau_t}^+$. Moreover,

$$(\iota \otimes \varphi\tau_t)\Delta(a) = \tau_{-t}\big((\iota \otimes \varphi)\Delta(\tau_t(a))\big) = \varphi(\tau_t(a))1.$$

So $\varphi\tau_t$ is a left invariant KMS weight on (A, Δ). Because σ and τ commute $\varphi\tau_t$ is invariant under σ. By Corollary 6.7 we get the existence of a number $\lambda_t > 0$ such that $\varphi\tau_t = \lambda_t\varphi$. This implies the existence of a number $\nu > 0$ such that $\varphi\tau_t = \nu^{-t}\varphi$ for all $t \in \mathbb{R}$ (*see* e.g. Result 2.15 of [24]). By applying R we obtain $\psi\tau_t = \nu^{-t}\psi$ for all $t \in \mathbb{R}$. So also σ' and τ commute. Then we get

$$\psi\sigma_t = \nu^{-t}\psi\tau_{-t}\sigma_t = \nu^{-t}\psi\sigma_t\tau_{-t} = \nu^{-t}\psi,$$

where the last equality follows from part i) of this proof. From this we get that σ' and σ commute. Also,

$$\varphi\sigma'_t = \varphi R\sigma_{-t}R = \psi\sigma_{-t}R = \nu^t\psi R = \nu^t\varphi.$$

This ends the proof of the first and the third statement.

(iii) Now, we prove the last statement of 2). Fix $t \in \mathbb{R}$. Observe that $S\sigma'_t = \sigma_{-t}S$ because $R\sigma'_t = \sigma_{-t}R$ and because τ and σ commute. Then we get for all $a, b \in \mathcal{N}_\varphi$

$$\begin{aligned}\sigma'_t\big((\iota\otimes\varphi)(\Delta(a^*)(1\otimes b))\big) &= (\iota\otimes\varphi)\big[(\sigma'_t\otimes\iota)(\Delta(a^*))(1\otimes b)\big]\\ &= \nu^{-t}(\iota\otimes\varphi)\big[(\sigma'_t\otimes\tau_{-t})(\Delta(a^*))(1\otimes\tau_{-t}(b))\big]\\ &= \nu^{-t}(\iota\otimes\varphi)\big[\Delta(\sigma'_t(a^*))(1\otimes\tau_{-t}(b))\big].\end{aligned}$$

So we get that $\sigma'_t((\iota\otimes\varphi)(\Delta(a^*)(1\otimes b)))$ belongs to $D(S)$ and

$$\begin{aligned}S\big(\sigma'_t\big((\iota\otimes\varphi)(\Delta(a^*)(1\otimes b))\big)\big) &= \nu^{-t}(\iota\otimes\varphi)\big[(1\otimes\sigma'_t(a^*))\Delta(\tau_{-t}(b))\big]\\ &= (\iota\otimes\varphi)\big[(1\otimes a^*)(\iota\otimes\sigma'_{-t})\Delta(\tau_{-t}(b))\big].\end{aligned}$$

On the other hand, $(\iota\otimes\varphi)(\Delta(a^*)(1\otimes b))$ belongs to $D(S)$ and

$$\begin{aligned}\sigma_{-t}\big(S\big((\iota\otimes\varphi)(\Delta(a^*)(1\otimes b))\big)\big) &= \sigma_{-t}\big((\iota\otimes\varphi)\big((1\otimes a^*)\Delta(b)\big)\big)\\ &= (\iota\otimes\varphi)\big((1\otimes a^*)(\sigma_{-t}\otimes\iota)\Delta(b)\big).\end{aligned}$$

Therefore,

$$(\iota\otimes\varphi)\big[(1\otimes a^*)(\iota\otimes\sigma'_{-t})\Delta(\tau_{-t}(b))\big] = (\iota\otimes\varphi)\big((1\otimes a^*)(\sigma_{-t}\otimes\iota)\Delta(b)\big).$$

Because φ is faithful we get

$$(\iota\otimes\sigma'_{-t})\Delta(\tau_{-t}(b)) = (\sigma_{-t}\otimes\iota)\Delta(b)$$

for all $b \in \mathcal{N}_\varphi$, so that $\Delta\tau_{-t} = (\sigma_{-t}\otimes\sigma'_t)\Delta$. □

Since $\varphi\tau_t = \nu^{-t}\varphi$, we can introduce the following positive operator in H (which is generally unbounded).

DEFINITION 6.9. – Define the strictly positive operator P in H such that $P^{it}\Lambda(a) = \nu^{\frac{t}{2}}\Lambda(\tau_t(a))$ for all $t \in \mathbb{R}$ and $a \in \mathcal{N}_\varphi$.

Then we get immediately that $P^{it}xP^{-it} = \widetilde{\tau}_t(x)$ for all $x \in \widetilde{A}$.

In Definition 1.2 of [64], Woronowicz introduces the notion of manageability of a multiplicative unitary. It is no big surprise that:

PROPOSITION 6.10. – *The multiplicative unitary W is manageable.*

Proof. – From Proposition 5.39, we get that for all $v \in D(N^{\frac{1}{2}})$, $w \in D(N^{-\frac{1}{2}})$ and $p, q \in H$

$$\big\langle I(\iota\otimes\omega_{q,p})(W)Iv, w\big\rangle = \big\langle(\iota\otimes\omega_{p,q})(W)N^{\frac{1}{2}}v, N^{-\frac{1}{2}}w\big\rangle.$$

Because $(\iota\otimes\omega_{p,q})(W) \in \widetilde{A}$ and $\widetilde{\tau}_t$ is implemented by N^{-it}, we can conclude that

$$(\iota\otimes\omega_{p,q})(W) \in D(\widetilde{\tau}_{-\frac{i}{2}}) \quad\text{and}\quad \widetilde{\tau}_{-\frac{i}{2}}\big((\iota\otimes\omega_{p,q})(W)\big) = I(\iota\otimes\omega_{q,p})(W)I.$$

Since $\widetilde{\tau}_t$ is also implemented by P^{it}, this implies that

$$\langle I(\iota\otimes\omega_{q,p})(W)Iv, w\rangle = \big\langle(\iota\otimes\omega_{p,q})(W)P^{-\frac{1}{2}}v, P^{\frac{1}{2}}w\big\rangle$$

for all $v \in D(P^{-\frac{1}{2}})$, $w \in D(P^{\frac{1}{2}})$ and $p, q \in H$. Because of Proposition 5.38 we can rewrite this as

$$\langle \Sigma W^* \Sigma(q \otimes v), p \otimes w \rangle = \langle \Sigma W \Sigma\big(Jp \otimes P^{-\frac{1}{2}} v\big), Jq \otimes P^{\frac{1}{2}} w \rangle.$$

If we can prove now that $\Sigma W^* \Sigma$ and $P \otimes P$ commute, we arrive at the manageability of $\Sigma W^* \Sigma$, which implies the manageability of W by Proposition 1.4 of [68]. It is clear we have to prove that W^* and $P \otimes P$ commute. But for $a, b \in \mathcal{N}_\varphi$ and $t \in \mathbb{R}$ we have

$$\begin{aligned} W^*(P^{it} \otimes P^{it})\big(\Lambda(a) \otimes \Lambda(b)\big) &= \nu^t W^*\big(\Lambda(\tau_t(a)) \otimes \Lambda(\tau_t(b))\big) \\ &= \nu^t (\Lambda \otimes \Lambda)\big(\Delta(\tau_t(b))(\tau_t(a) \otimes 1)\big) \\ &= \nu^t (\Lambda \otimes \Lambda)\big((\tau_t \otimes \tau_t)(\Delta(b)(a \otimes 1))\big) \\ &= (P^{it} \otimes P^{it}) W^*\big(\Lambda(a) \otimes \Lambda(b)\big). \end{aligned}$$

This concludes the proof of the proposition. □

COROLLARY 6.11. – *The linear spaces* $\Delta(A)(1 \otimes A)$ *and* $\Delta(A)(A \otimes 1)$ *are dense subsets of* $A \otimes A$.

Proof. – This follows immediately from Proposition 5.1 of [68]. □

7. The modular element of a C^*-algebraic quantum group

We will denote the W*-lift of ψ by $\tilde{\psi}$ and the W*-lift of σ' by $\tilde{\sigma}'$. So $\tilde{\psi}$ is a n.f.s. weight on $\tilde{A}$ with modular group $\tilde{\sigma}'$. Recall that $\tilde{\sigma}'_t \pi = \pi \sigma'_t$ for $t \in \mathbb{R}$. It is easy to see that $\tilde{\psi} = \tilde{\varphi} \tilde{R}$.

By Proposition 6.8, we have that $\varphi \sigma'_t = \nu^t \varphi$ for all $t \in \mathbb{R}$. So we get for all $t \in \mathbb{R}$ that $\tilde{\varphi} \tilde{\sigma}'_t = (\varphi \sigma'_t)\tilde{\ } = \nu^t \tilde{\varphi}$. Consequently, we can apply the adapted version of Radon–Nikodym (Theorem 1.16) to get hold of the modular element of $(\tilde{A}, \tilde{\Delta})$. This section revolves around the proof of the fact that this modular element is an unbounded group-like element.

For the notation used in the next definition, we refer to the second part of Section 1.4.

DEFINITION 7.1. – We define $\tilde{\delta}$ as the strictly positive element affiliated with the von Neumann algebra $\tilde{A}$ such that $\tilde{\sigma}_t(\tilde{\delta}) = \nu^t \tilde{\delta}$ for $t \in \mathbb{R}$ and $\tilde{\psi} = \tilde{\varphi}_{\tilde{\delta}}$.

Recall that the modular groups $\tilde{\sigma}$ and $\tilde{\sigma}'$ are related via the equality $\tilde{\sigma}'_t(x) = \tilde{\delta}^{it} \tilde{\sigma}_t(x) \tilde{\delta}^{-it}$ for $t \in \mathbb{R}$. So we have for $t \in \mathbb{R}$ that $\tilde{\sigma}'_t(\tilde{\delta}) = \nu^t \tilde{\delta}$.

Let us use this to introduce a special GNS-construction for $\tilde{\psi}$ (*see* the remarks before Proposition 1.15).

Notation 7.2. – We define $(H, \iota, \tilde{\Gamma})$ to be the GNS-construction for $\tilde{\psi} = \tilde{\varphi}_{\tilde{\delta}}$ constructed from $(H, \iota, \tilde{\Lambda})$ via $\tilde{\delta}$.

LEMMA 7.3. – *The operators* $\tilde{\Delta}(\tilde{\delta})$ *and* $\tilde{\delta} \otimes \tilde{\delta}$ *commute.*

Proof. – Fix $t \in \mathbb{R}$. Using Proposition 6.8, we get that

$$\begin{aligned} (\tilde{\sigma}_{-t} \tilde{\sigma}'_t \bar{\otimes} \tilde{\sigma}_{-t} \tilde{\sigma}'_t)\tilde{\Delta} &= (\tilde{\sigma}_{-t} \bar{\otimes} \tilde{\sigma}_{-t} \tilde{\sigma}'_t \tilde{\tau}_t)\tilde{\Delta} \tilde{\sigma}'_t \\ &= (\tilde{\sigma}_{-t} \tilde{\tau}_t \bar{\otimes} \tilde{\sigma}'_t \tilde{\tau}_t)\tilde{\Delta} \tilde{\sigma}_{-t} \tilde{\sigma}'_t = (\tilde{\sigma}_{-t} \bar{\otimes} \tilde{\sigma}'_t)\tilde{\Delta} \tilde{\tau}_t \tilde{\sigma}_{-t} \tilde{\sigma}'_t = \tilde{\Delta}\, \tilde{\sigma}_{-t} \tilde{\sigma}'_t. \end{aligned}$$

We have for all $x \in \tilde{A}$ that $(\tilde{\sigma}_{-t} \tilde{\sigma}'_t)(x) = \tilde{\delta}^{it} x \tilde{\delta}^{-it}$. Therefore we get

$$(\tilde{\delta}^{it} \otimes \tilde{\delta}^{it})\tilde{\Delta}(x)(\tilde{\delta}^{-it} \otimes \tilde{\delta}^{-it}) = \tilde{\Delta}(\tilde{\delta}^{it} x \tilde{\delta}^{-it})$$

for all $x \in \widetilde{A}$. This implies in particular for every $s \in \mathbb{R}$ that

$$(\widetilde{\delta}^{it} \otimes \widetilde{\delta}^{it})\widetilde{\Delta}(\widetilde{\delta}^{is})(\widetilde{\delta}^{-it} \otimes \widetilde{\delta}^{-it}) = \widetilde{\Delta}(\widetilde{\delta}^{it}\widetilde{\delta}^{is}\widetilde{\delta}^{-it}) = \widetilde{\Delta}(\widetilde{\delta}^{is}).$$

Hence $\widetilde{\Delta}(\widetilde{\delta})$ and $\widetilde{\delta} \otimes \widetilde{\delta}$ commute. □

Recall the following notation:

$$\mathcal{T}_{\widetilde{\psi}} = \{x \in \mathcal{N}_{\widetilde{\psi}} \cap \mathcal{N}_{\widetilde{\psi}}^* \mid x \text{ is analytic with respect to } \widetilde{\sigma}' \text{ and } \widetilde{\sigma}'_z(x) \in \mathcal{N}_{\widetilde{\psi}} \cap \mathcal{N}_{\widetilde{\psi}}^* \text{ for } z \in \mathbb{C}\}.$$

Remark 7.4. – Consider $a, b \in \mathcal{T}_\psi$. Using Proposition 5.20, we see that

$$\begin{aligned}
R\big((a\psi a^* \otimes \iota)\Delta(b^*b)\big) &= R\big((\psi \otimes \iota)((a^* \otimes 1)\Delta(b^*b)(a \otimes 1))\big) \\
&= R\big((\psi \otimes \iota)((\sigma'_i(a)\, a^* \otimes 1)\Delta(b^*b))\big) \\
&= (\psi \otimes \iota)\big(\Delta(\sigma'_{-\frac{i}{2}}(\sigma'_i(a)\, a^*))(\sigma'_{-\frac{i}{2}}(b^*b) \otimes 1)\big) \\
&= (\psi \otimes \iota)\big((\sigma'_{\frac{i}{2}}(b) \otimes 1)\Delta\big(\sigma'_{\frac{i}{2}}(a)\, \sigma'_{\frac{i}{2}}(a)^*\big)\big(\sigma'_{\frac{i}{2}}(b)^* \otimes 1\big)\big) \\
&= \big(\sigma'_{\frac{i}{2}}(b)^*\psi\sigma'_{\frac{i}{2}}(b) \otimes \iota\big)\Delta\big(\sigma'_{\frac{i}{2}}(a)\sigma'_{\frac{i}{2}}(a)^*\big).
\end{aligned}$$

We will also need this equation on the level of $\widetilde{A}$. In order to achieve this, we use an approximation procedure.

LEMMA 7.5. – *Consider $a, b \in \mathcal{T}_{\widetilde{\psi}}$. Then we have that*

$$\widetilde{R}\big((a\widetilde{\psi}a^* \,\overline{\otimes}\, \iota)\widetilde{\Delta}(b^*b)\big) = \big(\widetilde{\sigma}'_{\frac{i}{2}}(b)^*\widetilde{\psi}\widetilde{\sigma}'_{\frac{i}{2}}(b) \,\overline{\otimes}\, \iota\big)\widetilde{\Delta}\big(\widetilde{\sigma}'_{\frac{i}{2}}(a)\widetilde{\sigma}'_{\frac{i}{2}}(a)^*\big).$$

Proof. – By the previous remark, the formula is clear for $a, b \in \pi(\mathcal{T}_\psi)$. Now we define, for $x \in \widetilde{A}$ and $n \in \mathbb{N}$,

$$x(n) = \frac{n}{\sqrt{\pi}} \int \exp(-n^2t^2)\widetilde{\sigma}'_t(x)\, \mathrm{d}t.$$

Choose $a \in \mathcal{N}_{\widetilde{\psi}} \cap \mathcal{N}_{\widetilde{\psi}}^*$. By Proposition 2.22 of [30], we can take a net $(a_\alpha)_{\alpha \in I}$ in $\pi(\mathcal{N}_\psi \cap \mathcal{N}_\psi^*)$ such that $(a_\alpha)_{\alpha\in I} \to a$ strong* and bounded, $(\widetilde{\Gamma}(a_\alpha))_{\alpha \in I} \to \widetilde{\Gamma}(a)$, $(\widetilde{\Gamma}(a_\alpha^*))_{\alpha\in I} \to \widetilde{\Gamma}(a^*)$ in norm. Then we have, for every $n \in \mathbb{N}$ and $\alpha \in I$, that $a(n) \in \mathcal{T}_{\widetilde{\psi}}$ and $a_\alpha(n) \in \pi(\mathcal{T}_\psi)$. We see that, for every $n \in \mathbb{N}$ and every $z \in \mathbb{C}$,

$$\begin{aligned}
\big(\widetilde{\sigma}'_z(a_\alpha(n))\big)_{\alpha\in I} &\longrightarrow \widetilde{\sigma}'_z\big(a(n)\big) \quad \text{strong* and bounded,} \\
\big(\widetilde{\Gamma}(\widetilde{\sigma}'_z(a_\alpha(n)))\big)_{\alpha\in I} &\longrightarrow \widetilde{\Gamma}\big(\widetilde{\sigma}'_z(a(n))\big), \\
\big(\widetilde{\Gamma}(\widetilde{\sigma}'_z(a_\alpha(n)^*))\big)_{\alpha\in I} &\longrightarrow \widetilde{\Gamma}\big(\widetilde{\sigma}'_z(a(n)^*)\big).
\end{aligned}$$

Observe that $x\widetilde{\psi}x^* = \omega_{\widetilde{\Gamma}(x),\widetilde{\Gamma}(x)}$, for every $x \in \mathcal{N}_{\widetilde{\psi}}$. Then we may conclude that the formula is valid for $a(n)$ and $b(n)$ whenever $a, b \in \mathcal{N}_{\widetilde{\psi}} \cap \mathcal{N}_{\widetilde{\psi}}^*$ and $n \in \mathbb{N}$ by approximating a and b as above. Now choose $x \in \mathcal{T}_{\widetilde{\psi}}$. Then we have

$$\begin{aligned}
\big(\widetilde{\sigma}'_z(x(n))\big)_{n=1}^\infty &\longrightarrow \sigma'_z(x) \quad \text{strong* and bounded,} \\
\big(\widetilde{\Gamma}(\widetilde{\sigma}'_z(x(n)))\big)_{n=1}^\infty &\longrightarrow \widetilde{\Gamma}\big(\widetilde{\sigma}'_z(x)\big), \\
\big(\widetilde{\Gamma}(\widetilde{\sigma}'_z(x(n)^*))\big)_{n=1}^\infty &\longrightarrow \widetilde{\Gamma}\big(\widetilde{\sigma}'_z(x^*)\big).
\end{aligned}$$

Now we may conclude the formula is valid for $a,b \in \mathcal{T}_{\widetilde{\psi}}$, because we already know it is valid for all $a(n)$ and $b(n)$. □

The following result will be used in the proof of the group-like property of $\widetilde{\delta}$. It has some intrinsic interest because it gives a precise meaning to the algebraic property $(\iota \otimes \psi)\Delta(a) = \psi(a)\delta^{-1}$ appearing in [59].

RESULT 7.6. – *Consider $a \in \mathcal{M}^+_{\widetilde{\psi}}$ and $v \in D(\widetilde{\delta}^{-\frac{1}{2}})$. Then*

$$\widetilde{\psi}\big((\omega_{v,v} \,\overline{\otimes}\, \iota)\widetilde{\Delta}(a)\big) = \widetilde{\psi}(a)\langle \widetilde{\delta}^{-\frac{1}{2}} v, \widetilde{\delta}^{-\frac{1}{2}} v\rangle.$$

Proof. – Let $n \in \mathbb{N}$. Then we define the element $e_n \in \widetilde{A}$ by the following integral in the strong topology:

$$e_n = \frac{n}{\sqrt{\pi}} \int \exp(-n^2 t^2)\widetilde{\delta}^{it}\, \mathrm{d}t.$$

Using the fact that $\widetilde{\sigma}'_s(\delta^{it}) = \nu^{ist}\delta^{it}$ for $s,t \in \mathbb{R}$, it is then a standard exercise to check that:

(1) e_n is analytic with respect to $\widetilde{\sigma}'$;
(2) $\widetilde{\delta}^{-\frac{1}{2}} e_n$ is bounded and $\widetilde{\delta}^{-\frac{1}{2}} e_n$ is analytic with respect to $\widetilde{\sigma}'$;
(3) $\widetilde{\delta}^{-\frac{1}{2}} \widetilde{\sigma}'_{\frac{i}{2}}(e_n)$ is bounded and $\widetilde{\delta}^{-\frac{1}{2}} \widetilde{\sigma}'_{\frac{i}{2}}(e_n)$ is analytic with respect to $\widetilde{\sigma}'$;
(4) $\widetilde{\sigma}'_{\frac{i}{2}}(\widetilde{\delta}^{-\frac{1}{2}} e_n) = \nu^{-\frac{i}{4}} \widetilde{\delta}^{-\frac{1}{2}} \widetilde{\sigma}'_{\frac{i}{2}}(e_n)$.

Define $C = \langle e_n x \mid x \in \mathcal{T}_{\widetilde{\psi}},\ n \in \mathbb{N}\rangle$. Using the above properties of the elements e_n $(n \in \mathbb{N})$, it is not difficult to see that any element $c \in C$ satisfies the following properties:

(1) $c \in \mathcal{T}_{\widetilde{\psi}}$;
(2) $\widetilde{\delta}^{-\frac{1}{2}} c$ is bounded and $\widetilde{\delta}^{-\frac{1}{2}} c$ belongs to $D(\widetilde{\sigma}'_{\frac{i}{2}}) \cap \mathcal{N}_{\widetilde{\psi}}$;
(3) $\widetilde{\delta}^{-\frac{1}{2}} \widetilde{\sigma}'_{\frac{i}{2}}(c)$ is bounded and $\widetilde{\delta}^{-\frac{1}{2}} \widetilde{\sigma}'_{\frac{i}{2}}(c)$ belongs to $\mathcal{N}^*_{\widetilde{\psi}}$;
(4) $\widetilde{\sigma}'_{\frac{i}{2}}(\widetilde{\delta}^{-\frac{1}{2}} c) = \nu^{-\frac{i}{4}} \widetilde{\delta}^{-\frac{1}{2}} \widetilde{\sigma}'_{\frac{i}{2}}(c)$.

Choose $c \in C$ and $d \in \mathcal{T}_{\widetilde{\psi}}$.

Since $\widetilde{\delta}^{-\frac{1}{2}} \widetilde{\sigma}'_{\frac{i}{2}}(c)$ is bounded and $\widetilde{\delta}^{-\frac{1}{2}} \widetilde{\sigma}'_{\frac{i}{2}}(c)$ belongs to $\mathcal{N}^*_{\widetilde{\psi}}$, we get that $\widetilde{\sigma}'_{\frac{i}{2}}(c)^* \widetilde{\delta}^{-\frac{1}{2}}$ is bounded and

$$\overline{\widetilde{\sigma}'_{\frac{i}{2}}(c)^* \widetilde{\delta}^{-\frac{1}{2}}} = \big(\widetilde{\delta}^{-\frac{1}{2}}\, \widetilde{\sigma}'_{\frac{i}{2}}(c)\big)^* \in \mathcal{N}_{\widetilde{\psi}}.$$

Because $\widetilde{\varphi} = \widetilde{\psi}_{\widetilde{\delta}^{-1}}$, this implies that $\widetilde{\sigma}'_{\frac{i}{2}}(c)^*$ belongs to $\mathcal{N}_{\widetilde{\varphi}}$ and

$$\widetilde{\varphi}\big(\widetilde{\sigma}'_{\frac{i}{2}}(c)\widetilde{\sigma}'_{\frac{i}{2}}(c)^*\big) = \widetilde{\psi}\big(\overline{\widetilde{\sigma}'_{\frac{i}{2}}(c)^* \widetilde{\delta}^{-\frac{1}{2}}}^{\,*}\ \overline{\widetilde{\sigma}'_{\frac{i}{2}}(c)^* \widetilde{\delta}^{-\frac{1}{2}}}\big) = \widetilde{\psi}\big(\big(\widetilde{\delta}^{-\frac{1}{2}}\, \widetilde{\sigma}'_{\frac{i}{2}}(c)\big)\, \big(\widetilde{\delta}^{-\frac{1}{2}}\, \widetilde{\sigma}'_{\frac{i}{2}}(c)\big)^*\big).$$

We know moreover that $\widetilde{\delta}^{-\frac{1}{2}} c$ belongs to $D(\widetilde{\sigma}'_{\frac{i}{2}}) \cap \mathcal{N}_{\widetilde{\psi}}$ and that $\widetilde{\sigma}'_{\frac{i}{2}}(\widetilde{\delta}^{-\frac{1}{2}} c) = \nu^{-\frac{i}{4}} \widetilde{\delta}^{-\frac{1}{2}} \widetilde{\sigma}'_{\frac{i}{2}}(c)$. Plugging this in into the above equation, we find that:

$$\widetilde{\varphi}\big(\widetilde{\sigma}'_{\frac{i}{2}}(c)\widetilde{\sigma}'_{\frac{i}{2}}(c)^*\big) = \widetilde{\psi}\big(\widetilde{\sigma}'_{\frac{i}{2}}\big(\widetilde{\delta}^{-\frac{1}{2}} c\big)\, \widetilde{\sigma}'_{\frac{i}{2}}\big(\widetilde{\delta}^{-\frac{1}{2}} c\big)^*\big) = \widetilde{\psi}\big(\big(\widetilde{\delta}^{-\frac{1}{2}} c\big)^*\, \big(\widetilde{\delta}^{-\frac{1}{2}} c\big)\big). \tag{7.1}$$

By the previous lemma, we know that

$$\widetilde{R}\big((c\widetilde{\psi}c^* \,\overline{\otimes}\, \iota)\widetilde{\Delta}(d^*d)\big) = \big(\widetilde{\sigma}'_{\frac{i}{2}}(d)^* \widetilde{\psi} \widetilde{\sigma}'_{\frac{i}{2}}(d) \,\overline{\otimes}\, \iota\big)\widetilde{\Delta}\big(\widetilde{\sigma}'_{\frac{i}{2}}(c)\widetilde{\sigma}'_{\frac{i}{2}}(c)^*\big).$$

Therefore the left invariance of $\widetilde{\varphi}$ implies that

$$\begin{aligned}\widetilde{\psi}\big((c\widetilde{\psi}c^*\,\overline{\otimes}\,\iota)\widetilde{\Delta}(d^*d)\big) &= \widetilde{\varphi}\big(\widetilde{R}\big((c\widetilde{\psi}c^*\,\overline{\otimes}\,\iota)\widetilde{\Delta}(d^*d)\big)\big)\\ &= \widetilde{\varphi}\big((\widetilde{\sigma}'_{\frac{i}{2}}(d)^*\widetilde{\psi}\widetilde{\sigma}'_{\frac{i}{2}}(d)\,\overline{\otimes}\,\iota)\widetilde{\Delta}(\widetilde{\sigma}'_{\frac{i}{2}}(c)\widetilde{\sigma}'_{\frac{i}{2}}(c)^*)\big)\\ &= \widetilde{\psi}\big(\widetilde{\sigma}'_{\frac{i}{2}}(d)\,\widetilde{\sigma}'_{\frac{i}{2}}(d)^*\big)\widetilde{\varphi}\big(\widetilde{\sigma}'_{\frac{i}{2}}(c)\widetilde{\sigma}'_{\frac{i}{2}}(c)^*\big)\\ &= \widetilde{\psi}(d^*d)\widetilde{\psi}\big((\widetilde{\delta}^{-\frac{1}{2}}c)^*(\widetilde{\delta}^{-\frac{1}{2}}c)\big),\end{aligned}$$

where we used Eq. (7.1) in the last equality.

Since $c\in\mathcal{N}_{\widetilde{\psi}}$ and $\widetilde{\delta}^{-\frac{1}{2}}c\in\mathcal{N}_{\widetilde{\psi}}$, we have that $\widetilde{\Gamma}(c)$ belongs to $D(\widetilde{\delta}^{-\frac{1}{2}})$ and $\widetilde{\delta}^{-\frac{1}{2}}\widetilde{\Gamma}(c) = \widetilde{\Gamma}(\widetilde{\delta}^{-\frac{1}{2}}c)$. So the above equation becomes

$$\widetilde{\psi}\big((\omega_{\widetilde{\Gamma}(c),\widetilde{\Gamma}(c)}\,\overline{\otimes}\,\iota)\widetilde{\Delta}(d^*d)\big) = \widetilde{\psi}(d^*d)\big\langle\widetilde{\delta}^{-\frac{1}{2}}\widetilde{\Gamma}(c),\widetilde{\delta}^{-\frac{1}{2}}\widetilde{\Gamma}(c)\big\rangle. \tag{7.2}$$

We have for every $n\in\mathbb{N}$ and $x\in\mathcal{T}_{\widetilde{\psi}}$ that

$$\widetilde{\Gamma}(e_n x) = e_n\widetilde{\Gamma}(x).$$

Because $\widetilde{\Gamma}(\mathcal{T}_{\widetilde{\psi}})$ is dense in H, we infer from this that $\widetilde{\Gamma}(C)$ is a core for $\widetilde{\delta}^{-\frac{1}{2}}$.

Appealing to Lemma A.1, Eq. (7.2) now implies that

$$\widetilde{\psi}\big((\omega_{v,v}\,\overline{\otimes}\,\iota)\widetilde{\Delta}(d^*d)\big) = \widetilde{\psi}(d^*d)\big\langle\widetilde{\delta}^{-\frac{1}{2}}v,\widetilde{\delta}^{-\frac{1}{2}}v\big\rangle$$

for all $v\in D(\widetilde{\delta}^{-\frac{1}{2}})$ and $d\in\mathcal{T}_{\widetilde{\psi}}$.

Since $\mathcal{T}_{\widetilde{\psi}}$ is a 'bounded' σ-strong* core for $\widetilde{\Gamma}$, we can again use Lemma A.1 to conclude that

$$\widetilde{\psi}\big((\omega_{v,v}\,\overline{\otimes}\,\iota)\widetilde{\Delta}(b^*b)\big) = \widetilde{\psi}(b^*b)\big\langle\widetilde{\delta}^{-\frac{1}{2}}v,\widetilde{\delta}^{-\frac{1}{2}}v\big\rangle$$

for all $v\in D(\widetilde{\delta}^{-\frac{1}{2}})$ and $b\in\mathcal{N}_{\widetilde{\psi}}$. □

In the next results, we will use the normal *-homomorphism $\widetilde{\Delta}^{(2)}:\widetilde{A}\to\widetilde{A}\,\overline{\otimes}\,\widetilde{A}\,\overline{\otimes}\,\widetilde{A}$ given by $\widetilde{\Delta}^{(2)} = (\widetilde{\Delta}\,\overline{\otimes}\,\iota)\widetilde{\Delta} = (\iota\,\overline{\otimes}\,\widetilde{\Delta})\widetilde{\Delta}$.

LEMMA 7.7. – *Consider* $a\in\mathcal{M}^+_{\widetilde{\psi}}$, $v\in D(\widetilde{\delta}^{-\frac{1}{2}}\otimes\widetilde{\delta}^{-\frac{1}{2}})$. *Then*

$$\widetilde{\psi}\big((\omega_{v,v}\,\overline{\otimes}\,\iota)\widetilde{\Delta}^{(2)}(a)\big) = \widetilde{\psi}(a)\big\langle\big(\widetilde{\delta}^{-\frac{1}{2}}\otimes\widetilde{\delta}^{-\frac{1}{2}}\big)v,\big(\widetilde{\delta}^{-\frac{1}{2}}\otimes\widetilde{\delta}^{-\frac{1}{2}}\big)v\big\rangle.$$

Proof. – Choose $p_1,p_2,q_1,q_2\in D(\widetilde{\delta}^{-\frac{1}{2}})$. By Result 7.6 (and polarization), we have that $(\omega_{p_1,q_1}\,\overline{\otimes}\,\iota)\widetilde{\Delta}(a)$ belongs to $\mathcal{M}_{\widetilde{\psi}}$ and

$$\widetilde{\psi}\big((\omega_{p_1,q_1}\,\overline{\otimes}\,\iota)\widetilde{\Delta}(a)\big) = \widetilde{\psi}(a)\big\langle\widetilde{\delta}^{-\frac{1}{2}}p_1,\widetilde{\delta}^{-\frac{1}{2}}q_1\big\rangle.$$

Applying Result 7.6 once more, we see that $(\omega_{p_2,q_2}\,\overline{\otimes}\,\iota)\widetilde{\Delta}((\omega_{p_1,q_1}\,\overline{\otimes}\,\iota)\widetilde{\Delta}(a))$ belongs to $\mathcal{M}_{\widetilde{\psi}}$ and

$$\begin{aligned}
&\tilde{\psi}\big((\omega_{p_2,q_2}\,\overline{\otimes}\,\iota)\tilde{\Delta}((\omega_{p_1,q_1}\,\overline{\otimes}\,\iota)\tilde{\Delta}(a))\big)\\
&\quad=\tilde{\psi}\big((\omega_{p_1,q_1}\,\overline{\otimes}\,\iota)\tilde{\Delta}(a)\big)\big\langle\tilde{\delta}^{-\frac{1}{2}}p_2,\tilde{\delta}^{-\frac{1}{2}}q_2\big\rangle\\
&\quad=\tilde{\psi}(a)\big\langle\tilde{\delta}^{-\frac{1}{2}}p_1,\tilde{\delta}^{-\frac{1}{2}}q_1\big\rangle\big\langle\tilde{\delta}^{-\frac{1}{2}}p_2,\tilde{\delta}^{-\frac{1}{2}}q_2\big\rangle\\
&\quad=\tilde{\psi}(a)\big\langle\big(\tilde{\delta}^{-\frac{1}{2}}\otimes\tilde{\delta}^{-\frac{1}{2}}\big)(p_1\otimes p_2),\big(\tilde{\delta}^{-\frac{1}{2}}\otimes\tilde{\delta}^{-\frac{1}{2}}\big)(q_1\otimes q_2)\big\rangle.
\end{aligned}$$

Because $(\omega_{p_2,q_2}\,\overline{\otimes}\,\iota)\tilde{\Delta}((\omega_{p_1,q_1}\,\overline{\otimes}\,\iota)\tilde{\Delta}(a))=(\omega_{p_1\otimes p_2,q_1\otimes q_2}\,\overline{\otimes}\,\iota)\tilde{\Delta}^{(2)}(a)$, we see that

$$(\omega_{p_1\otimes p_2,q_1\otimes q_2}\,\overline{\otimes}\,\iota)\tilde{\Delta}^{(2)}(a)$$

belongs to $\mathcal{M}_{\tilde{\psi}}$ and

$$\tilde{\psi}\big((\omega_{p_1\otimes p_2,q_1\otimes q_2}\,\overline{\otimes}\,\iota)\tilde{\Delta}^{(2)}(a)\big)=\tilde{\psi}(a)\big\langle\big(\tilde{\delta}^{-\frac{1}{2}}\otimes\tilde{\delta}^{-\frac{1}{2}}\big)(p_1\otimes p_2),\big(\tilde{\delta}^{-\frac{1}{2}}\otimes\tilde{\delta}^{-\frac{1}{2}}\big)(q_1\otimes q_2)\big\rangle.$$

Therefore we get for all $w\in D(\tilde{\delta}^{-\frac{1}{2}})\odot D(\tilde{\delta}^{-\frac{1}{2}})$ that

$$\tilde{\psi}\big((\omega_{w,w}\,\overline{\otimes}\,\iota)\tilde{\Delta}^{(2)}(a)\big)=\tilde{\psi}(a)\big\langle\big(\tilde{\delta}^{-\frac{1}{2}}\otimes\tilde{\delta}^{-\frac{1}{2}}\big)w,\big(\tilde{\delta}^{-\frac{1}{2}}\otimes\tilde{\delta}^{-\frac{1}{2}}\big)w\big\rangle.$$

Because $D(\tilde{\delta}^{-\frac{1}{2}})\odot D(\tilde{\delta}^{-\frac{1}{2}})$ is a core for $\tilde{\delta}^{-\frac{1}{2}}\otimes\tilde{\delta}^{-\frac{1}{2}}$, the lemma follows now by using Lemma A.1. □

LEMMA 7.8. – *Consider* $a\in\mathcal{M}_{\tilde{\psi}}^{+}$, $v\in D(\tilde{\Delta}(\tilde{\delta}^{-\frac{1}{2}}))$. *Then*

$$\tilde{\psi}\big((\omega_{v,v}\,\overline{\otimes}\,\iota)\tilde{\Delta}^{(2)}(a)\big)=\tilde{\psi}(a)\big\langle\tilde{\Delta}\big(\tilde{\delta}^{-\frac{1}{2}}\big)v,\tilde{\Delta}\big(\tilde{\delta}^{-\frac{1}{2}}\big)v\big\rangle.$$

Proof. – Let $(e_i)_{i\in I}$ be an orthonormal basis for H.

Fix $i\in I$ for the moment and define $\theta_i\in B(H,\mathbb{C})$ by $\theta_i(u)=\langle u,e_i\rangle$ for all $u\in H$. Put $w_i=(\theta_i\otimes 1)Wv$. It is easy to see that $(\theta_i\otimes 1)(1\otimes\tilde{\delta}^{-\frac{1}{2}})\subseteq\tilde{\delta}^{-\frac{1}{2}}(\theta_i\otimes 1)$. Since $\tilde{\Delta}(\tilde{\delta}^{-\frac{1}{2}})=W^*(1\otimes\tilde{\delta}^{-\frac{1}{2}})W$, we have that $Wv\in D(1\otimes\tilde{\delta}^{-\frac{1}{2}})$. So the previous inclusion implies that $w_i\in D(\tilde{\delta}^{-\frac{1}{2}})$ and

$$(\theta_i\otimes 1)\big(1\otimes\tilde{\delta}^{-\frac{1}{2}}\big)Wv=\tilde{\delta}^{-\frac{1}{2}}(\theta_i\otimes 1)Wv=\tilde{\delta}^{-\frac{1}{2}}w_i.$$

Now observe that for every $x\in\tilde{A}$

$$\begin{aligned}
\omega_{v,v}(\tilde{\Delta}(x))&=\big\langle(1\otimes x)Wv,Wv\big\rangle=\sum_{i\in I}\big\langle(\theta_i^*\theta_i\otimes 1)(1\otimes x)Wv,Wv\big\rangle=\sum_{i\in I}\langle xw_i,w_i\rangle\\
&=\sum_{i\in I}\omega_{w_i,w_i}(x).
\end{aligned}$$

Because $\tilde{\psi}$ is normal we have

$$\begin{aligned}
\tilde{\psi}\big((\omega_{v,v}\,\overline{\otimes}\,\iota)\tilde{\Delta}^{(2)}(a)\big)&=\tilde{\psi}\big((\omega_{v,v}\tilde{\Delta}\,\overline{\otimes}\,\iota)\tilde{\Delta}(a)\big)\\
&=\sum_{i\in I}\tilde{\psi}\big((\omega_{w_i,w_i}\,\overline{\otimes}\,\iota)\tilde{\Delta}(a)\big)\\
&\overset{(*)}{=}\sum_{i\in I}\tilde{\psi}(a)\big\|\tilde{\delta}^{-\frac{1}{2}}w_i\big\|^2
\end{aligned}$$

$$= \sum_{i\in I} \widetilde{\psi}(a)\big\|(\theta_i\otimes 1)(1\otimes \widetilde{\delta}^{-\frac{1}{2}})Wv\big\|^2$$
$$= \widetilde{\psi}(a)\big\|(1\otimes \widetilde{\delta}^{-\frac{1}{2}})Wv\big\|^2$$
$$= \widetilde{\psi}(a)\langle \widetilde{\Delta}(\widetilde{\delta}^{-\frac{1}{2}})v, \widetilde{\Delta}(\widetilde{\delta}^{-\frac{1}{2}})v\rangle,$$

where we used Result 7.6 in equality (*). □

These last lemmas allow us to prove easily that $\widetilde{\delta}$ is a corepresentation of $(\widetilde{A},\widetilde{\Delta})$:

PROPOSITION 7.9. – *We have that* $\widetilde{\Delta}(\widetilde{\delta}) = \widetilde{\delta}\otimes\widetilde{\delta}$.

Proof. – Take an element $a \in \mathcal{M}^+_{\widetilde{\psi}}$ such that $\widetilde{\psi}(a) = 1$.

Then Lemmas 7.7 and 7.8 imply for all $v \in D(\widetilde{\delta}^{-\frac{1}{2}}\otimes\widetilde{\delta}^{-\frac{1}{2}}) \cap D(\widetilde{\Delta}(\widetilde{\delta}^{-\frac{1}{2}}))$ that

$$\big\langle (\widetilde{\delta}^{-\frac{1}{2}}\otimes\widetilde{\delta}^{-\frac{1}{2}})v, (\widetilde{\delta}^{-\frac{1}{2}}\otimes\widetilde{\delta}^{-\frac{1}{2}})v\big\rangle = \widetilde{\psi}\big((\omega_{v,v}\,\overline{\otimes}\,\iota)\widetilde{\Delta}^{(2)}(a)\big) = \big\langle \widetilde{\Delta}(\widetilde{\delta}^{-\frac{1}{2}})v, \widetilde{\Delta}(\widetilde{\delta}^{-\frac{1}{2}})v\big\rangle. \tag{7.3}$$

By Lemma 7.3, we know that $\widetilde{\Delta}(\widetilde{\delta})$ and $\widetilde{\delta}\otimes\widetilde{\delta}$ commute. This implies the existence of a subspace C of $H\otimes H$ such that C is a core for both $\widetilde{\delta}^{-\frac{1}{2}}\otimes\widetilde{\delta}^{-\frac{1}{2}}$ and $\widetilde{\Delta}(\widetilde{\delta}^{-\frac{1}{2}})$. If we combine this with Eq. (7.3), it is not difficult to see that $D(\widetilde{\delta}^{-\frac{1}{2}}\otimes\widetilde{\delta}^{-\frac{1}{2}}) = D(\widetilde{\Delta}(\widetilde{\delta}^{-\frac{1}{2}}))$ and

$$\big\|(\widetilde{\delta}^{-\frac{1}{2}}\otimes\widetilde{\delta}^{-\frac{1}{2}})v\big\| = \big\|\widetilde{\Delta}(\widetilde{\delta}^{-\frac{1}{2}})v\big\|$$

for $v \in D(\widetilde{\delta}^{-\frac{1}{2}}\otimes\widetilde{\delta}^{-\frac{1}{2}})$. Consequently, $\widetilde{\Delta}(\widetilde{\delta}) = \widetilde{\delta}\otimes\widetilde{\delta}$. □

A standard trick allows us to prove easily that $\widetilde{\delta}$ is affiliated with $\pi(A)$ in the C*-algebra sense.

PROPOSITION 7.10. – *The operator $\widetilde{\delta}$ is affiliated with $\pi(A)$ in the* C*-*algebra sense.*

Proof. – The previous proposition implies that $W^*(1\otimes\widetilde{\delta})W = \widetilde{\delta}\otimes\widetilde{\delta}$. Therefore we have for every $t\in\mathbb{R}$ that $(1\otimes\widetilde{\delta}^{it})W(1\otimes\widetilde{\delta}^{-it}) = W(\widetilde{\delta}^{it}\otimes 1)$. This implies for every $t\in\mathbb{R}$ and $\omega\in \mathrm{B}_0(H)^*$ that

$$(\iota\otimes\omega)(W)\widetilde{\delta}^{it} = (\iota\otimes\widetilde{\delta}^{-it}\omega\widetilde{\delta}^{it})(W).$$

So we find for every $\omega\in \mathrm{B}_0(H)^*$ that:
(1) $(\iota\otimes\omega)(W)\widetilde{\delta}^{it}\in\pi(A)$ for every $t\in\mathbb{R}$;
(2) the mapping $\mathbb{R}\to\pi(A): t\mapsto (\iota\otimes\omega)(W)\widetilde{\delta}^{it}$ is norm continuous.
Because such elements $(\iota\otimes\omega)(W)$ are dense in $\pi(A)$, we get for any $a\in\pi(A)$ that:
(1) $a\,\widetilde{\delta}^{it}\in\pi(A)$ for every $t\in\mathbb{R}$;
(2) the mapping $\mathbb{R}\to\pi(A): t\mapsto a\,\widetilde{\delta}^{it}$ is norm continuous.
This implies that (*see* e.g. Proposition 3.12 of [25]) $\widetilde{\delta}$ is affiliated with $\pi(A)$ in the C*-algebra sense. □

Now we use the faithful *-isomorphism $\pi: A\to\pi(A)$ to pull $\widetilde{\delta}$ down to A:

DEFINITION 7.11. – We define the strictly positive element δ affiliated with A such that $\pi(\delta) = \widetilde{\delta}$.

In the next proposition, we collect all the basic properties concerning δ. Recall that we discussed the extension of S to the multiplier algebra in Remark 5.44.

PROPOSITION 7.12. – *We have the following properties*:

(1) $\Delta(\delta)=\delta\otimes\delta$;
(2) $\tau_t(\delta)=\delta$ *for* $t\in\mathbb{R}$ *and* $R(\delta)=\delta^{-1}$;
(3) *let* $t\in\mathbb{R}$. *Then* δ^{it} *belongs to* $D(\overline{S})$ *and* $S(\delta^{it})=\delta^{-it}$;
(4) $\sigma_t(\delta)=\sigma'_t(\delta)=\nu^t\delta$ *for all* $t\in\mathbb{R}$;
(5) $\sigma'_t(a)=\delta^{it}\sigma_t(a)\delta^{-it}$ *for all* $t\in\mathbb{R}$ *and* $a\in A$;
(6) $\psi=\varphi_\delta$.

Proof. – The results concerning σ and σ' follow from the corresponding results on the W*-level. We only have to consider the statements concerning S, R and τ.

Choose $s,t\in\mathbb{R}$. Then we have that

$$\tau_t(\delta^{is})\otimes\sigma_t(\delta^{is})=(\tau_t\otimes\sigma_t)\Delta(\delta^{is})=\Delta(\sigma_t(\delta^{is}))=\nu^{ist}\Delta(\delta^{is})=\nu^{ist}\delta^{is}\otimes\delta^{is}=\delta^{is}\otimes\sigma_t(\delta^{is}).$$

Thus $\tau_t(\delta^{is})=\delta^{is}$.

Because $1\otimes\delta^{it}=(\delta^{-it}\otimes 1)\Delta(\delta^{it})$ and $1\otimes\delta^{-it}=\Delta(\delta^{-it})(\delta^{it}\otimes 1)$, Remark 5.44 about Proposition 5.33 implies immediately that $\delta^{it}\in D(\overline{S})$ and $S(\delta^{it})=\delta^{-it}$. Since $\tau_s(\delta^{it})=\delta^{it}$ for $s,t\in\mathbb{R}$, we get that $\delta^{it}\in D(\bar\tau_{-\frac{i}{2}})$ and $\tau_{-\frac{i}{2}}(\delta^{it})=\delta^{it}$. Combining this with the fact that $S=R\tau_{-\frac{i}{2}}$, we get that $R(\delta^{it})=\delta^{-it}$. □

Let us also fix a particular GNS-construction for ψ (see the remarks before Proposition 1.14).

Notation 7.13. – We define (H,π,Γ) to be the GNS-construction for $\psi=\varphi_\delta$ constructed from (H,π,Λ) via δ.

We end this section with the general uniqueness theorem for invariant weights.

THEOREM 7.14. – *Consider a left invariant proper weight* η *on* (A,Δ). *Then there exists a number* $r>0$ *such that* $\eta=r\varphi$.

Proof. – Result 6.5 implies that η is a KMS weight. Denote its modular group by κ. Proposition 5.37 implies that

$$(\tau_t\otimes\kappa_t)\Delta=\Delta\kappa_t\quad\text{for } t\in\mathbb{R}.\tag{7.4}$$

Fix $s\in\mathbb{R}$ for the moment. Using Proposition 6.8 and the above commutation, we get that

$$(\iota\otimes\sigma_{-s}\kappa_s)\Delta=\Delta\sigma_{-s}\kappa_s.$$

Arguing as in the beginning of proof of Proposition 6.8, we get that ψ is invariant under $\sigma_{-s}\kappa_s$. So $\sigma_{-s}\kappa_s$ commutes with σ'. Since σ_s commutes with σ', we get that κ_s commutes with σ'.

Choose $s,t\in\mathbb{R}$. Using the commutation mentioned above, the fact that $\tau_t(\delta)=\delta$ and $\Delta(\delta)=\delta\otimes\delta$, we find that

$$\Delta(\kappa_t(\delta^{is}))=(\tau_t\otimes\kappa_t)\Delta(\delta^{is})=\tau_t(\delta^{is})\otimes\kappa_t(\delta^{is})=\delta^{is}\otimes\kappa_t(\delta^{is}).$$

Hence

$$\Delta(\kappa_t(\delta^{is})\delta^{-is})=1\otimes\kappa_t(\delta^{is})\delta^{-is}.$$

Consequently, Result 5.13 implies the existence of a complex number $u\in\mathbb{C}$ with $|u|=1$ such that $\kappa_t(\delta^{is})=u\delta^{is}$.

Take $s\in\mathbb{R}$. Because $\sigma_t(x)=\delta^{-it}\sigma'_t(x)\delta^{it}$ for all $t\in\mathbb{R}$ and $x\in A$ and because we have already proven that σ' and κ_s commute, we conclude that σ and κ_s commute. So $\varphi\kappa_s$ is a KMS weight which is invariant under σ. The relation (7.4) above implies that $\varphi\kappa_s$ is left invariant. Therefore, Corollary 6.7 implies the existence of a number $\lambda_t > 0$ such that $\varphi\kappa_t = \lambda_t\varphi$. As usual, this implies the existence of a number $\lambda > 0$ such that $\lambda_t = \lambda^t$ for $t \in \mathbb{R}$.

The theorem follows now from corollary 6.7 (it is clear that we can reverse the roles of η and φ in this corollary). □

Combining this theorem with the equality $\chi(R \otimes R)\Delta = \Delta R$, uniqueness for right invariant weights is an immediate consequence.

THEOREM 7.15. – *Consider a right invariant proper weight η on (A, Δ). Then there exists a number $r > 0$ such that $\eta = r\psi$.*

Since we have established the uniqueness of the left invariant proper weights up to scalar, we can introduce some new terminology.

Terminology 7.16. –

- The number ν, determined by either one of the four conditions in Proposition 6.8 (3), is called the scaling constant of (A, Δ).
- The element δ, determined by Proposition 7.12 (6), is called the modular element of (A, Δ).

8. The reduced dual of a reduced C^*-algebraic quantum group

In this section, we construct the dual of a reduced C^*-algebraic quantum group following Chapter 3 of [15] (*see* also [6] and [33]) and relying on [68] (in which most of the hard work has already been done). The construction of the dual is a generalization of the construction of the reduced group C^*-algebra of a locally compact group to the quantum world.

We model the definition of the dual weight on Definition 3.5.2 of [15] (and even more on Proposition 3.5.4 of [15]). We will however follow a different route (not using left Hilbert algebras and the somewhat tricky *-operation) to define it, but this is not very essential. We give however a more elementary proof of the left invariance of the dual weight.

In the rest of this section, Σ denotes the flip map on $H \otimes H$.

DEFINITION 8.1. – We define:

- the set $\widehat{A} = [(\omega \otimes \iota)(W) \mid \omega \in \mathrm{B}_0(H)^*]$;
- the injective linear map $\widehat{\Delta} : \widehat{A} \to \mathrm{B}(H \otimes H)$ such that $\widehat{\Delta}(x) = \Sigma W(x \otimes 1)W^*\Sigma$ for all $x \in \widehat{A}$.

The flip map Σ is introduced to guarantee that the dual weight, constructed from the left invariant weight φ in the next part of this section, is again left invariant and not right invariant.

Thanks to Theorem 1.5 and Proposition 5.1 of [68], we can sit back and relax for a moment, allowing someone else to do the hard work for us.

THEOREM 8.2. – *The set $\widehat{A}$ is a non-degenerate sub-C^*-algebra of $\mathrm{B}(H)$ and the mapping $\widehat{\Delta}$ is a non-degenerate *-homomorphism from $\widehat{A}$ into $\mathrm{M}(\widehat{A} \otimes \widehat{A})$ such that:*

- $(\widehat{\Delta} \otimes \iota)\widehat{\Delta} = (\iota \otimes \widehat{\Delta})\widehat{\Delta}$;
- $\widehat{\Delta}(\widehat{A})(\widehat{A} \otimes 1)$ *and* $\widehat{\Delta}(\widehat{A})(1 \otimes \widehat{A})$ *are dense subsets of* $\widehat{A} \otimes \widehat{A}$.

Following Notation 3.19 and Proposition 3.21, we define the unitary element $V \in \mathcal{L}(A \otimes H) = \mathrm{M}(A \otimes \mathrm{B}_0(H))$ such that $V^*(a \otimes \Lambda(b)) = (\iota \otimes \Lambda)(\Delta(b))\, a$ for $a \in A$ and $b \in \mathcal{N}_\varphi$. Then it is not so difficult to check that

$$(\iota \otimes \omega_{\Lambda(a),\Lambda(b)})(V) = (\iota \otimes \varphi)\big(\Delta(b^*)(1 \otimes a)\big) \tag{8.1}$$

for $a,b\in\mathcal{N}_\varphi$. By Result 2.10, this implies that $(\pi\otimes\iota)(V)=W$. So, because π is faithful, V is just a copy of W, but with its first leg standing firmly in A.

The same Theorem 1.5 of [68] implies that $V\in \mathrm{M}(A\otimes\widehat{A})$. Since $W_{12}W_{13}W_{23}=W_{23}W_{12}$, we have moreover that $(\Delta\otimes\iota)(V)=V_{13}V_{23}$ and $(\iota\otimes\widehat{\Delta})(V)=V_{13}V_{12}$. The element V is called the left regular corepresentation of (A,Δ). It is at the same time the right regular corepresentation of $(\widehat{A},\widehat{\Delta})$.

The antipode has the following characterization in terms of V. This also shows that our antipode essentially coincides with the antipode defined in [68] (*cf.* Theorem 1.4.5 of [68]).

PROPOSITION 8.3. – *For all $\omega\in \mathrm{B}_0(H)^*$, we have that $(\iota\otimes\omega)(V)$ belongs to $D(S)$ and*

$$S\big((\iota\otimes\omega)(V)\big)=(\iota\otimes\omega)(V^*).$$

Moreover, the set

$$\{(\iota\otimes\omega)(V)\mid \omega\in \mathrm{B}_0(H)^*\}$$

is a core for S.

Proof. – Take $a,b\in\mathcal{N}_\varphi$. Eq. (8.1) implies that $(\iota\otimes\omega_{\Lambda(a),\Lambda(b)})(V^*)=(\iota\otimes\varphi)((1\otimes b^*)\Delta(a))$ and $(\iota\otimes\omega_{\Lambda(a),\Lambda(b)})(V)=(\iota\otimes\varphi)(\Delta(b^*)(1\otimes a))$. So Proposition 5.40 implies that $(\iota\otimes\omega_{\Lambda(a),\Lambda(b)})(V)$ belongs to $D(S)$ and

$$S\big((\iota\otimes\omega_{\Lambda(a),\Lambda(b)})(V)\big)=(\iota\otimes\omega_{\Lambda(a),\Lambda(b)})(V^*).$$

Therefore, the closedness of S implies easily for all $v,w\in H$ that $(\iota\otimes\omega_{v,w})(V)$ belongs to $D(S)$ and $S((\iota\otimes\omega_{v,w})(V))=(\iota\otimes\omega_{v,w})(V^*)$. Since any element of $\mathrm{B}_0(H)^*$ can be written as a norm convergent sum of vector functionals, the closedness of S implies for every $\omega\in\mathrm{B}_0(H)^*$ that $(\iota\otimes\omega)(V)$ belongs to $D(S)$ and $S((\iota\otimes\omega)(V))=(\iota\otimes\omega)(V^*)$.

The statement concerning the core follows immediately from Proposition 5.40. □

Let us strengthen the analogy with the group case by introducing the closed subspace $\mathrm{L}^1(A)$ of A^*:

$$\mathrm{L}^1(A)=[a\varphi b^*\mid a,b\in\mathcal{N}_\varphi].$$

By Proposition 3.6, it is clear that we also could have chosen ψ in order to define $\mathrm{L}^1(A)$. Notice that the linear mapping $\pi_*:\widetilde{A}_*\to\mathrm{L}^1(A):\omega\to\omega\pi$ is an isometric isomorphism. So we get also that $\mathrm{L}^1(A)=\{\omega\pi\mid\omega\in\mathrm{B}_0(H)^*\}$. Notice that $\widehat{A}=[(\omega\otimes\iota)(V)\mid\omega\in\mathrm{L}^1(A)]$

Define the contractive linear mapping $\lambda:\mathrm{L}^1(A)\to\widehat{A}$ such that

$$\lambda(\omega)=(\omega\otimes\iota)(V) \tag{8.2}$$

for $\omega\in\mathrm{L}^1(A)$. Notice that Eq. (8.1) implies that λ is an injection. Result 2.10 tells us that

$$(\omega\otimes\iota)(V^*)\Lambda(x)=\Lambda\big((\omega\otimes\iota)\Delta(x)\big) \tag{8.3}$$

for all $x\in\mathcal{N}_\varphi$ and $\omega\in\mathrm{L}^1(A)$.

The topological dual A^* is a Banach algebra under the multiplication $A^*\times A^*\to A^*:(\omega,\theta)\mapsto\omega\theta$ given by $(\omega\theta)(x)=(\omega\otimes\theta)\Delta(x)$ for all ω and θ in A^* and $x\in A$.

In the group case, this corresponds to the convolution product. It should be pointed out that, since the antipode can be unbounded, A^* carries no appropriate $*$-structure. It is however possible to define a $*$-operation on a large subalgebra of A^*. We will not go further into this.

Take $\omega \in A^*_+$ and $a, b \in \mathcal{N}_\varphi$. Fix also a cyclic GNS-construction (K, θ, u) for ω. Then we have for all $x \in A$ that

$$\begin{aligned}\big(\omega(a\varphi b^*)\big)(x) &= \big(\omega \otimes (a\varphi b^*)\big)(\Delta(x)) = (\omega \otimes \omega_{\Lambda(a),\Lambda(b)})\big(V^*(1 \otimes \pi(x))V\big) \\ &= \big\langle (1 \otimes \pi(x))(\theta \otimes \iota)(V)(u \otimes \Lambda(a)), (\theta \otimes \iota)(V)(u \otimes \Lambda(b))\big\rangle,\end{aligned}$$

which implies that $\omega(a\varphi b^*) \in \mathrm{L}^1(A)$. We conclude that $\mathrm{L}^1(A)$ is a left ideal in A^*. Because $\widetilde{R}$ is implemented by I we see that $\omega R \in \mathrm{L}^1(A)$ whenever $\omega \in \mathrm{L}^1(A)$. By using Proposition 5.26, we get that $\mathrm{L}^1(A)$ is a two-sided ideal in A^*, so it is certainly a subalgebra of A^*. It is also clear that for every $\omega \in \mathrm{L}^1(A)$ and every $a \in \mathrm{M}(A)$, the elements ωa and $a\omega$ belong to $\mathrm{L}^1(A)$.

The equality $(\Delta \otimes \iota)(V) = V_{13}V_{23}$ implies that $\lambda : \mathrm{L}^1(A) \to \mathrm{B}(H)$ is multiplicative.

In a next natural step, we want to turn $(\widehat{A}, \widehat{\Delta})$ into a reduced C*-algebraic quantum group by constructing a left and a right invariant weight on $(\widehat{A}, \widehat{\Delta})$. Copying Definition 2.1.6 of [15], we introduce the following subset of $\mathrm{L}^1(A)$.

Notation 8.4. – We define the subset $\mathcal{I}$ of $\mathrm{L}^1(A)$ as follows:

$$\mathcal{I} = \big\{\omega \in \mathrm{L}^1(A) \mid \text{there exists a number } M \geqslant 0 \text{ such that } |\omega(x^*)| \leqslant M\|\Lambda(x)\| \text{ for all } x \in \mathcal{N}_\varphi\big\}.$$

It is clear that $\mathcal{I}$ is a subspace of $\mathrm{L}^1(A)$. By Riesz' theorem for Hilbert spaces, there exists for every $\omega \in \mathcal{I}$ a unique element $\xi(\omega) \in H$ such that $\omega(x^*) = \langle \xi(\omega), \Lambda(x)\rangle$ for $x \in \mathcal{N}_\varphi$.

The linear map $\mathcal{I} \to H : \omega \mapsto \xi(\omega)$ should be thought of as a GNS-map (more precisely, a restriction of such a map) of a still to be constructed weight. It is not so difficult to see that this linear map is closed.

It is easy to construct an enormous amount of elements belonging to $\mathcal{I}$ (Proposition 2.1.7 (i) of [15]).

LEMMA 8.5. – *Let $a, b \in \mathcal{T}_\varphi$. Then $a\varphi b^*$ belongs to $\mathcal{I}$ and $\xi(a\varphi b^*) = \Lambda(a\,\sigma_i(b)^*)$.*

Proof. – We have for all $x \in \mathcal{N}_\varphi$ that

$$(a\varphi b^*)(x^*) = \varphi(b^*x^*a) = \varphi(x^*a\sigma_i(b)^*) = \big\langle \Lambda(a\sigma_i(b)^*), \Lambda(x)\big\rangle,$$

implying that $a\varphi b^*$ belongs to $\mathcal{I}$ and $\xi(a\varphi b^*) = \Lambda(a\sigma_i(b)^*)$. □

This implies immediately that $\{\xi(\omega) \mid \omega \in \mathcal{I}\}$ is dense in H and that $\mathcal{I}$ is dense in $\mathrm{L}^1(A)$.

Let us collect two basic properties in the next result (Proposition 2.1.7(iii) and Proposition 3.5.1(iii) of [15]).

RESULT 8.6. – *The following holds.*
- *The set $\mathcal{I}$ is a left ideal in $\mathrm{L}^1(A)$ such that $\xi(\omega\theta) = \lambda(\omega)\xi(\theta)$ for $\omega \in \mathrm{L}^1(A)$ and $\theta \in \mathcal{I}$.*
- *We have for $a \in \mathrm{M}(A)$ and $\omega \in \mathcal{I}$ that $a\omega$ belongs to $\mathcal{I}$ and $\xi(a\omega) = \pi(a)\xi(\omega)$.*

Proof. –
- We have for $x \in \mathcal{N}_\varphi$ that

$$\begin{aligned}(\omega\,\theta)(x^*) &= (\omega \otimes \theta)\Delta(x^*) = \theta\big((\omega \otimes \iota)\Delta(x^*)\big) = \theta\big((\overline{\omega} \otimes \iota)(\Delta(x))^*\big) \\ &= \big\langle \xi(\theta), \Lambda((\overline{\omega} \otimes \iota)\Delta(x))\big\rangle = \big\langle \xi(\theta), (\overline{\omega} \otimes \iota)(V^*)\Lambda(x)\big\rangle \\ &= \big\langle (\omega \otimes \iota)(V)\xi(\theta), \Lambda(x)\big\rangle.\end{aligned}$$

This implies that $\omega\theta \in \mathcal{I}$ and $\xi(\omega\theta) = \lambda(\omega)\xi(\theta)$.

- It is clear that for $x \in \mathcal{N}_\varphi$

$$\begin{aligned}(a\omega)(x^*) &= \omega(x^*a) = \omega((a^*x)^*) = \langle \xi(\omega), \Lambda(a^*x)\rangle \\ &= \langle \xi(\omega), \pi(a^*)\Lambda(x)\rangle = \langle \pi(a)\xi(\omega), \Lambda(x)\rangle.\end{aligned}$$

This implies that $a\omega \in \mathcal{I}$ and $\xi(a\omega) = \pi(a)\xi(\omega)$. □

In a next step, we introduce the one-parameter representation on $\mathrm{L}^1(A)$ which will implement the modular group of the dual weight.

Notation 8.7. – Define the norm continuous one-parameter representation ρ of $\mathbb{R}$ on $\mathrm{L}^1(A)$ such that $\rho_t(\omega)(x) = \omega(\delta^{-it}\tau_{-t}(x))$ for $\omega \in \mathrm{L}^1(A)$, $x \in A$ and $t \in \mathbb{R}$.

The fact that ρ is a representation follows immediately from the fact that $\tau_t(\delta) = \delta$ for $t \in \mathbb{R}$. Since any element of $\mathrm{L}^1(A)$ arises from an element in $\mathrm{B}_0(H)^*$ and since τ_t is implemented by P^{it} (*see* Definition 6.9), the norm continuity is easily verified.

Result 5.12 and Proposition 7.12 (1) imply immediately that ρ_t is an algebra automorphism of $\mathrm{L}^1(A)$ for every $t \in \mathbb{R}$.

Now we start to change our route a little bit in comparison with the route followed in [15]. We will not use left or right Hilbert algebra theory. Recall that we introduced the strictly positive operator P in Definition 6.9. So $P^{it}\Lambda(a) = \nu^{\frac{t}{2}}\Lambda(\tau_t(a))$ and $J\pi(\delta)^{it}J\Lambda(a) = \nu^{-\frac{t}{2}}\Lambda(a\delta^{-it})$ for all $t \in \mathbb{R}$ and $a \in \mathcal{N}_\varphi$. Since $\tau_t(\delta) = \delta$ for all $t \in \mathbb{R}$, we get immediately that $J\pi(\delta)J$ and P commute.

LEMMA 8.8. – *We have for all $t \in \mathbb{R}$ and $\omega \in \mathcal{I}$ that $\rho_t(\omega) \in \mathcal{I}$ and $\xi(\rho_t(\omega)) = P^{it}J\pi(\delta)^{it}J\xi(\omega)$.*

Proof. – Choose $x \in \mathcal{N}_\varphi$. The remarks before this lemma imply that

$$\begin{aligned}\rho_t(\omega)(x^*) &= \omega(\delta^{-it}\tau_{-t}(x^*)) = \omega((\tau_{-t}(x)\delta^{it})^*) = \langle \xi(\omega), \Lambda(\tau_{-t}(x)\delta^{it})\rangle \\ &= \nu^{-\frac{t}{2}}\langle \xi(\omega), J\pi(\delta)^{-it}J\Lambda(\tau_{-t}(x))\rangle = \langle \xi(\omega), J\pi(\delta)^{-it}JP^{-it}\Lambda(x)\rangle \\ &= \langle P^{it}J\pi(\delta)^{it}J\xi(\omega), \Lambda(x)\rangle.\end{aligned}$$

This implies that $\rho_t(\omega) \in \mathcal{I}$ and $\xi(\rho_t(\omega)) = P^{it}J\pi(\delta)^{it}J\xi(\omega)$. □

The next proposition is a simple consequence of Lemma 8.8 and the multiplicativity of ρ_t for every $t \in \mathbb{R}$.

PROPOSITION 8.9. – *There exists a unique norm continuous one-parameter group $\widehat{\sigma}$ on $\widehat{A}$ such that $\widehat{\sigma}_t(\lambda(\omega)) = \lambda(\rho_t(\omega))$ for all $t \in \mathbb{R}$ and $\omega \in \mathrm{L}^1(A)$. We have moreover that $\widehat{\sigma}_t(x) = P^{it}J\pi(\delta)^{it}JxJ\pi(\delta)^{-it}JP^{-it}$ for $t \in \mathbb{R}$ and $x \in \widehat{A}$.*

We will need the following two norm continuous one-parameter representations δ^* and τ^* of $\mathbb{R}$ on $\mathrm{L}^1(A)$ defined in such a way that:
- $\delta^*_t(\omega)(x) = \omega(\delta^{it}x)$,
- $\tau^*_t(\omega)(x) = \omega(\tau_t(x))$

for all $t \in \mathbb{R}$, $\omega \in \mathrm{L}^1(A)$ and $x \in A$. So δ^* and τ^* commute and $\rho_t = \tau^*_{-t}\delta^*_{-t}$ for $t \in \mathbb{R}$. This implies that $\tau^*_{-\frac{i}{2}}\delta^*_{-\frac{i}{2}}$ is closable and that its closure is equal to $\rho_{\frac{i}{2}}$ (*see* e.g. Proposition 4.9 of [25]).

Consider $z \in \mathbb{C}$, $x \in D(\tau_z)$ and $\omega \in D(\tau^*_z)$. Since the two functions

$$S(z) \to \mathbb{C}\colon y \mapsto \tau^*_y(\omega)(x) \quad \text{and} \quad S(z) \to \mathbb{C}\colon y \mapsto \omega(\tau_z(x))$$

are continuous on $S(z)$, analytic on $S(z)^\circ$ and agree on the real line, they are equal on $S(z)$. Hence $\tau_z^*(\omega)(x) = \omega(\tau_z(x))$.

LEMMA 8.10. – *We have for all $\omega \in D(\tau^*_{-\frac{i}{2}})$ that $(\omega R \otimes \iota)(V^*) = (\tau^*_{-\frac{i}{2}}(\omega) \otimes \iota)(V)$.*

Proof. – Choose $\theta \in \mathrm{B}_0(H)^*$. Proposition 8.3 implies that $(\iota \otimes \theta)(V)$ belongs to $D(S)$ and

$$S\big((\iota \otimes \theta)(V)\big) = (\iota \otimes \theta)(V^*).$$

So $(\iota \otimes \theta)(V)$ belongs to $D(\tau_{-\frac{i}{2}})$ and

$$\tau_{-\frac{i}{2}}\big((\iota \otimes \theta)(V)\big) = R\big((\iota \otimes \theta)(V^*)\big).$$

Applying ω to this equation, using the remark before this proposition and rearranging things, we get

$$\theta\big((\tau^*_{-\frac{i}{2}}(\omega) \otimes \iota)(V)\big) = \theta\big((\omega R \otimes \iota)(V^*)\big). \qquad \square$$

In Notation 7.13, we introduced the GNS-construction (H, π, Γ) for ψ. Since $\psi = \varphi R$, there exists an anti-unitary $U : H \to H$ such that $U\Gamma(x) = \Lambda(R(x^*))$ for $x \in \mathcal{N}_\psi$.

PROPOSITION 8.11. – *Consider $\omega \in \mathcal{I}$ and $\theta \in D(\rho_{\frac{i}{2}})$. Then*

$$\omega\theta \in \mathcal{I} \quad \textit{and} \quad \xi(\omega\theta) = U^* \lambda(\rho_{\frac{i}{2}}(\theta))^* U \xi(\omega).$$

Proof. – Since the mapping $\mathcal{I} \to H : \eta \to \xi(\eta)$ is closed and $D(\tau^*_{-\frac{i}{2}} \delta^*_{-\frac{i}{2}})$ is a core for $\rho_{\frac{i}{2}}$, it is sufficient to prove the result under the extra assumption that $\theta \in D(\tau^*_{-\frac{i}{2}} \delta^*_{-\frac{i}{2}})$.

For $n \in \mathbb{N}$, we define the element $e_n \in \mathrm{M}(A)$ as

$$e_n = \frac{n}{\sqrt{\pi}} \int \exp(-n^2 t^2) \delta^{it} \,\mathrm{d}t.$$

Then e_n is analytic with respect to σ, implying that $\mathcal{N}_\varphi e_n \subseteq \mathcal{N}_\varphi$.

Take $x \in \mathcal{N}_\varphi$. Since $\psi = \varphi_\delta$ and $\Gamma = \Lambda_\delta$, we get that $x(\delta^{-\frac{1}{2}} e_n)$ belongs to $\mathcal{N}_\psi$ and $\Gamma(x(\delta^{-\frac{1}{2}} e_n)) = \Lambda(x e_n)$. Therefore the right invariance of ψ implies that

$$(\iota \otimes \overline{\delta^*_{-\frac{i}{2}}(\theta)})\Delta\big(x(\delta^{-\frac{1}{2}} e_n)\big)$$

belongs to $\mathcal{N}_\psi$ and, using the remarks before this proposition,

$$\begin{aligned}
\Gamma\big((\iota \otimes \overline{\delta^*_{-\frac{i}{2}}(\theta)})\Delta(x(\delta^{-\frac{1}{2}} e_n))\big) &= U^* \Lambda\big(R\big((\iota \otimes \overline{\delta^*_{-\frac{i}{2}}(\theta)})(\Delta(x(\delta^{-\frac{1}{2}} e_n)))^*\big)\big) \\
&= U^* \Lambda\big(R\big((\iota \otimes \delta^*_{-\frac{i}{2}}(\theta))(\Delta(x(\delta^{-\frac{1}{2}} e_n))^*)\big)\big).
\end{aligned}$$

Using the equality $\chi(R \otimes R)\Delta = \Delta R$, this implies

$$\begin{aligned}
\Gamma\big((\iota \otimes \overline{\delta^*_{-\frac{i}{2}}(\theta)})\Delta(x(\delta^{-\frac{1}{2}} e_n))\big) &= U^* \Lambda\big((\delta^*_{-\frac{i}{2}}(\theta) R \otimes \iota)\Delta(R((x(\delta^{-\frac{1}{2}} e_n))^*))\big) \\
&= U^* \big(\delta^*_{-\frac{i}{2}}(\theta) R \otimes \iota\big)(V^*)\Lambda\big(R((x(\delta^{-\frac{1}{2}} e_n))^*)\big) \\
&= U^* \big(\delta^*_{-\frac{i}{2}}(\theta) R \otimes \iota\big)(V^*)\, U\Gamma\big(x(\delta^{-\frac{1}{2}} e_n)\big).
\end{aligned}$$

Using the previous lemma, this gives

$$\begin{aligned}(8.4)\quad \Gamma\big((\iota\otimes\overline{\delta^*_{-\frac{i}{2}}(\theta)})\Delta(x(\delta^{-\frac12}e_n))\big) &= U^*\big(\tau^*_{-\frac{i}{2}}(\delta^*_{-\frac{i}{2}}(\theta))\otimes\iota\big)(V)\,U\Lambda\big(x(\delta^{-\frac12}e_n)\delta^{\frac12}\big)\\ &= U^*(\rho_{\frac{i}{2}}(\theta)\otimes\iota)(V)\,U\Lambda(xe_n).\end{aligned}$$

Define the function $f:S(\frac{i}{2})\to A:z\mapsto(\iota\otimes\overline{\delta^*_{\bar z}(\theta)})\Delta(x(\delta^{iz}e_n))$. Then f is continuous on $S(\frac{i}{2})$, analytic on $S(\frac{i}{2})^\circ$ and

$$f(t)=(\iota\otimes\overline{\delta^*_t(\theta)})\Delta(x(\delta^{it}e_n))=(\iota\otimes\overline{\theta})\big(\Delta(xe_n\delta^{it})(1\otimes\delta^{-it})\big)=(\iota\otimes\overline{\theta})(\Delta(xe_n))\delta^{it}$$

for all $t\in\mathbb{R}$. This implies (*see* e.g. Corollary 6.8 of [25]) that $(\iota\otimes\overline{\theta})(\Delta(xe_n))$ is a left multiplier of $\delta^{-\frac12}$ and

$$(\iota\otimes\overline{\theta})(\Delta(xe_n))\delta^{-\frac12}=(\iota\otimes\overline{\delta^*_{-\frac{i}{2}}(\theta)})\Delta(x(\delta^{-\frac12}e_n))\in\mathcal{N}_\psi.$$

Because $\varphi=\psi_{\delta^{-1}}$ and $\Lambda=\Gamma_{\delta^{-1}}$, this implies that $(\iota\otimes\overline{\theta})(\Delta(xe_n))$ belongs to $\mathcal{N}_\varphi$ and

$$\Lambda\big((\iota\otimes\overline{\theta})(\Delta(xe_n))\big)=\Gamma\big((\iota\otimes\overline{\delta^*_{-\frac{i}{2}}(\theta)})\Delta(x(\delta^{-\frac12}e_n))\big)$$

so that Eq. (8.4) implies that

$$\Lambda\big((\iota\otimes\overline{\theta})(\Delta(xe_n))\big)=U^*\lambda(\rho_{\frac{i}{2}}(\theta))\,U\Lambda(xe_n).$$

Now

$$\begin{aligned}(\omega\theta)((xe_n)^*)&=(\omega\otimes\theta)\Delta((xe_n)^*)=\omega((\iota\otimes\theta)\Delta((xe_n)^*))=\langle\xi(\omega),\Lambda((\iota\otimes\overline{\theta})\Delta(xe_n))\rangle\\ &=\langle\xi(\omega),U^*\lambda(\rho_{\frac{i}{2}}(\theta))\,U\,\Lambda(xe_n)\rangle=\langle U^*\lambda(\rho_{\frac{i}{2}}(\theta))^*U\xi(\omega),\Lambda(xe_n)\rangle.\end{aligned}$$

Since $(xe_n)_{n=1}^\infty$ converges to x and $(\Lambda(xe_n))_{n=1}^\infty$ converges to $\Lambda(x)$ (*cf.* the proof of Lemma 5.7), we get that

$$(\omega\theta)(x^*)=\langle U^*\lambda(\rho_{\frac{i}{2}}(\theta))^*U\xi(\omega),\Lambda(x)\rangle.$$

This implies by definition that $\omega\theta$ belongs to $\mathcal{I}$ and that

$$\xi(\omega\theta)=U^*\lambda(\rho_{\frac{i}{2}}(\theta))^*U\xi(\omega).\qquad\square$$

Remark 8.12. – Since the mapping $\mathcal{I}\to H:\omega\mapsto\xi(\omega)$ is closed, Lemma 8.8 implies (*see* e.g. Lemma 1.1 of [24]) for every $\omega\in\mathcal{I}$ that $\frac{n}{\sqrt{\pi}}\int\exp(-n^2(t+\frac{i}{2})^2)\rho_t(\omega)\,\mathrm{d}t$ belongs to $\mathcal{I}$ and

$$\begin{aligned}&\xi\bigg(\frac{n}{\sqrt{\pi}}\int\exp\bigg(-n^2\Big(t+\frac{i}{2}\Big)^2\bigg)\rho_t(\omega)\,\mathrm{d}t\bigg)\\ &\quad=\frac{n}{\sqrt{\pi}}\int\exp\bigg(-n^2\Big(t+\frac{i}{2}\Big)^2\bigg)P^{it}J\pi(\delta)^{it}J\xi(\omega)\,\mathrm{d}t.\end{aligned}$$

Also notice that

$$\rho_{\frac{i}{2}}\bigg(\frac{n}{\sqrt{\pi}}\int\exp\bigg(-n^2\Big(t+\frac{i}{2}\Big)^2\bigg)\rho_t(\omega)\,\mathrm{d}t\bigg)=\frac{n}{\sqrt{\pi}}\int\exp(-n^2t^2)\rho_t(\omega)\,\mathrm{d}t.$$

So we see that $\rho_{\frac{i}{2}}(D(\rho_{\frac{i}{2}})\cap\mathcal{I})$ is dense in $\mathrm{L}^1(A)$.

We have now all the necessary information to construct easily the dual weight on $\widehat{A}$.

PROPOSITION 8.13. – *There exists a unique closed densely defined linear map $\widehat{\Lambda}$ from $D(\widehat{\Lambda}) \subseteq \widehat{A}$ into H such that $\lambda(\mathcal{I})$ is a core for $\widehat{\Lambda}$ and $\widehat{\Lambda}(\lambda(\omega)) = \xi(\omega)$ for $\omega \in \mathcal{I}$.*

Proof. – Take a sequence $(\omega_n)_{n=1}^{\infty}$ in $\mathcal{I}$, $w \in H$ such that $(\lambda(\omega_n))_{n=1}^{\infty}$ converges to 0 and $(\xi(\omega_n))_{n=1}^{\infty}$ converges to w.

Choose $\theta \in D(\rho_{\frac{i}{2}}) \cap \mathcal{I}$. Then we have by Proposition 8.11 that

$$\lambda(\omega_n)\xi(\theta) = U^*\lambda(\rho_{\frac{i}{2}}(\theta))^*U\xi(\omega_n)$$

for all $n \in \mathbb{N}$. Letting n tend to ∞, this equality gives $0 = U^*\lambda(\rho_{\frac{i}{2}}(\theta))^*Uw$. By Remark 8.12, this implies that $w = 0$. □

PROPOSITION 8.14. – *There exists a unique KMS weight $\widehat{\varphi}$ on $\widehat{A}$ such that $(H, \iota, \widehat{\Lambda})$ is a GNS-construction for $\widehat{\varphi}$. We have moreover that $\widehat{\varphi}$ is faithful and that $\widehat{\sigma}$ is the modular group for $\widehat{\varphi}$.*

Proof. –

(1) Since $\mathcal{I}$ is dense in $\mathrm{L}^1(A)$, the boundedness of λ implies that $\lambda(\mathcal{I})$ is dense in $\widehat{A}$. By Result 8.6, we have for all $x \in \lambda(\mathrm{L}^1(A))$ and $y \in \lambda(\mathcal{I})$ that $xy \in \lambda(\mathcal{I})$ and $\widehat{\Lambda}(xy) = x\widehat{\Lambda}(y)$. Using this and the closedness of $\widehat{\Lambda}$, it is easy to check that $D(\widehat{\Lambda})$ is a left ideal in $\widehat{A}$ and that $\widehat{\Lambda}(xy) = x\widehat{\Lambda}(y)$ for all $x \in \widehat{A}$ and $y \in D(\widehat{\Lambda})$.

(2) Take $t \in \mathbb{R}$. By Proposition 8.9 and Lemma 8.8, we have for all $x \in \lambda(\mathcal{I})$ that $\widehat{\sigma}_t(x) \in D(\widehat{\Lambda})$ and $\widehat{\Lambda}(\widehat{\sigma}_t(x)) = P^{it}J\pi(\delta)^{it}J\widehat{\Lambda}(x)$. The closedness of $\widehat{\Lambda}$ now implies easily for every $x \in D(\widehat{\Lambda})$ that $\widehat{\sigma}_t(x) \in D(\widehat{\Lambda})$ and $\widehat{\Lambda}(\widehat{\sigma}_t(x)) = P^{it}J\pi(\delta)^{it}J\widehat{\Lambda}(x)$.

(3) Choose $\omega \in D(\rho_{\frac{i}{2}})$. Then clearly $\lambda(\omega) \in D(\widehat{\sigma}_{\frac{i}{2}})$ and $\widehat{\sigma}_{\frac{i}{2}}(\lambda(\omega)) = \lambda(\rho_{\frac{i}{2}}(\omega))$. This implies by Proposition 8.11 for every $x \in \lambda(\mathcal{I})$ that $x\lambda(\omega) \in D(\widehat{\Lambda})$ and $\widehat{\Lambda}(x\lambda(\omega)) = U^*\widehat{\sigma}_{\frac{i}{2}}(\lambda(\omega))^*U\widehat{\Lambda}(x)$. Again, the closedness of $\widehat{\Lambda}$ gives for every $x \in D(\widehat{\Lambda})$ that

$$x\lambda(\omega) \in D(\widehat{\Lambda}) \quad \text{and} \quad \widehat{\Lambda}(x\lambda(\omega)) = U^*\widehat{\sigma}_{\frac{i}{2}}(\lambda(\omega))^*U\widehat{\Lambda}(x). \tag{8.5}$$

Because $\hat{\pi}(D(\rho_{\frac{i}{2}}))$ is dense in $D(\hat{\sigma}_{\frac{i}{2}})$ and invariant under $\hat{\sigma}$ we get that $\hat{\pi}(D(\rho_{\frac{i}{2}}))$ is a core for $\widehat{\sigma}_{\frac{i}{2}}$. Combining this with Eq. (8.5) and the closedness of $\widehat{\Lambda}$, we get for $x \in D(\widehat{\Lambda})$ and $y \in D(\widehat{\sigma}_{\frac{i}{2}})$ that $xy \in D(\widehat{\Lambda})$ and

$$\widehat{\Lambda}(xy) = U^*\widehat{\sigma}_{\frac{i}{2}}(y)^*U\widehat{\Lambda}(x).$$

Definition 5.12 and proposition 5.14 of [24] imply the existence of a KMS weight $\widehat{\varphi}$ on $\widehat{A}$ such that $(H, \iota, \widehat{\Lambda})$ is a GNS-construction for $\widehat{\varphi}$ and $\widehat{\sigma}$ is a modular group for $\widehat{\varphi}$. By the last equality of (3), we have $x\widehat{\Lambda}(y) = U^*\widehat{\sigma}_{\frac{i}{2}}(y)^*U\widehat{\Lambda}(x)$ for $x \in \mathcal{N}_{\widehat{\varphi}}$ and $y \in D(\widehat{\sigma}_{\frac{i}{2}}) \cap \mathcal{N}_{\widehat{\varphi}}$. Faithfulness follows easily from this. □

It turns out to be not so difficult to establish the left invariance of our dual weight.

PROPOSITION 8.15. – *The weight $\widehat{\varphi}$ is left invariant.*

Proof. – By the remarks after Theorem 8.2, we get that $V \in \mathrm{M}(A \otimes \widehat{A})$ and $(\iota \otimes \widehat{\Delta})(V) = V_{13}V_{12}$.

Choose $\omega \in \mathcal{I}$ and $\theta \in \widehat{A}^*$. Then $(\iota \otimes \theta)(V)$ belongs to $\mathrm{M}(A)$. Moreover,

$$\begin{aligned}(\theta\otimes\iota)\widehat{\Delta}(\lambda(\omega)) &= (\theta\otimes\iota)\widehat{\Delta}\big((\omega\otimes\iota)(V)\big) = (\omega\otimes\theta\otimes\iota)\big((\iota\otimes\widehat{\Delta})(V)\big)\\ &= (\omega\otimes\theta\otimes\iota)(V_{13}V_{12}) = (\omega\otimes\iota)\big(V((\iota\otimes\theta)(V)\otimes 1)\big)\\ &= \lambda\big([(\iota\otimes\theta)(V)]\omega\big).\end{aligned}$$

Hence Result 8.6 implies that $(\theta\otimes\iota)\widehat{\Delta}(\lambda(\omega))\in\mathcal{N}_{\widehat{\varphi}}$ and

$$\widehat{\Lambda}\big((\theta\otimes\iota)\widehat{\Delta}(\lambda(\omega))\big) = \pi\big((\iota\otimes\theta)(V)\big)\widehat{\Lambda}(\lambda(\omega)).$$

Therefore the closedness of $\widehat{\Lambda}$ implies for every $x\in\mathcal{N}_{\widehat{\varphi}}$ that $(\theta\otimes\iota)\widehat{\Delta}(x)\in\mathcal{N}_{\widehat{\varphi}}$ and

$$\widehat{\Lambda}\big((\theta\otimes\iota)\widehat{\Delta}(x)\big) = \pi\big((\iota\otimes\theta)(V)\big)\widehat{\Lambda}(x). \tag{8.6}$$

Now take $\eta\in\widehat{A}^*_+$ and $x\in\mathcal{N}_{\widehat{\varphi}}$. Fix also a cyclic GNS-construction (H_η,π_η,v_η) for η and an orthonormal basis $(e_i)_{i\in I}$ for H_η. By Lemma A.6, we know that

$$\widehat{\varphi}\big((\eta\otimes\iota)\widehat{\Delta}(x^*x)\big) = \widehat{\varphi}\big((\omega_{v_\eta,v_\eta}\otimes\iota)\widehat{\Delta}(x^*x)\big) = \sum_{i\in I}\widehat{\varphi}\big((\omega_{v_\eta,e_i}\otimes\iota)(\widehat{\Delta}(x))^*(\omega_{v_\eta,e_i}\otimes\iota)(\widehat{\Delta}(x))\big).$$

Hence, using Eq. (8.6) above,

$$\begin{aligned}\widehat{\varphi}\big((\eta\otimes\iota)\widehat{\Delta}(x^*x)\big) &= \sum_{i\in I}\big\langle\widehat{\Lambda}((\omega_{v_\eta,e_i}\otimes\iota)\widehat{\Delta}(x)),\widehat{\Lambda}((\omega_{v_\eta,e_i}\otimes\iota)\widehat{\Delta}(x))\big\rangle\\ &= \sum_{i\in I}\big\langle\pi((\iota\otimes\omega_{v_\eta,e_i})(V))\widehat{\Lambda}(x),\pi((\iota\otimes\omega_{v_\eta,e_i})(V))\widehat{\Lambda}(x)\big\rangle\\ &= \sum_{i\in I}\big\langle\pi((\iota\otimes\omega_{v_\eta,e_i})(V)^*(\iota\otimes\omega_{v_\eta,e_i})(V))\widehat{\Lambda}(x),\widehat{\Lambda}(x)\big\rangle.\end{aligned}$$

Using Lemma A.6 once again, the unitarity of V gives

$$\widehat{\varphi}\big((\eta\otimes\iota)\widehat{\Delta}(x^*x)\big) = \big\langle\pi((\iota\otimes\omega_{v_\eta,v_\eta})(V^*V))\widehat{\Lambda}(x),\widehat{\Lambda}(x)\big\rangle = \eta(1)\widehat{\varphi}(x^*x). \qquad\square$$

It is also easy to get an expression for W in terms of $\widehat{\varphi}$.

PROPOSITION 8.16. – *We have for all $x,y\in\mathcal{N}_{\widehat{\varphi}}$ that*

$$(\Sigma W\Sigma)\big(\widehat{\Lambda}(x)\otimes\widehat{\Lambda}(y)\big) = (\widehat{\Lambda}\otimes\widehat{\Lambda})\big(\widehat{\Delta}(y)(x\otimes 1)\big).$$

Proof. – Define the isometry W' on $H\otimes H$ such that

$$W'\big(\widehat{\Lambda}(x)\otimes\widehat{\Lambda}(y)\big) = (\widehat{\Lambda}\otimes\widehat{\Lambda})\big(\widehat{\Delta}(y)(x\otimes 1)\big)$$

for all $x,y\in\mathcal{N}_{\widehat{\varphi}}$. Take $\omega\in \mathrm{B}_0(H)^*$ and $x\in\mathcal{N}_{\widehat{\varphi}}$. Then

$$(\omega\otimes\iota)(W')\widehat{\Lambda}(x) = \widehat{\Lambda}\big((\omega\otimes\iota)\widehat{\Delta}(x)\big) \overset{(*)}{=} \pi\big((\iota\otimes\omega)(V)\big)\widehat{\Lambda}(x) = (\omega\otimes\iota)(\Sigma W\Sigma)\widehat{\Lambda}(x),$$

where in equality (*), we used Eq. (8.6) of the previous proposition. So $W'=\Sigma W\Sigma$. $\square$

Let us quickly prove that $(\widehat{A},\widehat{\Delta})$ is indeed a reduced C*-algebraic quantum group. So we have to prove the existence of a right invariant KMS weight on $(\widehat{A},\widehat{\Delta})$. This can be easily done by introducing the unitary antipode in the usual way (*see* Proposition 3.3.1 of [15]). For the sake of completeness (and because it is easy to do so), we include a proof.

PROPOSITION 8.17. – *There exists a unique *-antiautomorphism $\widehat{R}$ on $\widehat{A}$ such that $\widehat{R}(\lambda(\omega)) = \lambda(\omega R)$ for all $\omega \in \mathrm{L}^1(A)$. We have moreover that $\widehat{R}(x) = Jx^*J$ for $x \in \widehat{A}$.*

Proof. – Choose $\omega \in \mathrm{L}^1(A)$. Take $\theta \in \mathrm{B}_0(H)^*$ such that $\theta\pi = \omega$. So $\theta(I\pi(a)^*I) = (\omega R)(a)$ for $a \in A$. Using the commutation relation $(I \otimes J)W^*(I \otimes J) = W$ (*see* Proposition 5.38), we get that

$$J\lambda(\omega R)^*J = J(\omega R \otimes \iota)(V)^*J = (\theta \otimes \iota)\big((I \otimes J)W^*(I \otimes J)\big) = (\theta \otimes \iota)(W) = \lambda(\omega),$$

so $\lambda(\omega R) = J\lambda(\omega)^*J$. The rest of the proof is now obvious. □

Since $(I \otimes J)W^*(I \otimes J) = W$ and R is implemented by I, we get that $(R \otimes \widehat{R})(V) = V$.

PROPOSITION 8.18. – *We have that $\chi(\widehat{R} \otimes \widehat{R})\widehat{\Delta} = \widehat{\Delta}\,\widehat{R}$.*

Proof. – Choose $\omega \in \mathrm{L}^1(A)$. The remark before this proposition and the fact that $(\iota \otimes \widehat{\Delta})(V) = V_{13}V_{12}$ imply that

$$\widehat{\Delta}\big(\widehat{R}((\omega \otimes \iota)(V))\big) = \widehat{\Delta}\big((\omega R \otimes \iota)(V)\big) = (\omega R \otimes \iota \otimes \iota)\big((\iota \otimes \widehat{\Delta})(V)\big) = (\omega R \otimes \iota \otimes \iota)(V_{13}V_{12}).$$

So the antimultiplicativity of R and $\widehat{R}$ and the remark before this proposition imply that

$$\begin{aligned}(\chi(\widehat{R} \otimes \widehat{R})\widehat{\Delta})\big(\widehat{R}((\omega \otimes \iota)(V))\big) &= \chi(\omega \otimes \iota \otimes \iota)\big((R \otimes \widehat{R} \otimes \widehat{R})(V_{13}V_{12})\big) \\ &= \chi(\omega \otimes \iota \otimes \iota)\big((R \otimes \widehat{R})(V)_{12}(R \otimes \widehat{R})(V)_{13}\big) \\ &= \chi(\omega \otimes \iota \otimes \iota)(V_{12}\,V_{13}) \\ &= (\omega \otimes \iota \otimes \iota)(V_{13}\,V_{12}) = (\omega \otimes \iota \otimes \iota)\big((\iota \otimes \widehat{\Delta})(V)\big) \\ &= \widehat{\Delta}\big((\omega \otimes \iota)(V)\big)\end{aligned}$$

and the proposition follows. □

COROLLARY 8.19. – *The weight $\widehat{\varphi}\,\widehat{R}$ is a right invariant faithful KMS weight on $(\widehat{A}, \widehat{\Delta})$.*

Therefore we can conclude that:

THEOREM 8.20. – *The pair $(\widehat{A}, \widehat{\Delta})$ is a reduced* C*-*algebraic quantum group. It is called the reduced dual of (A, Δ).*

Notice that Proposition 8.16 implies that $\Sigma W^*\Sigma$ is the multiplicative unitary of the dual $(\widehat{A}, \widehat{\Delta})$ in the GNS-construction $(H, \iota, \widehat{\Lambda})$ for $\widehat{\varphi}$.

It does not take much extra work to identify the antipodal triple of $(\widehat{A}, \widehat{\Delta})$. Let us first introduce a symbol for the antipode of the dual.

Notation 8.21. – We denote the antipode of $(\widehat{A}, \widehat{\Delta})$ by $\hat{S}$.

Since $\Sigma W^*\Sigma$ is the multiplicative unitary for $(\widehat{A}, \widehat{\Delta})$, Proposition 8.3 gets in the dual setting the following form:

PROPOSITION 8.22. – *For all $\omega \in \mathrm{L}^1(A)$, we have that $(\omega \otimes \iota)(V^*)$ belongs to $D(\hat{S})$ and*

$$\hat{S}\big((\omega \otimes \iota)(V^*)\big) = (\omega \otimes \iota)(V).$$

Moreover, the set

$$\{(\omega \otimes \iota)(V^*) \mid \omega \in \mathrm{L}^1(A)\}$$

is a core for $\hat{S}$.

Now it is time to introduce the scaling group. Arguing as in Lemma 8.8, one gets for every $t \in \mathbb{R}$ and $\omega \in \mathcal{I}$ that

$$\tau_t^*(\omega) \in \mathcal{I} \quad \text{and} \quad \xi(\tau_t^*(\omega)) = \nu^{-\frac{t}{2}} P^{-it} \xi(\omega). \tag{8.7}$$

The equality $(\tau_t \otimes \tau_t)\Delta = \Delta\tau_t$ implies that $\tau_t^* : \mathrm{L}^1(A) \to \mathrm{L}^1(A)$ is multiplicative. As in the case of Proposition 8.9, this implies the following one.

PROPOSITION 8.23. – *There exists a unique norm continuous one-parameter group $\widehat{\tau}$ on $\widehat{A}$ such that $\widehat{\tau}_t(\lambda(\omega)) = \lambda(\omega\tau_{-t})$ for $t \in \mathbb{R}$. We have moreover that $\widehat{\tau}_t(x) = P^{it} x P^{-it}$ for $t \in \mathbb{R}$ and $x \in \hat{A}$.*

Using Eq. (8.7) and arguing as in part 2 of the proof of Proposition 8.14, we arrive at the following conclusion (first check the second statement).

PROPOSITION 8.24. – *We have that $\widehat{\varphi}\widehat{\tau}_t = \nu^t \widehat{\varphi}$. Moreover, $\widehat{\Lambda}(\widehat{\tau}_t(x)) = \nu^{\frac{t}{2}} P^{it} \widehat{\Lambda}(x)$ for $t \in \mathbb{R}$ and $x \in \mathcal{N}_{\widehat{\varphi}}$.*

From the proof of Proposition 6.10, we know that $(P^{it} \otimes P^{it}) W (P^{-it} \otimes P^{-it}) = W$ for all $t \in \mathbb{R}$. In other words, $(\tau_t \otimes \widehat{\tau}_t)(V) = V$ for all $t \in R$.

PROPOSITION 8.25. – *The following holds*:
(1) *$\widehat{R}$ is the unitary antipode of $(\widehat{A}, \widehat{\Delta})$,*
(2) *$\widehat{\tau}$ is the scaling group of $(\widehat{A}, \widehat{\Delta})$.*

Proof. – Let us start off with the second statement. Call τ' the scaling group of $(\widehat{A}, \widehat{\Delta})$. Take $t \in \mathbb{R}$ and $x \in \widehat{A}$. Then Proposition 8.9 and the remarks before this proposition imply that

$$\begin{aligned}
\widehat{\Delta}(\widehat{\sigma}_t(x)) &= \Sigma W(\widehat{\sigma}_t(x) \otimes 1) W^* \Sigma = \Sigma W \big(P^{it} J\pi(\delta)^{it} J x J \pi(\delta)^{-it} J P^{-it} \otimes 1\big) W^* \Sigma \\
&= \Sigma W \big(P^{it} J\pi(\delta)^{it} J x J \pi(\delta)^{-it} J P^{-it} \otimes P^{it} P^{-it}\big) W^* \Sigma \\
&= \big(P^{it} \otimes P^{it}\big) \Sigma W \big(J\pi(\delta)^{it} J x J \pi(\delta)^{-it} J \otimes 1\big) W^* \Sigma \big(P^{-it} \otimes P^{-it}\big).
\end{aligned}$$

Tomita–Takesaki theory tells us that $J\pi(\delta)^{it} J$ belongs to $\pi(A)'$ (this follows in fact easily from Proposition 1.12.2). Since W belongs to $\mathrm{M}(\pi(A) \otimes \mathrm{B}_0(H))$, we conclude that

$$\widehat{\Delta}(\widehat{\sigma}_t(x)) = \big(P^{it} \otimes P^{it} J\pi(\delta)^{it} J\big) \Sigma W(x \otimes 1) W^* \Sigma \big(P^{-it} \otimes J\pi(\delta)^{-it} J P^{-it}\big) = (\widehat{\tau}_t \otimes \widehat{\sigma}_t)\widehat{\Delta}(x).$$

Combining this with Proposition 5.38 (3), we see that $(\widehat{\tau}_t \otimes \iota)\widehat{\Delta}(x) = (\tau'_t \otimes \iota)\widehat{\Delta}(x)$. Arguing in the usual way, the density conditions imply that $\widehat{\tau} = \tau'$.

Let R' denote the unitary antipode of $(\widehat{A}, \widehat{\Delta})$. Choose $\omega \in D(\tau^*_{-\frac{i}{2}})$. Lemma 8.10 and the remark before Proposition 8.18 imply that

$$\widehat{R}\big((\omega \otimes \iota)(V^*)\big) = (\omega R \otimes \iota)(V^*) = (\tau^*_{-\frac{i}{2}}(\omega) \otimes \iota)(V).$$

On the other hand, Proposition 8.22 and the fact that $\hat{S} = \widehat{\tau}_{-\frac{i}{2}} R'$ imply that $(\omega \otimes \iota)(V) \in D(\widehat{\tau}_{\frac{i}{2}})$ and

$$R'\big((\omega \otimes \iota)(V^*)\big) = \widehat{\tau}_{\frac{i}{2}}\big((\omega \otimes \iota)(V)\big).$$

The two functions

$$S(\tfrac{i}{2}) \to \widehat{A} : y \mapsto (\tau^*_{-y}(\omega) \otimes \iota)(V) \quad \text{and} \quad S(\tfrac{i}{2}) \to \widehat{A} : y \mapsto \widehat{\tau}_y((\omega \otimes \iota)(V))$$

are continuous on $S(\frac{i}{2})$, analytic on $S(\frac{i}{2})^\circ$ and agree on the real line by the remark before this proposition. Hence they are equal. In particular $(\tau^*_{-\frac{i}{2}}(\omega) \otimes \iota)(V) = \widehat{\tau}_{\frac{i}{2}}((\omega \otimes \iota)(V))$ and we get that $\widehat{R}((\omega \otimes \iota)(V^*)) = R'((\omega \otimes \iota)(V^*))$. We conclude that $\widehat{R} = R'$. □

Notice that the first statement of this proposition and Proposition 8.24 imply that ν^{-1} is the scaling constant of $(\widehat{A}, \widehat{\Delta})$.

We defined the dual with the aid of the left invariant weight φ. In the next part, we show that the dual could have been easily defined using the right invariant weight $\psi = \varphi R$. Let us define a GNS-construction $(H, \pi_\psi, \Lambda_\psi)$ for ψ in such a way that:

- $\Lambda_\psi(a) = J\Lambda(R(a)^*)$ for $a \in \mathcal{N}_\psi$,
- $\pi_\psi(x) = J\pi(R(x)^*)J$ for $x \in A$,

where J still denotes the modular conjugation of φ in the GNS-construction (H, π, Λ).

If one would work with another GNS-construction for ψ, one would end up with a reduced C*-algebraic quantum group which is unitarily equivalent to the dual $(\widehat{A}, \widehat{\Delta})$. The same remark applies of course to φ itself.

It is then customary to define a unitary element $\widetilde{W} \in B(H \otimes H)$ such that

$$\widetilde{W}\big(\Lambda_\psi(a) \otimes \Lambda_\psi(b)\big) = (\Lambda_\psi \otimes \Lambda_\psi)\big(\Delta(a)(1 \otimes b)\big)$$

for $a, b \in \mathcal{N}_\psi$. This is again a multiplicative unitary, i.e. $\widetilde{W}_{12}\widetilde{W}_{13}\widetilde{W}_{23} = \widetilde{W}_{23}\widetilde{W}_{12}$. But this time,

(1) $\pi_\psi(A) = [(\omega \otimes \iota)(\widetilde{W}) \mid \omega \in B_0(H)^*]$,
(2) $(\pi_\psi \otimes \pi_\psi)(\Delta(a)) = \widetilde{W}(\pi_\psi(a) \otimes 1)\widetilde{W}^*$ for all $a \in A$.

LEMMA 8.26. – *The unitaries W and $\widetilde{W}$ are related in the following way*:

$$\widetilde{W} = (J \otimes J)\Sigma W^* \Sigma (J \otimes J).$$

Proof. – Choose $a, b \in \mathcal{N}_\psi$. Then $R(a)^*$ and $R(b)^*$ belong to $\mathcal{N}_\varphi$ and the formula $\chi(R \otimes R)\Delta = \Delta R$ implies that

$$\begin{aligned}
W^*\Sigma(J \otimes J)\big(\Lambda_\psi(a) \otimes \Lambda_\psi(b)\big) &= W^*\Sigma\big(\Lambda(R(a)^*) \otimes \Lambda(R(b)^*)\big) \\
&= W^*\big(\Lambda\big(R(b)^*\big) \otimes \Lambda\big(R(a)^*\big)\big) \\
&= (\Lambda \otimes \Lambda)\big(\Delta\big(R(a)^*\big)\big(R(b)^* \otimes 1\big)\big) \\
&= (\Lambda \otimes \Lambda)\big(\chi((R \otimes R)(\Delta(a)(1 \otimes b))^*)\big) \\
&= \Sigma(\Lambda \otimes \Lambda)\big((R \otimes R)\big(\Delta(a)(1 \otimes b)\big)^*\big) \\
&= \Sigma(J \otimes J)(\Lambda_\psi \otimes \Lambda_\psi)\big(\Delta(a)(1 \otimes b)\big) \\
&= \Sigma(J \otimes J)\widetilde{W}\big(\Lambda_\psi(a) \otimes \Lambda_\psi(b)\big). \qquad \Box
\end{aligned}$$

PROPOSITION 8.27. – *We have the following characterization for* $(\widehat{A}, \widehat{\Delta})$:

(1) $\widehat{A} = [(\iota \otimes \omega)(\widetilde{W}) \mid \omega \in B_0(H)^*]$,
(2) $\widehat{\Delta}(x) = \Sigma\widetilde{W}^*(1 \otimes x)\widetilde{W}\Sigma$ *for all* $x \in \widehat{A}$.

Proof. – Recall that the anti-unitary I has been fixed in the beginning of Section 6. Define U to be the unitary element in $B(H)$ such that $U = JI$. Then Proposition 5.38 and the previous lemma imply that $\widetilde{W} = \Sigma(U \otimes 1)W(U^* \otimes 1)\Sigma$. So we get for every $\omega \in B_0(H)^*$ that

$$(\iota \otimes \omega)(\widetilde{W}) = (U^*\omega U \otimes \iota)(W). \tag{8.8}$$

Consequently, Definition 8.1 implies immediately the first equality. Let us turn to the second one.

Therefore choose $v, w \in H$. Take an orthonormal basis $(e_i)_{i\in I}$ for H. Using Eq. (8.8) and Lemma A.5, we get that

$$\begin{aligned}\widehat{\Delta}\big((\iota \otimes \omega_{v,w})(\widetilde{W})\big) &= \widehat{\Delta}\big((\omega_{U^*v,U^*w} \otimes \iota)(W)\big) = (\omega_{U^*v,U^*w} \otimes \iota \otimes \iota)\big((\iota \otimes \widehat{\Delta})(W)\big)\\ &= (\omega_{U^*v,U^*w} \otimes \iota \otimes \iota)(W_{13}W_{12})\\ &= \sum_{i\in I}(\omega_{U^*v,e_i} \otimes \iota)(W) \otimes (\omega_{e_i,U^*w} \otimes \iota)(W).\end{aligned}$$

Using Eq. (8.8) and Lemma A.5 once more, we conclude

$$\begin{aligned}\widehat{\Delta}\big((\iota \otimes \omega_{v,w})(\widetilde{W})\big) &= \sum_{i\in I}(\omega_{U^*v,U^*(Ue_i)} \otimes \iota)(W) \otimes (\omega_{U^*(Ue_i),U^*w} \otimes \iota)(W)\\ &= \sum_{i\in I}(\iota \otimes \omega_{v,Ue_i})(\widetilde{W}) \otimes (\iota \otimes \omega_{Ue_i,w})(\widetilde{W})\\ &= (\iota \otimes \iota \otimes \omega_{v,w})(\widetilde{W}_{23}\widetilde{W}_{13}).\end{aligned}$$

Therefore the pentagonal equation for $\widetilde{W}$ implies that

$$\begin{aligned}\widehat{\Delta}\big((\iota \otimes \omega_{v,w})(\widetilde{W})\big) &= \Sigma(\iota \otimes \iota \otimes \omega_{v,w})(\widetilde{W}_{13}\widetilde{W}_{23})\Sigma\\ &= \Sigma(\iota \otimes \iota \otimes \omega_{v,w})(\widetilde{W}_{12}^*\widetilde{W}_{23}\widetilde{W}_{12})\Sigma = \Sigma\widetilde{W}^*\big(1 \otimes (\iota \otimes \omega_{v,w})(\widetilde{W})\big)\widetilde{W}\Sigma.\end{aligned}$$

The second equality follows now immediately from the first one. □

Remark 8.28. – Because $\lambda(L^1(A)) \odot \lambda(L^1(A))$ is a dense subset of $\widehat{A} \otimes \widehat{A}$ and because $L^1(A) \subseteq A^*$, we can consider a lot of elements in $M(\widehat{A} \otimes \widehat{A})$ as functionals on $A \otimes A$. So we can formally look at $\widehat{\Delta}(\lambda(\omega))(x \otimes y)$ for $\omega \in L^1(A)$ and $x, y \in A$. If $v, w \in H$ and $(e_i)_{i\in I}$ is an orthonormal basis for H we can make a calculation similar to the proof of the previous proposition, and obtain:

$$\widehat{\Delta}(\lambda(\omega_{v,w})) = \sum_{i\in I}\lambda(\omega_{v,e_i}) \otimes \lambda(\omega_{e_i,w}).$$

Hence we can write in some sense

$$\widehat{\Delta}(\lambda(\omega_{v,w}))(x \otimes y) = \sum_{i\in I}\omega_{v,e_i}(x)\omega_{e_i,w}(y) = \omega_{v,w}(yx).$$

This is precisely a formula one obtains in an algebraic theory (*see* e.g. [59]). It is possible to give a more precise meaning to the formula above, but we do not go into that.

We end this section with the Pontryagin duality theorem. As in the first part of this section, we can construct the dual reduced C*-algebraic quantum group of $(\widehat{A}, \widehat{\Delta})$ with respect to the GNS-construction $(H, \iota, \widehat{\Lambda})$ of $\widehat{\varphi}$. Let us denote the resulting reduced C*-algebraic quantum group by $(\widehat{\widehat{A}}, \widehat{\widehat{\Delta}})$.

From the remark after Theorem 8.20, we know that $\Sigma W^*\Sigma$ is the multiplicative unitary of $(\widehat{A},\widehat{\Delta})$ in the GNS-construction $(H,\iota,\widehat{\Lambda})$ for $\widehat{\varphi}$. Therefore, Definition 8.1 and the discussion in the second part of Section 4 imply easily the following theorem.

THEOREM 8.29. – *The reduced* C*-*algebraic quantum groups* (A,Δ) *and* $(\widehat{\widehat{A}},\widehat{\widehat{\Delta}})$ *are isomorphic. More specific, the mapping* $\pi : A \to \widehat{\widehat{A}}$ *is a* *-*isomorphism such that* $(\pi\otimes\pi)\Delta = \widehat{\widehat{\Delta}}\pi$.

Denote the dual weight of $\widehat{\varphi}$, as defined in Proposition 8.14, by $\widehat{\widehat{\varphi}}$. Let $\widehat{\mathcal{I}}$ be the left ideal in $\mathrm{L}^1(\widehat{A})$ constructed from $\widehat{\varphi}$ as in Notation 8.4 and let $\widehat{\widehat{\Lambda}}$ be the corresponding closed linear map as defined in Proposition 8.13. By definition of $\widehat{\widehat{\varphi}}$, the triple $(H,\iota,\widehat{\widehat{\Lambda}})$ is a GNS-construction for $\widehat{\widehat{\varphi}}$.

By uniqueness of the left invariant weight on (A,Δ), we get that $\widehat{\widehat{\varphi}}\pi$ is proportional to φ. But to our big surprise, we can easily prove that:

PROPOSITION 8.30. – *The weights* $\widehat{\widehat{\varphi}}\pi$ *and* φ *are equal. Moreover,* $\widehat{\widehat{\Lambda}}\pi = \Lambda$.

Proof. – Let $\widehat{\lambda}$ denote the canonical representation of $\mathrm{L}^1(\widehat{A})$ into $\widehat{\widehat{A}} = \pi(A)$ (as defined in Eq. (8.2)). Since $\Sigma W^*\Sigma$ is the left regular corepresentation of $(\widehat{A},\widehat{\Delta})$, we get that $\widehat{\lambda}(\omega) = (\iota\otimes\omega)(W^*)$ for $\omega\in \mathrm{L}^1(\widehat{A})$.

Notice that, since $\widehat{\widehat{\varphi}}\pi$ and φ are proportional, $\pi^{-1}(\mathcal{N}_{\widehat{\widehat{\varphi}}}) = \mathcal{N}_\varphi$. Choose $\omega\in\widehat{\mathcal{I}}$ and put $a = \pi^{-1}(\widehat{\lambda}(\omega)) = (\iota\otimes\omega)(V^*)\in\mathcal{N}_\varphi$. Then we have for $\theta\in\mathcal{I}$ that

$$\begin{aligned}\omega(\lambda(\theta)^*) &= \omega\big((\theta\otimes\iota)(V)^*\big) = \omega\big((\overline{\theta}\otimes\iota)(V^*)\big) = \overline{\theta}\big((\iota\otimes\omega)(V^*)\big)\\ &= \overline{\theta(a^*)} = \overline{\langle\xi(\theta),\Lambda(a)\rangle} = \big\langle\Lambda(a),\widehat{\Lambda}(\lambda(\theta))\big\rangle.\end{aligned}$$

Since $\lambda(\mathcal{I})$ is a core for $\widehat{\Lambda}$, this implies that

$$\omega(x^*) = \big\langle\Lambda(a),\widehat{\Lambda}(x)\big\rangle \quad \text{for all } x\in\mathcal{N}_{\widehat{\varphi}}. \tag{8.9}$$

Hence Proposition 8.13 implies that $\widehat{\widehat{\Lambda}}(\widehat{\lambda}(\omega)) = \Lambda(a) = \Lambda(\pi^{-1}(\widehat{\lambda}(\omega)))$.

Because $\widehat{\lambda}(\widehat{\mathcal{I}})$ is a core for $\widehat{\widehat{\Lambda}}$ and Λ is closed, this implies that $\widehat{\widehat{\Lambda}}(y) = \Lambda(\pi^{-1}(y))$ for all $y\in\mathcal{N}_{\widehat{\widehat{\varphi}}}$. Remembering that $\pi^{-1}(\mathcal{N}_{\widehat{\widehat{\varphi}}}) = \mathcal{N}_\varphi$, the proposition follows. □

Remark 8.31. – Consider $\theta\in\mathrm{L}^1(A)$ such that $\lambda(\theta)\in\mathcal{N}_{\widehat{\varphi}}$. Choose $\omega\in\widehat{\mathcal{I}}$. From Eq. (8.9) in the proof of the previous proposition, we know that $\langle\Lambda(\pi^{-1}(\widehat{\lambda}(\omega))),\widehat{\Lambda}(x)\rangle = \omega(x^*)$ for all $x\in\mathcal{N}_{\widehat{\varphi}}$.

In particular, $\langle\Lambda(\pi^{-1}(\widehat{\lambda}(\omega))),\widehat{\Lambda}(\lambda(\theta))\rangle = \omega(\lambda(\theta)^*)$. We can rewrite this as

$$\big\langle\widehat{\Lambda}(\lambda(\theta)),\Lambda(\pi^{-1}(\widehat{\lambda}(\omega)))\big\rangle = \theta(\pi^{-1}(\widehat{\lambda}(\omega))^*).$$

Because $\pi^{-1}(\widehat{\lambda}(\widehat{\mathcal{I}}))$ is a core for Λ, this implies that $\langle\widehat{\Lambda}(\lambda(\theta)),\Lambda(a)\rangle = \theta(a^*)$ for all $a\in\mathcal{N}_\varphi$. Consequently, θ belongs to $\mathcal{I}$.

So we arrive at the conclusion that $\mathcal{I} = \{\theta\in\mathrm{L}^1(A)\mid\lambda(\theta)\in\mathcal{N}_{\widehat{\varphi}}\}$.

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Appendix A: Some technical results concerning weights

LEMMA A.1. – *Consider a von Neumann algebra M, a normal semi-finite weight η on M and*

- *a vector space V with a subspace W,*
- *a normed space X and a linear mapping $\Lambda : V \to X$,*
- *a sesquilinear mapping $T : V \times V \to M$ such that $T(v,v) \geqslant 0$ for $v \in V$,*
- *a number $K \geqslant 0$,*

such that:

- *$\eta(T(v,v)) = K\|\Lambda(v)\|^2$ for $v \in W$;*
- *for every $v \in V$, there exists a net $(v_i)_{i\in I}$ in W such that $(T(v_i,v_i))_{i\in I}$ converges σ-strongly to $T(v,v)$ and $(\Lambda(v_i))_{i\in I}$ converges to $\Lambda(v)$.*

Then $\eta(T(v,v)) = K\|\Lambda(v)\|^2$ for $v \in V$.

Proof. – Choose $v \in V$. By assumption, there exists a net $(v_i)_{i\in I}$ in W such that $(T(v_i,v_i))_{i\in I}$ converges σ-strongly to $T(v,v)$ and $(\Lambda(v_i))_{i\in I}$ converges to $\Lambda(v)$.

Because η is normal, we have that

$$\eta(T(v,v)) \leqslant \liminf\big(\eta(T(v_i,v_i))\big)_{i\in I} = \liminf\big(K\|\Lambda(v_i)\|^2\big)_{i\in I} = K\|\Lambda(v)\|^2.$$

So we get that

$$T(v,v) \in \mathcal{M}_\eta^+ \quad\text{and}\quad \eta(T(v,v)) \leqslant K\|v\|^2 \quad \text{for all } v \in V. \tag{A.1}$$

By polarization, we get for every $v, w \in V$ that $T(v,w)$ belongs to $\mathcal{M}_\eta$. This gives us a semi-inner product $V \times V \to \mathbb{C}$: $(v,w) \mapsto \eta(T(v,w))$. This implies that the mapping $V \to \mathbb{R}^+$: $v \mapsto \eta(T(v,v))^{\frac{1}{2}}$ is a semi-norm on V.

Therefore we get for every v, $w \in V$ that

$$\big|\eta(T(v,v))^{\frac{1}{2}} - \eta(T(w,w))^{\frac{1}{2}}\big| \leqslant \eta\big(T(v-w,v-w)\big)^{\frac{1}{2}} \leqslant K^{\frac{1}{2}}\|\Lambda(v) - \Lambda(w)\|, \tag{A.2}$$

where we used Eq. (A.1) in the last inequality.

Take $u \in V$. By assumption, there exists a net $(u_j)_{j\in J}$ in W such that $(\Lambda(u_j))_{j\in J}$ converges to $\Lambda(u)$. Then inequality (A.2) implies that $(\eta(T(u_j,u_j)))_{j\in J}$ converges to $\eta(T(u,u))$.

But we assumed that $\eta(T(u_j,u_j)) = K\|\Lambda(u_j)\|^2$ for $j \in J$. Consequently, the net $(\eta(T(u_j,u_j)))_{j\in J}$ also converges to $K\|\Lambda(u)\|^2$. Therefore $\eta(T(u,u)) = K\|\Lambda(u)\|^2$. □

A.1. KSGNS-construction for a slice weight

Throughout this subsection, we will fix two C*-algebras A and B together with a proper weight φ on B. Let $(H_\varphi, \pi_\varphi, \Lambda_\varphi)$ be a GNS-construction for φ.

RESULT A.2. – *Consider $x \in \overline{\mathcal{N}}_{\iota\otimes\varphi}$ and $v \in H_\varphi$. Then there exists a unique element $q \in \mathrm{M}(A)$ such that $\theta(q) = \langle \Lambda_\varphi((\theta \otimes \iota)(x)), v\rangle$ for $\theta \in A^*$.*

Proof. – For $a \in \mathcal{N}_\varphi$, we put $q(a) = (\iota \otimes \varphi)((1 \otimes a^*)x) \in \mathrm{M}(A)$.

Using Proposition 1.24, we see for all $a, b \in \mathcal{N}_\varphi$ that

$$\begin{aligned}\|q(a) - q(b)\|^2 &= \|q(a-b)^* q(a-b)\| \\ &= \big\|(\iota \otimes \varphi)\big((1 \otimes (a-b)^*)x\big)^* (\iota \otimes \varphi)\big((1 \otimes (a-b)^*)\, x\big)\big\| \\ &\leqslant \big\|(\iota \otimes \varphi)\big([1 \otimes (a-b)]^*[1 \otimes (a-b)]\big)\big\| \|(\iota \otimes \varphi)(x^* x)\| \\ &= \|\Lambda_\varphi(a) - \Lambda_\varphi(b)\|^2 \|(\iota \otimes \varphi)(x^* x)\|.\end{aligned}$$

Now take a sequence $(a_n)_{n=1}^\infty$ in $\mathcal{N}_\varphi$ such that $(\Lambda_\varphi(a_n))_{n=1}^\infty$ converges to v. Then the previous inequality implies that $(q(a_n))_{n=1}^\infty$ is Cauchy and hence convergent in $\mathrm{M}(A)$. So there exists an element $q \in \mathrm{M}(A)$ such that $(q(a_n))_{n=1}^\infty$ converges in norm to q.

We have for every $\theta \in A^*$ and $n \in \mathbb{N}$ that

$$\theta(q(a_n)) = \varphi(a_n^*(\theta \otimes \iota)(x)) = \langle \Lambda_\varphi((\theta \otimes \iota)(x)), \Lambda_\varphi(a_n)\rangle,$$

which implies that $(\theta(q(a_n)))_{n=1}^\infty$ converges to $\langle \Lambda_\varphi((\theta \otimes \iota)(x)), v\rangle$. So $\theta(q)$ must be equal to $\langle \Lambda_\varphi((\theta \otimes \iota)(x)), v\rangle$. □

We borrowed the next result from [43].

LEMMA A.3. – *Consider an increasing net $(a_i)_{i\in I}$ in A^+ and an element $a \in A^+$ such that $\omega(a) = \sup\{\omega(a_i) \mid i \in I\}$ for all $\omega \in A_+^*$. Then the net $(a_i)_{i\in I}$ converges to a.*

Proof. – Define $S = \{\omega \in A_+^* \mid \|\omega\| \leqslant 1\}$ and equip S with the weak*-topology. Then S becomes a compact Hausdorff space.

For every $i \in I$, we define the function $f_i \in \mathrm{C}(S)^+$ such that $f_i(\omega) = \omega(a_i)$ for $\omega \in A_+^*$. So $(f_i)_{i\in I}$ is an increasing net in $\mathrm{C}(S)^+$. We also define the function $f \in \mathrm{C}(S)^+$ such that $f(\omega) = \omega(a)$ for $\omega \in A_+^*$.

By assumption, the net $(f_i)_{i\in I}$ converges pointwise to f. Therefore Dini's theorem implies that $(f_i)_{i\in I}$ converges uniformly to f.

Because $\|x\| = \sup\{|\omega(x)| \mid \omega \in S\}$ for $x \in A^+$, this implies that $(a_i)_{i\in I}$ converges in norm to a. □

It is not difficult to prove the following strict version.

LEMMA A.4. – *Consider an increasing net $(a_i)_{i\in I}$ in $\mathrm{M}(A)^+$ and an element $a \in \mathrm{M}(A)^+$ such that $\omega(a) = \sup\{\omega(a_i) \mid i \in I\}$ for all $\omega \in A_+^*$. Then the net $(a_i)_{i\in I}$ converges strictly to a.*

Proof. – Take $b \in A$. By the previous lemma, we get that the net $(b^* a_i b)_{i\in I}$ converges in norm to $b^* ab$.

Notice that $a_i \leqslant a$ for $i \in I$. So we get for $i \in I$ that

$$\|ab - a_i b\|^2 = \|b^*(a - a_i)^2 b\| \leqslant \|a - a_i\|\, \|b^*(a - a_i)b\| \leqslant 2\|a\|\, \|b^*(a - a_i)b\|,$$

which implies that $(a_i b)_{i\in I}$ converges in norm to ab. □

LEMMA A.5. – *Consider a non-degenerate *-representation π of A on a Hilbert space H and an orthonormal basis $(e_i)_{i\in I}$ for H. Let $x, y \in \mathrm{M}(A \otimes B)$ and $v, w \in H$. Then the net*

$$\left(\sum_{i\in J} (\omega_{w,e_i} \otimes \iota)(y)^* (\omega_{v,e_i} \otimes \iota)(x)\right)_{J \in F(I)}$$

is bounded and converges strictly to $(\omega_{v,w} \otimes \iota)(y^* x)$.

Proof. – By polarization, it is enough to prove the result for $y = x$ and $v = w$.

Choose $\mu \in B^*_+$ and take a cyclic GNS-construction (K, θ, u) for μ. Fix also an orthonormal basis $(f_\ell)_{\ell \in L}$ for K. Then

$$\begin{aligned}
&\sum_{i\in I} \mu\big((\omega_{v,e_i} \otimes \iota)(x)^*(\omega_{v,e_i} \otimes \iota)(x)\big)\\
&\quad = \sum_{i\in I} \big\langle \theta((\omega_{v,e_i} \otimes \iota)(x))u, \theta((\omega_{v,e_i} \otimes \iota)(x))u \big\rangle\\
&\quad = \sum_{i\in I}\sum_{\ell\in L} \big\langle \theta((\omega_{v,e_i} \otimes \iota)(x))u, f_\ell \big\rangle \big\langle f_\ell, \theta((\omega_{v,e_i} \otimes \iota)(x))u \big\rangle\\
&\quad = \sum_{i\in I}\sum_{\ell\in L} \big\langle (\pi \otimes \theta)(x)(v \otimes u), e_i \otimes f_\ell \big\rangle \big\langle e_i \otimes f_\ell, (\pi \otimes \theta)(x)(v \otimes u) \big\rangle\\
&\quad = \big\langle (\pi \otimes \theta)(x)(v \otimes u), (\pi \otimes \theta)(x)(v \otimes u) \big\rangle = (\omega_{v,v} \otimes \omega_{u,u})(x^*x) = \mu\big((\omega_{v,v} \otimes \iota)(x^*x)\big).
\end{aligned}$$

Consequently, the lemma follows from Lemma A.4. □

The next simple lemma will turn out to be useful at several instances.

LEMMA A.6. – *Consider a non-degenerate *-representation θ of A on a Hilbert space K and an orthonormal basis $(e_i)_{i\in I}$ for K. Let $v, w \in K$ and $x, y \in \overline{\mathcal{N}}_{\iota\otimes\varphi}$. Then the net*

$$\left(\sum_{i\in J} \varphi\big((\omega_{w,e_i} \otimes \iota)(y)^*(\omega_{v,e_i} \otimes \iota)(x)\big)\right)_{J\in F(I)}$$

converges to $\varphi((\omega_{v,w} \otimes \iota)(y^*x))$.

Proof. – By polarization, it is enough to prove this lemma for $x = y$ and $v = w$. We have for every $J \in F(I)$ that

$$\sum_{i\in J} \varphi\big((\omega_{v,e_i} \otimes \iota)(x)^*(\omega_{v,e_i} \otimes \iota)(x)\big) = \varphi\left(\sum_{i\in J} (\omega_{v,e_i} \otimes \iota)(x)^*(\omega_{v,e_i} \otimes \iota)(x)\right).$$

By Lemma A.5, we know that $\big(\sum_{i\in J} (\omega_{v,e_i} \otimes \iota)(x)^*(\omega_{v,e_i} \otimes \iota)(x)\big)_{J\in F(I)}$ is an increasing net which converges strictly to $(\omega_{v,v} \otimes \iota)(x^*x)$. Hence the strict lower semi-continuity of φ implies that the net

$$\left(\sum_{i\in J} \varphi\big((\omega_{v,e_i} \otimes \iota)(x)^*(\omega_{v,e_i} \otimes \iota)(x)\big)\right)_{J\in F(I)}$$

converges to $\varphi\big((\omega_{v,v} \otimes \iota)(x^*x)\big)$. □

We will now apply these results to get a sort of KSGNS-construction for the 'C^*-valued weight' $\iota \otimes \varphi$:

PROPOSITION A.7. – *There exists a unique linear map $\Lambda : \overline{\mathcal{N}}_{\iota\otimes\varphi} \to \mathcal{L}(A, A \otimes H_\varphi)$ such that*

$$\Lambda(x)^*\big(a \otimes \Lambda_\varphi(b)\big) = (\iota \otimes \varphi)\big(x^*(a \otimes b)\big)$$

for $a \in A$ and $b \in \mathcal{N}_\varphi$.

For $x \in \overline{\mathcal{N}}_{\iota\otimes\varphi}$, we put $(\iota \otimes \Lambda_\varphi)(x) = \Lambda(x)$. Then we have the following properties:

- *we have for all $x, y \in \overline{\mathcal{N}}_{\iota\otimes\varphi}$ that $(\iota \otimes \Lambda_\varphi)(y)^*(\iota \otimes \Lambda_\varphi)(x) = (\iota \otimes \varphi)(y^*x)$;*

- *consider* $a \in \mathrm{M}(A)$ *and* $b \in \overline{\mathcal{N}}_\varphi$, *then* $(\iota \otimes \Lambda_\varphi)(a \otimes b) = a \otimes \Lambda_\varphi(b)$;
- *we have for* $x \in \mathrm{M}(A \otimes B)$ *and* $y \in \overline{\mathcal{N}}_{\iota\otimes\varphi}$ *that* $(\iota \otimes \Lambda_\varphi)(xy) = (\iota \otimes \pi_\varphi)(x)(\iota \otimes \Lambda_\varphi)(y)$.

Proof. – Fix an orthonormal basis $(e_i)_{i\in I}$ for H_φ.

Take $x \in \overline{\mathcal{N}}_{\iota\otimes\varphi}$. Using Result A.2, we define for every $i \in I$ the element $q_i \in \mathrm{M}(A)$ such that $\eta(q_i) = \langle \Lambda_\varphi((\eta \otimes \iota)(x)), e_i \rangle$ for $\eta \in A^*$.

Choose $\mu \in A^*_+$ and a cyclic GNS-construction (K, θ, v) for μ. Fix also an orthonormal basis $(f_\ell)_{\ell\in L}$ for K. Then we have that

$$\begin{aligned}\sum_{i\in I} \mu(q_i^* q_i) &= \sum_{i\in I} \langle \theta(q_i)v, \theta(q_i)v\rangle = \sum_{i\in I}\sum_{\ell\in L} \langle \theta(q_i)v, f_\ell\rangle \langle f_\ell, \theta(q_i)v\rangle \\ &= \sum_{i\in I}\sum_{\ell\in L} |\omega_{v,f_\ell}(q_i)|^2 = \sum_{\ell\in L}\sum_{i\in I} |\langle \Lambda_\varphi((\omega_{v,f_\ell}\otimes\iota)(x)), e_i\rangle|^2 \\ &= \sum_{\ell\in L} \varphi\big((\omega_{v,f_\ell}\otimes\iota)(x)^*(\omega_{v,f_\ell}\otimes\iota)(x)\big).\end{aligned}$$

Therefore Lemma A.6 implies that

$$\sum_{i\in I} \mu(q_i^* q_i) = \varphi\big((\mu\otimes\iota)(x^*x)\big) = \mu\big((\iota\otimes\varphi)(x^*x)\big).$$

Hence Lemma A.4 implies that the net $(\sum_{i\in J} q_i^* q_i)_{J\in F(I)}$ converges strictly to $(\iota\otimes\varphi)(x^*x)$.

Take finite subsets J and K of I such that $J \subseteq K$ and $a \in A$. Then we have that

$$\left\| \sum_{i\in K} q_i a \otimes e_i - \sum_{i\in J} q_i a \otimes e_i \right\|^2 = \left\| \sum_{i\in K\setminus J} a^* q_i^* q_i a \right\|,$$

implying that the net

$$\left(\sum_{i\in J} q_i a \otimes e_i \right)_{J\in F(I)}$$

is Cauchy and hence convergent in $A \otimes H_\varphi$.

So we can define an A-linear operator $F_x : A \to A \otimes H_\varphi$ such that $F_x(a) = \sum_{i\in I} q_i a \otimes e_i$ for $a \in A$.

Because $\sum_{i\in I} q_i^* q_i = (\iota\otimes\varphi)(x^*x)$ in the strict topology, it follows that

$$\langle F_x(a), F_x(a)\rangle = a^*(\iota\otimes\varphi)(x^*x)a \quad \text{for } a \in A. \tag{A.3}$$

Choose $a \in A$, $b \in A$ and $c \in \mathcal{N}_\varphi$. Then we have for $\omega \in A^*$ that

$$\begin{aligned}\omega\big(\langle F_x(a), b\otimes\Lambda_\varphi(c)\rangle\big) &= \sum_{i\in I} \omega\big(\langle q_i a\otimes e_i, b\otimes \Lambda_\varphi(c)\rangle\big) = \sum_{i\in I} \omega(b^* q_i a)\langle e_i, \Lambda_\varphi(c)\rangle \\ &= \sum_{i\in I} \big\langle \Lambda_\varphi((a\omega b^*\otimes\iota)(x)), e_i\big\rangle \big\langle e_i, \Lambda_\varphi(c)\big\rangle \\ &= \big\langle \Lambda_\varphi((a\omega b^*\otimes\iota)(x)), \Lambda_\varphi(c)\big\rangle \\ &= \varphi\big(c^*(a\omega b^*\otimes\iota)(x)\big) = (a\omega b^*)\big((\iota\otimes\varphi)((1\otimes c^*)x)\big) \\ &= \omega\big((\iota\otimes\varphi)((b^*\otimes c^*)x)a\big).\end{aligned}$$

So we see that

$$\big\langle F_x(a), b\otimes\Lambda_\varphi(c)\big\rangle = (\iota\otimes\varphi)\big((b^*\otimes c^*)x\big)\, a. \tag{A.4}$$

The Cauchy–Schwarz inequality in Proposition 1.24 implies for $y\in A\odot\mathcal{N}_\varphi$ that

$$\|(\iota\otimes\varphi)(x^*y)\|^2 \leqslant \|(\iota\otimes\varphi)(x^*x)\|\,\|(\iota\otimes\varphi)(y^*y)\| = \|(\iota\otimes\varphi)(x^*x)\|\,\|(\iota\odot\Lambda_\varphi)(y)\|^2$$

This implies that there exists a unique bounded A-linear map $G_x : A\otimes H_\varphi\to A$ such that $G_x(b\otimes\Lambda_\varphi(c)) = (\iota\otimes\varphi)(x^*(b\otimes c))$ for $b\in A$ and $c\in\mathcal{N}_\varphi$. Eq. (A.4) then implies that $\langle F_x(a), v\rangle = G_x(v)^*a$ for $a\in A$ and $v\in A\otimes H_\varphi$. So F_x belongs to $\mathcal{L}(A, A\otimes H_\varphi)$ and $F_x = G_x^*$.

Now define the map $\Lambda : \overline{\mathcal{N}}_{\iota\otimes\varphi}\to\mathcal{L}(A, A\otimes H_\varphi)$ such that $\Lambda(x) = F_x$ for $x\in\overline{\mathcal{N}}_{\iota\otimes\varphi}$. So we have that $\Lambda(x)^*(b\otimes\Lambda_\varphi(c)) = (\iota\otimes\varphi)(x^*(b\otimes c))$ for $b\in A$, $c\in\mathcal{N}_\varphi$ and $x\in\overline{\mathcal{N}}_{\iota\otimes\varphi}$.

This implies immediately that Λ is linear and that $\Lambda(a\otimes b) = a\otimes\Lambda_\varphi(b)$ for $a\in \mathrm{M}(A)$ and $b\in\overline{\mathcal{N}}_\varphi$.

Eq. (A.3) implies that $\Lambda(x)^*\Lambda(x) = (\iota\otimes\varphi)(x^*x)$ for $x\in\overline{\mathcal{N}}_{\iota\otimes\varphi}$, so polarization yields that $\Lambda(y)^*\Lambda(x) = (\iota\otimes\varphi)(y^*x)$ for $x,y\in\overline{\mathcal{N}}_{\iota\otimes\varphi}$.

Choose $b\in A$ and $c\in\mathcal{N}_\varphi$. Then we have for $z\in A\otimes B$ that

$$\begin{aligned}\big\|(\iota\otimes\Lambda_\varphi)\big(z(b\otimes c)\big)\big\|^2 &= \big\|(\iota\otimes\Lambda_\varphi)\big(z(b\otimes c)\big)^*(\iota\otimes\Lambda_\varphi)\big(z(b\otimes c)\big)\big\|\\ &= \big\|(\iota\otimes\varphi)\big((b\otimes c)^*z^*z(b\otimes c)\big)\big\|\\ &\leqslant \|z^*z\|\,\big\|(\iota\otimes\varphi)\big((b\otimes c)^*(b\otimes c)\big)\big\| = \|z\|^2\,\|b\|^2\,\varphi(c^*c).\end{aligned}$$

So the linear mapping $A\otimes B\to\mathcal{L}(A, A\otimes H_\varphi)$: $z\mapsto(\iota\otimes\Lambda_\varphi)(z(b\otimes c))$ is bounded. This is of course also true for the linear mapping $A\otimes B\to\mathcal{L}(A, A\otimes H_\varphi)$: $z\mapsto(\iota\otimes\pi_\varphi)(z)(b\otimes\Lambda_\varphi(c))$.

It is easy to see that both mappings above agree on $A\odot B$ so they agree on $A\otimes B$, i.e. $(\iota\otimes\Lambda_\varphi)(z(b\otimes c)) = (\iota\otimes\pi_\varphi)(z)(b\otimes\Lambda_\varphi(c))$ for $z\in A\otimes B$.

Now choose $x\in\overline{\mathcal{N}}_{\iota\otimes\varphi}$ and $z\in\mathrm{M}(A\otimes B)$. Take an approximate unit $(u_j)_{j\in J}$ for $A\otimes B$. Then we have for $j\in I$ and $b\in A$, $c\in\mathcal{N}_\varphi$ that

$$\begin{aligned}&\big[(\iota\otimes\pi_\varphi)(u_j)(\iota\otimes\pi_\varphi)(z)(\iota\otimes\Lambda_\varphi)(x)\big]^*\big(b\otimes\Lambda_\varphi(c)\big)\\ &\quad= \big[(\iota\otimes\pi_\varphi)(u_jz)(\iota\otimes\Lambda_\varphi)(x)\big]^*\big(b\otimes\Lambda_\varphi(c)\big)\\ &\quad= (\iota\otimes\Lambda_\varphi)(x)^*(\iota\otimes\pi_\varphi)(z^*u_j^*)\big(b\otimes\Lambda_\varphi(c)\big) = (\iota\otimes\Lambda_\varphi)(x)^*(\iota\otimes\Lambda_\varphi)\big(z^*u_j^*(b\otimes c)\big)\\ &\quad= (\iota\otimes\varphi)\big(x^*z^*u_j^*(b\otimes c)\big) = (\iota\otimes\Lambda_\varphi)(zx)^*(\iota\otimes\Lambda_\varphi)\big(u_j^*(b\otimes c)\big)\\ &\quad= (\iota\otimes\Lambda_\varphi)(zx)^*(\iota\otimes\pi_\varphi)(u_j^*)\big(b\otimes\Lambda_\varphi(c)\big) = \big[(\iota\otimes\pi_\varphi)(u_j)(\iota\otimes\Lambda_\varphi)(zx)\big]^*\big(b\otimes\Lambda_\varphi(c)\big).\end{aligned}$$

Hence $(\iota\otimes\pi_\varphi)(u_j)(\iota\otimes\pi_\varphi)(z)(\iota\otimes\Lambda_\varphi)(x) = (\iota\otimes\pi_\varphi)(u_j)(\iota\otimes\Lambda_\varphi)(zx)$ for all $j\in J$, so $(\iota\otimes\pi_\varphi)(z)(\iota\otimes\Lambda_\varphi)(x) = (\iota\otimes\Lambda_\varphi)(zx)$. □

A.2. Partial GNS-construction for the tensor product of two weights

Throughout this subsection, we will fix C*-algebras A and B, a proper weight φ on A and a proper weight ψ on B. At the same time, we fix a GNS-construction $(H_\varphi,\pi_\varphi,\Lambda_\varphi)$ for φ and a GNS-construction $(H_\psi,\pi_\psi,\Lambda_\psi)$ for ψ. For used notations, we refer to Subsection 1.6.

PROPOSITION A.8. – *The following properties hold*:

- *the mapping* $\Lambda_\varphi\otimes\Lambda_\psi : \mathcal{N}(\varphi,\psi)\to H_\varphi\otimes H_\psi$ *is a linear map which is closed with respect to the strict topology on* $A\otimes B$ *and the norm topology on* $H_\varphi\otimes H_\psi$;
- *the mapping* $\Lambda_\varphi\otimes\Lambda_\psi : \mathcal{N}(\varphi,\psi)\to H_\varphi\otimes H_\psi$ *is closable with respect to the strict topology on* $\mathrm{M}(A\otimes B)$ *and the norm topology on* $H_\varphi\otimes H_\psi$. *Denote its closure by* $\overline{\Lambda_\varphi\otimes\Lambda_\psi}$. *Then* $D(\overline{\Lambda_\varphi\otimes\Lambda_\psi}) = \overline{\mathcal{N}}(\varphi,\psi)$ *and we put* $(\Lambda_\varphi\otimes\Lambda_\psi)(a) = (\overline{\Lambda_\varphi\otimes\Lambda_\psi})(a)$ *for* $a\in\overline{\mathcal{N}}(\varphi,\psi)$;

- $\langle(\Lambda_\varphi \otimes \Lambda_\psi)(x), (\Lambda_\varphi \otimes \Lambda_\psi)(y)\rangle = (\varphi \otimes \psi)(y^* x)$ *for* $x, y \in \overline{\mathcal{N}}(\varphi, \psi)$;
- $\overline{\mathcal{N}}(\varphi, \psi)$ *and* $\mathcal{N}(\varphi, \psi)$ *are left ideals in* $\mathrm{M}(A \otimes B)$ *and* $(\pi_\varphi \otimes \pi_\psi)(x)(\Lambda_\varphi \otimes \Lambda_\psi)(a) = (\Lambda_\varphi \otimes \Lambda_\psi)(x\,a)$ *for* $x \in \mathrm{M}(A \otimes B)$ *and* $a \in \overline{\mathcal{N}}(\varphi, \psi)$.

Proof. – Define a mapping $\Gamma : \overline{\mathcal{N}}(\varphi, \psi) \to H_\varphi \otimes H_\psi$ as follows.

Consider $x \in \overline{\mathcal{N}}(\varphi, \psi)$. Then we define $\Gamma(x) \in H_\varphi \otimes H_\psi$ such that

$$\langle \Gamma(x), \Lambda_\varphi(a) \otimes \Lambda_\psi(b) \rangle = (\varphi \otimes \psi)\big((a^* \otimes b^*)x\big)$$

for $a \in \mathcal{N}_\varphi$, $b \in \mathcal{N}_\psi$. So $\Lambda_\varphi \otimes \Lambda_\psi$ is the restriction of Γ to $\mathcal{N}(\varphi, \psi)$.

Fix a GNS-construction (H, π, Λ) for $\varphi \otimes \psi$ and define the isometry $U : H_\varphi \otimes H_\psi \to H$ such that $U(\Lambda_\varphi(a) \otimes \Lambda_\psi(b)) = \Lambda(a \otimes b)$ for $a \in \mathcal{N}_\varphi$ and $b \in \mathcal{N}_\psi$. We have for all $x \in \overline{\mathcal{N}}_{\varphi \otimes \psi}$ and $a \in \mathcal{N}_\varphi$ and $b \in \mathcal{N}_\psi$ that

$$\begin{aligned}\langle U^* \Lambda(x), \Lambda_\varphi(a) \otimes \Lambda_\psi(b) \rangle &= \langle \Lambda(x), U(\Lambda_\varphi(a) \otimes \Lambda_\psi(b)) \rangle \\ &= \langle \Lambda(x), \Lambda(a \otimes b) \rangle = (\varphi \otimes \psi)\big((a^* \otimes b^*)x\big).\end{aligned}$$

This implies for every $x \in \overline{\mathcal{N}}_{\varphi \otimes \psi}$ that $x \in \overline{\mathcal{N}}(\varphi, \psi) \Leftrightarrow \|U^* \Lambda(x)\|^2 = (\varphi \otimes \psi)(x^* x) \Leftrightarrow \|U^* \Lambda(x)\| = \|\Lambda(x)\| \Leftrightarrow \Lambda(x) \in U(H_\varphi \otimes H_\psi)$. Moreover, $\Gamma(x) = U^* \Lambda(x)$ and $U\Gamma(x) = \Lambda(x)$ for all $x \in \overline{\mathcal{N}}(\varphi, \psi)$.

This implies easily that $\overline{\mathcal{N}}(\varphi, \psi)$ is a subspace of $\mathrm{M}(A \otimes B)$ and that the map Γ is linear. We also have for every $x, y \in \overline{\mathcal{N}}(\varphi, \psi)$ that

$$\langle \Gamma(x), \Gamma(y) \rangle = \langle U\Gamma(x), U\Gamma(y) \rangle = \langle \Lambda(x), \Lambda(y) \rangle = (\varphi \otimes \psi)(y^* x).$$

Choose $x \in \mathrm{M}(A \otimes B)$ and $a \in \overline{\mathcal{N}}(\varphi, \psi)$. It is easy to see that $U(\pi_\varphi \otimes \pi_\psi)(x) = \pi(x)U$ (by checking it first on simple tensors and then using continuity arguments). Then $xa \in \overline{\mathcal{N}}_{\varphi \otimes \psi}$ and

$$\Lambda(xa) = \pi(x)\Lambda(a) = \pi(x)U\Gamma(a) = U(\pi_\varphi \otimes \pi_\psi)(x)\Gamma(a) \in U(H_\varphi \otimes H_\psi).$$

Hence

$$xa \in \overline{\mathcal{N}}(\varphi, \psi) \quad \text{and} \quad \Gamma(xa) = U^* \Lambda(xa) = (\pi_\varphi \otimes \pi_\psi)(x)\Gamma(a). \tag{A.5}$$

Choose a net $(x_i)_{i \in I}$ in $\overline{\mathcal{N}}(\varphi, \psi)$, $x \in \mathrm{M}(A \otimes B)$ and $v \in H_\varphi \otimes H_\psi$ such that $(x_i)_{i \in I}$ converges strictly to x and $(\Gamma(x_i))_{i \in I}$ converges to v. Because $U\Gamma(x_i) = \Lambda(x_i)$ for $i \in I$, this implies that $(\Lambda(x_i))_{i \in I}$ converges to Uv. So the strict closedness of $\overline{\Lambda}$ (*see* Proposition 1.9) implies that $x \in \overline{\mathcal{N}}_{\varphi \otimes \psi}$ and $\Lambda(x) = Uv$. Since $\Lambda(x) \in U(H_\varphi \otimes H_\psi)$, we get that $x \in \overline{\mathcal{N}}(\varphi, \psi)$ and $\Gamma(x) = U^* \Lambda(x) = v$. Hence Γ is strict-norm closed. Since $\Lambda_\varphi \otimes \Lambda_\psi$ is the restriction of Γ to $\overline{\mathcal{N}}(\varphi, \psi) \cap A$, it follows that $\Lambda_\varphi \otimes \Lambda_\psi$ is also strict-norm closed as a map on $A \otimes B$.

Using Eq. (A.5) and an approximate unit for A, we see that $\mathcal{N}(\varphi, \psi)$ is a strict core for Γ. □

PROPOSITION A.9. – *Consider an orthonormal basis* $(e_i)_{i \in I}$ *for* H_φ. *Let* $x \in \overline{\mathcal{N}}_{\iota \otimes \psi}$ *and* $y \in \overline{\mathcal{N}}_\varphi$. *Then* $x\,(y \otimes 1)$ *belongs to* $\overline{\mathcal{N}}(\varphi, \psi)$,

$$\begin{aligned}\sum_{i \in I} \big\| \Lambda_\psi\big((\omega_{\Lambda_\varphi(y), e_i} \otimes \iota)(x)\big) \big\|^2 &= (\varphi \otimes \psi)\big((y^* \otimes 1)\, x^* x\, (y \otimes 1)\big) \\ &= \varphi\big(y^* (\iota \otimes \psi)(x^* x) y\big) < \infty\end{aligned}$$

and

$$(\Lambda_\varphi \otimes \Lambda_\psi)\big(x(y \otimes 1)\big) = \sum_{i \in I} e_i \otimes \Lambda_\psi\big((\omega_{\Lambda_\varphi(y), e_i} \otimes \iota)(x)\big).$$

Proof. – We have that

$$(\varphi\otimes\psi)\big((y^*\otimes 1)x^*x(y\otimes 1)\big)=\sup_{\omega\in\mathcal{G}_\varphi,\theta\in\mathcal{G}_\psi}(\omega\otimes\theta)\big((y^*\otimes 1)x^*x(y\otimes 1)\big)$$
$$=\sup_{\omega\in\mathcal{G}_\varphi}\Big(\sup_{\theta\in\mathcal{G}_\psi}\omega(y^*(\iota\otimes\theta)(x^*x)y)\Big).$$

Since $((\iota\otimes\theta)(x^*x))_{\theta\in\mathcal{G}_\psi}$ is an increasing net which converges strictly to $(\iota\otimes\psi)(x^*x)$, the above equality implies that

$$\text{(A.6)}\quad (\varphi\otimes\psi)\big((y^*\otimes 1)x^*x(y\otimes 1)\big)=\sup_{\omega\in\mathcal{G}_\varphi}\omega\big(y^*(\iota\otimes\psi)(x^*x)y\big)=\varphi\big(y^*(\iota\otimes\psi)(x^*x)y\big).$$

So we have in particular that $x(y\otimes 1)\in\overline{\mathcal{N}}_{\varphi\otimes\psi}$. Now

$$\sum_{i\in I}\big\|\Lambda_\psi\big((\omega_{\Lambda_\varphi(y),e_i}\otimes\iota)(x)\big)\big\|^2=\sum_{i\in I}\psi\big((\omega_{\Lambda_\varphi(y),e_i}\otimes\iota)(x)^*(\omega_{\Lambda_\varphi(y),e_i}\otimes\iota)(x)\big).$$

Using Lemma A.6, the above equality implies that

$$\sum_{i\in I}\big\|\Lambda_\psi\big((\omega_{\Lambda_\varphi(y),e_i}\otimes\iota)(x)\big)\big\|^2=\psi\big((\omega_{\Lambda_\varphi(y),\Lambda_\varphi(y)}\otimes\iota)(x^*x)\big)$$
$$=\omega_{\Lambda_\varphi(y),\Lambda_\varphi(y)}\big((\iota\otimes\psi)(x^*x)\big)$$
$$=(\varphi\otimes\psi)\big((y^*\otimes 1)\,x^*x\,(y\otimes 1)\big).$$

Put $v=\sum_{i\in I}e_i\otimes\Lambda_\psi((\omega_{\Lambda_\varphi(y),e_i}\otimes\iota)(x))$. Then it is clear that

$$\|v\|^2=(\varphi\otimes\psi)\big((y^*\otimes 1)x^*x(y\otimes 1)\big).$$

Now, choose $a\in\mathcal{N}_\varphi$ and $b\in\mathcal{N}_\psi$. Then we have that

$$\text{(A.7)}\qquad \big\langle v,\Lambda_\varphi(a)\otimes\Lambda_\psi(b)\big\rangle=\sum_{i\in I}\langle e_i,\Lambda_\varphi(a)\rangle\big\langle\Lambda_\psi\big((\omega_{\Lambda_\varphi(y),e_i}\otimes\iota)(x)\big),\Lambda_\psi(b)\big\rangle.$$

We have for $w\in H_\varphi$ that

$$\big\|\Lambda_\psi\big((\omega_{\Lambda_\varphi(y),w}\otimes\iota)(x)\big)\big\|^2=\psi\big((\omega_{\Lambda_\varphi(y),w}\otimes\iota)(x)^*(\omega_{\Lambda_\varphi(y),w}\otimes\iota)(x)\big)$$
$$\leqslant\|w\|^2\psi\big((\omega_{\Lambda_\varphi(y),\Lambda_\varphi(y)}\otimes\iota)(x^*x)\big).$$

So we get that the antilinear map $H_\varphi\to H_\psi: w\mapsto\Lambda_\psi((\omega_{\Lambda_\varphi(y),w}\otimes\iota)(x))$ is bounded. Therefore Eq. (A.7) implies that

$$\big\langle v,\Lambda_\varphi(a)\otimes\Lambda_\psi(b)\big\rangle=\big\langle\Lambda_\psi((\omega_{\Lambda_\varphi(y),\Lambda_\varphi(a)}\otimes\iota)(x)),\Lambda_\psi(b)\big\rangle$$
$$=\psi\big(b^*(\omega_{\Lambda_\varphi(y),\Lambda_\varphi(a)}\otimes\iota)(x)\big)=\omega_{\Lambda_\varphi(y),\Lambda_\varphi(a)}\big((\iota\otimes\psi)((1\otimes b^*)x)\big)$$
$$=(\varphi\otimes\psi)\big((a^*\otimes b^*)x(y\otimes 1)\big),$$

where the last equality follows in a similar way as Eq. (A.6) by using polarization. □

Note added in proof. – Recently, the second author proved that in fact for arbitrary proper weights φ and ψ on C*-algebras A and B, one has that $\mathcal{N}(\varphi,\psi)=\mathcal{N}_{\varphi\otimes\psi}$, and hence also that $(H_\varphi\otimes H_\psi,\pi_\varphi\otimes\pi_\psi,\Lambda_\varphi\otimes\Lambda_\psi)$ is a GNS-construction for $\varphi\otimes\psi$. Moreover one can prove that $\mathcal{N}_\varphi\odot\mathcal{N}_\psi$ is a core for $\Lambda_\varphi\otimes\Lambda_\psi$.

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