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# INTEGRAL LATTICES IN TQFT

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ABSTRACT. – We find explicit bases for naturally defined lattices over a ring of algebraic integers in the $SO(3)$-TQFT-modules of surfaces at roots of unity of odd prime order. Some applications relating quantum invariants to classical 3-manifold topology are given.


RÉSUMÉ. – Nous trouvons des bases explicites pour certains réseaux naturels définis sur un anneau d'entiers algébriques dans les modules associés aux surfaces par les $SO(3)$-TQFT aux racines de l'unité d'ordre premier impair. Nous donnons quelques applications reliant invariants quantiques et topologie classique des variétés de dimension trois.


**Contents**



## 1. Introduction

Let $p$ be an odd prime. Based on integrality results for the quantum $SO(3)$-invariant of closed oriented 3-manifolds [22,19], an integral TQFT-functor $\mathcal{S}_p$ was defined in [8] and [10]. It

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associates to a closed surface $\Sigma$ a free lattice [2] $\mathcal{S}_p(\Sigma)$ over the cyclotomic ring

$$\mathcal{O} = \begin{cases} \mathbb{Z}[\zeta_p] & \text{if } p \equiv -1 \pmod 4, \\ \mathbb{Z}[\zeta_p, i] = \mathbb{Z}[\zeta_{4p}] & \text{if } p \equiv 1 \pmod 4, \end{cases}$$

where $\zeta_n$ is a primitive $n$-th root of unity. The lattice $\mathcal{S}_p(\Sigma)$ carries an $\mathcal{O}$-valued non-degenerate hermitian form which will be denoted by $(\ ,\ )_\Sigma$. Here, non-degenerate means that the form induces an injective adjoint map $\mathcal{S}_p(\Sigma) \to \mathcal{S}_p(\Sigma)^*$. The lattice $\mathcal{S}_p(\Sigma)$ also carries a linear representation of an appropriately extended mapping class group of $\Sigma$; moreover, this representation preserves the hermitian form on $\mathcal{S}_p(\Sigma)$.

The integral lattices $\mathcal{S}_p(\Sigma)$ have a natural definition in terms of the vector-valued quantum $SO(3)$-invariants for 3-manifolds with boundary (see below). Their existence thus reflects interesting structural properties of quantum invariants and TQFT's. The main aim of the present paper is to give an explicit description of these lattices for surfaces of arbitrary genus, possibly disconnected, by describing bases for them.

Here, we allow surfaces to be equipped with a (possibly empty) collection of colored banded points (a banded point is an embedded oriented arc), where a color is an integer $i \in \{0, 1, \ldots, p-2\}$. But we require the sum of colors on the colored points of any component of a surface to be even (this is a feature of the $SO(3)$-theory we consider). Before, explicit bases of $\mathcal{S}_p(\Sigma)$ were known only in the case where the surface $\Sigma$ is connected and has genus one or two with no colored points [10].

The bases we find display some nice "graph-like" structure which we believe should generalize to other TQFT's, at least for those associated to integral modular categories as defined in [21]. One can ask whether the graph-like bases we have found might be related to canonical bases in representation theory.

An interesting new feature is that the usual tensor product axiom of TQFT holds only with some modification for the lattices $\mathcal{S}_p(\Sigma)$: it turns out that the lattice associated to a disjoint union of surfaces is sometimes bigger than the tensor product of the lattices associated to the individual components, although this phenomenon does not happen for surfaces without colored points.

We remark that Kerler [15] has announced an integral version of the $SO(3)$ TQFT at $p = 5$, but details have not yet appeared. Chen and Le have recently constructed integral TQFTs from quantum groups [6,5], although without describing explicit bases for the modules associated to surfaces.

We give two topological applications of our theory. In Section 15, we prove a conjecture which relates the cut number of a 3-manifold $M$ to divisibility properties of its quantum invariants. The cut number is the same as the co-rank of $\pi_1(M)$. Thus this result relates quantum invariants to the fundamental group of a 3-manifold.

Our second application concerns the Frohman and Kania-Bartoszynska ideal of 3-manifolds with boundary [7] which can be used to show that one 3-manifold does not embed in another. This ideal is hard to compute directly from its definition, except in very special circumstances, as its definition involves the quantum invariants of infinitely many 3-manifolds. However our integral bases allow us to give an explicit finite set of generators for this ideal (see Section 16). We apply this method to exhibit a family of examples of 3-manifolds with the homology of a solid torus which cannot embed in $S^3$.

The integral TQFT-functor $\mathcal{S}_p$ is a refinement of the $SO(3)$-TQFT-functor $V_p$ constructed in [3], which in turn is, in some sense, an alternative version of a special case of the Reshetikhin–

[2] A lattice over a Dedekind domain is a finitely generated torsion-free module. In general, a lattice need not be free, and the freeness of the lattices $\mathcal{S}_p(\Sigma)$ is a non-trivial fact.

Turaev theory [25] of quantum invariants of 3-manifolds. In particular, the lattice $\mathcal{S}_p(\Sigma)$ is defined in [8,10] as an $\mathcal{O}$-submodule of the TQFT-vector space $V_p(\Sigma)$. The latter is defined over the quotient field of $\mathcal{O}$ and exists also when $p$ is not prime. In fact, $V_p(\Sigma)$ can be defined (and is a free module) already over the ring $\mathcal{O}[\frac{1}{p}]$, see [3], and this is the version of $V_p$ that we refer to in the rest of the paper.

As a submodule of $V_p(\Sigma)$, the lattice $\mathcal{S}_p(\Sigma)$ is simply defined as the $\mathcal{O}$-span of the vectors associated to 3-manifolds $M$ with boundary $\partial M = \Sigma$, with the condition that $M$ should have no closed components. It is clear from this definition that the extended mapping class group acts on $\mathcal{S}_p(\Sigma)$. The fact that $\mathcal{S}_p(\Sigma)$ is a free lattice of rank equal to the dimension of $V_p(\Sigma)$ is shown in [8]. The hermitian form $(\ ,\ )_\Sigma$ is obtained by rescaling the natural form on $V_p(\Sigma)$ given by gluing manifolds together along their boundary and computing the invariant of the closed manifold so obtained. That such a rescaling is possible depends crucially on the integrality result for this invariant of closed 3-manifolds due to H. Murakami [22] and Masbaum and Roberts [19] (see [8] for more details). [3]

Bases of the free $\mathcal{O}[\frac{1}{p}]$-module $V_p(\Sigma)$ are well understood in terms of admissible colorings of uni-trivalent graphs. Here, any uni-trivalent graph which is the spine of a handlebody with boundary $\Sigma$, and with the univalent vertices meeting $\Sigma$ in the colored points, may be used. The $\mathcal{O}$-span of such a *graph basis* is a sublattice of $\mathcal{S}_p(\Sigma)$, but this sublattice is almost never invariant under the mapping class group, and hence cannot be equal to the whole integral lattice $\mathcal{S}_p(\Sigma)$. One might hope that a basis of $\mathcal{S}_p(\Sigma)$ could be obtained by rescaling the elements of a graph basis in some way, but this is not the case. Still, the situation is actually rather nice. We will show that the lattice $\mathcal{S}_p(\Sigma)$ admits what we call *graph-like* bases associated to a special kind of uni-trivalent graph which we call a *lollipop tree*. Roughly speaking, a graph-like basis is obtained from the usual graph basis associated to the lollipop tree by taking certain linear combinations, followed by some overall rescaling depending on the colors. The nice thing is that the linear combinations are taken independently in each handle. For precise definitions and a statement of the result in the case of connected surfaces, see Section 4.

We remark here that for connected surfaces, $\mathcal{S}_p(\Sigma)$ has a simple skein-theoretical description shown in [10], namely as the $\mathcal{O}$-span in $V_p(\Sigma)$ of banded links and graphs in a handlebody colored in a certain way. This description will be given below in Proposition 2.2. For the purpose of the present paper, this description can be taken as the definition of $\mathcal{S}_p(\Sigma)$. Moreover, our method of constructing a basis will give an independent proof that $\mathcal{S}_p(\Sigma)$ is indeed a free lattice.

For disconnected surfaces, we describe a basis of $\mathcal{S}_p(\Sigma)$ in Section 12, where we also discuss the modified tensor product axiom.

In the case of surfaces of genus one and two without colored points, the natural hermitian form $(\ ,\ )_\Sigma$ was shown in [10] to be unimodular (here, unimodular means that the adjoint map is not only injective but is an isomorphism). As already observed there, this property no longer holds in higher genus. It is then natural to consider the dual lattice $\mathcal{S}_p^\sharp(\Sigma)$, defined as

$$\mathcal{S}_p^\sharp(\Sigma) = \big\{ x \in V_p(\Sigma) \mid (x,y)_\Sigma \in \mathcal{O} \text{ for all } y \in \mathcal{S}_p(\Sigma) \big\}.$$

Note that $\mathcal{S}_p(\Sigma) \subset \mathcal{S}_p^\sharp(\Sigma)$, with equality if and only if the form is unimodular. [4]

The main result about the dual lattices $\mathcal{S}_p^\sharp(\Sigma)$ is that they also admit graph-like bases. In fact, such bases can be described as rescalings of graph-like bases for $\mathcal{S}_p(\Sigma)$ (see again Section 4 for

[3] Using deep results of Habiro, integrality of the $SO(3)$-invariant for all 3-manifolds and any odd $p$ was proven in 2006 by Beliakova and Le [2] (see also Le [16]).

[4] It is easy to check that since $\mathcal{S}_p(\Sigma)$ is a free lattice, so is $\mathcal{S}_p^\sharp(\Sigma)$. In fact, $\mathcal{S}_p^\sharp(\Sigma)$ is isomorphic as an $\mathcal{O}$-module to the dual of the conjugate of $\mathcal{S}_p(\Sigma)$, justifying the terminology.

a precise statement). Moreover, the dual lattices play an important role in the proof that our bases are indeed bases, as the proof proceeds by studying the two lattices simultaneously.

The dual lattice $\mathcal{S}_p^\sharp(\Sigma)$ is, of course, also preserved by the action of the mapping class group. Note that when one expresses the mapping class group representation on $\mathcal{S}_p(\Sigma)$ or on $\mathcal{S}_p^\sharp(\Sigma)$ in our integral bases, all matrix coefficients will be algebraic integers.

One also obtains a representation by isometries (of the associated torsion "linking form") on the quotient $\mathcal{S}_p^\sharp(\Sigma)/\mathcal{S}_p(\Sigma)$, which admits a simple description, at least if $p \equiv -1 \pmod 4$: When $\Sigma$ is connected and has no colored points, $\mathcal{S}_p^\sharp(\Sigma)/\mathcal{S}_p(\Sigma)$ is a skew-symmetric inner product space over the finite field $\mathbb{F}_p$. If $p \equiv 1 \pmod 4$, a similar statement holds with a little more effort for a refined theory, see Section 14 for details. It should be mentioned, however, that here we merely begin the study of these representations of mapping class groups on torsion modules; we hope to return to this matter elsewhere.

*Note added March 2006.* As a further application of the integral lattices $\mathcal{S}_p(\Sigma)$, it will be shown in [18] that when one expresses the mapping class group representations in our graph-like bases associated to lollipop trees, then the representations (at least when restricted to the Torelli group) have a perturbative limit as $p \to \infty$, in much the same sense as Ohtsuki's power series invariant [23] of homology spheres is a limit of the quantum $SO(3)$-invariants at roots of prime order.

*Notations and conventions.* Throughout the paper, $p \geqslant 5$ will be a prime integer, and we put $d = (p-1)/2$.[5] Recall that our ground ring $\mathcal{O}$ contains a primitive $p$-th root of unity $\zeta_p$. We define $h \in \mathcal{O}$ by

$$h = 1 - \zeta_p. \tag{1}$$

One has that $h^{p-1}$ is a unit times $p$, so that $\mathcal{O}[\frac{1}{p}] = \mathcal{O}[\frac{1}{h}]$. For skein theory purposes, we put $A = -\zeta_p^{d+1}$. This is a primitive $2p$-th root of unity such that $A^2 = \zeta_p$. The quantum integer $[n]$ is defined by $[n] = (\zeta_p^n - \zeta_p^{-n})/(\zeta_p - \zeta_p^{-1})$. We also fix $\mathcal{D} \in \mathcal{O}$ such that

$$\mathcal{D}^2 = \frac{-p}{(\zeta_p - \zeta_p^{-1})^2}. \tag{2}$$

One has that $\mathcal{D}$ is a unit times $h^{d-1}$.

Unless otherwise stated all manifolds that we consider are assumed to be compact and oriented. If a surface $\Sigma$ is equipped with a particular collection of colored banded points, we denote the latter by $\ell(\Sigma)$. When writing $V_p(\Sigma)$, $\mathcal{S}_p(\Sigma)$, etc., we let $\Sigma$ stand for the surface equipped with its given colored points.

As usual in TQFT, our surfaces and 3-manifolds are equipped with an additional structure to resolve the "framing anomaly". As it is well-known how to do this [3,29,28], and the additional structure is basically irrelevant for integrality questions, we postpone further discussion of this "framing" issue until Section 13 where some details of the construction will be needed.

## 2. Skein theory and the definition of $\mathcal{S}_p(\Sigma)$

Unless otherwise stated, by skein module we will mean the Kauffman Bracket skein module over $\mathcal{O}[\frac{1}{p}] = \mathcal{O}[\frac{1}{h}]$. Recall that this skein module of a 3-manifold $M$ is the free $\mathcal{O}[\frac{1}{h}]$ module on the banded links in $M$ modulo the well-known Kauffman relations and isotopy. We denote

[5] We assume $p \geqslant 5$ because the color 2, which will play an important role in our construction, does not exist for $p = 3$. But the theory for $p = 3$ is trivial anyway [3].

this module by $K(M)$. Elements of $K(M)$ can be described by colored trivalent banded graphs, which can be expanded to linear combinations of banded links in the familiar way. In particular, a strand colored, say $a$, is replaced by $a$ parallel strands with the Jones–Wenzl idempotent, denoted $f_a$, inserted. This is a particular linear combination of $a$–$a$ tangles, and is sometimes denoted as a rectangle with $a$ inputs on each of the long sides. See for instance [20]. We also need to consider the relative skein module of a 3-manifold whose boundary is equipped with some banded colored points $\ell(\partial M)$. Here one takes the free $\mathcal{O}[\frac{1}{h}]$ module on those links (or rather, tangles) which are expansions of colored graphs which meet the boundary nicely in the colored points. In this case we only use isotopy relative the boundary in the relations. This relative skein module is denoted $K(M,\ell(\partial M))$. An element of this module is often denoted $(M,L)$ where $L$ stands for the colored link or graph in $M$.

Suppose the surface $\Sigma$ is equipped with a (possibly empty) collection of colored points $\ell(\Sigma)$. The $\mathcal{O}[\frac{1}{p}]$-module $V_p(\Sigma)$ can be described as follows. For any 3-manifold $M$ with boundary $\Sigma$, there is a surjective map from $K(M,\ell(\Sigma))$ to $V_p(\Sigma)$.[6] The image of the skein class represented by $(M,L)$ is denoted $[(M,L)]$. Here we think of $(M,L)$ as a cobordism from $\emptyset$ to $\Sigma$. Suppose $M'$ is a second 3-manifold with boundary $\Sigma$, then a sesquilinear form $\langle\ ,\ \rangle_{M,M'}: K(M,\ell(\Sigma)) \times K(M',\ell(\Sigma)) \to \mathcal{O}[\frac{1}{p}]$ is defined by

$$\big\langle (M,L),(M',L')\big\rangle_{M,M'} = \big\langle (M \cup_\Sigma -M', L \cup_{\ell(\Sigma)} -L')\big\rangle_p.$$

Here $\langle\ \rangle_p$ denotes the quantum invariant of a closed 3-manifold, and the minus sign indicates reversal of orientation. The kernel of the map $K(M,\ell(\Sigma)) \to V_p(\Sigma)$ is the left radical of the form $\langle\ ,\ \rangle_{M,M'}$. Moreover, this form induces the canonical nonsingular hermitian form $\langle\ ,\ \rangle_\Sigma: V_p(\Sigma) \times V_p(\Sigma) \to \mathcal{O}[\frac{1}{p}]$ (which is independent of $M$ and $M'$). All the results of this paragraph appear in [3].

DEFINITION 2.1. – Given a closed surface $\Sigma$ with possibly a collection of colored banded points $\ell(\Sigma)$, we define $\mathcal{S}_p(\Sigma)$ to be the $\mathcal{O}$-submodule of $V_p(\Sigma)$ generated by all vectors $[(M,L)]$ where $M$ is any 3-manifold with boundary $\Sigma$ having no closed connected components, and the colored graph $L \subset M$ meets $\Sigma$ nicely in $\ell(\Sigma)$.

As shown in [10], $\mathcal{S}_p(\Sigma)$ has a skein-theoretical description, as follows. First, we let $v$ denote the skein element in $K(S^1 \times D^2)$ described by $h^{-1}(2+z)$ where $z$ is $S^1 \times \{0\}$ with standard framing. Thus $v$ denotes a skein class and also the element in $V_p(S^1 \times S^1)$ which it represents, depending on context. (The $v$ used in [10] differs by a unit from the $v$ used here.) Coloring a link component $v$ is shorthand for replacing this component by the linear combination $h^{-1}(2+z)$ and expanding linearly.

Next, by a *$v$-graph* in 3-manifold with possibly some colored points in the boundary, we will mean a banded colored graph in $M$ which agrees with $\ell(\partial M)$ on the boundary together with possibly some other banded link components which have been colored with $v$. By [10, Prop. 5.6 and Cor. 7.5 ], we have that:

PROPOSITION 2.2. – *Suppose $\Sigma$ is a connected surface. Choose a connected 3-manifold $M$ with boundary $\Sigma$. Then $\mathcal{S}_p(\Sigma)$ is generated over $\mathcal{O}$ by elements represented by $v$-graphs in $M$ which meet the boundary in the colored points of $\Sigma$.*

[6] There is no connectivity hypothesis on $M$ since $V_p$ satisfies the tensor product axiom. Here we use our assumption that the sum of the colors of the banded points on every connected component of every surface is even.

For connected surfaces, the above can be taken as an alternative definition of $\mathcal{S}_p(\Sigma)$. There is also a version of this for surfaces which are not connected but we delay stating it until Section 12 when it is needed.

## 3. Lollipop trees and the small graph basis of $V_p(\Sigma)$

We let $\Sigma$ denote the boundary of a genus $g$ handlebody $H_g$ and fix a particular collection of colored banded points in $\Sigma$ which we denote by $\ell(\Sigma)$. A basis of $V_p(\Sigma)$ can be described by the $p$-admissible (see below) colorings of any uni-trivalent banded graph $G$ having the same homotopy type as the handlebody $H_g$ and which meets the boundary at $\ell(\Sigma)$ in the univalent vertices of $G$. Moreover, the colorings should extend the given colorings at these banded colored points [3]. As usual, a colored graph with an edge colored zero is identified with the same graph without that edge (similarly for zero-colored points in $\ell(\Sigma)$).

For example, we can take the graph in Fig. 1 when $g=5$ and $\ell(\Sigma)$ has 6 points. The banding of this graph lies in the plane. The points $\ell(\Sigma)$ are depicted at the bottom of the diagram. The x's in this and later diagrams denote holes in $H_g$.

A $p$-admissible coloring of $G$ is an assignment of colors to the edges of $G$ such that at every vertex of $G$, the three colors $i,j,k$, say, meeting at that vertex satisfy the conditions

$$\begin{aligned} &i+j+k \equiv 0 \pmod 2,\\ &|i-j| \leqslant k \leqslant i+j,\\ &i+j+k \leqslant 2p-4 = 4d-2. \end{aligned}$$

To a $p$-admissible coloring of $G$ one associates in the usual way a skein element in the handlebody $H_g$, by replacing the edges of $G$ with appropriate Jones–Wenzl idempotents. Identifying now the boundary of the handlebody with our surface $\Sigma$, this skein element defines in turn a vector in $V_p(\Sigma)$. The vectors associated to $p$-admissible colorings where, in addition, the colors satisfy a parity condition, form a basis of $V_p(\Sigma)$ [3, Theorem 4.14]. [7]

In Proposition 3.2 below, we will describe a different basis of $V_p(\Sigma)$, where the parity condition is replaced by a "smallness" condition. For this, we must restrict our graph to be a lollipop tree, defined as follows.

DEFINITION 3.1. – Let $G$ be a uni-trivalent graph as above. Let $g$ be the first Betti number of $G$ and let $s$ be the number of its univalent vertices. (Of course, $g$ is the genus of $\Sigma$ and $s$ is

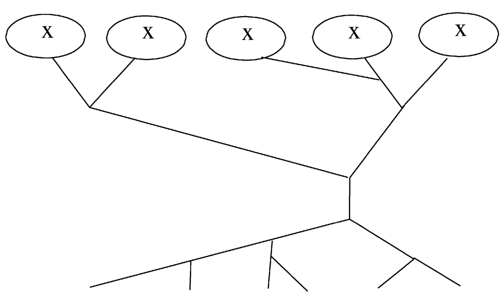



Fig. 1.

[7] In the exceptional case where $\Sigma$ is a two-sphere, with only one even-colored point, there is no such graph. Thus $V_p(\Sigma)$ is zero if the color is nonzero, and $\mathcal{O}[\frac{1}{p}]$ if the color is zero.

the number of colored points in $\ell(\Sigma)$.) Then $G$ is called a *lollipop tree* provided it satisfies the following conditions.

(i) $G$ has exactly $g$ loop edges, so that the complement of these loop edges in $G$ is a tree $T$.

(ii) If $s > 0$, there must be a single edge of the tree $T$ called the *trunk edge* (or simply the *trunk*) with the property that if we remove the interior of the trunk from $T$, we obtain the disjoint union of two trees: one which meets every loop edge and one which contains every uni-valent vertex. (If $s = 1$, this second tree consists of a single point.)

Note that a lollipop tree is not actually a tree. We chose to call it so because of the special case where there is only one loop edge and the tree $T$ consists of just one edge; in this case the graph $G$ looks somewhat like a lollipop.

For example, the graph of Fig. 1 is a lollipop tree.

PROPOSITION 3.2. – *The vectors associated to $p$-admissible colorings of a lollipop tree $G$, where the loop edges are assigned colors in the interval $[0, d-1]$, form a basis of $V_p(\Sigma)$.*

We will refer to this basis as the *small graph basis* and denote it by $\mathcal{G}$. Here, the adjective "small" refers to the colors on the loop edges.

*Proof.* – By [3, Theorem 4.14], we have a basis by taking all $p$-admissible colorings with even colors on the loop edges. (Observe that the parity of the colors of the edges of the sub-graph $T$ is imposed by the colors of $\ell(\Sigma)$.) Lemma 8.2 of [10] shows how to replace even colors on the loop edges by small ones, i.e., colors in $[0, d-1]$. Specifically, the lemma dealt with the case $g = 2, s = 0$, but the same argument works in general. □

## 4. Bases for $\mathcal{S}_p(\Sigma)$ and $\mathcal{S}_p^\sharp(\Sigma)$ for connected surfaces

We are now ready to state our results concerning graph-like bases of $\mathcal{S}_p(\Sigma)$ and $\mathcal{S}_p^\sharp(\Sigma)$. Fix a lollipop tree $G$ and define $g$ and $s$ as in the preceding section.

We will use the following labelling of the edges of $G$. Recall that the complement of the loop edges of $G$ is a tree $T$. We call an edge of $T$ *a stick edge* if it is incident with a loop edge of $G$. An edge is called *ordinary* otherwise. We denote their colors by $2a_1, \ldots, 2a_g$ for the stick edges (which will always have an even color), and by $c_1, c_2, \ldots$ for the ordinary edges. Here

$$0 \leqslant a_i \leqslant d-1. \tag{3}$$

The case $g = 2, s = 0$ is special as there is only one stick edge. In this case we put $a_1 = a_2$ but both $a_1$ and $a_2$ are to be entered in Equations (5), (6), and (7) below.

The color of the trunk is always even; we denote it by $2e$. If $s = 0$, we also set $e$ to be zero. We note that the trunk is usually an ordinary edge but it is a stick edge when $g = 1$ and $s \geqslant 1$.

We denote the colors of the loop edges by $a_i + b_i$ ($i = 1, \ldots, g$). Here, the loop edge colored $a_i + b_i$ is incident to the stick edge of $T$ labelled $2a_i$. Note $b_i \geqslant 0$ by $p$-admissibility. Moreover, since loop edges should have small colors, we have

$$0 \leqslant b_i \leqslant d-1-a_i. \tag{4}$$

The elements of the small graph basis $\mathcal{G}$ will be denoted by $\mathfrak{g}(a,b,c)$, where $a = (a_1, \ldots, a_g)$, $b = (b_1, \ldots, b_g)$, and $c = (c_1, c_2, \ldots)$. The index set is precisely the set of $(a,b,c)$ satisfying conditions (3) and (4), and such that $(2a, c)$ is a $p$-admissible coloring of the tree $T$ extending the given coloring of $\ell(\Sigma)$. We will refer to $(a,b,c)$ as a small coloring of $G$.

*The basis of* $\mathcal{S}_p(\Sigma)$. Our basis of $\mathcal{S}_p(\Sigma)$ will be denoted by $\mathcal{B}$. It consists of vectors $\mathfrak{b}(a,b,c)$ indexed by the same set as the small graph basis $\mathcal{G}$. They are defined as follows. Recall that $h = 1-\zeta_p$. For $x \in \mathbb{R}$, we use the notation $\lfloor x \rfloor$ (resp. $\lceil x \rceil$) to denote the greatest integer $\leqslant x$ (resp. the smallest integer $\geqslant x$).

If $b=0$, the vector $\mathfrak{b}(a,0,c)$ is just a rescaling of $\mathfrak{g}(a,0,c)$:

$$\mathfrak{b}(a,0,c) = h^{-\lfloor \frac{1}{2}(-e+\Sigma_i a_i)\rfloor} \mathfrak{g}(a,0,c). \tag{5}$$

(Observe that one always has $e \leqslant \sum_i a_i$.) If $b>0$, the vector $\mathfrak{b}(a,b,c)$ is conveniently described using multiplicative notation, as follows.

Think of the handlebody $H_g$ as $P \times I$ where $P$ is a $g$-holed disk thought of as a regular neighborhood of a planar embedding of the banded graph $G$. This endows the absolute skein module of $H_g$ with an algebra structure, where multiplication is given by putting one skein element on top of the other. Similarly, relative skein modules of $H_g$ are modules over the absolute skein module. Note that this multiplication is non-commutative in general. This also induces an algebra structure on $V_p(\Sigma)$ in the case $\ell(\Sigma)$ is empty. Although these algebra/module structures are not canonically associated to the surface $\Sigma$, they are well-defined once $\Sigma$ has been identified with the boundary of $P \times I$ with all colored points, say, at level one-half. Similarly, we obtain $\mathcal{O}$-algebra/module structures on the lattices $\mathcal{S}_p(\Sigma)$, which will be used throughout the paper.

For $i=1,\dots,g$, let $z_i$ denote the skein element represented by a circle around the $i$-th hole of the $g$-holed disk $P$ and with framing parallel to $P$. We have

$$z_i = \mathfrak{g}\big((0,\dots,0),(0,\dots,0,1,0,\dots,0),(0,\dots)\big).$$

(The only nonzero coefficient sits at the $i$-th entry of the $g$-tuple $b$.) We define

$$v_i = h^{-1}(2+z_i)$$

which is, of course, the same as $z_i$ cabled by the skein element $v$ defined in Section 2.

Using the module structure on $\mathcal{S}_p(\Sigma)$ discussed above, we can now define the elements of our basis $\mathcal{B}$ as follows:

$$\begin{aligned}\mathfrak{b}(a,b,c) &= v_1^{b_1}\cdots v_g^{b_g}\,\mathfrak{b}(a,0,c)\\ &= h^{-\Sigma_i b_i-\lfloor\frac{1}{2}(-e+\Sigma_i a_i)\rfloor}(2+z_1)^{b_1}\cdots(2+z_g)^{b_g}\,\mathfrak{g}(a,0,c).\end{aligned} \tag{6}$$

THEOREM 4.1. – *$\mathcal{B}$ is a basis of $\mathcal{S}_p(\Sigma)$.*

The proof will be completed in Section 9. Note that it is enough to show that these vectors lie in $\mathcal{S}_p(\Sigma)$ and generate it over $\mathcal{O}$. As their number is equal to the dimension of $V_p(\Sigma)$, they must then form a basis, and $\mathcal{S}_p(\Sigma)$ must be a free lattice.

We call $\mathcal{B}$ a *graph-like* basis. Note that $\mathfrak{b}(a,b,c)$ is a linear combination of vectors $\mathfrak{g}(a,b',c)$ with $b'_i \leqslant b_i$ for all $i=1,\dots,g$. Moreover, the multiplicative expression in Equation (6) shows that the linear combinations are taken independently in each handle. Observe also that the rescaling factor is a power of $h$ whose exponent depends on the numbers $a_i$ and $b_i$ and the trunk color $2e$, but does not explicitly depend on the colors $c_i$.

In genus one with no colored points, the basis $\mathcal{B}$ is up to units the same as the second integral basis $\{1,v,\dots,v^{d-1}\}$ of [10]. By functoriality, it follows that the vectors $v_i$ and their products lie in $\mathcal{S}_p(\Sigma)$. However, it is not obvious *a priori* that the vectors $\mathfrak{b}(a,0,c)$ lie in $\mathcal{S}_p(\Sigma)$. (Even in genus two the basis $\mathcal{B}$ is different from the one obtained in [10].)

COROLLARY 4.2. – *The extended mapping class group* (*see Section* 13) *acts on* $\mathcal{S}_p(\Sigma)$ *in the basis* $\mathcal{B}$ *by matrices with coefficients in the cyclotomic ring* $\mathcal{O}$. *Moreover, these matrices preserve a non-degenerate* $\mathcal{O}$*-valued hermitian form.*

*The basis of* $\mathcal{S}_p^\sharp(\Sigma)$. Recall that $\mathcal{S}_p^\sharp(\Sigma) \subset V_p(\Sigma)$ is the dual lattice to $\mathcal{S}_p(\Sigma)$ with respect to the $\mathcal{O}$-valued hermitian form $(\ ,\ )_\Sigma$. (We will review the definition of this form in Section 8.) Our basis of $\mathcal{S}_p^\sharp(\Sigma)$ will be denoted by $\mathcal{B}^\sharp$. Using the algebra/module structure discussed above, we define the elements of $\mathcal{B}^\sharp$ as follows:

$$\mathfrak{b}^\sharp(a,b,c) = h^{-\sum_i b_i - \lceil \frac{1}{2}(e+\sum_i a_i)\rceil}(2+z_1)^{b_1}\cdots(2+z_g)^{b_g}\,\mathfrak{g}(a,0,c). \tag{7}$$

as $(a,b,c)$ vary over the same index set. Note that the only difference between Equations (6) and (7) is that the trunk half-color $e$ appears with the opposite sign, and $\lfloor\ \rfloor$ is replaced with $\lceil\ \rceil$. Thus $\mathfrak{b}^\sharp(a,b,c)$ is a rescaling of $\mathfrak{b}(a,b,c)$.

THEOREM 4.3. – $\mathcal{B}^\sharp$ *is a basis of* $\mathcal{S}_p^\sharp(\Sigma)$.

The following sections until Section 9 will be devoted to the proofs of Theorems 4.1 and 4.3.

## 5. The 3-ball lemma

Let $K_\mathcal{O}(D^3, \ell_{n,2})$ denote the skein module of $D^3$ relative $n$ points in $S^2 = \partial D^3$ colored 2. Here, the notation $K_\mathcal{O}$ means that we consider skein modules with coefficients in $\mathcal{O}$, not in $\mathcal{O}[\frac{1}{p}]$. (But the result is actually more general; see the remark at the end of this section.) The following result will be needed in proving that the vectors $\mathfrak{b}(a,b,c)$ lie in $\mathcal{S}_p(\Sigma)$.

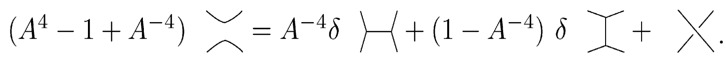
THEOREM 5.1 (3-ball lemma). – *If* $n$ *is even,* $K_\mathcal{O}(D^3, \ell_{n,2})$ *is generated by skein elements which can be represented by a collection of* $n/2$ *disjoint arcs colored* 2. *If* $n$ *is odd,* $K_\mathcal{O}(D^3, \ell_{n,2})$ *is generated by skein elements which can be represented by a collection of* $(n-3)/2$ *disjoint arcs colored* 2 *union one Y shaped component also colored* 2.

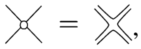
Recall that the module $K_\mathcal{O}(D^3, \ell_{n,2})$ is generated by colored graphs which meet the boundary nicely in the given $n$ points colored 2. We remark here that the required Jones–Wenzl idempotents are all defined over $\mathcal{O}$, because the only denominators needed in their definition are the quantum integers $[n]$ ($n \leqslant p-2$) which are invertible in $\mathcal{O}$.

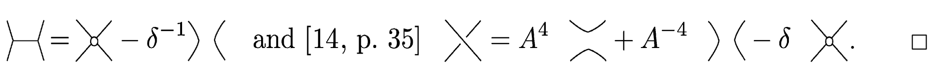
The proof of Theorem 5.1 proceeds by a series of lemmas. The first is an exercise in skein theory whose proof is left to the reader (represent elements of $K_\mathcal{O}(D^3, \ell_{n,2})$ by diagrams in a disk and apply the usual fusion formulas [14,20]).

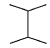
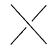
LEMMA 5.2. – $K_\mathcal{O}(D^3, \ell_{n,2})$ *is generated by unions of tree graphs where all edges are colored* 2.

In the remainder of this section, unless otherwise stated, unlabeled arcs in figures are assumed to be colored 2. As usual, we put $\delta = -A^2 - A^{-2}$. The following two lemmas are simple skein-theoretical calculations.

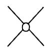
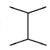
LEMMA 5.3. – *One has* 

*Proof.* – We use the abbreviation: , where the arcs in the right-hand diagram are colored 1. To prove 5.3, it is enough to expand the right-hand side using



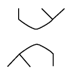
LEMMA 5.4. – *One has*  $= (A^4 - 1)\delta^{-1}$  $+ A^{-4}\delta^{-1}$  $- \delta^{-1}$ .

*Proof.* – Rewrite the right-hand side and simplify using the equations in the proof of Lemma 5.3:

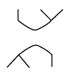
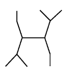
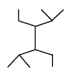
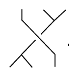
$\delta^{-1}\Big(A^4$  $+ A^{-4}$  $-$ $\Big) - \delta^{-1}$  $=$  $- \delta^{-1}$  $=$ . □

We refer to the diagram on the left-hand side of Lemma 5.4 as an I-bar. Using this lemma repeatedly to expand I-bars in tree graphs colored 2, we see from Lemma 5.2 that $K_{\mathcal{O}}(D^3, \ell_{n,2})$ is generated by skein elements which can be represented as disjoint unions of arcs colored 2 and $Y$-shaped graphs colored 2. Here we use that $\delta = -[2]$ is invertible in $\mathcal{O}$.

The crucial step is now the following lemma which shows how to replace two $Y$-shaped graphs colored 2 by three arcs; clearly this will be enough to complete the proof of Theorem 5.1.

LEMMA 5.5. – *The element*  *of* $K_{\mathcal{O}}(D^3, \ell_{6,2})$ *is an* $\mathcal{O}$*-linear combination of diagrams consisting of three arcs colored* 2.

*Proof.* – Extending all the diagrams in Lemma 5.3 by the same wiring, we obtain:

$(A^4 - 1 + A^{-4})$  $= A^{-4}\delta$  $+ (1 - A^{-4})\delta$  $+$ .

Using Lemma 5.4 to expand two I-bars in the first diagram on the right-hand side, two I-bars in the second diagram on the right-hand side, and one I-bar in the third diagram on the right-hand side, we see that the right-hand side is an $\mathcal{O}$-linear combination of diagrams consisting of 3 arcs colored 2. But $A^4 - 1 + A^{-4}$ (which, up to units, is the sixth cyclotomic polynomial in $A^4$) is easily seen to be invertible in $\mathcal{O}$. This proves the lemma, and completes the proof of Theorem 5.1. □

*Remark* 5.6. – We never used that $A$ is a root of unity in this proof. Thus the result also holds for the skein module with coefficients in $\mathbb{Q}(A)$, the ring of rational functions in $A$. In fact, it would be enough to work with the subring $\mathbb{Z}[A, A^{-1}]$ with inverses of the relevant quantum integers and of $A^4 - 1 + A^{-4}$ adjoined to it.

## 6. Proof that $\mathfrak{b}(a, b, c) \in \mathcal{S}_p(\Sigma)$

We fix some notation and terminology. Let $\Sigma_g$ denote the boundary of the handlebody $H_g$ with no colored points. The colored graph that represents $\mathfrak{g}((1,1),(0,0)) \in \mathcal{S}_p(\Sigma_2)$ is called an *eyeglass* and the colored graph that represents $\mathfrak{g}((1,1,1),(0,0,0)) \in \mathcal{S}_p(\Sigma_3)$ is called a *tripod*. See Fig. 2.

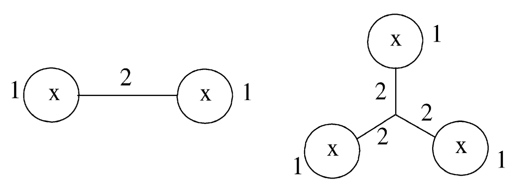



Fig. 2. An eyeglass and a tripod.

LEMMA 6.1. – *Eyeglasses and tripods are divisible by $h$ in $\mathcal{S}_p$.*

Since by definition $\mathfrak{b}((1,1),(0,0)) = h^{-1}\mathfrak{g}((1,1),(0,0))$ and $\mathfrak{b}((1,1,1),(0,0,0)) = h^{-1}\mathfrak{g}((1,1,1),(0,0,0))$, it follows that $\mathfrak{b}((1,1),(0,0)) \in \mathcal{S}_p(\Sigma_2)$ and $\mathfrak{b}((1,1,1),(0,0,0)) \in \mathcal{S}_p(\Sigma_3)$.

*Proof.* – Recall $z_i$ denotes a simple loop which encloses the $i$-th hole. Let $z_{i,j}$ denote a simple loop which encloses just the $i$-th and $j$-th hole, etc. If we want these curves colored $v$, we just change the $z$ to a $v$. A scalar denotes this scalar times the empty link.

If we expand the idempotent $f_2$ in $\mathfrak{g}((1,1),(0,0))$, we get $z_{1,2} + [2]^{-1}z_1z_2$. Making the substitution $z = hv - 2$, we get

$$\mathfrak{g}\big((1,1),(0,0)\big) = hv_{1,2} + [2]^{-1}\big(h^2v_1v_2 - 2hv_1 - 2hv_2 - 2\zeta_p^{-1}h^2\big).$$

(We have used $2 - [2] = 2 - \zeta_p - \zeta_p^{-1} = -\zeta_p^{-1}h^2$.) As $[2]$ is a unit of $\mathcal{O}$ and $v$-graphs are in $\mathcal{S}_p$ (Proposition 2.2), this shows $\mathfrak{g}((1,1),(0,0))$ is divisible by $h$ in $\mathcal{S}_p(\Sigma_2)$. The divisibility of tripods is proved in the same way. □

Now consider again an arbitrary connected surface $\Sigma$, possibly with colored points $\ell(\Sigma)$. Fix a lollipop tree $G$ and consider the elements $\mathfrak{b}(a,b,c)$ defined in Section 4.

PROPOSITION 6.2. – *One has* $\mathfrak{b}(a,b,c) \in \mathcal{S}_p(\Sigma)$.

*Proof.* – As already observed in Section 4, it is enough to show this for $b = 0$, since $\mathfrak{b}(a,b,c)$ is obtained from $\mathfrak{b}(a,0,c)$ by multiplying it by some $v$-colored curves. In other words, we need to show that $\mathfrak{g}(a,0,c)$ is divisible by $h^{\lfloor\frac{1}{2}(-e+\Sigma_i a_i)\rfloor}$ in $\mathcal{S}_p(\Sigma)$.

The colored graph representing $\mathfrak{g}(a,0,c)$ contains $g$ lollipops, where we mean by lollipop a subgraph consisting of a colored loop joined to a stick at a single point. We now perform a local change at the $g$ lollipops and obtain a new skein element $w(a,0,c)$, as is done in Fig. 3. To do this, we can view the $i$-th lollipop in $\mathfrak{g}(a,0,c)$ as $a_i$ arcs starting and ending at the idempotent $f_{2a_i}$ and looping around the $i$-th hole. Here we refer to the usual device of representing an idempotent by a rectangle. Specifically, the $j$-th arc connects the $j$-th and the $(2a_i - j + 1)$-th input of the idempotent. The modification from $\mathfrak{g}(a,0,c)$ to $w(a,0,c)$ consists of inserting a braid so that the $a_i$ arcs now connect consecutive inputs on the $f_{2a_i}$. By a well-known property of the idempotent $f_{2a_i}$, this changes a given skein element only by multiplying it by some power of $A$. Hence it is enough to show that $w(a,0,c)$ is divisible by the above-mentioned power of $h$.

We can draw $w(a,0,c)$ slightly differently: we also insert $a_i$ idempotents $f_2$ (as is also done in Fig. 3). This does not change the skein element at all (the $f_2$'s are "absorbed" by the $f_{2a_i}$ by another well-known property of the idempotent). We refer to this last operation as *spawning off*

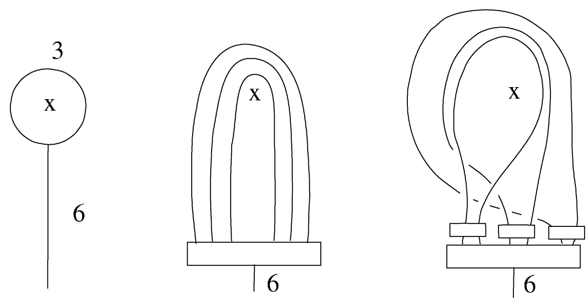



Fig. 3. A lollipop in $\mathfrak{g}(a,0,c)$, its expansion as a skein diagram, and the corresponding portion of $w(a,0,c)$.

$f_2$'s from $f_{2a_i}$. If $\ell(\Sigma)$ is nonempty, we also spawn off $e$ idempotents $f_2$ from the $f_{2e}$ on the trunk in the diagram for $w(a,0,c)$.

There is a 3-ball in $H_g$ whose boundary intersects $w(a,0,c)$ in $e+\sum_i a_i$ points colored 2 corresponding to the idempotents we spawned off. By the 3-ball Lemma 5.1, we can replace this part of the diagram with a linear combination over $\mathcal{O}$ of diagrams each with $\lfloor \frac{1}{2}(e+\sum_i a_i)\rfloor$ arcs and Y's colored two. At most $e$ of these arcs and Y's meet the trunk. Thus the rest of the arcs and Y's are completed to eyeglasses and tripods in the larger diagram. Thus $w(a,0,c)$ is also represented as a linear combination over $\mathcal{O}$ of diagrams with

$$\left\lfloor \frac{1}{2}\left(e+\sum_i a_i\right)\right\rfloor - e = \left\lfloor \frac{1}{2}\left(-e+\sum_i a_i\right)\right\rfloor$$

eyeglasses and tripods. As each eyeglass and tripod is divisible by $h$ this gives exactly the required divisibility of $w(a,0,c)$. This completes the proof. □

## 7. The lollipop lemma

The lollipop lemma (Theorem 7.1 below) will be used to show that the elements $\mathfrak{b}^\sharp(a,b,c)$ lie in $\mathcal{S}_p^\sharp(\Sigma)$. Recall that a $v$-graph in a 3-manifold is a usual colored graph together with some banded link components colored $v$, where $v=h^{-1}(2+z)$ (see Section 2). As before, a subgraph in a $v$-graph consisting of a colored loop meeting an edge (called the stick) is called a lollipop. A stick can take part in two lollipops. The color of the loop edge is at least one half of the stick color, but is allowed to be greater than that. Note that we have imposed no condition on how the loop edge of a lollipop is embedded into the ambient manifold.

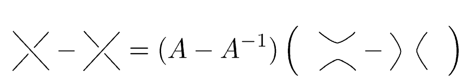
THEOREM 7.1 (Lollipop lemma). – *Let $L$ be a $v$-graph in $S^3$ containing $N$ lollipops with stick colors $2a_1, 2a_2, \ldots, 2a_N$, then its evaluation $\langle L\rangle$ is divisible by $h^{\lceil \frac{1}{2}\Sigma_{i=1}^N a_i\rceil}$.*

Here, the evaluation $\langle L\rangle$ of a skein element $L$ in $S^3$ is defined to be the ordinary Kauffman bracket. Thus the empty link evaluates to 1 and a zero-framed unknotted loop colored one evaluates to $-\zeta_p-\zeta_p^{-1}$, for example. As shown in [10], the evaluation of a $v$-graph in $S^3$ lies in $\mathcal{O}$.

Let us first prove a special case. A *basic lollipop* is a lollipop where the stick is colored 2 and the loop edge is colored 1.

LEMMA 7.2. – *The evaluation of a $v$-graph in $S^3$ with $N$ basic lollipops is divisible by $h^{\lceil N/2\rceil}$ in $\mathcal{O}$.*

*Proof.* – If we have a basic lollipop whose loop spans a disk which misses the rest of the $v$-graph $L$ then $\langle L\rangle = 0$, by a well-known property of the Jones–Wenzl idempotents. More generally, if $L$ intersects a 3-ball only in a basic lollipop, then $\langle L\rangle=0$. Let us show that we may reduce to this case by changing crossings using the relation



and all the error terms are divisible by $h^{\lceil N/2\rceil}$. (In this figure, strands are ordinary strands, i.e. colored 1.)

By expanding all idempotents except one at the stick of each lollipop, we may assume that all arcs of the graph are colored 1. When we change a crossing between a $v$-colored link component and a strand colored 1, the skein relation shows that the error term is given by $h^{-1}(A - A^{-1}) = -A^{-1}$ times the difference of two evaluations of $v$-graphs, each of which satisfies again the hypothesis of the lemma but has one less $v$-colored link component. By induction it follows that we can assume that all $v$-colored link components are unlinked from the rest of the graph. Since a $v$-colored link evaluates to something integral, this shows that we may assume that there are no $v$-colored link components.

Next if we change crossings between a loop colored 1 of a basic lollipop and any arc colored 1, the error term is $-A^{-1}h$ times a $v$-graph with at least $N-2$ basic lollipops. By induction on the number of lollipops the error term satisfies the conclusion of the lemma. Hence we may reduce to the case where one basic lollipop is not linked with anything. □

*Remark* 7.3. – In a similar way, we can give a new proof of the fact that the evaluation of a $v$-graph lies in $\mathcal{O}$. Namely, one checks that it is true for $v$-colored unlinks and then reduces to this case by changing crossings, observing that error terms always lie in $\mathcal{O}$.

*Proof of Theorem 7.1.* – We may expand the loop idempotents over $\mathcal{O}$ into terms where each stick idempotent has $a_i$ arcs that meet it at $2a_i$ points along an edge of the idempotent. Here we refer again to the device of representing an idempotent by a rectangle. As in the previous section, we insert a braid in the strands that meet this edge so that the arcs now join points to their immediate neighbors. This only changes the evaluation by a power of $A$. Then without changing the evaluation one can spawn from each $2a_i$-stranded idempotent $a_i$ 2-stranded idempotents. Now each term is a $v$-graph with $\sum_{i=1}^{N} a_i$ basic lollipops so the result follows from Lemma 7.2. □

## 8. Proof that $\mathfrak{b}^\sharp(a,b,c) \in \mathcal{S}_p^\sharp(\Sigma)$

In this section, we use the Lollipop Lemma to show that the elements $\mathfrak{b}^\sharp(a,b,c)$ defined in Section 4 lie in $\mathcal{S}_p^\sharp(\Sigma)$. Let us first review the definition of this module in more detail.

The hermitian form $(\ ,\ )_\Sigma$ is simply a rescaling of the canonical hermitian form $\langle\ ,\ \rangle_\Sigma$ on $V_p(\Sigma)$ (see Section 2):

$$(x,y)_\Sigma = \mathcal{D}^{\beta_0(\Sigma)}\langle x,y\rangle_\Sigma.$$

Here $\beta_0(\Sigma)$ is the number of components of $\Sigma$, and $\mathcal{D}$ is defined in Equation (2) in Section 1. In fact, $\mathcal{D}$ is the inverse of the quantum invariant $\langle S^3\rangle_p$.

When restricted to $\mathcal{S}_p(\Sigma) \subset V_p(\Sigma)$, the form $(\ ,\ )_\Sigma$ takes values in $\mathcal{O}$. This follows from the definition of $\mathcal{S}_p(\Sigma)$ and the integrality result for quantum invariants of closed connected manifolds [22,19]. (See [8,10] for more details.)

DEFINITION 8.1. – The lattice $\mathcal{S}_p^\sharp(\Sigma)$ is the dual lattice to $\mathcal{S}_p(\Sigma)$ with respect to the hermitian form $(\ ,\ )_\Sigma$.

Now assume $\Sigma$ is connected. If $M$ has boundary $\Sigma$, and $M$ is also connected, let $K_v(M,\ell(\Sigma))$ denote the $\mathcal{O}$-submodule of $K(M,\ell(\Sigma))$ spanned by $v$-graphs in $M$. Proposition 2.2 is equivalent to saying that for connected $M$ and $\Sigma$, the natural map

$$K_v\big(M,\ell(\Sigma)\big) \to \mathcal{S}_p(\Sigma)$$

is surjective.

*Remark* 8.2. – The inclusions $K_{\mathcal{O}}(M,\ell(\Sigma)) \subset K_v(M,\ell(\Sigma)) \subset K(M,\ell(\Sigma))$ are strict in general.

The following Proposition 8.3 is a useful device for describing $\mathcal{S}_p^\sharp(\Sigma)$. Let $H$ and $H'$ be two complementary handlebodies in $S^3$ with $\partial H = \Sigma = -\partial H'$. We define a bilinear form

$$((\ ,\ ))_{H,H'}: K\big(H,\ell(\Sigma)\big) \times K\big(H',-\ell(\Sigma)\big) \longrightarrow \mathcal{O}\left[\frac{1}{h}\right]$$

by

$$((L,L'))_{H,H'} = \langle L \cup_{\ell(\Sigma)} L' \rangle.$$

Here $\langle\ \rangle$ is the usual Kauffman bracket of a colored graph in $S^3$. When restricted to $K_v(H,\ell(\Sigma)) \times K_v(H',-\ell(\Sigma))$, this form takes values in $\mathcal{O}$.

PROPOSITION 8.3. – *A skein element $x \in K(H,\ell(\Sigma))$ represents an element of $\mathcal{S}_p^\sharp(\Sigma)$ if and only if $((x,x'))_{H,H'} \in \mathcal{O}$ for all $x' \in K_v(H',-\ell(\Sigma))$.*

*Proof.* – The proof is essentially a standard argument in the skein-theoretical approach to TQFT's. The skein element $x$ defines the vector $[(H,x)] \in V_p(\Sigma)$. The hermitian form is given by

$$\big([(H,x)],[(H,y)]\big)_\Sigma = \mathcal{D}\big\langle (H \cup_\Sigma -H, x \cup_{\ell(\Sigma)} y^\star)\big\rangle_p, \tag{8}$$

where $y^\star$ denotes the skein element in $-H$ obtained from $y$ by reversing orientation. This can also be viewed as a bilinear pairing of $[(H,x)] \in V_p(\Sigma)$ with $[(-H,y^\star)] \in V_p(-\Sigma)$. But $[(-H,y^\star)]$ can also be represented by some skein element $y' \in K(H',-\ell(\Sigma))$. Thus the hermitian pairing (8) is equal to

$$\mathcal{D}\big\langle (H \cup_\Sigma H', x \cup_{\ell(\Sigma)} y')\big\rangle_p = \langle x \cup_{\ell(\Sigma)} y'\rangle = \big((x,y')\big)_{H,H'}$$

since $H \cup_\Sigma H' = S^3$. Now

$$\big[(H,y)\big] \in \mathcal{S}_p(\Sigma) \quad \Longleftrightarrow \quad \big[(-H,y^\star)\big] \in \mathcal{S}_p(-\Sigma) \quad \Longleftrightarrow \quad y' \in K_v\big(H',-\ell(\Sigma)\big).$$

Thus one has $[(H,x)] \in \mathcal{S}_p^\sharp(\Sigma)$ if and only if $((x,y'))_{H,H'} \in \mathcal{O}$ for all $y' \in K_v(H',-\ell(\Sigma))$. $\square$

We are now ready to prove the following.

PROPOSITION 8.4. – *The elements $\mathfrak{b}^\sharp(a,b,c)$ lie in $\mathcal{S}_p^\sharp(\Sigma)$.*

*Proof.* – Embed the handlebody $H_g$ into $S^3$ so that its exterior is also a handlebody $H_g'$. By Proposition 8.3, we only need show that if $\mathfrak{b}^\sharp(a,b,c)$ in $H_g$ is completed by any $v$-graph in $H_g'$, then the evaluation of the result lies in $\mathcal{O}$. However we may isotope the $v$-colored curves $v_i^{b_i}$ $(i=1,\ldots,g)$ in $\mathfrak{b}^\sharp(a,b,c)$ across $\Sigma$. Thus we only need to prove this statement in the case $b=0$. In other words, we only need to show that if $\mathfrak{g}(a,0,c)$ is completed by any $v$-graph in the complementary handlebody then the evaluation of the result is divisible by $h^{\lceil \frac{1}{2}(e+\Sigma_i a_i)\rceil}$ in $\mathcal{O}$.

If $e=0$, this follows immediately from the Lollipop Lemma 7.1, as in any completion of $\mathfrak{g}(a,0,c)$ we have the $g$ lollipops of $\mathfrak{g}(a,0,c)$ with the sum of the stick half-colors equal to $\sum_i a_i$.

If $e>0$, then in any completion of $\mathfrak{g}(a,0,c)$ we first modify the part in $H_g'$ which is glued to the trunk edge of $\mathfrak{g}(a,0,c)$ in a now familiar way. To do this, we represent the idempotent $f_{2e}$ on the trunk edge diagrammatically by the usual rectangle and expand all the idempotents in the glued-on part of the graph into strands colored 1. In every term of this expansion, we have $e$ arcs that start and end at the “bottom” of this rectangle. Inserting an appropriate braid we can arrange that the arcs now join consecutive points on the "bottom" of the rectangle, and then we spawn off $e$ idempotents $f_2$. As before, we can compensate for these changes by changing the coefficients in the expansion over $\mathcal{O}$ by some powers of $A$. In each term, we now see $e$ basic lollipops with stick color 2 below the trunk edge. Using these and the $g$ lollipops in $\mathfrak{g}(a,0,c)$ itself, we see that any completion of $\mathfrak{g}(a,0,c)$ can be expanded over $\mathcal{O}$ as a linear combination of $v$-graphs containing lollipops with the sum of the stick half-colors equal to $e+\sum_i a_i$. By the Lollipop Lemma 7.1 we are done. □

## 9. Index counting

Let $\mathbb{B}$ be the $\mathcal{O}$-lattice in $V_p(\Sigma)$ spanned by the $\mathfrak{b}(a,b,c)$. Let $\mathbb{B}'$ be the $\mathcal{O}$-lattice spanned by the $\mathfrak{b}^\sharp(a,b,c)$.[8] We know $\mathbb{B}\subset\mathcal{S}_p(\Sigma)$ and $\mathbb{B}'\subset\mathcal{S}_p^\sharp(\Sigma)$ by Propositions 6.2 and 8.4. In this section, we will use order ideals to show that both inclusions are equalities (Theorem 9.3). This will prove Theorems 4.1 and 4.3. We will sometimes abbreviate $\mathcal{S}_p(\Sigma)$ by $\mathcal{S}$ and $\mathcal{S}_p^\sharp(\Sigma)$ by $\mathcal{S}^\sharp$.

Let $\mathbb{G}$ denote the $\mathcal{O}$-lattice spanned by the small graph basis $\mathcal{G}$ of $V_p(\Sigma)$, i.e. the elements $\mathfrak{g}(a,b,c)$. One has $\mathbb{G}\subset\mathbb{B}\subset\mathbb{B}'$ and all three of these lattices are free.

If $L\subset L'$ is an inclusion of free lattices of the same rank over $\mathcal{O}$, define their *index* $[L':L]$ to be the determinant up to units in $\mathcal{O}$ of a matrix representing a basis for $L$ in terms of a basis for $L'$.

PROPOSITION 9.1. – $[\mathbb{B}:\mathbb{G}][\mathbb{B}':\mathbb{G}]=[\mathbb{G}^\sharp:\mathbb{G}]$.

*Proof.* – By [10, 5.4], we have

$$[\mathbb{G}^\sharp:\mathbb{G}]=h^{g(d-1)\dim(V_p(\Sigma))}.$$

Actually [10, 5.4] only deals with the case $\ell(\Sigma)=\emptyset$, but this equation holds in general by the same argument. By construction: $[\mathbb{B}:\mathbb{G}]=h^N$ where $N=\sum_{(a,b,c)}(\sum_i b_i+\lfloor\frac12(-e+\sum_i a_i)\rfloor)$. Also by construction: $[\mathbb{B}':\mathbb{G}]=h^{N'}$ where $N'=\sum_{(a,b,c)}(\sum_i b_i+\lceil\frac12(e+\sum_i a_i)\rceil)$. Of course, $e$ depends on $(a,b,c)$ in these expressions. Luckily, $e$ drops out when we look at $N+N'$.

We need to show $N+N'=g(d-1)\dim(V_p(\Sigma))$, or

$$\sum_{(a,b,c)}\left(2\sum_i b_i+\sum_i a_i\right)=g(d-1)\sum_{(a,b,c)}1.$$

It suffices to prove this with $a$ and $c$ held constant:

$$\sum_b\left(2\sum_{i=1}^g b_i+\sum_{i=1}^g a_i\right)=g(d-1)\sum_b 1.$$

Recall that the set of $b$ being summed over is the set of $(b_1,\ldots,b_g)$ such that $0\leqslant b_i\leqslant d-1-a_i$, and the cardinality of this set is $\prod_{j=1}^g(d-a_j)$. So it suffices to show:

$$2\sum_b\sum_{i=1}^g b_i+\left(\sum_{i=1}^g a_i\right)\prod_{j=1}^g(d-a_j)=g(d-1)\prod_{j=1}^g(d-a_j).$$

[8] We don't use the notation $\mathbb{B}^\sharp$ because it is not clear a priori that $\mathbb{B}'$ is the dual lattice of $\mathbb{B}$.

Noting that $g(d-1)=\sum_{i=1}^{g}(d-1)$, we only need to show

$$2\sum_{b}\sum_{i=1}^{g} b_i=\sum_{i=1}^{g}(d-1-a_i)\prod_{j=1}^{g}(d-a_j).$$

This equation expresses the fact that the average value of $\sum_{i=1}^{g} b_i$ over the index of summation $b$ is $\frac{1}{2}\sum_i(d-1-a_i)$. To see this, we consider the involution on the index set $\{b\}$ defined by sending $b_i$ to $d-1-a_i-b_i$ for each $i$. On each orbit the average value of $\sum_{i=1}^{g} b_i$ is $\frac{1}{2}\sum_i(d-1-a_i)$. This establishes the identity. $\square$

If $M$ is a finitely generated torsion module over $\mathcal{O}$, we denote its order ideal by $\mathfrak{o}(M)$. If $L\subset L'$ is an inclusion of free lattices of the same rank over $\mathcal{O}$, then $\mathfrak{o}(L'/L)$ is the principal ideal in $\mathcal{O}$ generated by $[L':L]$.

We will need the following result (here $\overline{M}$ denotes the conjugate module of $M$).

PROPOSITION 9.2. – $\mathbb{G}^\sharp/\mathcal{S}^\sharp\cong\overline{\mathcal{S}/\mathbb{G}}$.

Proposition 9.2 is a general fact about inclusions of (not necessarily free) lattices of the same rank equipped with a non-singular hermitian form over a Dedekind domain. We were unable to find this fact stated in the literature, but it is not hard to deduce it from [24, Theorem 4.12]. We omit the details.

The proof of Theorems 4.1 and 4.3 is completed with the following

THEOREM 9.3. – *$\mathcal{S}_p(\Sigma)=\mathbb{B}$ and $\mathcal{S}_p^\sharp(\Sigma)=\mathbb{B}'$. In particular $\mathcal{S}_p(\Sigma)$ and $\mathcal{S}_p^\sharp(\Sigma)$ are free.*

*Proof.* – As $\mathbb{G}\subset\mathbb{B}\subset\mathcal{S}$, we have that

$$\mathfrak{o}(\mathcal{S}/\mathbb{G})=\mathfrak{o}(\mathcal{S}/\mathbb{B})\mathfrak{o}(\mathbb{B}/\mathbb{G}). \tag{9}$$

Similarly $\mathbb{G}\subset\mathbb{B}'\subset\mathcal{S}^\sharp$ gives us:

$$\mathfrak{o}(\mathcal{S}^\sharp/\mathbb{G})=\mathfrak{o}(\mathcal{S}^\sharp/\mathbb{B}')\mathfrak{o}(\mathbb{B}'/\mathbb{G}). \tag{10}$$

$$\begin{aligned}
\mathfrak{o}(\mathbb{G}^\sharp/\mathbb{G}) &= \mathfrak{o}(\mathbb{G}^\sharp/\mathcal{S}^\sharp)\mathfrak{o}(\mathcal{S}^\sharp/\mathbb{G})\\
&= \overline{\mathfrak{o}(\mathcal{S}/\mathbb{G})}\mathfrak{o}(\mathcal{S}^\sharp/\mathbb{G}) \quad \text{by Proposition 9.2}\\
&= \overline{\mathfrak{o}(\mathcal{S}/\mathbb{B})\mathfrak{o}(\mathbb{B}/\mathbb{G})}\mathfrak{o}(\mathcal{S}^\sharp/\mathbb{B}')\mathfrak{o}(\mathbb{B}'/\mathbb{G}) \quad \text{by Equations (9) and (10)}\\
&= \overline{\mathfrak{o}(\mathcal{S}/\mathbb{B})}\mathfrak{o}(\mathbb{B}/\mathbb{G})\mathfrak{o}(\mathcal{S}^\sharp/\mathbb{B}')\mathfrak{o}(\mathbb{B}'/\mathbb{G}) \quad \text{as } h \text{ is self-conjugate up to units}\\
&= \overline{\mathfrak{o}(\mathcal{S}/\mathbb{B})}\mathfrak{o}(\mathcal{S}^\sharp/\mathbb{B}')\mathfrak{o}(\mathbb{G}^\sharp/\mathbb{G}) \quad \text{by Proposition 9.1.}
\end{aligned}$$

Thus $\mathfrak{o}(\mathcal{S}/\mathbb{B})=\mathfrak{o}(\mathcal{S}^\sharp/\mathbb{B}')=1$, hence $\mathcal{S}=\mathbb{B}$ and $\mathcal{S}^\sharp=\mathbb{B}'$. $\square$

## 10. The necessity of a trunk

In this section we give an example which explains why we have insisted that lollipop trees have trunks. Let $\Sigma$ be the boundary of a regular neighborhood of a lollipop tree $G$. A graph-like basis for $G$ is a basis for $\mathcal{S}_p(\Sigma)$ that is obtained from the small graph basis for $V_p(\Sigma)$ by "peeling" off any excess $b$ from each loop color of a small basis vector, inserting a compensating $(z+2)^b$ and then rescaling the resulting elements by some factor.


Fig. 4. The graphs $G$ and $G'$.


Fig. 5. The basis elements $\mathfrak{g}_0$, $\mathfrak{g}_1$, $\mathfrak{g}_0'$ and $\mathfrak{g}_1'$.

Now let $\Sigma$ denote a surface of genus two with two banded points colored 2. Consider the graphs pictured in Fig. 4.

While $G$ is a lollipop tree, $G'$ is not, as it does not have a trunk. Still, admissible colorings with small colors on $G'$ also yield a basis for $V_p(\Sigma)$, and the notion of graph-like basis makes sense for $G'$. However, we have the following

THEOREM 10.1. – $\mathcal{S}_5(\Sigma)$ *does not have a graph-like basis associated to the graph* $G'$.

*Proof.* – Fig. 5 illustrates two basis elements from $G$ and two from $G'$. In this figure, the loops are all colored 1 and the edges all colored 2.

Using the fact that $\mathcal{S}_p$ of a 2-sphere with four points colored 2 is two-dimensional, one can see that

$$\mathfrak{g}_0 = a\mathfrak{g}_0' + b\mathfrak{g}_1', \tag{11}$$

$$\mathfrak{g}_1 = \alpha\mathfrak{g}_0' + \beta\mathfrak{g}_1', \tag{12}$$

for some $a, b, \alpha, \beta \in \mathcal{O}$. The exact values of these are not important, but we will need to use that $a, b \in \mathcal{O}^\star$ which is easily checked.

As $G$ is a lollipop tree, we know that

$$h^{-1}\mathfrak{g}_0 \in \mathcal{S}_5(\Sigma) \quad \text{and} \quad h^{-1}\mathfrak{g}_1 \notin \mathcal{S}_5(\Sigma). \tag{13}$$

Let us try to express $h^{-1}\mathfrak{g}_0$ in terms of a graph-like basis for $G'$, assuming such a basis exists. We can modify the elements $\mathfrak{g}_0'$ and $\mathfrak{g}_1'$ only by a rescaling, as no peeling off is possible for these elements. Hence the required expression for $h^{-1}\mathfrak{g}_0$ would be

$$h^{-1}\mathfrak{g}_0 = ah^{-1}\mathfrak{g}_0' + bh^{-1}\mathfrak{g}_1'.$$

Since $a$ and $b$ are units of $\mathcal{O}$, both $h^{-1}\mathfrak{g}_0'$ and $h^{-1}\mathfrak{g}_1'$ would have to exist in $\mathcal{S}_5(\Sigma)$. But then Equation (12) implies that $h^{-1}\mathfrak{g}_1$ would also exist in $\mathcal{S}_5(\Sigma)$, which contradicts the second half of statement (13). Thus $\mathcal{S}_5(\Sigma)$ does not have a graph-like basis for $G'$. $\square$

## 11. Integral modular functor

The collection of $\mathcal{O}[\frac{1}{p}]$-modules $V_p(\Sigma)$ associated to surfaces with colored points form a modular functor (see e.g. [28]). This means that the $V_p(\Sigma)$ satisfy certain axioms describing their behaviour when surfaces are cut into pieces in various ways. These axioms reflect the semi-simplicity of the underlying modular category [28].

The integral lattices $\mathcal{S}_p(\Sigma)$ should be a guiding example towards the notion of an "integral modular functor". But their behaviour under such "cut and paste" operations is considerably more complicated. In some sense, the reason is that the integral theory is no longer semi-simple.

We hope to develop these ideas elsewhere. Here, we limit ourselves to describing one axiom in a particularly simple situation, where a rescaling of the usual modular functor axiom suffices.

Let $\gamma$ be a separating simple closed curve on the connected surface $\Sigma$. The curve $\gamma$ cuts $\Sigma$ into two subsurfaces $\Sigma'$ and $\Sigma''$. If $\Sigma$ has colored points, we assume $\gamma$ misses these points. For simplicity, we also assume that the sum of the colors on $\Sigma'$ (hence also on $\Sigma''$) is even. Let $\Sigma'_i$ and $\Sigma''_i$ be the subsurfaces with their boundary capped off by a disk containing an additional $i$-colored banded point. Then [3, 1.14] the obvious gluing map along the disks induces an isomorphism

$$\bigoplus_{i=0}^{d-1} V_p(\Sigma'_{2i}) \otimes V_p(\Sigma''_{2i}) \xrightarrow{\approx} V_p(\Sigma). \tag{14}$$

This induces an injective map

$$\bigoplus_{i=0}^{d-1} \mathcal{S}_p(\Sigma'_{2i}) \otimes \mathcal{S}_p(\Sigma''_{2i}) \longrightarrow \mathcal{S}_p(\Sigma), \tag{15}$$

whose image is a free sublattice of $\mathcal{S}_p(\Sigma)$.

THEOREM 11.1. – *If at most one of $\Sigma'$ and $\Sigma''$ has colored points, then there exist bases $\mathcal{B}'_{2i}$ of $\mathcal{S}_p(\Sigma'_{2i})$, $\mathcal{B}''_{2i}$ of $\mathcal{S}_p(\Sigma''_{2i})$, and $\mathcal{B}$ of $\mathcal{S}_p(\Sigma)$, such that $\mathcal{B}$ is a rescaling of the tensor product basis $\bigsqcup_i \mathcal{B}'_{2i} \otimes \mathcal{B}''_{2i}$. (Here, $\bigsqcup_i \mathcal{B}'_{2i} \otimes \mathcal{B}''_{2i}$ is viewed as a basis of $V_p(\Sigma)$ via the map* (14).)

*More precisely, there are functions $\varphi_i : \mathcal{B}'_{2i} \times \mathcal{B}''_{2i} \to \mathbb{Z}_{\geqslant 0}$ such that*

$$\mathcal{B} = \bigsqcup_i \{ h^{-\varphi_i(\mathfrak{b}',\mathfrak{b}'')} \mathfrak{b}' \otimes \mathfrak{b}'' \mid \mathfrak{b}' \in \mathcal{B}'_{2i}, \mathfrak{b}'' \in \mathcal{B}''_{2i} \}. \tag{16}$$

*Proof.* – Write $\Sigma$ as the boundary of a handlebody $H$ such that the curve $\gamma$ bounds a disk $D$ in $H$. This disk cuts $H$ in handlebodies $H'$ and $H''$ with boundary $\Sigma' \cup D$ and $\Sigma'' \cup D$. We can find a graph-like basis $\mathcal{B}$ of $\mathcal{S}_p(\Sigma)$ with respect to a lollipop tree $G$ in $H$ such that $G$ meets the disk $D$ transversely in one edge, and such that, moreover, $G' = G \cap H'$ is a lollipop tree in $H'$, and $G'' = G \cap H''$ is a lollipop tree in $H''$. Here we use the hypothesis that at most one of $\Sigma'$ and $\Sigma''$ has colored points. Taking for $\mathcal{B}'_{2i}$ and $\mathcal{B}''_{2i}$ the graph-like bases associated to $G'$ and $G''$, with color $2i$ on the edge meeting the disk $D$, the result follows. $\square$

It is easy to write down an explicit formula for the rescaling factors $\varphi_i : \mathcal{B}'_{2i} \times \mathcal{B}''_{2i} \to \mathbb{Z}_{\geqslant 0}$ in this situation; this is left to the reader.

*Remark* 11.2. – (i) If both $\Sigma'$ and $\Sigma''$ have colored points, then in general there is no basis of $\mathcal{S}_p(\Sigma)$ which is just a rescaling of a tensor product basis as in (16). This follows from the example in the previous section.

(ii) If $\gamma$ is non-separating, the modular functor axiom also needs more modification than just a rescaling.

## 12. Disconnected surfaces and the tensor product axiom

Let $\Sigma$ and $\Sigma'$ be two closed surfaces. We have compatible natural maps $\mathcal{S}_p(\Sigma)\otimes\mathcal{S}_p(\Sigma')\to\mathcal{S}_p(\Sigma\sqcup\Sigma')$ and $V_p(\Sigma)\otimes V_p(\Sigma')\to V_p(\Sigma\sqcup\Sigma')$ specified by sending $[(M,L)]\otimes[(M',L')]$ to $[(M\sqcup M',L\sqcup L')]$ where $(M,L)$ and $(M',L')$ are 3-manifolds with colored links whose boundary is $\Sigma$ and $\Sigma'$, respectively.

The map $V_p(\Sigma)\otimes V_p(\Sigma')\to V_p(\Sigma\sqcup\Sigma')$ is an isomorphism [3], and this property is called the tensor product axiom. We are interested in the extent that $\mathcal{S}_p(\Sigma)\otimes\mathcal{S}_p(\Sigma')\to\mathcal{S}_p(\Sigma\sqcup\Sigma')$ is also an isomorphism. It follows from Corollary 12.3 below that this map is an isomorphism if $\Sigma$ and $\Sigma'$ have no colored points. If there are colored points, the image of this map may only be a sublattice of $\mathcal{S}_p(\Sigma\sqcup\Sigma')$. But we shall see that a basis of $\mathcal{S}_p(\Sigma\sqcup\Sigma')$ can always be obtained as a rescaling of a tensor product basis.

The following definition allows for a convenient expression of the needed rescaling. Let $\mathcal{B}$ be the basis of $\mathcal{S}_p(\Sigma)$, for a connected surface $\Sigma$, associated to some lollipop tree $G$ in a handlebody $H$. We define the *oddity* $\varepsilon(\mathfrak{b})$ of a basis element $\mathfrak{b}\in\mathcal{B}$ as follows.[9] Let $2a_1,\dots,2a_k$ be the colors of the stick edges, and let $2e$ be the trunk color of $\mathfrak{b}$. Let $A(\mathfrak{b})=\sum_i a_i$, and $e(\mathfrak{b})=e$. Define $\varepsilon(\mathfrak{b})=1$ if $e(\mathfrak{b})=d-1$ and $A(\mathfrak{b})-e(\mathfrak{b})$ is odd. Otherwise, $\varepsilon(\mathfrak{b})=0$.

THEOREM 12.1. – *Let $\Sigma_1,\dots,\Sigma_n$ be connected surfaces, and let $\mathcal{B}_1,\dots,\mathcal{B}_n$ be graph-like bases of $\mathcal{S}_p(\Sigma_1),\dots,\mathcal{S}_p(\Sigma_n)$. Then the set*

$$\mathcal{B}=\left\{h^{-\lfloor\frac{1}{2}\sum_i\varepsilon(\mathfrak{b}_i)\rfloor}\mathfrak{b}_1\otimes\cdots\otimes\mathfrak{b}_n\mid\mathfrak{b}_i\in\mathcal{B}_i\right\}\tag{17}$$

*is a basis of $\mathcal{S}_p(\bigsqcup_i\Sigma_i)$.*

*Remark* 12.2. – If $\Sigma$ is connected, the lattice $\mathcal{S}_p(\Sigma)$ has basis vectors with non-trivial oddity if and only if $\Sigma$ has genus at least two and the sum of the colors of the banded points on $\Sigma$ is at least $2(d-1)=p-3$. For example, the rightmost diagram in Fig. 7 represents an element with non-trivial oddity in $\mathcal{S}_5$ of a genus 2 surface with one colored point colored 2.

COROLLARY 12.3. – *The natural map $\mathcal{S}_p(\Sigma)\otimes\mathcal{S}_p(\Sigma')\to\mathcal{S}_p(\Sigma\sqcup\Sigma')$ is always injective with a cokernel isomorphic to a direct sum of cyclic modules $\mathcal{O}/h\mathcal{O}$. It is an isomorphism if one of the surfaces has no colored points.*

*Remark* 12.4. – In fact, the map $\mathcal{S}_p(\Sigma)\otimes\mathcal{S}_p(\Sigma')\to\mathcal{S}_p(\Sigma\sqcup\Sigma')$ has a non-trivial cokernel if and only if both surfaces have a connected component whose $\mathcal{S}_p$-lattice has basis vectors with non-trivial oddity. This follows from Theorem 12.1.

Since the tensor product axiom for $\mathcal{S}_p$ does not always hold for surfaces with colored points, it seems that any proof of the tensor product axiom for surfaces without colored points must ultimately depend on detailed knowledge of bases for $\mathcal{S}_p$ for connected surfaces.

For the proof of Theorem 12.1, we need to state the analog of Proposition 2.2 for disconnected surfaces. The proof is the same as in the connected case. A little terminology is convenient. Let $\pi_0(\Sigma)$ be the set of connected components of $\Sigma$. We say that a 3-manifold $M$ with boundary $\Sigma$ represents a partition $P$ of $\pi_0(\Sigma)$ if $P$ coincides with the partition given by the fibers of the natural map $\pi_0(\Sigma)\to\pi_0(M)$.

PROPOSITION 12.5. – *If $\Sigma$ is not connected, choose any collection $\mathcal{M}$ of 3-manifolds $M$ with boundary $\Sigma$ such that every partition of $\pi_0(\Sigma)$ is represented by some $M\in\mathcal{M}$. Then $\mathcal{S}_p(\Sigma)$ is generated over $\mathcal{O}$ by elements in $V_p(\Sigma)$ represented by $v$-graphs in the 3-manifolds $M\in\mathcal{M}$, where, as before, the $v$-graphs must meet the boundary in the colored points of $\Sigma$.*

[9] A different notion of parity will be defined in Section 14.

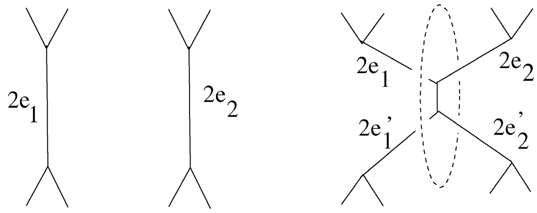



Fig. 6. Lollipop trees $G_1$ in $H_{g_1}$, $G_2$ in $H_{g_2}$ and $G$ in $H_{g_1}\#H_{g_2}$. We say $G$ is the grafting of $G_1$ and $G_2$ along their trunks (only the parts near the trunks are shown, and all loop edges are above the trunks in this figure).

*Proof of Theorem 12.1 in the case* $n=2$. – We suppose $\Sigma_i$ is the boundary of $H_{g_i}$. We wish to find a basis for $\mathcal{S}_p(\Sigma_1 \sqcup \Sigma_2)$ which we recall is defined as a subset of $V_p(\Sigma_1 \sqcup \Sigma_2)$ which we can identify with $V_p(\Sigma_1) \otimes V_p(\Sigma_2)$. Thus any element of $\mathcal{S}_p(\Sigma_1 \sqcup \Sigma_2)$ may be described as a linear combinations over $\mathcal{O}[\frac{1}{h}]$ of elements of the form $\mathfrak{b}_1 \otimes \mathfrak{b}_2$. Here $\mathfrak{b}_i$ is defined with respect to some lollipop tree $G_i$ in $H_{g_i}$.

Let $\#$ denote the operation of (interior) connected sum of connected 3-manifolds. $\Sigma_1 \sqcup \Sigma_2$ is the boundary of $H_{g_1}\#H_{g_2}$. According to Proposition 12.5, $\mathcal{S}_p(\Sigma_1 \sqcup \Sigma_2)$ is generated by $v$-graphs in $H_{g_1} \sqcup H_{g_2}$ and $v$-graphs in $H_{g_1}\#H_{g_2}$. The $v$-graphs in $H_{g_1} \sqcup H_{g_2}$ generate the subspace of $\mathcal{S}_p(\Sigma_1 \sqcup \Sigma_2)$ that has as basis the set of $\mathfrak{b}_1 \otimes \mathfrak{b}_2$ associated to small colorings of $G_1$ and $G_2$. We want to see if $v$-graphs in $H_{g_1}\#H_{g_2}$ can give any elements not in this subspace.

Let $\#_\partial$ denote the operation of boundary connected sum of connected 3-manifolds with connected boundaries. $H_{g_1}\#H_{g_2}$ is homeomorphic to $H_{g_1}\#_\partial H_{g_2}$ with a 2-handle attached along the curve which bounds the disk used to form the boundary connected sum. We indicate such curves in figures as dotted curves. Thus dotted curves always indicate that a 2-handle should be attached to a handlebody along the curve. The handlebodies are not actually drawn in our figures but are regular neighborhoods of the lollipop trees. We identify $H_{g_1}\#_\partial H_{g_2}$ with $H_{g_1+g_2}$.

A $v$-graph in $H_{g_1}\#H_{g_2}$ can be isotoped into $H_{g_1+g_2}$ where we know a graph-like basis for $\mathcal{S}_p(\partial(H_{g_1+g_2}))$ associated to the lollipop tree $G$ obtained by the grafting of $G_1$ and $G_2$ along their trunks. See Fig. 6. If $\mathfrak{b}$ is such a basis element of $\mathcal{S}_p(\partial(H_{g_1+g_2}))$, we let $\hat{\mathfrak{b}}$ denote the element that $\mathfrak{b}$ represents in $\mathcal{S}_p(\Sigma_1 \sqcup \Sigma_2)$. Our task is to compute $\hat{\mathfrak{b}}$.

Suppose the coloring of $G$ that gives $\mathfrak{b}$ is as shown on the right of Fig. 6. Note that the 2-sphere composed of the disk spanning the dotted curve and the core of the 2-handle which is attached along this dotted curve can be isotoped to intersect the colored lollipop tree in either two points colored $2e_1$ and $2e_1'$ or two points colored $2e_2$ and $2e_2'$. As $V_p$ of a 2-sphere with two points with distinct even colors is the zero module, we see that $\hat{\mathfrak{b}}$ is zero unless $e_1=e_1'$ and $e_2=e_2'$. In this case, we may consider the small colorings of the $G_i$ which agree with the colorings of $G$ except on the edges already labelled $2e_i$ in Fig. 6. Let $\mathfrak{b}_i$ denote the basis elements of $\mathcal{S}_p(\Sigma_i)$ indexed by these small colorings of $G_i$. In this case, $\hat{\mathfrak{b}}$ is up to units $h^E$ $\mathfrak{b}_1 \otimes \mathfrak{b}_2$ where

$$E=d-1-\left\lfloor \frac{1}{2}\bigl(A(\mathfrak{b})-e(\mathfrak{b})\bigr)\right\rfloor+\left\lfloor \frac{1}{2}\bigl(A(\mathfrak{b}_1)-e(\mathfrak{b}_1)\bigr)\right\rfloor+\left\lfloor \frac{1}{2}\bigl(A(\mathfrak{b}_2)-e(\mathfrak{b}_2)\bigr)\right\rfloor.$$

To see this one uses fusion and the fact that $V_p$ of a 2-sphere with a single point with a nonzero even color is the zero module. (When applying fusion, one encounters certain coefficients, but these are products of quantum integers and their inverses, hence units in $\mathcal{O}$.) One also must take into account the powers of $h$ in the definitions of $\mathfrak{b}, \mathfrak{b}_1$, and $\mathfrak{b}_2$. The $d-1$ term comes from

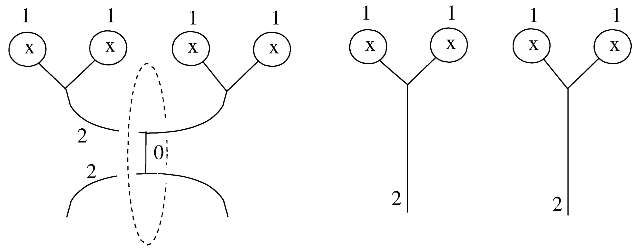



Fig. 7.

the surgery axiom (S1) of [3] [10] which implies that attaching a 1-handle to a 3-manifold has the effect of multiplying its quantum invariant by $\mathcal{D}$ which we recall, up to units, is $h^{d-1}$.

One checks that if $E$ is negative, it must be $-1$, and this happens precisely when $e(\mathfrak{b}) = 0$, $e(\mathfrak{b}_1) = e(\mathfrak{b}_2) = d-1$, and $A(\mathfrak{b}_1) \equiv A(\mathfrak{b}_2) \equiv d \pmod 2$, in other words, when $\varepsilon(\mathfrak{b}_1) = \varepsilon(\mathfrak{b}_2) = 1$. Conversely, if $\varepsilon(\mathfrak{b}_1) = \varepsilon(\mathfrak{b}_2) = 1$ we can find a $\mathfrak{b}$ such that $\hat{\mathfrak{b}}$ is $h^{-1}\mathfrak{b}_1 \otimes \mathfrak{b}_2$ up to units. Thus the elements $h^{-\lfloor \frac{1}{2}(\varepsilon(\mathfrak{b}_1)+\varepsilon(\mathfrak{b}_2))\rfloor}\mathfrak{b}_1 \otimes \mathfrak{b}_2$ form indeed a basis of $\mathcal{S}_p(\Sigma_1 \sqcup \Sigma_2)$. $\square$

*Example* 12.6. – Let $\Sigma = \partial H_2$ with one point colored 2. On the left of Fig. 7 is pictured a skein element in $H_2 \# H_2$ representing an element of $\mathcal{S}_5(\Sigma \sqcup \Sigma)$ which is divisible by $h^2$. On the right of the figure is a skein element in $H_2 \sqcup H_2$ representing an element of $\mathcal{S}_5(\Sigma \sqcup \Sigma)$ which (by the argument in the proof above) is $h$ times the element on the left (up to units) and thus is divisible by $h$ in $S_5(\Sigma \sqcup \Sigma)$. This latter element is of the form $\mathfrak{b} \otimes \mathfrak{b}$ where $\mathfrak{b} \in \mathcal{S}_5(\Sigma)$ is not divisible by $h$. This shows that the cokernel of the homomorphism from $S_5(\Sigma) \otimes S_5(\Sigma) \to S_5(\Sigma \sqcup \Sigma)$ contains a non-trivial element which is annihilated by $h$. Of course, one has $\varepsilon(\mathfrak{b}) = 1$.

*Proof of Theorem 12.1 for* $n > 2$. – As before, we suppose $\Sigma_i$ is the boundary of $H_{g_i}$. Given a partition of $\pi_0(\bigsqcup_i \Sigma_i)$, we can take a sequence of internal connected sums of the $H_{g_i}$ together to form a manifold, say $H$, that represents the given partition. Using partitions into at most pairs and singletons, it follows from the $n = 2$ case that rescaled tensor products of the form

$$h^{-\lfloor \frac{1}{2}\sum_i \varepsilon(\mathfrak{b}_i)\rfloor}\mathfrak{b}_1 \otimes \cdots \otimes \mathfrak{b}_n \quad (\mathfrak{b}_i \in \mathcal{B}_i)$$

lie in $\mathcal{S}_p(\bigsqcup_i \Sigma_i)$. Thus we only have to show that other partitions do not give rise to elements not in the $\mathcal{O}$-span of these ones. By induction on $n$, it is enough to consider the case where $H$ is connected. For this, we apply the same strategy as used in the proof for $n = 2$.

Let $G_i$ be lollipop trees in $H_{g_i}$. We identify the boundary connected sum of all the $H_{g_i}$ with $H_g$ where $g = \sum g_i$. The boundary of $H_g$ is $\#_i\Sigma_i$, the connected sum of all the $\Sigma_i$. Our $H$ is the interior connected sum $\# H_g$ with boundary $\bigsqcup_i \Sigma_i$; it is obtained from $H_g$ by adding $n-1$ two-handles along the $n-1$ connecting circles in $\#_i\Sigma_i$ .

We graft the $G_i$ together as indicated in Fig. 8 to form a lollipop tree $G$ in $H_g$. As in the case $n = 2$, if $\mathfrak{b}$ is a basis element of $\mathcal{S}_p(\#_i\Sigma_i)$ given by a small coloring of $G$, we denote by $\hat{\mathfrak{b}}$ the induced element of $\mathcal{S}_p(\bigsqcup_i \Sigma_i)$ *via* the inclusion $H_g \subset H$. Our task is to compute the space spanned by $v$-graphs in $H$, but since they all can be isotoped into $H_g$, we only have to compute the span of the elements $\hat{\mathfrak{b}}$ as $\mathfrak{b}$ ranges over the basis associated to the small colorings of $G$.

Again as in the case $n = 2$, $\hat{\mathfrak{b}}$ will be zero unless $n-1$ specified pairs of edges have the same color. If these are the same, then we may define a related small coloring on each $G_i$. Let $\mathfrak{b}_i$

[10] In [3], $\mathcal{D}$ is denoted by $\eta^{-1}$.

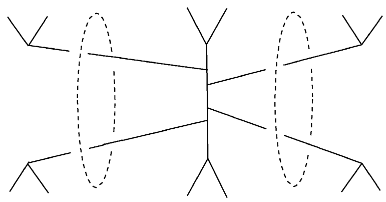


Fig. 8. Grafting three lollipop trees. If $n > 3$, one generalizes this pattern in the obvious way.

denote the basis vector for $\mathcal{S}_p(\Sigma_i)$ associated to this coloring. As above, if $\hat{\mathfrak{b}}$ does not represent zero, $\hat{\mathfrak{b}}$ is, up to units of $\mathcal{O}$, given by $h^E \mathfrak{b}_1 \otimes \cdots \otimes \mathfrak{b}_n$, where

$$\begin{aligned}
E &= (n-1)(d-1) - \left\lfloor \frac{1}{2}\big(A(\mathfrak{b}) - e(\mathfrak{b})\big) \right\rfloor + \sum_i \left\lfloor \frac{1}{2}(A(\mathfrak{b}_i) - e(\mathfrak{b}_i)) \right\rfloor \\
&\geqslant (n-1)(d-1) - \frac{1}{2}\big(A(\mathfrak{b}) - e(\mathfrak{b})\big) - \frac{n}{2} + \sum_i \frac{1}{2}\big(A(\mathfrak{b}_i) - e(\mathfrak{b}_i)\big) \\
&= (n-1)(d-1) - \frac{n}{2} + \frac{1}{2}\bigg(e(\mathfrak{b}) - \sum_i e(\mathfrak{b}_i)\bigg), \quad \text{as } A(\mathfrak{b}) = \sum_i A(\mathfrak{b}_i) \\
&\geqslant (n-1)(d-1) - \frac{n}{2} - \frac{n(d-1)}{2}, \quad \text{as } e(\mathfrak{b}) \geqslant 0, \text{ and } e(\mathfrak{b}_i) \leqslant d-1 \\
&= \frac{(n-2)(d-2)}{2} - 1.
\end{aligned}$$

If $p > 5$, then $d > 2$, and $E$ is non-negative (since $n \geqslant 3$ here). Thus $v$-graphs in $H$ do not give rise to any new elements of $\mathcal{S}_p(\bigsqcup_i \Sigma_i)$.

We are reduced to the case that $p = 5$ and $d = 2$, when $E$ perhaps can be negative but is never less than $-1$. One has $E = -1$ exactly when all $\geqslant$ signs in the computation above are equalities. This requires in particular that $A(\mathfrak{b}_i) - e(\mathfrak{b}_i)$ must be odd for all $i$, and that $e(\mathfrak{b}_i) = d - 1$ for all $i$. In other words, we must have $\varepsilon(\mathfrak{b}_i) = 1$ for all $i$.

From this we conclude that for $p = 5$ the connect-summing of $n \geqslant 3$ handlebodies together may imply the divisibility of an element of the form $\mathfrak{b}_1 \otimes \cdots \otimes \mathfrak{b}_n$ by $h$ but in every case, one can also see at least this divisibility arising from a partitioning of these handlebodies into at most pairs and singletons. This completes the proof. □

*Remark* 12.7. – It might have occurred to the reader that perhaps we should have defined $\mathcal{S}_p$ of a disconnected surface by a tensor product formula, rather than as in Definition 2.1. This would have definite drawbacks when we consider how cobordisms act in the $\mathcal{S}_p$-theory. Here is an example.

Suppose we let $\Sigma = \partial H_2$ with a point colored $p - 3$ and let $C$ be the cobordism from $\Sigma \# \Sigma$ to $\Sigma \sqcup \Sigma$ constructed by adding a 2-handle as in Example 12.6. The TQFT associates to $C$ a linear map $Z_p(C)$ from $V_p(\Sigma \# \Sigma)$ to $V_p(\Sigma \sqcup \Sigma)$. If we had defined $\mathcal{S}_p(\Sigma \sqcup \Sigma)$ to be $\mathcal{S}_p(\Sigma) \otimes \mathcal{S}_p(\Sigma)$, then $Z_p(C)$ would not map $\mathcal{S}_p(\Sigma \# \Sigma)$ into $\mathcal{S}_p(\Sigma \sqcup \Sigma)$.

On the other hand, with Definition 2.1 of $\mathcal{S}_p$, for any cobordism $C$ from, say $\Sigma'$ to $\Sigma''$ such that $\beta_0(C, \Sigma'') = 0$, the induced map $Z_p(C)$ will map $\mathcal{S}_p(\Sigma')$ into $\mathcal{S}_p(\Sigma'')$. Such cobordisms are called *targeted* in [8]. More generally, we have that if $C$ is any cobordism $C$ from $\Sigma'$ to $\Sigma''$, then $Z_p(C)$ maps $\mathcal{S}_p(\Sigma')$ into $\mathcal{D}^{-\beta_0(C, \Sigma'')} \mathcal{S}_p(\Sigma'')$.

We remark that our theory is not half-projective with parameter $\mathcal{D}$ in the sense of Kerler [15]. Nor does it seem possible to make it so by rescaling.

## 13. Extra structure, mapping class group representations, and $\mathcal{S}_p^+(\Sigma)$

The TQFTs we have been studying require that surfaces and 3-manifolds be equipped with some extra structure in order to avoid having a framing anomaly. Until now this extra structure did not play any essential role in our arguments. Thus a detailed discussion was not required. However some of the results in the following sections are improved by using the refined theory $V_p^+$ and its integral version $\mathcal{S}_p^+$ which are defined and discussed in [8,10]. For this reason, we now discuss the extra structures which we employ. For simplicity, we will focus on the case where $\Sigma$ is connected in this section and the next one (but we allow $\Sigma$ to have colored points).

As in [8,11], we follow the method originally developed by Walker [29] and Turaev [28]. This approach equips 3-manifolds with integer weights, and surfaces with Lagrangian subspaces of their first real homology. The cobordisms from $\Sigma$ to $\Sigma$ which are mapping cylinders form a central extension, denoted $\widetilde{\Gamma}(\Sigma)$, of the mapping class group. Forgetting the weight on these cobordisms defines a projection onto the ordinary mapping class group $\Gamma(\Sigma)$. The kernel of this homomorphism is the group of integers $\mathbb{Z}$.

The extended mapping class group $\widetilde{\Gamma}(\Sigma)$ acts on $\mathcal{S}_p(\Sigma)$ preserving the form $(\ ,\ )_\Sigma$. Thus $\widetilde{\Gamma}(\Sigma)$ acts on $\mathcal{S}_p^\sharp(\Sigma)$ as well. We note that elements of $\widetilde{\Gamma}(\Sigma)$ which map to the identity of $\Gamma(\Sigma)$ act on $\mathcal{S}_p(\Sigma)$ by multiplication by some power of $\zeta_p$, if $p \equiv -1 \pmod 4$, and by some power of $\zeta_{4p}$, if $p \equiv 1 \pmod 4$.

By a standard handlebody, we mean one equipped with the weight $0$. The boundary of a standard handlebody $H$ will be equipped with the Lagrangian given by the kernel of the inclusion to the homology of $H$.

In [8, §7], the notion of an even cobordism is defined. The even cobordisms from $\Sigma$ to $\Sigma$ which are mapping cylinders form an index two subgroup of $\widetilde{\Gamma}(\Sigma)$ denoted $\widetilde{\Gamma}(\Sigma)^+$ which still surjects onto $\Gamma(\Sigma)$.

In the remainder of this section, we assume $p \equiv 1 \pmod 4$. Even cobordisms were used in [8] to reduce coefficients from $\mathcal{O} = \mathbb{Z}[\zeta_{4p}]$ to $\mathbb{Z}[\zeta_p]$ which we will denote by $\mathcal{O}^+$. One obtains free $\mathcal{O}^+$-modules $\mathcal{S}_p^+(\Sigma)$ so that

$$\mathcal{S}_p^+(\Sigma) \otimes_{\mathcal{O}^+} \mathcal{O} = \mathcal{S}_p(\Sigma).$$

The module $\mathcal{S}_p^+(\Sigma)$ is again generated by $v$-graphs in a standard handlebody $H$, but coefficients are now required to lie in $\mathcal{O}^+$. (Here we are using the fact that a standard handlebody $H$ when viewed as a morphism from the empty set to $\Sigma$ is even.) Thus our $\mathfrak{b}(a,b,c)$ also define elements of $\mathcal{S}_p^+(\Sigma)$. Moreover the set of $\mathfrak{b}(a,b,c)$ associated to small colorings forms a basis for $\mathcal{S}_p^+(\Sigma)$ by the same proof.

There is a sesquilinear form

$$(\ ,\ )_\Sigma^+ : \mathcal{S}_p^+(\Sigma) \times \mathcal{S}_p^+(\Sigma) \to \mathcal{O}^+$$

obtained by multiplying the form $(\ ,\ )_\Sigma$ by $i^{\delta(\Sigma)}$, where $\delta(\Sigma)$ is zero or one depending on whether the genus of $\Sigma$ is even, or odd. The reason why this form takes values in $\mathcal{O}^+$ is explained in [10, Remark 9.6]. Note that $(\ ,\ )_\Sigma^+$ is hermitian or skew-hermitian, depending on the parity of the genus.

We define $\mathcal{S}_p^{+\sharp}(\Sigma)$ in the same way as $\mathcal{S}_p^\sharp(\Sigma)$ but using the form $(\ ,\ )_\Sigma^+$. Then the $\mathfrak{b}^\sharp(a,b,c)$ form a basis for $\mathcal{S}_p^{+\sharp}(\Sigma)$.

The even extended mapping class group $\widetilde{\Gamma}(\Sigma)^+$ acts on $\mathcal{S}_p^+(\Sigma)$ preserving the form $(\ ,\ )_\Sigma^+$, and therefore also acts on $\mathcal{S}_p^{+\sharp}(\Sigma)$. Elements of $\widetilde{\Gamma}(\Sigma)^+$ which map to the identity of $\Gamma(\Sigma)$ act by multiplication by some power of $\zeta_p$.

## 14. Associated finite torsion modules

The extended mapping class group $\widetilde{\Gamma}(\Sigma)$ acts on $\mathcal{S}_p(\Sigma)/h^N\mathcal{S}_p(\Sigma)$ $(N \geqslant 1)$ and also on $\mathcal{S}_p^\sharp(\Sigma)/\mathcal{S}_p(\Sigma)$. These are finitely generated torsion $\mathcal{O}$-modules. The hermitian form $(\ ,\ )_\Sigma$ on $\mathcal{S}_p(\Sigma)$ induces in the obvious way an $\mathcal{O}/h^N\mathcal{O}$-valued form on $\mathcal{S}_p(\Sigma)/h^N\mathcal{S}_p(\Sigma)$ and an $\mathcal{O}(\frac{1}{h})/\mathcal{O}$-valued form on $\mathcal{S}_p^\sharp(\Sigma)/\mathcal{S}_p(\Sigma)$, and $\widetilde{\Gamma}(\Sigma)$ acts preserving these forms. The structure of these modules and forms follows easily from our bases. Let us describe them in some interesting cases. If $p \equiv 1 \pmod 4$, we will mainly look at the modules coming from the refined theory $\mathcal{S}_p^+$.

Let $G$ be a lollipop tree and consider the basis vectors $\mathfrak{b}(a,b,c)$ for $\mathcal{S}_p(\Sigma)$, where $(a,b,c)$ runs through the small colorings of $G$. Clearly, $\mathcal{S}_p(\Sigma)/h^N\mathcal{S}_p(\Sigma)$ is a free $\mathcal{O}/h^N\mathcal{O}$-module with the induced basis.

PROPOSITION 14.1. – *One has an orthogonal decomposition*

$$\mathcal{S}_p(\Sigma) = \bigoplus_{a,c} \mathcal{S}_p(\Sigma)_{G;a,c}$$

*with respect to the form $(\ ,\ )_\Sigma$, where $\mathcal{S}_p(\Sigma)_{G;a,c}$ is the $\mathcal{O}$-span of the basis vectors $\mathfrak{b}(a,b,c)$ where $b$ varies so that $(a,b,c)$ is a small coloring of $G$.*

*Proof.* – This follows from the fact that the small graph basis vectors $\mathfrak{g}(a,b,c)$ are an orthogonal basis of $V_p(\Sigma)$ [3, Theorem 4.11]. □

Clearly, this also induces an orthogonal decomposition of the induced form on $\mathcal{S}_p(\Sigma)/h^N\mathcal{S}_p(\Sigma)$.

Let us now look at $\mathcal{S}_p^\sharp(\Sigma)/\mathcal{S}_p(\Sigma)$. One has $\mathfrak{b}^\sharp(a,b,c) = h^{-n(a,c)}\mathfrak{b}(a,b,c)$, where $n(a,c) = e$, if $\sum a_i$ is even, and $n(a,c) = e+1$, if $\sum a_i$ is odd (here, as always, $e$ is the trunk half-color of $(a,b,c)$). One has an orthogonal decomposition

$$\mathcal{S}_p^\sharp(\Sigma)/\mathcal{S}_p(\Sigma) = \bigoplus_{(a,c)} \mathcal{S}_p^\sharp(\Sigma)_{G;a,c}/\mathcal{S}_p(\Sigma)_{G;a,c},$$

and each summand is a free $\mathcal{O}/h^{n(a,c)}\mathcal{O}$-module.

If $p \equiv 1 \pmod 4$, similar statements hold for the $\mathcal{O}^+$-modules $\mathcal{S}_p^+(\Sigma)/h^N\mathcal{S}_p^+(\Sigma)$ $(N \geqslant 1)$ and $\mathcal{S}_p^{+\sharp}(\Sigma)/\mathcal{S}_p^+(\Sigma)$.

We will say $(a,b,c)$ is an *odd* (*even*) small coloring if $\sum a_i$ is odd (even respectively).

If $p \equiv -1 \pmod 4$, then $\mathcal{O} = \mathbb{Z}[\zeta_p]$ and $\mathcal{O}/h\mathcal{O} = \mathbb{F}_p$, the finite field with $p$ elements. If $p \equiv 1 \pmod 4$, then $\mathcal{O} = \mathbb{Z}[\zeta_{4p}] = \mathbb{Z}[\zeta_p, i]$ and $\mathcal{O}/h\mathcal{O} = \mathbb{F}_p[i]$. In this case, we mainly consider $\mathcal{O}^+/h\mathcal{O}^+$ which is again equal to $\mathbb{F}_p$.

PROPOSITION 14.2. – *If $p \equiv -1 \pmod 4$, the hermitian form $(\ ,\ )_\Sigma$ induces a symmetric form on the $\mathbb{F}_p$-vector space $\mathcal{S}_p(\Sigma)/h\mathcal{S}_p(\Sigma)$. If $p \equiv 1 \pmod 4$, the $(-1)^g$-hermitian form $(\ ,\ )_\Sigma^+$ induces a $(-1)^g$-symmetric form on the $\mathbb{F}_p$-vector space $\mathcal{S}_p^+(\Sigma)/h\mathcal{S}_p^+(\Sigma)$. (Here $g$ is the genus of $\Sigma$.) In both cases, one has an action of the (ordinary) mapping class group $\Gamma(\Sigma)$ preserving these forms.*

*Proof.* – The symmetry properties of the induced forms follow from the fact that the conjugation on $\mathbb{Z}[\zeta_p]$ induces the trivial involution on $\mathbb{F}_p$. The action of the (even) extended mapping class group descends to the ordinary mapping class group $\Gamma(\Sigma)$ because $\zeta_p$ acts as the identity on $\mathbb{F}_p$. □

*Remark* 14.3. – If $\gamma \subset \Sigma$ is a separating simple closed curve, and all the colored points of $\Sigma$ lie on one side of $\Gamma$, then the associated Dehn twist $T_\gamma \in \Gamma(\Sigma)$ acts trivially on the $\mathbb{F}_p$-vector spaces above. This follows from Theorem 11.1. Indeed, $T_\gamma$ acts diagonally in the basis given by that theorem, and its eigenvalues are the twist coefficients $\mu_{2e}$ (see e.g. [3, Remark 7.6(ii)]) which are powers of $\zeta_p$, hence congruent to $1 \pmod{h}$. This argument is due to Kerler.

The forms given by Proposition 14.2 are singular in general. Of course the radical of such a form is preserved by the mapping class group. In this way we obtain, as well, a subrepresentation of characteristic $p$ given by the radical, and a quotient representation of characteristic $p$. The quotient representation preserves the induced non-singular inner product space structure. We denote the radical of the form by $\mathrm{rad}_p(\Sigma)$.

For the rest of this section, we consider only the important special case that $\Sigma$ has no colored points. Then the radical is spanned by the images of the $\mathfrak{b}(a,b,c)$ associated to odd small colorings. The quotient representation has dimension given by the number of even small colorings.

Observe that $\mathcal{S}_p^\sharp(\Sigma)/\mathcal{S}_p(\Sigma)$ is also the free $\mathcal{O}/h\mathcal{O}$-module on the odd small colorings of $G$.

PROPOSITION 14.4. – *If $p \equiv -1 \pmod{4}$, there is an isomorphism between the $\mathbb{F}_p$-vector spaces $\mathcal{S}_p^\sharp(\Sigma)/\mathcal{S}_p(\Sigma)$ and $\mathrm{rad}_p(\Sigma)$ intertwining $\Gamma(\Sigma)$ representations. If $p \equiv 1 \pmod{4}$, there is an isomorphism between $\mathcal{S}_p^{+\sharp}(\Sigma)/\mathcal{S}_p^{+}(\Sigma)$ and $\mathrm{rad}_p(\Sigma)$ intertwining the $\Gamma(\Sigma)$ representations.*

*Proof.* – In the first case, consider the composition of the map from $\mathcal{S}_p^\sharp(\Sigma)$ to $\mathcal{S}_p(\Sigma)$ given by multiplication by $h$ followed by the quotient map to $\mathcal{S}_p(\Sigma)/h\mathcal{S}_p(\Sigma)$. This is onto $\mathrm{rad}_p(\Sigma)$, and has $\mathcal{S}_p(\Sigma)$ as kernel. The second case is seen similarly. □

Recall that the hermitian form $(\ ,\ )_\Sigma$ on $\mathcal{S}_p(\Sigma)$ induces a non-singular $\mathcal{O}(\frac{1}{h})/\mathcal{O}$-valued form on $\mathcal{S}_p^\sharp(\Sigma)/\mathcal{S}_p(\Sigma)$. In our special situation, it suffices to multiply this form by $h$ to get an $\mathcal{O}/h\mathcal{O}$-valued form on $\mathcal{S}_p^\sharp(\Sigma)/\mathcal{S}_p(\Sigma)$. We denote this new form on $\mathcal{S}_p(\Sigma)$ by $h.(\ ,\ )_\Sigma$.

PROPOSITION 14.5. – *If $p \equiv -1 \pmod{4}$, the form $h.(\ ,\ )_\Sigma$ induces a skew-symmetric form on the $\mathbb{F}_p$-vector space $\mathcal{S}_p^\sharp(\Sigma)/\mathcal{S}_p(\Sigma)$. If $p \equiv 1 \pmod{4}$, the form $h.(\ ,\ )_\Sigma^{+}$ induces a $(-1)^{g+1}$-symmetric form on the $\mathbb{F}_p$-vector space $\mathcal{S}_p^{+\sharp}(\Sigma)/\mathcal{S}_p^{+}(\Sigma)$. In both cases, the forms are non-degenerate, and one has an action of the mapping class group $\Gamma(\Sigma)$ preserving these forms.*

*Proof.* – The proof is basically the same as for Proposition 14.2. One just needs to observe that $\bar{h} = 1 - \zeta_p^{-1} = -\zeta_p^{-1} h$. Assume $p \equiv -1 \pmod{4}$. For $x, y \in \mathcal{S}_p^\sharp(\Sigma)$ one has

$$h.(y,x)_\Sigma = h.\overline{(x,y)_\Sigma} = -\zeta_p \overline{h.(x,y)_\Sigma}$$

as elements of $\mathcal{O}$. As before, $\zeta_p$ acts trivially on $\mathbb{F}_p$ and the induced conjugation is the identity. Only the minus sign remains. It is the reason why one gets a skew-symmetric form on $\mathcal{S}_p^\sharp(\Sigma)/\mathcal{S}_p(\Sigma)$ while the form in Proposition 14.2 was symmetric. The case $p \equiv 1 \pmod{4}$ is proved in the same way. □

Via the isomorphism of Proposition 14.4, one gets in this way an inner product structure on $\mathrm{rad}_p(\Sigma)$ as well.

## 15. An upper bound for the cut number of a 3-manifold

The co-rank of a group is the maximal $k$ such that there is an epimorphism of that group onto the free group on $k$ letters [26]. If $M$ is an oriented connected closed 3-manifold, define $c(M)$, the *cut number* of $M$, to be the maximal number of closed oriented surfaces that one can place in $M$ and still have a connected complement. Jaco [13] showed that the co-rank of $\pi_1(M)$ is the cut number of $M$. It is easy to see that $c(M) \leqslant \beta_1(M)$.

In this section, we show that another upper bound on the cut number is given by quantum $SO(3)$-invariants. Let $M$ be an oriented connected closed 3-manifold and $L$ a banded colored graph in $M$. We use the normalization $I_p(M,L) = \mathcal{D}\langle (M,L)\rangle_p$ where $\langle\ \rangle_p$ is the invariant defined in [3], and $M$ is given the weight zero. This is the same normalization as in [19]. For example, $I_p(S^3) = 1$ (since $\mathcal{D} = \langle S^3\rangle_p^{-1}$). The invariant $I_p$ takes values in $\mathcal{O}$ by [22,19]. We let $\mathfrak{o}_p(M,L)$ denote the highest power of $h$ which divides $I_p(M,L)$.

THEOREM 15.1. – *Let $M$ be an oriented connected closed 3-manifold, $L$ a banded colored graph $L$ in $M$, and $p = 2d+1 \geqslant 5$ a prime. Then*

$$c(M) \leqslant \frac{\mathfrak{o}_p(M,L)}{d-1}.$$

Varying $L$, we obtain many upper bounds on the cut number of $M$. So far, no example has been found where an interesting upper bound on $c(M)$ is obtained by choosing $L$ nonempty that cannot also be obtained by choosing $L$ empty.

Theorem 15.1 was conjectured by the first author. He and Kerler then obtained Theorem 15.1 for $p = 5$ and $L$ empty. The proof we give of the more general result incorporates some ideas that were used in the proof of this special case.

Cochran and Melvin showed [4] that $\beta_1(M)/3 \leqslant \mathfrak{o}_p(M)/(d-1)$. Three recent papers show that $\beta_1(M)/3$ can be larger than the cut number $c(M)$ [12,17,27]. In this case the result of Cochran and Melvin gives a better lower bound on $\mathfrak{o}_p(M)$ than Theorem 15.1. On the other hand, when $\beta_1(M)/3 < c(M)$ (which may also happen), Theorem 15.1 gives a better lower bound on $\mathfrak{o}_p(M)$.

In the proof, we will use the following easy proposition which follows from the functoriality properties of the TQFT. We will also use it in Section 16.

PROPOSITION 15.2. – *Suppose $N \subset M$ and $N'$ is the exterior of $N$ in $M$. Further suppose $[N] \in \mathcal{S}_p(\partial N)$ can be written as a linear combination $\sum_i a_i[N_i]$. We have that $I_p(M) = \sum_i a_i I_p(N_i \cup_{\partial N} N')$.*

*Proof of Theorem 15.1.* – Let $c$ be the cut number of $M$. We can find $c$ disjoint connected oriented surfaces in $M$ which do not disconnect $M$. As these surfaces do not disconnect, we may tube up parallel copies of these surfaces to find a copy of the connected sum of these surfaces disjoint from the original surfaces but homologous to the union of these surfaces. If we add this new surface to the collection, we obtain a collection of $c+1$ connected oriented surfaces which disconnect but no sub-collection of $c$ of them disconnects. These $c+1$ surfaces disconnect $M$ into two components $Y$ and $Y'$ and each of the $c+1$ surfaces is a boundary component of the closures of both $Y$ and $Y'$.

Let $\Sigma_i$ for $1 \leqslant i \leqslant c+1$ denote the individual surfaces. We can find $c$ disjoint arcs $\alpha_i$ in $Y$ for $1 \leqslant i \leqslant c$ which join points $x_i \in \Sigma_i$ to points $y_i \in \Sigma_{i+1}$. Similarly we can find $c$ disjoint arcs $\alpha_i'$ in $Y'$ which join the points $x_i \in \Sigma_i$ to the points $y_i \in \Sigma_{i+1}$. A neighborhood of the union of the $\Sigma_i$ and the $\alpha_i$ and the $\alpha_i'$ is a cobordism $P$ from the connected sum of the $\Sigma_i$ to itself. We isotope $L$ so that it is transverse to the $\Sigma_i$'s. We assign colored points to each $\Sigma_i$ according to this intersection.

We denote the connected sum of the $\Sigma_i$ by $\Sigma$. On $\Sigma$ we have $c$ simple closed curves $\gamma_i$ such that if we perform surgery along them we recover the disjoint union $\bigsqcup_i \Sigma_i$. The cobordism $P$

has a nice handle decomposition as follows: $\Sigma \times I$ union $c$ 2-handles along $\{\gamma_i\}_{1 \leqslant i \leqslant c}$ union $c$ 1-handles which "reconnect" the $\Sigma_i$. The core of each 1-handle can be completed to a circle which meets exactly one core of a 2-handle transversely in a single point. We can identify $P$ with the result of framed surgery on $\Sigma \times I$ along $\{\gamma_i \times \frac{1}{2}\}_{1 \leqslant i \leqslant c}$ with the framing along $\Sigma \times \frac{1}{2}$. Let $\gamma(\omega)$ denote $\{\gamma_i \times \frac{1}{2}\}_{1 \leqslant i \leqslant c}$ colored with the skein element $\omega$ appearing in the surgery axiom (S2) of [3]. The surgery axiom says that $(\Sigma \times I, \gamma(\omega))$ induces the same endomorphism of $V_p(\Sigma)$ under the TQFT as does $P$.

Place $\Sigma \times I$ in $S^3$ so that its complement is the disjoint union of two handlebodies $H$ and $H'$ and the $\gamma_j$ bound disks $D_j$ in $H$. Express $[Y] \in \mathcal{S}_p(\Sigma)$ in terms of the basis $\mathcal{B}$ associated to a lollipop tree $G$ for $H$ with respect to $\ell(\Sigma)$. [11] Similarly express $[Y'] \in \mathcal{S}_p(-\Sigma)$ in terms of the basis $\mathcal{B}'$ associated to a lollipop tree $G'$ for $H'$ with respect to $\ell(\Sigma)$.

Applying Proposition 15.2 twice, once to expand $Y$ and once to expand $Y'$, we have that $I_p(M, L)$ is a linear combination over $\mathcal{O}$ of evaluations of skein classes in $S^3$. Recall that every basis element $\mathfrak{b} = \mathfrak{b}(a, b, c) \in \mathcal{B}$ is given by $h^{-\lfloor \frac{1}{2}(A(\mathfrak{b}) - e(\mathfrak{b})) \rfloor} \mathfrak{g}(a, 0, c)$ union some $v$-colored curves. Thus the term corresponding to $(\mathfrak{b}, \mathfrak{b}') \in \mathcal{B} \times \mathcal{B}'$ in the expansion of $I_p(M, L)$ is an integral multiple of $h^{-\lfloor \frac{1}{2}(A(\mathfrak{b}) - e(\mathfrak{b})) \rfloor - \lfloor \frac{1}{2}(A(\mathfrak{b}') - e(\mathfrak{b}')) \rfloor}$ times the evaluation of the union of

(i) a colored trivalent graph, namely the gluing of the graphs $\mathfrak{g}(a, 0, c)$ in $H$ and $\mathfrak{g}(a', 0, c')$ in $H'$ along their univalent vertices (which correspond to $\ell(\Sigma)$),
(ii) the $\gamma_i$'s each colored by $\omega$, and
(iii) some extra $v$-colored curves.

We can do fusion along the strands which pass through the spanning disks $D_j$ for the $\omega$-colored $\gamma_j$'s (none of the $v$-colored curves do), and discard the terms arising with a single non-zero color passing through these disks, as for all these terms the color must be even. Then each $\omega$-colored $\gamma_j$ spanning a disk may be replaced by an extra scalar factor of $\mathcal{D}$. (This is the same argument as in [19, Proof of Lemma 4.1].) Up to units this gives an extra factor of $h^{(d-1)c}$. On the other hand, the Lollipop Lemma 7.1 gives an extra factor of $h^{\lceil \frac{1}{2}(A(\mathfrak{b}) + A(\mathfrak{b}')) \rceil}$ which compensates the above-mentioned negative power of $h$. Thus each term in the expansion of $I_p(M, L)$ is divisible by $h^{(d-1)c}$. The result follows. $\square$

*Example* 15.3. – Let $\Sigma$ be a closed surface of genus 2, and let $T$ be the Dehn twist along an essential separating curve $\gamma$. Let $M_n$ be the mapping torus of $T^n$. Essential curves disjoint from $\gamma$ sweep out tori in $M_n$. Picking one such curve on each side of $\gamma$, we see that the cut number $c(M_n) \geqslant 2$. On the other hand, computing the trace of the map induced by $T^n$ on $V_p(\Sigma)$, we find

$$I_p(M_n) = \mathcal{D} \sum_{j=0}^{d-1} \zeta_p^{2nj(j+1)} (d-j)^2.$$

If $n$ is not a multiple of $p$, this can be evaluated using Gauss sums. One finds that $\mathfrak{o}_p(M_n) = 2d - 2$, and $c(M_n) \leqslant 2$ by Theorem 15.1. [12] Varying $p$, this shows $c(M_n) = 2$ for every non-zero $n$. We remark that this can also be seen classically by computing the co-rank of $\pi_1(M_n)$.

[11] Warning: we cannot assume that $G$ meets each $D_j$ in a single point, as there may be colored points on each $\Sigma_i$. But this causes no problem in the argument.
[12] If $n$ is a multiple of $p$, then $I_p(M_n)$ is $\mathcal{D}$ times $\dim V_p(\Sigma) = d(d+1)(2d+1)/6$; this does not lead to a good upper bound for $c(M_n)$.

## 16. The Frohman Kania-Bartoszynska ideal invariant

DEFINITION 16.1 ([7]). – Given a connected 3-manifold $N$ with boundary, let $\mathcal{J}_p(N)$ be the ideal in $\mathcal{O}$ generated by

$$\{I_p(M) \mid M \text{is a closed connected oriented 3-manifold containing } N\}.$$

In the case $p \equiv 1 \pmod 4$, we also define $\mathcal{J}_p^+(N) = \mathcal{J}_p(N) \cap \mathcal{O}^+$.

*Remark* 16.2. – If $p \equiv 1 \pmod 4$, the ideal $\mathcal{J}_p(N)$ is generated by scalars which are either in $\mathcal{O}^+$ or in $i\mathcal{O}^+$. (In fact, $I_p(M)$ lies in $i^{\beta_1(M)}\mathcal{O}^+$ [19, Remark 5.2].) Thus $\mathcal{J}_p(N)$ is generated over $\mathcal{O}$ by $\mathcal{J}_p^+(N)$. This is why we prefer to use $\mathcal{J}_p^+(N)$ if $p \equiv 1 \pmod 4$.

This ideal is interesting because of the following immediate proposition.

PROPOSITION 16.3 ([7]). – *If $N_1$ embeds in $N_2$, then $\mathcal{J}_p(N_2) \subset \mathcal{J}_p(N_1)$, and $\mathcal{J}_p^+(N_2) \subset \mathcal{J}_p^+(N_1)$, if $p \equiv 1 \pmod 4$.*

*Remark* 16.4. – Frohman and Kania-Bartoszynska actually made this definition using the $SU(2)$ theory in place of the $SO(3)$ theory used here.

The ideal $\mathcal{J}_p(N)$ is hard to compute from its definition, because it involves the quantum invariants of infinitely many manifolds. Frohman and Kania-Bartoszynska were able to show that a related ideal (associated to the Turaev–Viro invariant at the third root of unity) is non-trivial [13] for the union of two solid tori glued together by identifying neighborhoods of $(2,1)$ curves on their boundary. But it seems that $\mathcal{J}_p(N)$ has never been computed exactly, except for manifolds $N$ with boundary a 2-sphere. [14]

We can give a finite set of generators for $\mathcal{J}_p(N)$ using our bases.

THEOREM 16.5. – *Let $N$ be an oriented connected 3-manifold with boundary $\Sigma$; then $\mathcal{J}_p(N)$ is generated by the scalar products $([N], \mathfrak{b})_\Sigma$ as $\mathfrak{b}$ varies over a basis for $\mathcal{S}_p(\Sigma)$.*

This follows immediately from Proposition 15.2. An example where $\mathcal{J}_p(N)$ is computed using Theorem 16.5 will be given below.

*Remark* 16.6. – In practice, if $N$ has connected boundary, we may present it as surgery on a link in a handlebody $H$ standardly embedded in $S^3$. Then we can just as well replace the form $([N], \mathfrak{b})_\Sigma$ with the generalized Hopf pairing $(([N], \mathfrak{b}'))_{H,H'}$ (see Section 8) where $\mathfrak{b}'$ are elements of the graph-like basis for $\mathcal{S}_p(-\Sigma)$ associated to a lollipop tree in the complementary handlebody $H'$. Thus $\mathcal{J}_p(N)$ can be computed from the evaluation of a finite number of explicitly given skein elements (consisting of $v$-graphs together with some $\omega$-colored curves) in $S^3$.

When one computes examples of $\mathcal{J}_p(N)$, one frequently gets the trivial ideal or some power of $(h)$. Here is an example where it is neither.

Let $L$ be the link in Fig. 9. Let $K$ be the knotted component ($K$ is a $(5,2)$ torus knot), and $J$ the unknotted component. Let $N(k)$ be the result of surgery to the exterior of $J$ along $K$ with framing $k$. If $k$ is odd, $N(k)$ is a homology circle.

PROPOSITION 16.7. – *One has*

$$\mathcal{J}_5^+\big(N(k)\big) = \begin{cases} (1+2\zeta_5^3) & \textit{if } k \equiv 0 \pmod 5, \\ (1) & \textit{otherwise.} \end{cases}$$

[13] We take the trivial ideal of a ring to be the ring itself.

[14] However, sometimes it is possible to compute that the ideal is trivial or a power of $(h)$, without using Theorem 16.5, by making use of known theorems about the quantum invariants of closed 3-manifolds. See [9] for examples of this.

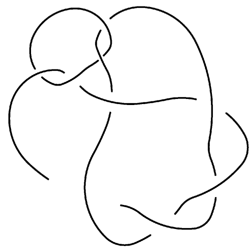


Fig. 9. L9a12 in Thistlethwaite's list of prime links [1].

*Proof.* – Proceeding as in Remark 16.6, the computation of the ideal $\mathcal{J}_5^+(N(k))$ in $\mathbb{Z}[\zeta_5]$ reduces to the evaluation of two skein elements in $S^3$ (since $\mathcal{S}_5$ of a torus has rank two). We used data in Bar-Natan's Knot Atlas and his Mathematica package KnotTheory [1] to help calculate this ideal. □

The ideal $(1+2\zeta_5^3)$ is a non-trivial ideal as the norm of $1+2\zeta_5^3$ is $11$. Thus one has the following immediate

COROLLARY 16.8. – *The manifold $N(5n)$ does not embed into any closed 3-manifold $M$ whose $I_5(M)$ is not divisible by $1+2\zeta_5^3$. In particular $N(5n)$ does not embed into the 3-sphere (since $I_p(S^3)=1$).*

Further such examples are explored in [9]. Some more general results about the ideals $\mathcal{J}_p(N)$ are given there as well.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