ANNALES SCIENTIFIQUES DE L'É.N.S.

ERIC M. FRIEDLANDER

Bloch-Ogus properties for topological cycle theory

Annales scientifiques de l'É.N.S. 4e série, tome 33, nº 1 (2000), p. 57-79
<http://www.numdam.org/item?id=ASENS_2000_4_33_1_57_0>

Ann. Scient. Éc. Norm. Sup.,
4e série, t. 33, 2000, p. 57 à 79.

BLOCH–OGUS PROPERTIES FOR TOPOLOGICAL CYCLE THEORY

BY ERIC M. FRIEDLANDER [1]

ABSTRACT. – We reformulate and extend "morphic cohomology" developed by the author and H.B. Lawson so that, together with "Lawson homology", it satisfies the axioms of S. Bloch and A. Ogus for a "Poincaré duality theory with supports" on complex quasi-projective varieties. © 2000 Éditions scientifiques et médicales Elsevier SAS

RÉSUMÉ. – Nous reformulons et étendons la cohomologie morphique développée par l'auteur et H.B. Lawson de telle sorte qu'elle satisfasse, avec l'homologie de Lawson, les axiomes de S. Bloch et A. Ogus pour une théorie de la dualité de Poincaré à supports sur les variétés complexes quasi-projectives. © 2000 Éditions scientifiques et médicales Elsevier SAS

In this paper, we re-formulate "morphic cohomology" as introduced by the author and H.B. Lawson [8] in such a way that it and "Lawson homology" satisfy the list of basic properties codified by S. Bloch and A. Ogus [3]. This reformulation enables us to clarify and unify our previous definitions and provides this *topological cycle theory* with foundational properties which have proved useful for other cohomology theories. One formal consequence of these Bloch–Ogus properties is the existence of a local-to-global spectral sequence which should prove valuable for computations (a simple example of which is given in Corollary 7.3).

The basic result of this paper is that *topological cycle cohomology theory* (which agrees with morphic cohomology for smooth varieties) in conjunction with *topological cycle homology theory* (which is shown to always agree with Lawson homology) do indeed satisfy the Bloch–Ogus properties for a "Poincaré duality theory with supports" on complex quasi-projective varieties. As we demonstrate in the final section of this paper, our techniques also suffice to prove that topological cycle theory satisfies the stronger axioms of H. Gillet [15] (other than homotopy invariance of cohomology on singular varieties).

We view the challenge of verification of the Bloch–Ogus properties as worthy for several reasons. First, the properties require certain definitions and constructions whose development add substance to morphic cohomology/Lawson homology. For example, considerable effort is required to extend earlier definitions to a cohomology theory defined and contravariantly functorial on all quasi-projective varieties. As another example, our cap product pairing (whose continuity is established in [14]) leads to a natural extension of earlier duality theorems of the author and Lawson [9,6,11].

Second, the properties constrain the formulation of our theory, thereby giving us a good basis for choosing the definitions we propose. Third, the fact that these properties can be verified tells

[1] Partially supported by the N.S.F. and the N.S.A.

us that this theory, although originating as it does in differential geometry and topology, behaves very much as other theories familiar to algebraic geometers and thus might be more readily applicable to geometric problems. Finally, the constructions we present are closely related to those involved in formulating a suitable motivic cohomology theory as in [13] (for example, we use V. Voevodsky's qfh-topology to extend definitions from normal varieties to all varieties), so that our topological point of view may serve as an accessible entry into that more algebraic theory.

Over the past decade, Lawson homology and morphic cohomology have been reformulated several times, with each reformulation providing either a simplification of definitions and/or an extension of the class of varieties for which the theory is applicable (cf. [5,6,8,9,12,16,17]). As now presented our topological cycle theory is an appropriate extension to all quasi-projective varieties of Lawson homology/morphic cohomology on smooth varieties.

Throughout this paper, "variety" will be taken to mean a locally closed subset (in the Zariski topology) of some complex projective space (i.e., a reduced but not-necessarily irreducible scheme of finite type over the complex field $\mathbf{C}$ which admits a Zariski locally closed embedding in a complex projective space).

I am grateful to Robert Lateveer for asking whether the Bloch–Ogus properties hold for topological cycle theory. I also thank my colleagues Andrei Suslin, Vladimir Voevodsky, and Mark Walker for valuable insights into this formalism. Finally, I thank the referee who suggested consideration of Gillet's axioms as well as those of Bloch and Ogus.

1. Spaces of continuous algebraic maps

In this section, we investigate the topological mapping space $\mathcal{M}or(X, \mathcal{C}_r(Y))$ of *continuous algebraic maps* from a variety X to the abelian monoid $\mathcal{C}_r(Y)$ of effective r-cycles on a projective variety Y. We begin by considering cycles on $X \times Y$ equi-dimensional of relative dimension r over a normal variety X as considered in [6]. We recall that such "effective cocycles" on a normal variety X can be reinterpreted as morphisms from X to $\mathcal{C}_r(Y)$. With the aid of the qfh topology, we are led to consider continuous algebraic maps from a general variety X to a projective variety W. This leads to a suitable topological monoid $\mathcal{M}or(X, \mathcal{C}_r(Y))$ for general X.

If Y is a projective variety provided with a given embedding $Y \subset \mathbf{P}^N$ into some projective space, then the Chow variety $C_{r,d}(Y)$ is a projective variety (i.e., a Zariski closed subset of some projective space) whose points naturally correspond to effective r-cycles on Y of degree d. We let $\mathcal{C}_r(Y)$ denote the abelian monoid $\coprod_{d \geqslant 0} C_{r,d}(Y)$. We provide $\mathcal{C}_r(Y)$ with the analytic topology; so defined, the topological abelian monoid $\mathcal{C}_r(Y)$ is independent of the projective embedding of Y. (Indeed, the algebraic structure of $\mathcal{C}_r(Y)$ does not depend upon the projective embedding of Y as shown by D. Barlet [2].) If $V \subset Y$ is a Zariski open subset of a projective variety Y with complement $Y_\infty = Y - V$, then we define $\mathcal{C}_r(V)$ to be the quotient topological monoid $\mathcal{C}_r(Y)/\mathcal{C}_r(Y_\infty)$; so defined $\mathcal{C}_r(V)$ is independent of the projective closure $V \subset Y$ (cf. [7,17]).

We begin with what we consider a good definition of the topological abelian monoid of equi-dimensional cycles over a normal variety X.

DEFINITION ([6, 1.6]). – *Let X be a normal quasi-projective variety of pure dimension m and Y a projective variety. The topological abelian monoid of effective cocycles on X of relative dimension r in Y is the following quotient monoid*

$$\mathcal{C}_r(Y)(X) =: \mathcal{E}_r(Y)(X)/\mathcal{C}_{r+m}(X_\infty \times Y).$$

Here, $X \subset \overline{X}$ is a projective closure, $X_\infty = \overline{X} - X$, and $\mathcal{E}_r(Y)(X) \subset \mathcal{C}_{r+m}(\overline{X} \times Y)$ is the constructible submonoid of effective $r+m$-cycles on $\overline{X} \times Y$ whose restrictions to $X \times Y$ are equi-dimensional over X.

To extend this to more general X, it will be useful to recall the following reinterpretation of $\mathcal{C}_r(Y)(X)$.

PROPOSITION 1.1 ([9, C.3]). – *Let X be a normal, quasi-projective variety and Y projective. The topological abelian monoid $\mathcal{C}_r(Y)(X)$ can be identified with the abelian monoid of morphisms from X to $\mathcal{C}_r(Y)$ provided with the following topology: a sequence $\{f_n : X \to \mathcal{C}_r(Y)\}$ converges to some morphism $f : X \to \mathcal{C}_r(Y)$ provided that it converges with respect to the compact open topology in $\mathrm{Hom}_{\mathrm{cont}}(X, \mathcal{C}_r(Y))$ (where $X, \mathcal{C}_r(Y)$ are provided with the analytic topology) and provided that for some projective closure $X \subset \overline{X}$ the closures of the graphs of f_n in $\overline{X} \times Y$ have bounded degree.*

We denote this topological abelian monoid of morphisms from X to $\mathcal{C}_r(Y)$ by $\mathcal{M}or(X, \mathcal{C}_r(Y))$. As shown in [6, 3.3], the map $\mathcal{M}or(X, \mathcal{C}_r(Y)) \to \mathcal{M}or(X', \mathcal{C}_r(Y))$ induced by a morphism $f : X' \to X$ is continuous (with respect to the topology described above).

We next recall V. Voevodsky's qfh-topology, a Grothendieck topology on the category of varieties. Recall that a continuous map $f : V \to Y$ of topological spaces is said to be a universal topological epimorphism if for all $Y' \to Y$ the pull-back $f' : V \times_Y Y' \to Y'$ satisfies the conditions that f' is surjective and that $U \subset Y'$ is open if and only if $f'^{-1}(U) \subset V \times_Y Y'$ is open.

DEFINITION ([18]). – *A qfh-covering of a variety X is a finite family of quasi-finite morphisms $\{p_i : X_i \to X\}$ such that $\coprod p_i : \coprod X_i \to X$ is a universal topological epimorphism in the Zariski topology.*

A presheaf F on the category of varieties is said to be a qfh-sheaf if it satisfies the sheaf axiom for all qfh-coverings $\{X_i \to X\}$:

$$F(X) = \text{equalizer}\left\{ \prod F(X_i) \rightrightarrows \prod F(X_i \times_X X_j) \right\}.$$

PROPOSITION 1.2. – *The contravariant functor $\mathcal{M}or(-, Y)$ sending a normal variety X to the set of morphisms to a fixed projective variety Y satisfies the qfh-sheaf condition for all qfh-coverings $\{X_i \to X\}$ with the property that X and each X_i is normal.*

Moreover, for any such qfh-covering, the inclusion $\mathcal{M}or(X, Y) \subset \prod \mathcal{M}or(X_i, Y)$ identifies $\mathcal{M}or(X, Y)$ as a subspace of $\prod \mathcal{M}or(X_i, Y)$.

Proof. – By [18, 10.3], it suffices to assume $\{X_i \to X\}$ is either a Zariski open covering or a covering of the form $X' \to X$ where X' is the normalization of X in some finite Galois extension of the field of fractions $k(X)$ of X. (Following [18], we call this second type of covering a pseudo-Galois covering.) The (set theoretic) sheaf condition for $\mathcal{M}or(-, Y)$ is evident for a Zariski open covering.

If $p : X' \to X$ is a pseudo-Galois covering with Galois group G, then

$$\text{equalizer}\{\mathcal{M}or(X', Y) \rightrightarrows \mathcal{M}or(X' \times_X X')\} = \mathcal{M}or(X', Y)^G.$$

To check the sheaf condition for such a pseudo covering, consider a morphism $f' : X' \to Y$ which is G-equivariant. Such a morphism determines a rational map $f : X \to Y$ since $k(X) = k(X')^G$. The graph Γ_f of this rational map (i.e., the closure in $X \times Y$ of the morphism defined on some Zariski open subset of X) has the property that $p^{-1}(\Gamma_f) = \Gamma_{f'}$ and is thus quasi-finite as well as birational over X. Since Y is projective, $\Gamma_f \to X$ is also finite and thus Zariski's Main Theorem implies that $\Gamma_f \to X$ is an isomorphism. This implies that f is a morphism as required to verify the sheaf condition.

To check that the topology on $\mathcal{M}or(X,Y)$ is the subspace topology inherited from $\prod \mathcal{M}or(X_i,Y)$, we must verify that a sequence $\{f_n : X \to Y\}$ converges to $f : X \to Y$ if each of the induced sequences $\{p_i \circ f_n : X_i \to Y\}$ converges to $p_i \circ f$. Once again, it suffices to check this when $\{f_n\}$ is a Zariski covering and when $X' \to X$ is a pseudo-Galois covering with finite Galois group G. In the case of a Zariski covering, this is clear in view of the fact that a Zariski open covering is also a covering by open subsets in the analytic topology (so convergence with respect to the compact-open topology of $\{f_n\}$ to $f : X \to Y$ is assured) and the finiteness of the covering $\{X_i \subset X\}$ (which assures boundedness of degree). In the case of a pseudo-Galois covering $X' \to X$, we observe that X is a quotient space of X' in the analytic topology so that convergence of $\{f_n\}$ in the compact-open topology is clear. Moreover, as argued in [7, 1.6], the boundedness of the degrees of the closures of the graphs of $\{p \circ f_n\}$ is equivalent to that of $\{f_n\}$. $\square$

The following extension technique is a technique of A. Suslin and V. Voevodsky.

PROPOSITION 1.3. – *Let F be a contravariant functor defined on normal varieties which satisfies the qfh-sheaf axiom when restricted to qfh-coverings of normal varieties by normal varieties. Then F admits a unique extension as a qfh-sheaf on all varieties.*

Moreover, if F takes values in topological spaces and satisfies the condition that the inclusion $F(X) \subset \prod F(X_i)$ identifies $F(X)$ as a subspace of $\prod F(X_i)$ for all qfh-coverings of normal varieties by normal varieties, then F becomes a qfh-sheaf defined on all varieties with values in topological spaces.

Proof. – The uniqueness of such an extension is clear, since the normalization $X^{\sim} \to X$ of a variety X is a qfh-covering. To construct the extension, we define $F(X)$ for a general variety X with normalization $X^{\sim} \to X$ as the equalizer

$$F(X) =: \text{equalizer}\{F(X^{\sim}) \rightrightarrows F(X^{\approx})\},$$

where $X^{\approx} \to X^{\sim} \times_X X^{\sim}$ is the normalization of $X^{\sim} \times_X X^{\sim}$. One readily checks that this satisfies the sheaf condition for any qfh-covering of any variety using the observation that an equalizer of a diagram of equalizers is the equalizer of the associated large diagram.

To provide a topology on $F(X)$ for X arbitrary, we give it the subspace topology with respect to the embedding $F(X) \subset F(X^{\sim})$, where $X^{\sim} \to X$ is the normalization of X. With this topology, if $\{X_i \to X\}$ is any qfh-covering, then $F(X)$ so defined has the subspace topology of $\prod F(X_i)$ in view of the following observation: if $T'' \subset T'$ and $T' \subset T$ are subspaces, then $T'' \subset T$ is also a subspace (i.e., has the subspace topology). $\square$

Recall that a *continuous algebraic map* $f : X \to Y$ with Y projective is a Zariski closed subset $\Gamma_f \subset X \times Y$ with the property that Γ_f is the graph of a set-theoretic map from X to Y [5]. So defined, a continuous algebraic map from X to Y is equivalent to a morphism from X^{wn} to Y, where $X^{wn} \to X$ is the weak normalization of X. (The weak normalization $p_w : X^{wn} \to X$ is a map which factors the normalization $p : X^{\sim} \to X$, which is a homeomorphism on underlying analytic topological spaces, and which satisfies the condition that the ring of analytic functions of X^{wn} consists of those continuous functions on X whose composition with p are analytic on $X^{\sim}$.)

The following proposition gives some insight into why continuous algebraic maps occurred in the formulation of morphic cohomology theory (cf. [8]).

PROPOSITION 1.4. – *Let Y be a projective variety and let $\mathcal{M}or(-,Y)$ denote the qfh-sheaf on varieties with values in topological spaces whose value on a normal variety X is the space of morphisms from X to Y as topologized in Proposition* 1.1. *Then for any X, the underlying set of $\mathcal{M}or(X,Y)$ is the set of continuous algebraic maps from X to Y.*

Proof. – It suffices to show for any variety X with normalization $X^{\tilde{}}$ that the set of continuous algebraic maps from $X \to Y$ is naturally identified with the equalizer of $\mathcal{M}or(X^{\tilde{}},Y) \rightrightarrows \mathcal{M}or(X^{\tilde{}\tilde{}},Y)$, where $X^{\tilde{}\tilde{}} \to X^{\tilde{}} \times_X X^{\tilde{}}$ is the normalization of $X^{\tilde{}} \times_X X^{\tilde{}}$. Since a morphism $X^{wn} \to Y$ induces a morphism $X^{\tilde{}} \to Y$, there is a natural inclusion of this set of continuous algebraic maps into the equalizer. On the other hand, given a morphism $f^{\tilde{}} : X^{\tilde{}} \to Y$ which lies in the equalizer, we readily verify that its graph $\Gamma_{f^{\tilde{}}} \subset X^{\tilde{}} \times Y$ projects to a closed subvariety of $X \times Y$ which maps birationally and bijectively to X and therefore represents a continuous algebraic map. □

DEFINITION. – *The topological abelian monoid of effective relative r-cycles on a variety X with values in a projective variety Y is defined to be*

$$\mathcal{M}or\big(X,\mathcal{C}_r(Y)\big) =: \coprod_d \mathcal{M}or\big(X,C_{r,d}(Y)\big).$$

Each $f : X \to C_{r,d}(Y)$ has an associated graph $\Gamma_f \subset X \times Y$ which is equi-dimensional of relative dimension r over X. Sending f to its graph Γ_f is clearly 1–1. On the other hand, if X^{wn} is not normal, then not every effective algebraic cycle on $X \times Y$ equi-dimensional of relative dimension r over X arises as the graph of such a continuous algebraic map. Thus, our qfh-sheafification of relative cycles does not give the group of equidimensional cycles over X on an arbitrary variety X.

2. Topological cycle cohomology and homology

In the preceding section, we constructed the qfh-sheaf $\mathcal{M}or(-,\mathcal{C}_r(Y))$ which takes values in topological abelian monoids. The purpose of this section is to convert $\mathcal{M}or(X,\mathcal{C}_r(Y))$ to a bivariant functor which will lead us to definitions of bivariant topological cycle cohomology, topological cycle cohomology (with supports), and topological cycle homology. As the reader will quickly see, this section consists primarily of formal definitions. Our somewhat sophisticated formulation of this theory is justified by the properties we prove in the next two sections.

Notation. – Let *Zar* denote the big Zariski site consisting of quasi-projective complex algebraic varieties (whose coverings are Zariski open coverings); for a given variety X, let Zar_X denote the small Zariski site consisting of Zariski open subsets of X. We denote by $\mathcal{P}$ (respectively, $\mathcal{P}_X$) the category of chain complexes bounded below of abelian presheaves on *Zar* (respectively, Zar_X); we denote by $\mathcal{S}$ (respectively, $\mathcal{S}_X$) the category of chain complexes bounded below of abelian sheaves on *Zar* (respectively, Zar_X).

If A is a topological abelian monoid, we denote by $A^{\tilde{}}$ the normalized chain complex associated to the group completion of the simplicial abelian monoid $Sing.A$ obtained by level-wise group completing the abelian monoids $Sing_k A$. Here, $Sing.T$ denotes the singular complex of a topological space T, a simplicial set which inherits the structure of a simplicial abelian monoid provided that T itself has the structure of a topological abelian monoid.

If C_* is a chain complex and n an integer, we denote by $C_*[n]$ the chain complex "shifted to the left by n"; in other words, $(C_*[n])_i = C_{i-n}$ for each $i \in \mathbf{Z}$. In other words, if C_* is viewed as a (co-)chain complex with differential of degree $+1$ and denoted C^* (with $C_i \equiv C^{-i}$), then $(C^*[n])^j = C^{n+j}$.

Example. – For any simplicial variety $\mathcal{U}$, let $\mathbf{Z}_{\mathcal{U}.}$ denote the presheaf defined to send V to the normalized chain complex of the simplicial abelian group whose value on $\underline{n} \in \Delta$ is the free abelian group on the set $\mathrm{Hom}(V, \mathcal{U}_n)$.

We now introduce the presheaves of chain complexes bounded below which we shall use to define topological cycle cohomology and homology.

DEFINITION. – *For a given non-negative integer s, we define the complex of abelian presheaves on Zar*

$$\mathcal{M}(-,s) =: U \mapsto \mathrm{cone}\{\mathcal{M}or(U, \mathcal{C}_0(\mathbf{P}^{s-1}))^{\tilde{}} \to \mathcal{M}or(U, \mathcal{C}_0(\mathbf{P}^s))^{\tilde{}}\}[-2s] \in \mathcal{P}.$$

Consider a variety X, closed subvariety $Y \subset X$, and a non-negative integer s. Let $\mathcal{M}(-,s)_{|X}$ denote the restriction of $\mathcal{M}(-,s)$ to Zar_X. We define the following complexes of abelian presheaves on Zar_X (i.e., objects of $\mathcal{P}_X$):

$$\mathcal{M}_Y(-,s) =: U \mapsto \mathrm{cone}\{\mathcal{M}(U,s)_{|X} \to \mathcal{M}(U \cap X - Y, s)_{|X}\}[-1],$$
$$\mathcal{L}(-,r) =: U \mapsto \mathrm{cone}\{\mathcal{C}_r(U_\infty)^{\tilde{}} \to \mathcal{C}_r(\overline{X})^{\tilde{}}\}[2r],$$

where $X \subset \overline{X}$ is a projective closure and $U_\infty \subset \overline{X}$ is the Zariski closed complement of $U \subset X \subset \overline{X}$.

Remark 2.1. – The presheaf $\mathcal{L}(-,r) \in \mathcal{P}_X$ applied to any $U \in Zar_X$ is a chain complex of torsion free abelian groups. This follows from the observation that $A^{\tilde{}}$ is a chain complex of torsion free abelian groups whenever the topological abelian monoid A is torsion free.

These presheaves are presheaves of morphic cohomology and Lawson homology groups as we observe in the following proposition.

PROPOSITION 2.2. – *For a normal variety X and any $n \in \mathbf{Z}$,*

$$\mathrm{H}_{-n}(\mathcal{M}(X,s)) = \mathrm{H}_{-n}(\mathcal{M}(X,s)_{|X}) = \mathrm{L}^s\mathrm{H}^n(X),$$

where $\mathrm{L}^s\mathrm{H}^n(X)$ denotes the morphic cohomology of X as introduced in [8] *and formulated in* [6].

For any variety X,

$$\mathrm{H}_m(\mathcal{L}(X,r)) = \mathrm{L}_r\mathrm{H}_m(X),$$

where $\mathrm{L}_r\mathrm{H}_m(X)$ is the Lawson homology of X as in [7].

Proof. – The formulation of morphic cohomology in [6, 2.4],

$$\mathrm{L}^s\mathrm{H}^{2s-j}(X,s) \equiv \pi_j\big((\mathcal{C}_0(\mathbf{P}^s)(X)/\mathcal{C}_0(\mathbf{P}^{s-1})(X))^+\big),$$

is in terms of the homotopy groups of the "naïve topological group completion" of the topological quotient monoid $\mathcal{C}_0(\mathbf{P}^s)(X)/\mathcal{C}_0(\mathbf{P}^{s-1})(X)$. On the other hand, if A is a topological abelian monoid, then $Sing.A \to (Sing.A)^+$ is a homotopy theoretic group completion [12, AppQ] and

$$\pi_i\big((Sing.A)^+\big) = \mathrm{H}_i(A^{\tilde{}}), \quad i \geqslant 0.$$

Proposition 1.2 provides a natural isomorphism of the topological abelian monoids $\mathcal{C}_0(\mathbf{P}^s)(X)$ and $\mathcal{M}or(X, \mathcal{C}_0(\mathbf{P}^s))$. Consequently, [6, 1.9, 2.2] establishes that the original formulation of morphic cohomology agrees with $\mathrm{H}_{-n}(\mathcal{M}(X,s))$.

Similarly, the formulation of Lawson homology in [7, 1.5],

$$\mathrm{L}_r\mathrm{H}_{2r+i}(X) = \pi_i\big(Z_r(\overline{X})/Z_r(\overline{X}-X)\big),$$

is in terms of the homotopy groups of the quotient topological group $Z_r(\overline{X})/Z_r(\overline{X}-X)$, where $X \subset \overline{X}$ is a projective closure and $Z_r(\overline{X})$ is the naïve topological group completion of the Chow monoid $\mathcal{C}_r(\overline{X})$. As argued above, the fact that $\mathcal{C}_r(\overline{X}) \to Z_r(\overline{X})$ is a homotopy theoretic group completion and that $Z_r(\overline{X}-X) \to Z_r(\overline{X}) \to Z_r(X)$ yields a distinguished triangle of chain complexes (cf. [7]) implies that

$$\pi_i\big(Z_r(X)\big) = \mathrm{H}_{i+2r}\big(\mathcal{L}(X,r)\big). \qquad \square$$

Our definition of cohomology and homology will be in terms of maps in the derived category $\mathcal{D}_X$ associated to $\mathcal{S}_X$ introduced in the following definition.

DEFINITION. – *Let $\mathcal{HP}$ denote the homotopy category of $\mathcal{P}$ obtained by identifying maps of chain complexes related by a chain homotopy. Then $\mathcal{HP}$ has the natural the structure of a triangulated category whose distinguished triangles are triples in $\mathcal{P}$, $P \to Q \to R \to P[1]$, which are isomorphic to a triple arising from a short exact sequence of chain complexes $0 \to P \to Q \to R \to 0$. We similarly define the homotopy categories $\mathcal{HP}_X, \mathcal{HS}, \mathcal{HS}_X$.*

Let $\mathcal{D}$ denote the localization of $\mathcal{P}$ with respect to the thick subcategory of those $P \in \mathcal{P}$ with the property that every stalk of P is acyclic. So defined, $\mathcal{D}$ is isomorphic to the derived category of $\mathcal{S}$. We similarly define $\mathcal{D}_X$, a localization of both $\mathcal{P}_X$ and $\mathcal{S}_X$.

We say that $f : P \to Q$ in either $\mathcal{P}, \mathcal{P}_X, \mathcal{S}$, or $\mathcal{S}_X$ is a *quasi-isomorphism* if each fibre of both the kernel and cokernel of f is acyclic.

DEFINITION. – *We define the bivariant topological cycle cohomology of X with values in Y and weight r to be*

$$\mathcal{H}^j(X;Y,r) =: \mathrm{Hom}_{\mathcal{D}_X}\big(\mathbf{Z}_X, \mathcal{M}(-;Y,r)[j]\big)$$

where $\mathcal{M}(-;Y,r)(U) = \mathcal{M}or(U,\mathcal{C}_r(Y))^{\tilde{}}[2r]$.

We define the topological cycle cohomology of X with supports in $Y \subset X$ closed to be

$$\mathcal{H}^j_Y(X,s) \equiv \mathrm{Hom}_{\mathcal{D}_X}\big(\mathbf{Z}_X, \mathcal{M}_Y(-,s)[j]\big).$$

In particular,

$$\mathcal{H}^j(X,s) \equiv \mathrm{Hom}_{\mathcal{D}_X}\big(\mathbf{Z}_X, \mathcal{M}(-,s)[j]\big).$$

We define the topological cycle homology of X in degree i to be

$$\mathcal{H}_i(X,r) \equiv \mathrm{Hom}_{\mathcal{D}_X}\big(\mathbf{Z}_X[i], \mathcal{L}(-,r)\big).$$

3. Topological cycle homology theory

We begin this section by verifying that topological cycle homology equals Lawson homology. This enables us to prove the Bloch–Ogus properties (including functoriality) of a "twisted homology theory" for topological cycle homology theory. On the other hand, our definition in terms of the derived category is so formulated to fit with topological cycle cohomology theory as we shall see in later sections.

DEFINITION. – *$P \in \mathcal{P}_X$ is said to be pseudo-flasque provided that*

$$P(U) \to P(U_1) \oplus P(U_2) \to P(U_1 \cap U_2)$$

is quasi-isomorphic to a distinguished triangle of chain complexes whenever U_1, U_2 are Zariski opens of $U \in Zar_X$ with $U = U_1 \cup U_2$.

The following theorem is the derived category analogue of a theorem of Brown–Gersten concerning simplicial presheaves.

THEOREM 3.1. – *Let $\mathcal{HP}_X$ denote the homotopy category of the category $\mathcal{P}_X$ of chain complexes (of presheaves of abelian groups on Zar_X). Then for any $P \in \mathcal{P}_X$, any $i \in \mathbf{Z}$*

$$\mathrm{H}_i\big(P(X)\big) = \mathrm{Hom}_{\mathcal{HP}_X}\big(\mathbf{Z}_X[i], P\big).$$

If $P \in \mathcal{P}_X$ is pseudo-flasque, then for any $i \in \mathbf{Z}$ the natural map

$$\mathrm{Hom}_{\mathcal{HP}_X}\big(\mathbf{Z}_X[i], P\big) \to \mathrm{Hom}_{\mathcal{D}_X}\big(\mathbf{Z}_X[i], P\big)$$

is an isomorphism.

Proof. – The equality $\mathrm{H}_i(P(X)) = \mathrm{Hom}_{\mathcal{HP}_X}(\mathbf{Z}_X[i], P)$ is essentially immediate.

Let $P \to I$ in $\mathcal{P}_X$ be a quasi-isomorphism with I injective (i.e., I_i is an injective presheaf on Zar_X for each i) and let $T = \mathrm{cone}\{P \to I\}$. Since an injective presheaf is flasque, the octahedral axiom implies that T is pseudo-flasque (assuming P is pseudo-flasque) as well as acyclic. Hence, the proof of [4, Theorem1′] implies that $T(U)$ is acyclic for all $U \in Zar_X$. In particular, $P(X) \to I(X)$ is a quasi-isomorphism of chain complexes, so that

$$\mathrm{Hom}_{\mathcal{HP}_X}\big(\mathbf{Z}_X[i], P\big) = \mathrm{Hom}_{\mathcal{HP}_X}\big(\mathbf{Z}_X[i], I\big) = \mathrm{Hom}_{\mathcal{D}_X}\big(\mathbf{Z}_X[i], I\big) = \mathrm{Hom}_{\mathcal{D}_X}\big(\mathbf{Z}_X[i], P\big).$$

□

Theorem 3.1 and the localization theorem in Lawson homology easily imply the following theorem.

THEOREM 3.2. – *Whenever $Y \subset X$ is a closed subvariety, there is a "localization" distinguished triangle in $\mathcal{HP}_X$*

$$\mathcal{L}(- \cap Y, r) \to \mathcal{L}(-, r) \to \mathcal{L}\big(- \cap (X - Y), r\big) \to \mathcal{L}(- \cap Y, r)[1] \tag{3.1}$$

for each $r \geqslant 0$. Consequently, the presheaf $\mathcal{L}(-, r)$ in $\mathcal{HP}_X$ is pseudo-flasque, so that

$$\mathcal{H}_i(X, r) = \mathrm{H}_i\big(\mathcal{L}(X, r)\big) = \mathrm{L}_r\mathrm{H}_i(X),$$

where $\mathrm{L}_r\mathrm{H}_i(X)$ is the Lawson homology of X as defined in earlier papers (e.g., [7]).

In particular, $\mathcal{H}_i(X, r) = 0$ for $i < 0$.

Proof. – For any $U \in Zar_X$, the localization theorem for Lawson homology [7, 1.6], [17] asserts that

$$\mathcal{L}(U \cap Y, r) \to \mathcal{L}(U, r) \to \mathcal{L}\big(U \cap (X - Y), r\big) \to \mathcal{L}(U \cap Y, r)[1] \tag{3.2}$$

is a distinguished triangle of chain complexes. (We implicitly appeal to Proposition 2.2.) This immediately implies that (3.1) is a distinguished triangle in $\mathcal{P}_X$.

To verify that $\mathcal{L}(-,r)$ is pseudo-flasque, it suffices to show for any $U_1 \subset U$, $U_2 \subset U$ in Zar_X with $U_1 \cup U_2 = U$ that

$$\begin{tikzcd} \mathcal{L}(U,r) \arrow[r] \arrow[d] & \mathcal{L}(U_1,r) \arrow[d] \\ \mathcal{L}(U_2,r) \arrow[r] & \mathcal{L}(U_1 \cap U_2, r) \end{tikzcd}$$

is homotopy Cartesian. This follows immediately from the localization sequence (3.2), since $U - U_2 = U_1 - U_1 - (U_1 \cap U_2)$. $\square$

As observed in [7], Lawson homology is contravariantly functorial for flat maps and covariantly functorial for proper maps, thereby giving us the Bloch–Ogus properties (1.2) and (1.2.1) for topological cycle homology.

The following compatibility of this functorial behaviour is (a generalization of) the Bloch–Ogus property (1.2.2).

PROPOSITION 3.3. – *Consider the following Cartesian square of varieties*

$$\begin{tikzcd} X' \arrow[r, "h"] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

with flat horizontal maps of relative dimension e *and proper vertical maps. Then for any* $i \in \mathbf{Z}$ *and* $r \geqslant 0$, *the following square commutes*

$$\begin{tikzcd} \mathcal{H}_i(X', r+e) \arrow[d, "p'_*"'] & \mathcal{H}_i(X,r) \arrow[l, "h^*"'] \arrow[d, "p_*"] \\ \mathcal{H}_i(Y', r+e) & \mathcal{H}_i(Y,r) \arrow[l, "g^*"'] \end{tikzcd}$$

Proof. – By Theorem 3.2, it suffices to prove the commutativity of the following square

$$\begin{tikzcd} \mathcal{C}_{r+e}(\overline{X}')/\mathcal{C}_{r+e}(\overline{X}' - X') \arrow[d, "p'_*"'] & \mathcal{C}_r(\overline{X})/\mathcal{C}_r(\overline{X} - X) \arrow[l, "h^*"'] \arrow[d, "p_*"] \\ \mathcal{C}_{r+e}(\overline{Y}')/\mathcal{C}_{r+e}(\overline{Y}' - Y') & \mathcal{C}_r(\overline{Y})/\mathcal{C}_r(\overline{Y} - Y) \arrow[l, "g^*"'] \end{tikzcd}$$

where $Y \subset \overline{Y}$ is a projective closure, $\overline{X}$ is the closure of X in a projective closure of $X \times_Y \overline{Y}$, $\overline{Y}'$ is the closure of Y' in a projective closure of $Y' \times_Y \overline{Y}$, and $\overline{X}'$ is the closure of X' in a projective closure of $X' \times_{Y' \times_Y X} \overline{Y}' \times_{\overline{Y}} \overline{X}$. This follows directly from the definitions of flat pull-back and proper push-forward of cycles. $\square$

Bloch–Ogus require of the localization sequence the following naturality, their property (1.2.4).

PROPOSITION 3.4. – *Consider the following commutative square*

$$\begin{tikzcd} Y' \arrow[r] \arrow[d, "f_Y"'] & X' \arrow[d, "f"] \\ Y \arrow[r] & X \end{tikzcd}$$

whose horizontal maps are closed immersions, whose vertical maps are proper, and satisfying the condition that $f_Y(Y') = Y$. Then the following is a commutative ladder of localization exact sequences

$$\begin{tikzcd} \cdots \arrow[r] & \mathcal{H}_i(Y',r) \arrow[r] \arrow[d, "f_{Y*}"'] & \mathcal{H}_i(X',r) \arrow[r] \arrow[d, "f_*"'] & \mathcal{H}_i(X'-Y',r) \arrow[r] \arrow[d, "f_* \circ g^*"] & \cdots \\ \cdots \arrow[r] & \mathcal{H}_i(Y,r) \arrow[r] & \mathcal{H}_i(X,r) \arrow[r] & \mathcal{H}_i(X-Y,r) \arrow[r] & \cdots \end{tikzcd}$$

where $g : X' - f^{-1}(Y) \subset X' - Y'$.

Proof. – Consider the following commutative diagram whose rows are distinguished triangles of chain complexes

$$\begin{tikzcd} \mathcal{L}(Y',r) \arrow[r] \arrow[d] & \mathcal{L}(X',r) \arrow[r] \arrow[d] & \mathcal{L}(X'-Y',r) \arrow[r] & \mathcal{L}(Y',r)[1] \\ \mathcal{L}(Y,r) \arrow[r] & \mathcal{L}(X,r) \arrow[r] & \mathcal{L}(X-Y,r) \arrow[r] & \mathcal{L}(X-Y,r)[1]. \end{tikzcd}$$

It suffices to show that the composition $\mathcal{L}(X',r) \to \mathcal{L}(X,r) \to \mathcal{L}(X-Y,r)$ factors through $\mathcal{L}(X',r) \to \mathcal{L}(X'-Y',r)$. This in turn follows from the observation that

$$\mathcal{C}_r(\overline{X}') \to \mathcal{C}_r(\overline{X}) \to \mathcal{C}_r(\overline{X})/\mathcal{C}_r\big(\overline{X} - (X-Y)\big)$$

factors through $\mathcal{C}_r(\overline{X}') \to \mathcal{C}_r(\overline{X}')/\mathcal{C}_r(\overline{X}' - (X'-Y'))$, where $X \subset \overline{X}$ is a projective closure and $\overline{X}'$ is the closure of X' in a projective closure of $X' \times_X \overline{X}$. $\square$

4. Topological cycle cohomology theory

We now proceed to consider topological cycle cohomology theory. The excision property (Theorem 4.3 below) justifies our formulation of this theory in terms of the derived category $\mathcal{D}_X$.

We begin with functoriality of topological cycle cohomology with supports.

PROPOSITION 4.1. – *A Cartesian square*

$$\begin{tikzcd} Y' \arrow[r] \arrow[d, "f_Y"'] & X' \arrow[d, "f_X"] \\ Y \arrow[r] & X \end{tikzcd}$$

of varieties whose horizontal arrows are closed immersions naturally induces a map of graded groups for all $j \in \mathbf{Z}$, $s \geqslant 0$:

$$(f_X, f_Y)^* : \bigoplus_j \mathcal{H}_Y^j(X, s) \to \bigoplus_j \mathcal{H}_{Y'}^j(X', s).$$

Frequently, f_Y will remain implicit, f_X will be simply denoted f, and $(f_X, f_Y)^*$ by f^*.

Proof. – Since $f^* : \mathcal{P}_X \to \mathcal{P}_{X'}$ is exact and preserves quasi-isomorphisms and since $\mathbf{Z}_{X'} = f^*\mathbf{Z}_X$, we obtain a natural map

$$\mathrm{Hom}_{\mathcal{D}_X}\big(\mathbf{Z}_X, \mathcal{M}_Y(-, s)[j]\big) \to \mathrm{Hom}_{\mathcal{D}_{X'}}\big(\mathbf{Z}_{X'}, f^*\mathcal{M}_Y(-, s)[j]\big).$$

The natural map

$$\begin{aligned} f^*\mathcal{M}\big(-, \mathcal{C}_0(\mathbf{P}^s)\big)(U') \equiv \operatorname*{colim}_{U' \subset f^{-1}(U)} \mathcal{M}\big(f^{-1}(U), \mathcal{C}_0(\mathbf{P}^s)\big) \\ \to \mathcal{M}\big(-, \mathcal{C}_0(\mathbf{P}^s)\big)(U') \end{aligned}$$

induces a map in $\mathcal{P}_{X'}$ of presheaves of chain complexes on $Zar_{X'}$, $f^*\mathcal{M}_Y(-, s) \to \mathcal{M}_{Y'}(-, s)$. Functoriality thus follows. $\square$

The next proposition is the assertion of the validity of the first two Bloch–Ogus properties for the cohomology theory [3, 1.1.1, 1.1.2].

PROPOSITION 4.2. – *Let $Z \to Y \to X$ be closed immersions. Then for any $s \geqslant 0$, there is a long exact sequence*

$$\cdots \mathcal{H}_Z^j(X, s) \to \mathcal{H}_Y^j(X, s) \to \mathcal{H}_{Y-Z}^j(X - Z, s) \to \mathcal{H}_Z^{j+1}(X, s) \cdots$$

Moreover, if $Z' \to Y' \to X'$ is another sequence of closed immersions and if

$$\begin{tikzcd} Z' \arrow[r] \arrow[d, "f_Z"'] & Y' \arrow[r] \arrow[d, "f_Y"'] & X' \arrow[d, "f_X"] \\ Z \arrow[r] & Y \arrow[r] & X \end{tikzcd} \tag{4.1}$$

is a commutative diagram with Cartesian squares, then $(f_X, f_Z)^*$, $(f_X, f_Y)^*$, and $(f_{X|}, f_{Y|})^*$ fit together to form a commutative ladder of long exact sequences, where

$$f_{X|}, f_{Y|} : (X' - Z', Y' - Z') \to (X - Z, Y - Z)$$

is the restriction of f_X, f_Y.

Proof. – We consider the following commutative diagram in $\mathcal{P}_X$ whose rows are distinguished triangles

$$\begin{tikzcd}
\mathcal{M}(-,s) \arrow[r] \arrow[d, "="'] & \mathcal{M}(-\cap(X-Z),s) \arrow[r] \arrow[d] & \mathcal{M}_Z(-,s)[-1] \arrow[d] \\
\mathcal{M}(-,s) \arrow[r] \arrow[d] & \mathcal{M}(-\cap(X-Y),s) \arrow[r] \arrow[d] & \mathcal{M}_Y(-,s)[-1] \arrow[d] \\
0 \arrow[r] & \mathcal{M}_{X-Y}(-\cap(X-Z),s)[-1] \arrow[r, "="] & \mathcal{M}_{X-Y}(-\cap(X-Z),s)[-1]
\end{tikzcd}$$

The octahedral axiom implies that the right vertical column is also a distinguished triangle, thereby implying the asserted long exact sequence. The naturality of this construction implies that the commutative diagram (4.1) determines a map of distinguished triangles and thus a commutative ladder of exact sequences. □

The following excision theorem is the Bloch–Ogus property (1.1.3). This is the property which requires our somewhat sophisticated definition of topological cycle cohomology in terms of maps in the derived category $\mathcal{D}_X$ of bounded complexes of presheaves.

THEOREM 4.3. – *Let $Y \subset X$ be a closed subvariety and $i: U \subset X$ be a Zariski open subset containing Y. Then the natural map*

$$\mathcal{H}^j_Z(X,s) \to \mathcal{H}^j_Z(U,s)$$

is an isomorphism for all $j \in \mathbf{Z}, s \geqslant 0$.

Proof. – Consider the following commutative square in $\mathcal{P}_X$:

$$\begin{tikzcd}
\mathcal{M}(-,s) \arrow[r] \arrow[d, "p"'] & \mathcal{M}(-\cap U,s) \arrow[d, "p_U"'] \\
\mathcal{M}(-\cap(X-Y),s) \arrow[r] & \mathcal{M}(-\cap(U-Y),s).
\end{tikzcd}$$

If $V \in Zar_X$ is sufficiently fine that V is either contained in U or $X-Y$, then the evaluation of this square on V leads either to a square whose horizontal maps are equalities of chain complexes or to a square whose vertical maps are equalities. This immediately implies that the map

$$\mathcal{M}_Z(-,s) = \mathrm{cone}(p)[1] \to \mathrm{cone}(p_U)[1] = \mathcal{M}_Z(-,s)$$

is a quasi-isomorphism and hence the statement of the theorem. □

5. Cap product and duality

In this section, we relate our cohomology and homology theories via a cap product and verify the final property of Bloch–Ogus: Poincaré duality for a smooth variety relating topological cycle cohomology with supports to topological cycle homology. Continuity of the cap product for possibly singular varieties is somewhat subtle; the necessary arguments can be found in [14]. On the other hand, the duality theorem (for smooth varieties) is a direct consequence of duality proved in [6] following [9]. As a consequence, we show that topological cycle cohomology of a smooth variety equals morphic cohomology.

We begin by sketching the construction of cap product, based on techniques developped in [14].

PROPOSITION 5.1 (cf. [14, 2.6]). – *There is a natural pairing of presheaves in $\mathcal{P}_X$*

$$\mathcal{M}(-,s)[2s]\otimes\mathcal{L}(-,r)\to\mathcal{L}(-\times\mathbf{A}^s,r), \tag{5.1}$$

where $\mathcal{L}(-\times\mathbf{A}^s,r)$ sends $U\in Zar_X$ to

$$\operatorname{cone}\{\mathcal{C}_r(U_\infty\times\mathbf{P}^s\cup\overline{X}\times\mathbf{P}^{s-1})^{\tilde{}}\to\mathcal{C}_r(\overline{X}\times\mathbf{P}^s)^{\tilde{}}\}[2r].$$

(Here, $X\subset\overline{X}$ is a projective closure and $U_\infty=\overline{X}-U$.)

Proof. – For $U\in Zar_X$, we consider the pairing

$$\Gamma_{()|}:\mathcal{M}or(U,\mathcal{C}_0(\mathbf{P}^s))\times\mathcal{C}_r(U)\to\mathcal{C}(U\times\mathbf{P}^s) \tag{5.2}$$

which sends the pair $(f:U\to\mathcal{C}_0(\mathbf{P}^s),i_W:W\subset U)$ to the graph $\Gamma_{f|W}\subset U\times\mathbf{P}^s$ of $i_W\circ f$ for any irreducible r-cycle W on U. The continuity of this pairing is established (in much greater generality) in [14, 2.6].

The naturality of (5.2) with respect to U and $\mathbf{P}^{s-1}\subset\mathbf{P}^s$ implies that this pairing induces a pairing of the form (5.1). □

Using the localization property of Lawson homology, we obtain the following cap product pairing. We implicitly use the *homotopy invariance* property of Lawson homology [7] which asserts that flat pull-back

$$\mathcal{L}(-,r-s)[2s]\to\mathcal{L}(-\times\mathbf{A}^s,r) \tag{5.3}$$

is a quasi-isomorphism.

PROPOSITION 5.2. – *If $Y\subset X$ is closed, then the pairing* (5.1) *induces a natural pairing*

$$\mathcal{M}_Y(-,s)\otimes\mathcal{L}(-,r)\to\mathcal{L}(-\cap Y,r-s)$$

in $\mathcal{D}_X$ which yields the following cap product pairing on cohomology/homology

$$\cap:\mathcal{H}^j_Y(X,s)\otimes\mathcal{H}_n(X,r)\to\mathcal{H}_{n-j}(Y,r-s)$$

whenever $r\geqslant s\geqslant 0$, $n\geqslant j$.

Proof. – The naturality of (5.1) enables us to obtain the following commutative diagram in $\mathcal{P}_X$ whose columns are distinguished triangles:

$$\begin{tikzcd}
\mathcal{M}_Y(-,s)\otimes\mathcal{L}(-,r) \arrow[r] \arrow[d] & \mathcal{L}(-\cap Y\times\mathbf{A}^s,r)[-2s] \arrow[d] \\
\mathcal{M}(-,s)\otimes\mathcal{L}(-,r) \arrow[r] \arrow[d] & \mathcal{L}(-\times\mathbf{A}^s,r)[-2s] \arrow[d] \\
\mathcal{M}(-\cap(X-Y),s)\otimes\mathcal{L}(-,r) \arrow[r] & \mathcal{L}(-\cap(X-Y)\times\mathbf{A}^s,r)[-2s]
\end{tikzcd} \tag{5.4}$$

The asserted pairing on cohomology/homology is obtained from the top row by applying the following sublemma in conjunction with Remark 2.1. □

SUBLEMMA. – *A pairing $P \otimes Q \to R$ of presheaves in $\mathcal{P}_X$ induces a pairing*

$$\mathrm{Hom}_{\mathcal{D}_X}\big(\mathbf{Z}_X[i], P\big) \otimes \mathrm{Hom}_{\mathcal{D}_X}\big(\mathbf{Z}_X[j], Q\big) \to \mathrm{Hom}_{\mathcal{D}_X}\big(\mathbf{Z}_X[i+j], R\big),$$

provided that $Q(U)$ is a complex of abelian groups without torsion for every $U \in Zar_X$.

Proof. – If $\mathcal{P} \to P$, $\mathcal{Q} \to Q$ are quasi-isomorphisms in $\mathcal{P}_X$ with $\mathcal{P}, \mathcal{Q}$ projective, then $\mathcal{P} \otimes \mathcal{Q} \to P \otimes Q$ is also a quasi-isomorphism since each $Q(U)$ is torsion free. Thus, the asserted pairing is given by the natural pairing

$$\mathrm{Hom}_{\mathcal{HP}_X}\big(\mathbf{Z}_X[i], \mathcal{P}\big) \otimes \mathrm{Hom}_{\mathcal{HP}_X}\big(\mathbf{Z}_X[j], \mathcal{Q}\big) \to \mathrm{Hom}_{\mathcal{HP}_X}\big(\mathbf{Z}_X[i+j], \mathcal{P} \otimes \mathcal{Q}\big).$$

□

Observe that if X is an irreducible variety of dimension d, then X itself viewed as an effective d-cycle determines a point of $\mathcal{C}_d(X)$ and thus a *fundamental class*

$$\eta_X \in \big(\mathcal{L}(X, d)\big)_{2d}. \tag{5.5}$$

(By an abuse of notation, we shall also use η_X to denote the associated homology class in $\mathrm{H}_{2d}(\mathcal{L}(X,d))$.) Moreover, as required by Bloch–Ogus (1.3.4), $\alpha^*(\eta_X) = \eta_{X'}$ whenever $\alpha : X' \to X$ is an etale morphism because the flat pull-back of the d-cycle X on X is the d-cycle X' on X'.

Observe that if X is an irreducible variety of pure dimension d, then the pairing of (5.2) restricted to $\eta_X \in (\mathcal{L}(X,d))_{2d}$,

$$- \cap \eta_X : \mathcal{M}(-, s) \to \mathcal{L}(-, d-s)[-2d],$$

is induced by sending a continuous algebraic map $f : U \to \mathcal{C}_0(\mathbf{P}^s)$ to its graph $\Gamma_f \in \mathcal{C}_d(U \times \mathbf{P}^s)$ for any $U \in Zar_X$. Hence, whenever X satisfies the condition that topological cycle cohomology equals morphic cohomology (e.g., X smooth by Corollary 5.6 below), the induced map is the duality map of [6,9]

$$\mathcal{D} : \mathcal{H}^j(X, s) \to \mathcal{H}_{2d-j}(X, d-s). \tag{5.6}$$

PROPOSITION 5.3. – *Consider the following Cartesian square of varieties*

$$\begin{tikzcd} Y' \arrow[r] \arrow[d, "\beta"'] & X' \arrow[d, "\alpha"] \\ Y \arrow[r] & X \end{tikzcd}$$

whose horizontal arrows are closed immersions and whose vertical arrows are etale. Then the following square commutes

$$\begin{tikzcd} \mathcal{H}^j_Y(X, s) \otimes \mathcal{H}_n(X, r) \arrow[r] \arrow[d, "\alpha^* \otimes \alpha^*"'] & \mathcal{H}_{n-j}(Y, r-s) \arrow[d, "\beta^*"] \\ \mathcal{H}^j_{Y'}(X', s) \otimes \mathcal{H}_n(X', r) \arrow[r] & \mathcal{H}_{n-j}(Y', r-s) \end{tikzcd}$$

for all $j, n \in \mathbf{Z}$, $r \geqslant s \geqslant 0$.

Proof. – Because α^* is exact, preserves quasi-isomorphisms, and sends $\mathbf{Z}_X \in \mathcal{S}_X$ to $\mathbf{Z}_{X'} \in \mathcal{S}'_X$, there is a natural map $\mathrm{Hom}_{\mathcal{D}_X}(\mathbf{Z}_X[i], F) \to \mathrm{Hom}_{\mathcal{D}_{X'}}(\mathbf{Z}_{X'}[i], \alpha^* F)$ for any $F \in \mathcal{P}_X$. Thus, it suffices to prove the commutativity of the following square in $\mathcal{D}_{X'}$:

$$\begin{tikzcd} \alpha^*\mathcal{M}_Y(-,s)\otimes\alpha^*\mathcal{L}(-,r) \arrow[r, "\cap"] \arrow[d, "\alpha^*\otimes\alpha^*"'] & \alpha^*\mathcal{L}(-\cap Y, r) \arrow[d, "\beta^*"'] \\ \mathcal{M}_{Y'}(-s)\otimes\mathcal{L}(-,r) \arrow[r, "\cap"] & \mathcal{L}(-\cap Y', r) \end{tikzcd} \tag{5.7}$$

Using diagram (5.4), we readily see that it suffices to take $Y = X$, $Y' = X'$ and to prove the commutativity in $\mathcal{P}_{X'}$ of this special case of (5.7). To prove this, it suffices to prove the commutativity of the following square of topological abelian monoids whose horizontal maps are given by (5.2)

$$\begin{tikzcd} \mathcal{M}or(U, \mathcal{C}_0(\mathbf{P}^s)) \times \mathcal{C}_r(U) \arrow[r, "\Gamma_{()|-}"] \arrow[d, "\circ\alpha\circ j,\ \alpha^*\circ j^*"'] & \mathcal{C}_r(U \times \mathbf{P}^s) \arrow[d, "\alpha^*\circ j^*"] \\ \mathcal{M}or(U', \mathcal{C}_0(\mathbf{P}^s)) \times \mathcal{C}_r(U') \arrow[r, "\Gamma_{()|-}"] & \mathcal{C}_r(U' \times \mathbf{P}^s) \end{tikzcd} \tag{5.8}$$

where $j : U' \subset \alpha^{-1}(U)$. Finally, the commutativity of (5.8) is verified by observing that the graph of $f \circ \alpha \circ j : U' \to U \to \mathcal{C}_0(\mathbf{P}^s)$ restricted to $Z' \equiv U' \times_X Z$, $\Gamma_{f\circ\alpha\circ j|Z'} \in \mathcal{C}_r(U' \times \mathbf{P}^s)$, equals the pull-back via $(\alpha \circ j \times 1) : U' \times \mathbf{P}^s \to U \times \mathbf{P}^s$ of $\Gamma_{f|Z} \in \mathcal{C}_r(U \times \mathbf{P}^s)$. $\square$

The following "projection formula" is the Bloch–Ogus property (1.3.3).

PROPOSITION 5.4. – *Consider the Cartesian square of varieties*

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "g"'] & W \arrow[d, "f"] \\ Y \arrow[r] & X \end{tikzcd}$$

whose horizontal maps are closed immersions and whose vertical maps are proper. Then for any $a \in \mathcal{H}^j_Y(X, s)$, $z \in \mathcal{H}_n(W, r)$,

$$a \cap f_*(z) = g_*\big(f^*(a) \cap z\big).$$

Proof. – Observe that Theorem 3.1 implies that

$$\mathrm{Hom}_{\mathcal{D}_W}\big(\mathbf{Z}_W[i], f^*\mathcal{L}(-,r)\big) = \mathrm{H}_i\big(f^*\mathcal{L}(W,r)\big) = \mathrm{H}_i\big(\mathcal{L}(X,r)\big) = \mathcal{H}_i(X,r)$$

since $f^*\mathcal{L}(-,r) \in \mathcal{P}_W$ is pseudo-flasque. Hence, as in the proof of Proposition 5.3, it suffices to consider the diagram in $\mathcal{P}_W$ of natural maps

$$\begin{tikzcd} \mathcal{M}(-,s)\otimes & \mathcal{L}(-,r) \arrow[r, "\cap"] \arrow[d, "f_*"] & \mathcal{L}(-\times\mathbf{A}^s, r)[-2s] \arrow[d, "f_*"] \\ f^*\mathcal{M}(-,s)\otimes \arrow[u, "f^*"] & f^*\mathcal{L}(-,r) \arrow[r, "\cap"] & f^*\mathcal{L}(-\times\mathbf{A}^s, r)[-2s] \end{tikzcd} \tag{5.9}$$

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{D}_W}(\mathbf{Z}_W[i],\mathcal{M}(-,s))\otimes \mathrm{Hom}_{\mathcal{D}_W}(\mathbf{Z}_W[j],\mathcal{L}(-,r)) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}_W}(\mathbf{Z}_W[i+j],\mathcal{L}(-,r)) \\
\uparrow \qquad\qquad \downarrow & & \downarrow \\
\mathrm{Hom}_{\mathcal{D}_W}(\mathbf{Z}_W[i],f^*\mathcal{M}(-,s))\otimes \mathrm{Hom}_{\mathcal{D}_W}(\mathbf{Z}_W[j],f^*\mathcal{L}(-,r)) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}_W}(\mathbf{Z}_W[i+j],f^*\mathcal{L}(-,r)) \\
\uparrow \qquad\qquad {=}\uparrow & & \uparrow{=} \\
\mathrm{Hom}_{\mathcal{D}_X}(\mathbf{Z}_X[i],\mathcal{M}(-,s))\otimes \mathrm{Hom}_{\mathcal{D}_X}(\mathbf{Z}_X[j],\mathcal{L}(-,r)) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}_X}(\mathbf{Z}_X[i+j],\mathcal{L}(-,r))
\end{array}
$$

DIAGRAM D.

and to show for any $V \in Zar_W$, any $\alpha \in f^*\mathcal{M}(V,s)$, any $s \in \mathcal{L}(V,r)$ that

$$\alpha \cap f_*(z) = f_*(f^*\alpha \cap z).$$

This will provide the "commutativity" of Diagram (D). To analyze (5.9), it suffices to consider the following diagram of topological abelian monoids for any $U \in Zar_X$, $V \in Zar_W$ with $i:V \subset f^{-1}(U)$:

$$
\begin{array}{ccc}
\mathcal{M}or(V,\mathcal{C}_0(\mathbf{P}^s))\times \mathcal{C}_r(V) & \longrightarrow & \mathcal{C}_r(V\times \mathbf{P}^s) \\
{\scriptstyle \circ f\circ i}\uparrow \qquad\qquad \downarrow{\scriptstyle f_*\circ i_*} & & \downarrow{\scriptstyle (f\times 1)_*\circ(i\times 1)_*} \\
\mathcal{M}or(U,\mathcal{C}_0(\mathbf{P}^s))\times \mathcal{C}_r(U) & \longrightarrow & \mathcal{C}_r(U\times \mathbf{P}^s)
\end{array}
\tag{5.10}
$$

We readily verify for any $Z \in \mathcal{C}_r(V)$, $\phi \in \mathcal{M}or(U,\mathcal{C}_0(\mathbf{P}^s))$ that

$$(f \times 1)_* \circ (i \times 1)_*(\Gamma_{\phi\circ f\circ i|Z}) = \Gamma_{\phi|f_*i_*(Z)}.$$

We interpret this equality as asserting the "commutativity" of (5.10) and thus the analogous "commutativity" of (5.9). □

The next Bloch–Ogus property is the following assertion of Poincaré duality.

THEOREM 5.5. – *Assume that X is a smooth variety of dimension d and $Y \subset X$ is a closed subvariety. Then cap product with the fundamental class $\eta_X \in \mathrm{H}_{2d}(\mathcal{L}(X,d))$*

$$-\cap \eta_X : \mathcal{H}^j_Y(X,s) \to \mathcal{H}_{2d-j}(Y,d-s)$$

is an isomorphism.

Proof. – The duality theorem of [6,9] implies that

$$-\cap \eta_X : \mathcal{M}(-,s)[2s] \to \mathcal{L}(-\times \mathbf{A}^s,d)[-2d]$$

is a quasi-isomorphism. Applying homotopy invariance (cf. (5.3)), we conclude

$$-\cap \eta_X : \mathcal{M}(-,s) \to \mathcal{L}(-,d-s)[-2d] \tag{5.11}$$

is a quasi-isomorphism. This extends to a map of distinguished triangles

$$\begin{tikzcd}
\mathcal{M}_Y(-,s) \arrow[r] \arrow[d] & \mathcal{L}(-\cap Y, d-s)[-2d] \arrow[d] \\
\mathcal{M}(-,s) \arrow[r] \arrow[d] & \mathcal{L}(-, d-s)[-2d] \arrow[d] \\
\mathcal{M}(-\cap U, s) \arrow[r] & \mathcal{L}(-\cap U, d-s)[-2d].
\end{tikzcd}$$

whose horizontal maps are quasi-isomorphisms, where $U = X - Y$. □

The quasi-isomorphism (5.11) in conjunction with Corollary 3.2 immediately implies the following equality of morphic cohomology and topological cohomology for smooth varieties.

COROLLARY 5.6. – *If X is a smooth variety and Y a closed subvariety, then the presheaf $\mathcal{M}or_Y(-,s)$ in $\mathcal{S}_X$ is pseudo-flasque. In particular, if X is smooth, then*

$$\mathcal{H}^j(X,s) \simeq \mathrm{L}^s\mathrm{H}^j(X),$$

where the right hand side is morphic cohomology as defined in earlier papers.

The final Bloch–Ogus property [3, 1.5] verifies the triviality of the cycle class of a principal divisor.

PROPOSITION 5.7. – *If X is a smooth variety of pure dimension d and if $i : W \to X$ is a principal divisor, then the image $i_*(\eta_W) \in Z_{d-1}(X)$ of the fundamental class of W is homologically trivial:*

$$i_*(\eta_W) = 0 \in \mathcal{H}_{2d-2}(X, d-1).$$

Proof. – We recall the existence of a Gysin map $i_W^! : Z_r(X) \to Z_{r-1}(W)$ whose composition with $i_* : Z_r(W) \to Z_r(X)$ depends (in the derived category) only upon the line bundle of which W is the zero locus of a global section [7, 2.4]. If W is principal, then L is the trivial line bundle so that this composition is 0. On the other hand, $i_W^!(\eta_X) = \eta_W$ [7, 2.4]. □

6. Relationship to singular cohomology/homology

As observed in previous papers (esp., [8,12]), there are natural maps from morphic cohomology on normal varieties to singular cohomology and from Lawson homology to Borel–Moore homology. In this section, we confirm that these maps are naturally formulated in our context of derived categories of presheaves and thereby extend to our more general setting. As we see, these maps are often compatible with cap product in singular homology/cohomology theory (a result generalizing the main result of [11]).

We refer the reader to [11] for a quick sketch of integral cycles, Lipschitz neighborhood retracts, and related matters.

DEFINITION. – *For any variety X, we denote by*

$$\varepsilon : An_X \to Zar_X$$

the natural morphism of sites, where An_X is the site whose objects are analytic open subsets of X and whose morphisms are inclusions. (The functor ε is associated to the fact that every Zariski open sbset of X is also an analytic open subset.)

We let $\mathcal{P}_X^{\mathrm{an}}$ denote the triangluated category of presheaves on An_X with values in the abelian category of chain complexes bounded below of abelian groups. We denote by $\mathcal{D}_X^{\mathrm{an}}$ the localization of $\mathcal{P}_X^{\mathrm{an}}$ by the thick subcategory of those $P \in \mathcal{P}_X^{\mathrm{an}}$ each fibre of which is acyclic.

Observe that

$$\varepsilon^*: \mathcal{P}_X \to \mathcal{P}_X^{\mathrm{an}}$$

is an exact functor which induces

$$\varepsilon^*: \mathcal{D}_X \to \mathcal{D}_X^{\mathrm{an}}$$

since each fibre of $\varepsilon^* P \in \mathcal{P}_X^{\mathrm{an}}$ is acyclic whenever each fibre of $P \in \mathcal{P}_X$ is acyclic.

DEFINITION. – *We define the presheaf*

$$\mathcal{H}om(-,2s) =: U \mapsto \mathrm{cone}\{\mathrm{Hom}_{\mathrm{cont}}(U, Z_0(\mathbf{P}^{s-1}))\tilde{} \to \mathrm{Hom}_{\mathrm{cont}}(U, Z_0(\mathbf{P}^s))\tilde{}\}[-2s]$$

where $\mathrm{Hom}_{\mathrm{cont}}(U,P)$ denotes the mapping space of continuous maps from U to P (each with the analytic topology) given the compact-open topology and where $Z_0(P) = \mathcal{C}_0(P)^+$ denotes the naïve topological group completion of the topological monoid $\mathcal{C}_0(P)$.

We define the presheaf

$$\mathcal{Z}(-,m) =: U \mapsto \big(\mathcal{Z}_m(\overline{X})\tilde{}/\mathcal{Z}_m(\overline{X}-U)\big)\tilde{}[m]$$

where $X \subset \overline{X}$ is a projective closure and where $\mathcal{Z}_m(Y)$ denotes the topological abelian group of integral m-cycles on Y (in the sense of geometric measure theory) provided with the flat-norm topology.

The following proposition interprets singular cohomology/homology in terms similar to our formulation of topological cycle theory.

PROPOSITION 6.1. – *For any variety X, the presheaves*

$$\mathcal{H}om(-,2s), \quad \mathcal{Z}(-,m) \in \mathcal{P}_X^{\mathrm{an}}$$

are pseudo-flasque. Moreover, for any $U \in An_X$,

$$\mathrm{Hom}_{\mathcal{D}_X^{\mathrm{an}}}\big(\mathbf{Z}_X, \mathcal{H}om(-,2s)[j]\big) = \mathrm{H}_{-j}\big(\mathcal{H}om(U,2s)\big) = \mathrm{H}^j(X), \quad j \leqslant 2s, \tag{6.1}$$

$$\mathrm{Hom}_{\mathcal{D}_X^{\mathrm{an}}}\big(\mathbf{Z}_X[i], \mathcal{Z}(-,m)\big) = \mathrm{H}_i\big(\mathcal{Z}(U,m)\big) = \mathrm{H}_i^{BM}(U), \quad i \geqslant m, \tag{6.2}$$

where $\mathrm{H}^*(U)$ and $\mathrm{H}_*^{BM}(U)$ denote the singular cohomology and Borel–Moore homology of U (considered with its analytic topology).

Proof. – Let $K(\mathbf{Z},n)$ denote the Eilenberg–MacLane space with unique non-vanishing homotopy group $\mathbf{Z}$ in degree n, represented for example by the infinite symmetric power $SP(S^n)$ of the n-sphere. Since $Z_0(\mathbf{P}^{s-1}) \to Z_0(\mathbf{P}^s)$ can be identified with the natural embedding

$$\prod_{i=0}^{s-1} K(\mathbf{Z},2i) \to \prod_{i=0}^{s} K(\mathbf{Z},2i),$$

$\mathcal{H}om(-,2s)[2s]$ is isomorphic in $\mathcal{D}_X^{\mathrm{an}}$ to the presheaf $U \mapsto \mathrm{Hom}_{\mathrm{cont}}(U, K(\mathbf{Z},2s))$. As in the proof of Proposition 3.2, in order to prove (6.1) it suffices to show that

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{cont}}(U, K(\mathbf{Z},2s)) & \longrightarrow & \mathrm{Hom}_{\mathrm{cont}}(U_1, K(\mathbf{Z},2s)) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathrm{cont}}(U_2, K(\mathbf{Z},2s)) & \longrightarrow & \mathrm{Hom}_{\mathrm{cont}}(U_3, K(\mathbf{Z},2s)) \end{array} \tag{6.3}$$

is homotopy Cartesian, whenever $U_1, U_2 \subset U$ in An_X with $U = U_1 \cup U_2, U_3 = U_1 \cap U_2$. Since

$$\pi_j\big(\mathrm{Hom}_{\mathrm{cont}}(U, K(\mathbf{Z},2s))\big) = \mathrm{H}^{2s-j}(U),$$

the fact that (6.3) is homotopy Cartesian (and the validity of (6.1)) follows immediately from Mayer–Vietoris for singular cohomology.

A well known theorem of F. Almgren's [1] implies that

$$\pi_i\big(\mathcal{Z}_m(\overline{X})\big) = \mathrm{H}_{i+m}(\overline{X}).$$

As above, Mayer–Vietoris (this time for singular homology) implies that $\mathcal{Z}(-,m)$ is pseudo-flasque and provides the equalities of (6.2). □

PROPOSITION 6.2. – *For any variety X, there are naturally constructed maps in $\mathcal{P}_X^{\mathrm{an}}$*

$$\varepsilon^*\mathcal{M}(-,s) \to \mathcal{H}om(-,2s)$$

$$\varepsilon^*\mathcal{L}(-,r) \to \mathcal{Z}(-,2r)$$

which determine natural maps

$$\Phi^* : \mathcal{H}^j(X,s) \to \mathrm{H}^j(X), \qquad \Phi_* : \mathcal{H}_i(X,r) \to \mathrm{H}_i^{BM}(X).$$

Proof. – By Proposition 6.1 and the fact that ε^* is exact and preserves weak equivalences, in order to construct Φ^* and Φ_* it suffices to exhibit the asserted maps in $\mathcal{P}_X^{\mathrm{an}}$. For this, it suffices to exhibit natural maps for any $U \in Zar_X$

$$\mathcal{M}(U,s) \to \mathcal{H}om(U,2s), \qquad \mathcal{L}(U,r) \to \mathcal{Z}(U,2r).$$

The first is induced by the natural inclusion $\mathcal{M}or(U, \mathcal{C}_0(P)) \to \mathrm{Hom}_{\mathrm{cont}}(U, Z_0(P))$ (induced by $\mathcal{C}_0(P) \to \mathcal{C}_0(P)^+ = Z_0(P)$) and the second by the natural inclusion $Z_r(\overline{X}) \subset \mathcal{Z}_{2r}(\overline{X})$. □

The "duality map" relating morphic cohomology and Lawson homology is induced by the construction which sends a continuous algebraic map to its graph (cf. [9]). In view of (5.6), this can be interpreted as cap product in topological cycle theory with the fundamental class of X. The basic result of [11] (namely, [11, 5.5]) is the assertion that if X is a projective, normal variety then this duality map is compatible with cap product with the fundamental class from singular cohomolgy to singular homology. The following compatibility theorem (more general than that of [11]) is an easy consequence of [11, 5.5] in conjunction with Proposition 5.4.

PROPOSITION 6.3. – *Let X be a projective variety. Then the natural maps of (6.2) are compatible with cap product with fundamental classes of algebraic cycles in the sense that the* *following square commutes*

$$
\begin{array}{ccc}
\mathcal{H}^j(X,s)\otimes\mathcal{H}_{2r}(X,r) & \xrightarrow{\cap} & \mathcal{H}_{2r-2s+j}(X,r-s) \\
\Big\downarrow{\scriptstyle \Phi^*\otimes\Phi_*} & & \Big\downarrow{\scriptstyle \Phi_*} \\
\mathrm{H}^j(X)\otimes\mathrm{H}_{2r}(X) & \xrightarrow{\cap} & \mathrm{H}_{2r-2s+j}(X)
\end{array}
\tag{6.4}
$$

for $r\geqslant s\geqslant 0$, $j\leqslant 2s$.

Proof. – Let $j:Y\subset X$ be the closed embedding of an irreducible subvariety of dimension r, let $cl_X(Y)\in \mathrm{H}_{2r}(\mathcal{L}(X,r))$ denote its associated cycle class, and let $[Y]$ denote $cl_Y(Y)$. Let $p:Y^{\sim}\to Y$ denote the normalization of Y and observe that $i_*([Y^{\sim}])=cl_X(Y)$, where $i=j\circ p:Y^{\sim}\to X$. Then the commutativity of (6.4) follows from the following chain of equalities for any $\alpha\in\mathcal{H}^j(X,s)$:

$$\Phi_*\big(\alpha\cap cl_X(Y)\big)=\Phi_*\big(i_*\big(i^*\alpha\cap[Y^{\sim}]\big)\big)=i_*\big(\Phi^*(i^*\alpha)\cap\Phi_*([Y^{\sim}])\big)=\Phi^*(\alpha)\cap\Phi_*\big(cl_X(Y)\big)$$

where the first equality follows from Proposition 5.4, the second from [11, 5.5], and the third from the analogue in singular theory of Proposition 5.4. □

7. Local to global spectral sequence

Since we have shown that topological cycle theory satisfies all of the properties Bloch and Ogus require of a "Poincaré duality theory with supports", their analysis implies the validity of "Gersten's Conjecture" for topological cycle cohomology (Theorem 7.1) and thus a local to global spectral sequence (Theorem 7.2).

We denote by $\mathcal{H}^j(s)$ the sheaf associated to the presheaf on Zar_X sending U to $\mathcal{H}^j(U,s)$.

THEOREM 7.1 (cf. [3, 4.2.2]). – *For any smooth variety X, there is a resolution of sheaves on Zar_X*

$$0\to\mathcal{H}^j(s)\to\bigoplus_{\mathrm{codim}(x)=0} i_x\mathcal{H}^j\big(\mathrm{Spec}\,k(x),s\big)\to\bigoplus_{\mathrm{codim}(x)=1} i_x\mathcal{H}^{j-1}\big(\mathrm{Spec}\,k(x),s-1\big)$$
$$\to\cdots\to\bigoplus_{\mathrm{codim}(x)=j} i_x\mathcal{H}^0\big(\mathrm{Spec}\,k(x),s-j\big)\to 0,$$

where i_xA denotes the extension by zero from $\{\overline{x}\}$ to X of the constant sheaf with value the abelian group A on the closure $\{\overline{x}\}$ of the (Zariski) point $x\in X$.

THEOREM 7.2 (cf. [3, 6.2]). – *Let X be a smooth variety. Then there is a natural spectral sequence*

$$E_2^{p,q}=\mathrm{H}^p_{Zar}\big(X,\mathcal{H}^q(s)\big)\Rightarrow\mathcal{H}^{p+q}(X,s).$$

Moreover, the filtration $\{F^p\mathcal{H}^n(X,s)\}$ on $\mathcal{H}^n(X,s)$ associated to this spectral sequence is the arithmetic filtration given by

$$F^p\mathcal{H}^n(X,s)=\bigcup_Z Ker\big\{\mathcal{H}^n(X,s)\to\mathcal{H}^n(X-Z,s)\big\},$$

where the union is taken over all closed subvarieties $Z\subset X$ of codimension p.

We give one corollary of this theorem, a vanishing result confirming in this special case the expected vanishing range of Lawson homology. Namely, one expects that if X is a smooth variety of dimension d, then $\mathcal{H}_i(X,*)=0$ for $i>2d$. By duality, this is equivalent to the expectation that topological cycle cohomology should vanish in negative degrees.

COROLLARY 7.3. – *Let X be a smooth, rational 3-fold. Then*

$$\mathcal{H}^j(X,2)=0, \quad j<0.$$

Proof. – By Theorem 7.2, it suffices to prove that $\mathcal{H}^j(U,2)=0$, $j<0$, for all sufficiently small $U\in \mathit{Zar}_X$. Since X is rational, this is implied by the assertion that $\mathcal{H}^j(V,2)=0$, $j<0$, for all $V\subset \mathbf{P}^3$. By duality (i.e., Theorem 5.5), it suffices to verify that $\mathrm{H}_i(\mathcal{L}(V,1))=0$, $i>6$. Using the localization sequence (associated to the distinguished triangle of Theorem 3.2) in conjunction with the vanishing $\mathrm{H}_i(\mathcal{L}(\mathbf{P}^3,1))=0$, $i>6$, as shown in [16], we conclude that it suffices to prove that $\mathrm{H}_i(\mathcal{L}(Y,1))=0$, $i>6$, for any closed proper subvariety $Y\subset \mathbf{P}^3$.

Indeed, we show that $\mathrm{H}_i(\mathcal{L}(Y,1))=0$, $i>4$, for any closed proper subvariety $Y\subset \mathbf{P}^3$. Applying localization to the closed embedding of the singular locus $Y_{\mathrm{sing}}\subset Y$ with Zariski open complement Y_{ns} and observing that $Z_1(Y_{\mathrm{sing}})$ is discrete, we conclude that it suffices to assume that Y is smooth and possibly quasi-projective. Applying resolution of singularities to the projective closure $\overline{Y}$ of Y and localization once again, we conclude that it suffices to recall the computation of Lawson homology for divisors on a smooth, projective variety given in [6, 4.6]. □

The proof of Corollary 7.3 shows somewhat more. Namely, the vanishing of topological cycle cohomology in negative dimensions and weight two less than the dimension of X depends only upon the birational class of X.

8. Gillet's axioms

In this final section, we sketch how our topological cycle theory satisfies the axioms of H. Gillet [15], thereby presumably implying the existence of chern classes with values in topological cycle cohomology theory and a Riemman–Roch formula.

Gillet requires a graded complex $\underline{\Gamma}^*(*)$ of sheaves on the big Zariski site *Zar*. We take

$$\underline{\Gamma}^*(s)=\mathcal{M}(-,s)_{\mathit{Zar}}$$

the sheafification of the complex $\mathcal{M}(-,s)$ of presheaves on *Zar*. Gillet further requires the existence of an associative, (graded) commutative pairing

$$\underline{\Gamma}^*(*)\overset{L}{\otimes}\underline{\Gamma}^*(*)\to\underline{\Gamma}^*(*).$$

Such a pairing is established in [14, 3.1], using the join pairing of [8].

Gillet then postulates a list of axioms [15, 1.2]. The first is given by Proposition 3.3, the second by Proposition 3.4, and the third by Proposition 5.4. The existence of a functorial fundamental class η_X is stated in (5.3), whereas Theorem 5.5 provides the duality isomorphism required in Gillet's fifth axiom.

Gillet's sixth axiom is the condition that the duality isomorphism of Theorem 5.5 arises from a map of complexes (in the derived category); indeed, this is shown in the proof of Theorem 5.5.

Similarly, his seventh axiom is the condition that the projection formula arise from a diagram of complexes as shown in the proof of Theorem 5.5. There is no requirement given in axiom eight.

Gillet's ninth axiom, (1.2.ix), is the condition of homotopy invariance for cohomology. Since Lawson homology is homotopy invariant [7, 2.3], this homotopy invariance for topological cycle cohomology of smooth varieties is valid thanks to duality. More generally, this axiom does not appear to hold.

Axiom ten follows from the projective bundle theorem of [7, 2.5]. The final Gillet axiom, (1.2.xi) is essentially the existence of a first Chern class $Pic(-) \to \mathcal{H}^2(-,1)$. For a line bundle L generated by its global sections, $c_1(L)$ was exhibited in [8]; more generally, any line bundle L on a quasi-projective variety X locally closed in some projective space $\mathbf{P}^N$ has the property that $L \otimes \mathcal{O}(m)$ is generated by its global sections for m sufficiently large, so that we may define

$$c_1(L) = c_1\big(L \otimes \mathcal{O}(m)\big) - c_1\big(\mathcal{O}(m)\big).$$

A more precise version of Gillet's first Chern class axiom is formulated in terms of the existence of a morphism $\mathcal{O}^*[-1] \to \underline{\Gamma}^*(1)$ in $\mathcal{D}_X$ such that $[\mathcal{O}_{\mathbf{P}^1}(1)] \in \mathrm{H}^2(\mathbf{P}^1, \mathcal{O}^*[-1])$ is sent to $c_1(\mathcal{O}_{\mathbf{P}^1})$. Using [13], we observe that $\mathcal{O}^*[-1]$ can be realized in $\mathcal{D}_X$ as the sheaf of chain complexes associated to the group completion of the sheaf of simplicial abelian moniods

$$U \mapsto \mathcal{M}or\big(U \times \Delta^\bullet, \mathcal{C}_0(\mathbf{P}^1)\big)/\mathcal{M}or\big(U \times \Delta^\bullet, \mathcal{C}_0(\mathbf{P}^0)\big).$$

Thus, the natural map from the algebraic singular complex to the topological singular complex of the topological abelian monoid $\mathcal{M}or(U, \mathcal{C}_0(\mathbf{P}^1))$ determines a natural map from $\mathcal{O}^*[-1]$ to $\underline{\Gamma}^*(1)$.

REFERENCES

[1] F.J. ALMGREN, JR., Homotopy groups of the integral cyclic groups, *Topology* 1 (1962) 257–299.

[2] D. BARLET, *Espace Analytique Réduit des Cycles Analytiques Complexes Compacts d'un Espace Analytique Complexe de Dimension Finite, Fonctions de Plusieurs Variables*, II, Lecture Notes in Math., Vol. 482, Springer, Berlin, 1975, pp. 1–158.

[3] S. BLOCH and A. OGUS, Gersten's conjecture and the homology of schemes, *Ann. Éc. Norm. Sup.* 7 (1974) 181–202.

[4] K. BROWN and S. GERSTEN, *Algebraic K-Theory as Generalized Sheaf Cohomology, Algebraic K-Theory*, I, Lecture Notes in Math., Vol. 341, Springer, Berlin, 1973, pp. 266–292.

[5] E. FRIEDLANDER, Algebraic cocycles, Chow varieties, and Lawson homology, *Compositio Math.* 77 (1991) 55–93.

[6] E. FRIEDLANDER, Algebraic cocycles on normal, quasi-projective varieties, *Compositio Math.* 110 (1998) 127–162.

[7] E. FRIEDLANDER and O. GABBER, Cycles spaces and intersection theory, *Topological Methods in Modern Mathematics* (1993) 325–370.

[8] E. FRIEDLANDER and H.B. LAWSON, A theory of algebraic cocycles, *Annals of Math.* 136 (1992) 361–428.

[9] E. FRIEDLANDER and H.B. LAWSON, Duality relating spaces of algebraic cocycles and cycles, *Topology* 36 (1997) 533–565.

[10] E. FRIEDLANDER and H.B. LAWSON, Moving algebraic cycles of bounded degree, *Inventiones Math.* 132 (1998) 91–119.

[11] E. FRIEDLANDER and H.B. LAWSON, Graph mappings and Poincaré duality.

[12] E. FRIEDLANDER and B. MAZUR, *Filtration on the Homology of Algebraic Varieties*, Memoirs of AMS, Vol. 529, 1994.

[13] E. FRIEDLANDER and V. VOEVODSKY, Bivariant cycle cohomology, in: V. Voevodsky, A. Suslin and E. Friedlander, eds., *Cycles, Transfers, and Motivic Homology Theories*, Annals of Math. Studies, 1999.

[14] E. FRIEDLANDER and M. WALKER, Function spaces and continuous algebraic pairings for varieties, *Compositio Math.* (to appear).

[15] H. GILLET, Riemann–Roch theorems for higher algebraic K-theory, *Adv. in Math.* 40 (1981) 203–289.

[16] H.B. LAWSON, Algebraic cycles and homotopy theory, *Annals of Math.* 129 (1989) 253–291.

[17] P. LIMA-FILHO, Completions and fibrations for topological monoids, *Trans. Amer. Math. Soc.* 340 (1993) 127–147.

[18] A. SUSLIN and V. VOEVODSKY, Chow sheaves, in: V. Voevodsky, A. Suslin and E. Friedlander, eds., *Cycles, Transfers, and Motivic Homology Theories*, Annals of Math. Studies, 1999.

(Manuscript received May 11, 1998.)

Eric M. FRIEDLANDER
Department of Mathematics,
Northwestern University, Evanston,
IL 60208-2730, USA
e-mail: eric@math.nwu.edu