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STATISTICAL PROPERTIES OF TOPOLOGICAL COLLET–ECKMANN MAPS

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ABSTRACT. – We study geometric and statistical properties of complex rational maps satisfying a non-uniform hyperbolicity condition called "Topological Collet–Eckmann". This condition is weaker than the "Collet–Eckmann" condition. We show that every such map possesses a unique conformal probability measure of minimal exponent, and that this measure is non-atomic, ergodic, and that its Hausdorff dimension is equal to the Hausdorff dimension of the Julia set. Furthermore, we show that there is a unique invariant probability measure that is absolutely continuous with respect to this conformal measure, and that this invariant measure is exponentially mixing (it has exponential decay of correlations) and satisfies the Central Limit Theorem.

We also show that for a complex rational map the existence of such invariant measure characterizes the Topological Collet–Eckmann condition: a rational map satisfies the Topological Collet–Eckmann condition if, and only if, it possesses an exponentially mixing invariant measure that is absolutely continuous with respect to some conformal measure, and whose topological support contains at least 2 points.

RÉSUMÉ. – On étudie des propriétés géométriques et statistiques des fonctions rationnelles complexes qui satisfont une condition d'hyperbolicité non uniforme appelée « Collet–Eckmann topologique », laquelle est plus faible que la condition de « Collet–Eckmann ». On montre qu'une telle application possède une unique mesure de probabilité conforme d'exposant minimale, et que cette mesure ne possède pas d'atomes, est ergodique, et sa dimension de Hausdorff est égale à la dimension de Hausdorff de l'ensemble de Julia. De plus, on montre qu'il existe une unique mesure invariante qui est absolument continue par rapport à cette mesure, et que cette mesure invariante est exponentiellement mélangeante (la vitesse de décroissance des corrélations est exponentielle) et satisfait le Théorème Central Limite.

On montre aussi que pour une fonction rationnelle l'existence d'une telle mesure invariante caractérise la condition de Collet–Eckmann topologique : une fonction rationnelle satisfait la condition de Collet–Eckmann topologique si et seulement si elle possède une mesure invariante exponentiellement mélangeante qui est absolument continue par rapport à une mesure conforme, et telle que son support topologique contienne au moins 2 points.

1. Introduction

We consider complex rational maps $f:\overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of degree at least 2, viewed as dynamical systems acting on the Riemann sphere $\overline{\mathbb{C}}$. We provide a systematic approach to study geometric and statistical properties of rational maps. For simplicity we restrict to rational maps satisfying

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a non-uniform hyperbolicity condition called "Topological Collet–Eckmann". This condition is very natural and important, because it has several equivalent formulations [35] and because the set of (non-hyperbolic) rational maps satisfying this condition has positive Lebesgue measure in the space of all rational maps of a given degree [1], see also [39]. Some of our main results (Theorems B and C) extend without change to analogous results for smooth multimodal maps of the interval with negative Schwarzian derivative.

1.1. The Topological Collet–Eckmann condition

The Topological Collet–Eckmann condition, that we will abbreviate from now on by TCE, was originally formulated in topological terms. It has however many equivalent formulations. For example, the TCE condition is equivalent to the following strong form of Pesin's non-uniform hyperbolicity condition: The Lyapunov exponent of each invariant probability measure supported on the Julia set is larger than a positive constant, that is independent of the measure. In this paper we will mainly use the following equivalent formulation of the TCE condition.

EXPONENTIAL SHRINKING OF COMPONENTS (EXPSHRINK). – *There exist $\lambda_{\mathrm{Exp}} > 1$ and $r_0 > 0$ such that for every $x \in J(f)$, every integer $n \geqslant 1$, and every connected component W of $f^{-n}(B(x, r_0))$ we have*

$$\operatorname{diam}(W) \leqslant \lambda_{\mathrm{Exp}}^{-n}.$$

See [35] for the original formulation of the TCE condition, and for other equivalent formulations.

The TCE condition is closely related to the "Collet–Eckmann condition": A rational map f satisfies the *Collet–Eckmann condition* if every non-repelling periodic point of f is attracting, and if for every critical value v in the Julia set $J(f)$ of f that is not mapped to a critical point under forward iteration, the derivative $|(f^n)'(v)|$ growths exponentially with n. The Collet–Eckmann condition is stronger than the TCE condition, but it is not equivalent to it. In fact, there are rational maps f that satisfy the TCE condition and that have a critical value v in $J(f)$ that is not mapped to a critical point under forward iteration, and such that $\liminf_{n\to\infty} \frac{1}{n} \ln |(f')^n(v)| = -\infty$. See [36, §5].

The Collet–Eckmann condition was introduced in [8], in the context of unimodal maps. It has been extensively studied for complex rational maps, see [1,3,14,16–18,31–33,35,36,42] and references therein. In particular, M. Aspenberg recently proved in [1] that in the space of all rational maps of a given degree, there is a set of positive Lebesgue measure of (non-hyperbolic) rational maps that satisfy the Collet–Eckmann condition, and hence the TCE condition. See also [39].

1.2. Conformal measures

In general a Julia set has a fractal nature, as its Hausdorff dimension is larger than its topological dimension, see e.g. [47]. In this case, a natural geometric measure on the Julia set is the "conformal measures of minimal exponent": Given $t > 0$, a non-zero Borel measure μ is *conformal of exponent* t for f, if for every Borel subset U of $\overline{\mathbb{C}}$ where f is injective, we have

$$\mu\big(f(U)\big) = \int_U |f'|^t \, d\mu.$$

Every rational map admits a conformal measure [43] and the minimal exponent for which such a measure exists is equal to the "hyperbolic dimension" of the Julia set, see [11,30] and also [25,44,37]. For a rational map satisfying the TCE condition, the minimal exponent is equal to the Hausdorff dimension of the Julia set [34]. For a uniformly hyperbolic rational map f there is a unique conformal measure, up to a constant factor, which is the restriction to the Julia set of the Hausdorff measure of dimension $\mathrm{HD}(J(f))$ [43]. However, for certain rational maps this last measure might be zero or infinity. There are also examples of purely atomic conformal measures. See the survey article [44] for these and other results related to conformal measures.

Our first result is about the existence of a non-atomic conformal measure of minimal exponent.

THEOREM A. – *Every rational map satisfying the TCE condition admits a unique conformal probability measure of minimal exponent. This measure is non-atomic, ergodic, its Hausdorff dimension is equal to the Hausdorff dimension of the Julia set, and it is supported on the conical Julia set.*

We recall the definition of "conical Julia set" in Appendix B. This set is also called "radial Julia set". The conformal measures of a rational map without recurrent critical points in the Julia set are well understood [44]. The first result in the more subtle case when there is a recurrent critical point in the Julia set, was proved by the first named author in [31]. There it is shown that a rational map satisfying the Collet–Eckmann condition and an additional "Tsujii type" condition admits a non-atomic conformal measure of minimal exponent. This result was extended by J. Graczyk and S. Smirnov to all rational maps satisfying the Collet–Eckmann condition, and the weaker "summability condition" [17]. The methods employed in these articles breakdown for rational maps satisfying the TCE condition, as they use the growth of derivatives at critical values in an essential way.

1.3. Absolutely continuous invariant measures and their statistical properties

Having a non-atomic conformal measure of minimal exponent as a reference measure, it is natural to look for absolutely continuous invariant measures. Note that if a rational map satisfies the TCE condition and its Julia set is the whole sphere, then the measure given by Theorem A is the spherical area measure on $\overline{\mathbb{C}}$.

Recall that if $(X,\mathscr{B},\nu)$ is a probability space and if $f:X\to X$ is a measure preserving map, then the measure ν is said to be *mixing*, if for every pair of square integrable functions $\varphi,\psi:X\to\mathbb{R}$ we have

$$\int(\varphi\circ f^n)\cdot\psi\,d\nu-\int\varphi\,d\nu\int\psi\,d\nu\to 0,\quad \text{as } n\to+\infty.$$

THEOREM B. – *Let f be a rational map satisfying the TCE condition. Then there is a unique f invariant probability measure that is absolutely continuous with respect to the unique conformal probability measure of minimal exponent of f. Moreover this measure is ergodic, mixing, and its density with respect to the conformal measure of minimal exponent is bounded from below by a positive constant almost everywhere.*

There are several existence results of this type for rational maps with no recurrent critical points in their Julia set [44]. For other existence results in the case of recurrent critical points, see [1,3,17,31,32,39].

We next study the statistical properties of the invariant measure given by Theorem B. Given a metric space (X,dist) and a measurable map $f:X\to X$, we say that an invariant measure ν is *exponentially mixing*, or that it has *exponential decay of correlations*, if there are constants $C>0$ and $\rho\in(0,1)$ such that every $n\geqslant 1$, every bounded measurable function $\varphi:X\to\mathbb{R}$ and every Lipschitz continuous function $\psi : X \to \mathbb{R}$, we have

$$\left| \int (\varphi \circ f^n) \cdot \psi \, d\nu - \int \varphi \, d\nu \int \psi \, d\nu \right| \leqslant C \Big(\sup_{z \in X} \big| \varphi(z) \big| \Big) \|\psi\|_{\mathrm{Lip}} \cdot \rho^n. \tag{1}$$

Here $\|\psi\|_{\mathrm{Lip}} = \sup_{z,z' \in X, z \neq z'} \frac{|\psi(z) - \psi(z')|}{\mathrm{dist}(z,z')}$ denotes the best Lipschitz constant of ψ. Moreover we will say that *the Central Limit Theorem holds for* ν, if for every Lipschitz continuous function $\psi : X \to \mathbb{R}$ that is not a coboundary (i.e. it cannot be written in the form $\varphi \circ f - \varphi$ for a square integrable function φ) there is $\sigma > 0$ such that for every $x \in \mathbb{R}$ we have,

$$\nu \left\{ \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \psi \circ f^j < x \right\} \to \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} \exp\left(-\frac{u^2}{2\sigma^2} \right) du, \quad \text{as } n \to +\infty.$$

THEOREM C. – *If f is a rational map satisfying the TCE condition, then the invariant measure given by Theorem B is exponentially mixing and the Central Limit Theorem holds for this measure.*

To the best of our knowledge this is the first result of this type in the holomorphic setting. In the case of unimodal maps, a similar result was proved in [46,22]. In [5] this result was shown for Collet–Eckmann multimodal maps having all its critical points of the same critical order. The proof that we give here for rational maps extends without change to smooth multimodal maps with negative Schwarzian derivative that satisfy the TCE condition (and hence for those satisfying the Collet–Eckmann condition), with no restriction on the critical orders of critical points.

1.4. Further results

The following is essentially a converse of Theorems B and C.

THEOREM D. – *Let f be a rational map possessing an exponentially mixing invariant measure ν that is absolutely continuous with respect to some conformal measure of f. Then, either f satisfies the TCE condition, or ν is supported on an indifferent fixed point of f in $J(f)$.*

The second case in this theorem can indeed occur. In fact, let f be a rational map without critical points in $J(f)$ and with a parabolic fixed point p. Then $p \in J(f)$, $\mathrm{HD}(J(f)) < 2$, and for each $\delta > \mathrm{HD}(J(f))$ there is a purely atomic conformal measure μ of exponent δ that is supported on $\bigcup_{n \geqslant 1} f^{-n}(p)$, see for example [44, Theorem 3.7]. Thus, the Dirac measure at p is an exponentially mixing invariant measure that is absolutely continuous with respect to μ.

The following corollary, that was the original motivation of this paper, is a direct consequence of Theorems B, C and D.

COROLLARY 1.1. – *A rational map satisfies the TCE condition if and only if it possesses an exponentially mixing invariant measure that is absolutely continuous with respect to some conformal measure, and whose topological support contains at least* 2 *points.*

This result adds yet another equivalent formulation of the TCE condition to those obtained in [35]. For unimodal maps of the interval, a result analogous to Theorem D was shown by T. Nowicki and D. Sands in [28]. To prove Theorem D we first show, using the assumption that the measure is exponentially mixing, that its Lyapunov exponent is positive. Then we use the following result of N. Dobbs [13], based on the results of Ledrappier in [20,21]: Every ergodic invariant measure of positive Lyapunov exponent that is absolutely continuous with respect to a conformal measure μ, is in fact equivalent to μ and its density is bounded from below by a positive constant almost everywhere. Using this and again the assumption that the measure is exponentially mixing, we show as in [28], that the map is uniformly hyperbolic on periodic orbits: There is $\lambda > 1$ such that for every positive integer n and every repelling periodic point p of f of period n, we have $|(f^n)'(p)| \geqslant \lambda^n$. This finishes the proof, as for a complex rational map this last condition is equivalent to the TCE condition [35].

We will show now, relying on known facts, that for a rational map f satisfying the TCE condition the invariant measure given by Theorem B is characterized as the unique equilibrium state of f with potential $-\operatorname{HD}(J(f))\ln|f'|$ supported on $J(f)$, and also as the unique invariant measure supported on $J(f)$ whose Haussdorff dimension is equal to $\operatorname{HD}(J(f))$. Similar results for Collet–Eckmann unimodal maps where shown in [4], see also [29]. We note first that each invariant measure ν supported on $J(f)$ has a positive Lyapunov exponent $\chi(\nu) := \int \ln|f'|\,d\nu > 0$ [35]. So, if we denote by $\alpha(f) > 0$ the least value of $t > 0$ for which there exists a conformal measure of exponent t of f supported on $J(f)$, then [21, Theorem B] implies that the measure given by Theorem B is characterized as the unique equilibrium state of f with potential $-\alpha(f)\ln|f'|$. Thus the first characterization follows from the equality $\alpha(f) = \operatorname{HD}(J(f))$, shown in [34] for rational maps satisfying the TCE condition. The second characterization is an easy consequence of the first one, and of the fact that for each invariant measure supported on $J(f)$, the measure theoretic entropy $h(\nu)$ of ν satisfies $h(\nu) = \operatorname{HD}(\nu)\chi(\nu)$, see [21, Théorème 3] and also [23,37] (here we use again that for every invariant measure ν supported on $J(f)$ we have $\chi(\nu) > 0$).

Finally, we also show that for a rational map satisfying the TCE condition the Hausdorff dimension of the set of points in the Julia set that are not in the conical Julia set is equal to 0, compare with [41]. This is a direct consequence of Theorem E in §4.4 and of Lemma 7.2.

1.5. Strategy

We now explain the strategy of proof of our main results, and simultaneously describe the organization of the paper.

To prove Theorems A, B and C we use an inducing scheme. That is, we construct a (Markovian) induced map and then deduce properties of the rational map from properties of the induced map. To construct a conformal measure supported on the conical Julia set (Theorem A), we basically construct an induced map whose maximal invariant set has the largest possible Hausdorff dimension (equal to the Hausdorff dimension of the Julia set) and which is strongly regular in the sense of [26]. By the results of [26] this induced map has a conformal measure of exponent equal to the Hausdorff dimension of the Julia set. Then we spread this measure to obtain a conformal measure of the rational map. For the existence and statistical properties of the absolutely continuous measure (Theorems B and C) we do a "tail estimate" and use the results of L.-S. Young [46]. The main difference with previous approaches is that we estimate diameters of pull-backs directly using condition ExpShrink, and not through derivatives at critical values.

We now describe the structure of the paper in more detail. The core of this paper is divided into 2 independent parts. In the first part (§§3, 4) we construct, for a given rational map satisfying the TCE condition, an induced map which is uniformly hyperbolic in the sense that its derivative is exponentially big with respect to the return time, and that it satisfies some additional properties (Theorem E in §4.4). For this, we first show in §3 that the TCE condition implies a strong form of the "Backward Contraction" property of [40]. The construction of the induced map is based on the concept of "nice couple" introduced in [40], which is closely related to nice intervals of real one-dimensional dynamics (§§4.1, 4.2). To each nice couple we associate an induced map of the rational map (§4.3). Then we repeat in §4.4 the construction of nice couples of [40], using the Backward Contraction property that was shown in §3. The desired properties of the induced map associated to this nice couple follow easily from the Backward Contraction property.

In the second part (§§5–7) we give very simple conditions on a nice couple, for a rational map satisfying the TCE condition, so that the associated induced map has the following properties: Its maximal invariant set has the largest possible Hausdorff dimension (equal to the Hausdorff dimension of the Julia set) and there is $\alpha \in (0, \mathrm{HD}(J(f)))$ such that,

$$\sum_W \mathrm{diam}(W)^\alpha < +\infty, \tag{2}$$

where the sum is over all the connected components of the domain of the induced map. This is stated as the Key Lemma in §7. Both estimates are very important for the existence of the conformal measure supported on the conical Julia set, as well as for the existence and statistical properties of the absolutely continuous invariant measure. To prove (2) we introduce in §5 a "discrete density" that behaves well under univalent and unicritical pull-backs. Then we prove in §6 a general result estimating the discrete density of the domain of the first entry map to a nice set. The use of the discrete density greatly simplifies the proof of the Key Lemma, as it reduces considerably the combinatorial arguments.

To prove Theorem A (in §8.1) we show a general result (Theorem 2 in Appendix B) that roughly states that if a rational map admits a nice couple satisfying the conclusions of the Key Lemma, then the rational map has a conformal measure supported on the conical Julia set. The proof is very simple: The hypothesis imply that the induced map associated to the nice couple is (strongly) regular in the sense of [26]. Then the results of [26] imply that this induced map has a conformal measure whose dimension is equal to $\mathrm{HD}(J(f))$, and we spread this measure using the rational map, to obtain a conformal measure of f that is supported on the conical Julia set.

We prove Theorems B and C simultaneously in §8.2. We deduce these results from some results of Young in [46]. As usual the most difficult part is the "tail estimate". In our case it follows easily from Theorem E and from the Key Lemma.

In Appendix A we gather some general properties of the induced maps considered here, the most important being translations of results of [26] to our particular setting. The proof of Theorem D is in §8.3.

1.6. Notes and references

There are quadratic polynomials having no conformal measures supported on the conical Julia set [38]. Even when there exists such a measure, little is known in general about its geometric properties. For example, to the best of our knowledge it is not known if the Hausdorff dimension of such a measure is positive.

In §1.3 we have stated the property "exponential mixing" for Lipschitz functions ψ for simplicity. A result analogous to Theorem C holds when ψ is only Hölder continuous.

In [19] it is shown that for every rational map of degree at least 2, the measure of maximal entropy is exponentially mixing and that the Central Limit Theorem holds for this measure, see also [12] for further results. This last result was first shown in [10]. However, in most cases the measure of maximal entropy does not describe well the geometry of the Julia set, because its Hausdorff dimension is strictly smaller than the Hausdorff dimension of the Julia set [47].

If in Theorem D we assumed the stronger hypothesis that the invariant measure ν is equivalent to a conformal measure, as in Theorem B, then we would immediately get $\chi(\nu) > 0$ by Ruelle's inequality, since the entropy of ν is positive in this case. In fact, if we normalize ν to have total mass equal to 1, then the Jacobian of f with respect to ν must be almost everywhere larger than 1, so the entropy of ν is positive by Rochlin's formula.

In [6, §6] it is shown that there is a complex quadratic polynomial f such that for each $t \geqslant \mathrm{HD}_{\mathrm{hyp}}(f)$ there is no equilibrium state of f with potential $-t \ln|f'|$. Since the Lyapunov exponent of each invariant measure supported on the Julia set is non-negative [30], it follows that every invariant measure supported on $J(f)$, has positive Lyapunov exponent.

2. Preliminaries

For basic references on the iteration of rational maps, see [7,27].

An open subset of the Riemann sphere $\overline{\mathbb{C}}$ will be called *simply-connected* if it is connected and if its boundary is connected.

2.1. Spherical metric

We will identify the Riemann sphere $\overline{\mathbb{C}}$ with $\mathbb{C} \cup \{\infty\}$ and endow $\overline{\mathbb{C}}$ with the spherical metric. The spherical metric will be normalized in such a way that its density with respect to the Euclidean metric on $\mathbb{C}$ is equal to $z \mapsto (1 + |z|^2)^{-1}$. With this normalization the diameter of $\overline{\mathbb{C}}$ is equal to $\pi/2$.

Distances, balls, diameters and derivatives are all taken with respect to the spherical metric. For $z \in \overline{\mathbb{C}}$ and $r > 0$, we denote by $B(z,r) \subset \overline{\mathbb{C}}$ the ball centered at z and with radius r. Note that an open ball of radius $r > \frac{1}{2}\operatorname{diam}(\overline{\mathbb{C}})$ is equal to $\overline{\mathbb{C}}$.

2.2. Critical points

Fix a complex rational map f. We denote by $\mathrm{Crit}(f)$ the set of critical points of f and by $J(f)$ the Julia set of f. Moreover we put $\mathscr{C}(f) := \mathrm{Crit}(f) \cap J(f)$. When there is no danger of confusion we denote $\mathrm{Crit}(f)$ and $\mathscr{C}(f)$ just by Crit and $\mathscr{C}$.

For simplicity we will assume that no critical point in $\mathscr{C}$ is mapped to another critical point under forward iteration. The general case can be handled by treating whole blocks of critical points as a single critical point; that is, if the critical points $c_0, \ldots, c_k \in J(f)$ are such that c_i is mapped to c_{i+1} by forward iteration, and maximal with this property, then we treat this block of critical points as a single critical point.

2.3. Pull-backs

Given a subset V of $\overline{\mathbb{C}}$ and an integer $n \geqslant 1$, each connected component of $f^{-n}(V)$ will be called *pull-back of* V *by* f^n. Note that the set V is not assumed to be connected.

When considering a pull-back of a ball, we will implicitly assume that this ball is disjoint from the forward orbits of critical points not in $J(f)$. So such a pull-back can only contain critical points in $J(f)$. If f does not have indifferent cycles (e.g. if f satisfies the TCE condition), it follows by the Fatou–Sullivan classification of the connected components of the Fatou set [2,7, 27], that there is a neighborhood of $J(f)$ disjoint from the forward orbit of critical points not in $J(f)$. So in this case every ball centered at a point of $J(f)$ and of sufficiently small radius, will meet our requirement.

2.4. Distortion of univalent pull-backs

We will now state a version of Koebe Distortion Theorem, taking into account that derivatives are taken with respect to the spherical metric. Given $x \in J(f)$, $r > 0$ and an integer $n \geqslant 1$, let W be a connected component of $f^{-n}(B(x,r))$ on which f^n is univalent. Then for $\varepsilon \in (0,1)$ let $W(\varepsilon) \subset W$ be the preimage of $B(x, \varepsilon r)$ by f^n in W.

Fix two periodic orbits O_1 and O_2 of f of period at least 2 and let $r_K > 0$ be sufficiently small such that for every $x \in \overline{\mathbb{C}}$ the ball $B(x, r_K)$ is disjoint from either O_1 or O_2. So, for every positive integer n and every component W of $f^{-n}(B(x, r_K))$, we have,

$$\operatorname{diam}\big(\overline{\mathbb{C}} \setminus W\big) \geqslant \min\big\{\operatorname{diam}(O_1), \operatorname{diam}(O_2)\big\} > 0.$$

Hence the following version of Koebe Distortion Theorem holds (see also Lemma 1.2 of [32]).

KOEBE. – *For every $\varepsilon \in (0,1)$ there exists a constant $K(\varepsilon) > 1$ such that, if $x \in J(f)$, $r > 0$, $n \geqslant 1$, W and $W(\varepsilon)$ are as above with $r \in (0, r_K)$, then the distortion of f^n on $W(\varepsilon)$ is bounded by $K(\varepsilon)$. That is, for every z_1, $z_2 \in W(\varepsilon)$ we have,*

$$\big|(f^n)'(z_1)\big| / \big|(f^n)'(z_2)\big| \leqslant K(\varepsilon).$$

Moreover $K(\varepsilon) \to 1$ as $\varepsilon \to 0$.

3. TCE implies Backward Contraction

The purpose of this section is to prove the following property of rational maps satisfying ExpShrink, which is closely related to the Backward Contraction property of [40].

PROPOSITION 3.1. – *Let f be a rational map satisfying ExpShrink with constant $\lambda_{\mathrm{Exp}} > 1$. Then for every $\lambda \in (1, \lambda_{\mathrm{Exp}})$ there are constants $\theta \in (0,1)$ and $\alpha > 0$, such that for every $\delta > 0$ small and every $c \in \mathscr{C}$ there is a constant $\delta(c) \in [\delta, \delta^\theta]$ satisfying the following property.*

For every c, $c' \in \mathscr{C}$, every integer $n \geqslant 1$ and every pull-back W of $B(c, \delta^{-\alpha}\delta(c))$ by f^n,

$$\operatorname{dist}(W, c') \leqslant \delta(c') \textit{ implies } \operatorname{diam}(W) \leqslant \lambda^{-n}\delta(c').$$

After some preliminary lemmas in §§3.1, 3.2 and 3.3, the proof of this proposition is given in §3.4.

3.1. Distortion lemma for bounded degree maps

The following is a well-known general lemma, that is needed in the proof of Proposition 3.1.

LEMMA 3.2. – *Fix $\varepsilon > 0$ and $\hat{r} > 0$ small. Let $\widehat{W}$ be a simply-connected subset of $\overline{\mathbb{C}}$ such that $\operatorname{diam}(\overline{\mathbb{C}} \setminus \widehat{W}) > \varepsilon$. Let $z \in \overline{\mathbb{C}}$, $r \in (0, \hat{r})$, $\varphi : \widehat{W} \to B(z, r)$ be a holomorphic and proper map and let W be a connected component of $\varphi^{-1}(B(z, r/2))$. Then there is a constant $K > 1$, depending only on ε and on the degree of φ, such that for every connected set $U \subset B(z, r/2)$ and every connected component $V \subset W$ of $\varphi^{-1}(U)$, we have*

$$\frac{\operatorname{diam}(V)}{\operatorname{diam}(W)} \geqslant K \frac{\operatorname{diam}(U)}{r}.$$

Proof. – For a set $D \subset \overline{\mathbb{C}}$ conformally equivalent to a disc and for $\eta \subset D$ we denote by $\operatorname{diam}_D(\eta)$ the diameter of η with respect to the hyperbolic metric of D.

Since $r \in (0, \hat{r})$ there is $K_0 > 0$ such that

$$\operatorname{diam}(U)/r \leqslant K_0 \operatorname{diam}_{B(z,r)}(U). \tag{3}$$

Moreover the modulus of the annulus $B(z,r)\setminus\overline{B(z,r/2)}$ is bounded from below in terms of $\hat{r}$ only. Thus the modulus of the annulus $\widehat{W}\setminus\overline{W}$ is bounded in terms of $\hat{r}$ and of the degree of φ only. Let $w\in W$ be such that $\varphi(w)=z$ and consider a conformal representation $\psi:\mathbb{D}\to\widehat{W}$ such that $\psi(0)=w$. Let $W':=\psi^{-1}(W)$ and $V':=\psi^{-1}(V)$.

Since $\operatorname{diam}(\overline{\mathbb{C}}\setminus\widehat{W})>\varepsilon$, by Koebe Distortion Theorem the distortion of ψ on W' (where $\mathbb{D}\subset\mathbb{C}$ is endowed with the Euclidean metric) is bounded by some constant $\kappa_0>1$. Therefore we have,

$$\operatorname{diam}(V)/\operatorname{diam}(W)\geqslant\kappa_0\operatorname{diam}(V')/\operatorname{diam}(W')\geqslant\kappa_0\operatorname{diam}(V').$$

Since the modulus of $\mathbb{D}\setminus\overline{W'}$ is equal to that of $\widehat{W}\setminus\overline{W}$, which is bounded in terms of the degree of φ only, there is a constant $\kappa_1>0$ such that

$$\operatorname{diam}(V')\geqslant\kappa_1\operatorname{diam}_{\mathbb{D}}(V')\geqslant\kappa_1\operatorname{diam}_{B(z,r)}(U),$$

where the last inequality follows from Schwarz' Lemma. So by (3) we have,

$$\frac{\operatorname{diam}(V)}{\operatorname{diam}(W)}\geqslant\kappa_0\kappa_1\operatorname{diam}_{B(z,r)}(U)\geqslant K_0^{-1}\kappa_0\kappa_1\frac{\operatorname{diam}(U)}{r}.\qquad\square$$

3.2. Diameter of pull-backs

The following is analogous to Lemma 1.9 of [33].

LEMMA 3.3. – *Let f be a rational map satisfying ExpShrink with constants $\lambda_{\mathrm{Exp}}>1$ and $r_0>0$. Then the following assertions hold.*

1. *There are constants $C_0>0$ and $\theta_0\in(0,1)$ such that for every $r\in(0,r_0)$, every integer $n\geqslant1$, every $x\in J(f)$ and every connected component W of $f^{-n}(B(x,r))$, we have*

$$\operatorname{diam}(W)\leqslant C_0\lambda_{\mathrm{Exp}}^{-n}r^{\theta_0}.$$

2. *For each $\lambda\in(1,\lambda_{\mathrm{Exp}})$ and $\beta\geqslant0$ there is a constant $A_0:=A_0(\lambda,\beta)>0$, such that if in addition $n\geqslant A_0\ln r^{-1}$, then*

$$\operatorname{diam}(W)\leqslant\lambda^{-n}r^{1+\beta}.$$

Proof. – **1**. Put $M=\sup|f'|$. Given $r\in(0,r_0)$ and $x\in J(f)$ let $m\geqslant0$ be the integer such that $M^{-(m-1)}\leqslant r<M^{-m}$ and let W_0 be the connected component of $f^{-m}(B(f^m(x),r_0))$ that contains x. Thus,

$$B(x,r)\subset B(x,M^{-m})\subset W_0.$$

Let $n\geqslant1$ be an integer, let W be a pull-back of $B(x,r)$ by f^n and let W_n be the corresponding pull-back of W_0. Then, letting $\theta_0:=\frac{\ln M}{\ln\lambda_{\mathrm{Exp}}}$ we have

$$\operatorname{diam}(W)\leqslant\operatorname{diam}(W_n)\leqslant\lambda_{\mathrm{Exp}}^{-(n+m)}=\lambda_{\mathrm{Exp}}^{-n}M^{-\theta_0 m}\leqslant M^{\theta_0}\lambda_{\mathrm{Exp}}^{-n}r^{\theta_0}.$$

2. Let A_0 be a constant satisfying $A_0>(1+\beta-\theta_0)(\ln\frac{\lambda_{\mathrm{Exp}}}{\lambda})^{-1}$. So if $n\geqslant A_0\ln\frac{1}{r}$ and $r>0$ is small enough, we have

$$M^{\theta_0}\lambda_{\mathrm{Exp}}^{-n}r^{\theta_0}\leqslant\lambda^{-n}r^{1+\beta}.\qquad\square$$

3.3. Pull-backs and critical points

Fix a complex rational map f of degree at least 2. Recall that we assume that no critical point in $\mathscr{C} := \mathrm{Crit}(f) \cap J(f)$ is mapped to a critical point under forward iteration (cf. Preliminaries).

The following is Lemma 1 of [30].

LEMMA 3.4. – *Put $M = \sup_{\overline{\mathbb{C}}} |f'|$. Then there is a constant $\kappa > 0$ such that for every $c \in \mathscr{C}$ and every $n \geqslant 1$, we have*

$$\mathrm{dist}\big(c, f^{-n}(c)\big) \geqslant \kappa M^{-n}.$$

Proof. – For a critical point c of f denote by $d(c)$ the local degree of f at c. Let $C \geqslant 0$ be such that for $\delta > 0$ small and $c \in \mathscr{C}$, we have $\mathrm{diam}(f(B(c,\delta))) \leqslant C\delta^{d(c)}$. Given $n \geqslant 1$ let $w \in f^{-n}(c)$ be a point closest to c and put $r := \mathrm{dist}(c, w)$. Letting $C_0(c) := M^{-1}C3^{d(c)}$ and considering that $B(w, 2r) \subset B(c, 3r)$, we have

$$\mathrm{diam}\big(f^n\big(B(w,2r)\big)\big) \leqslant M^{n-1}\,\mathrm{diam}\big(f\big(B(w,2r)\big)\big) \leqslant C_0(c)M^n r^{d(c)}.$$

Thus $f^n(B(c,r)) \subset f^n(B(w,2r)) \subset B(c, C_0(c)M^n r^{d(c)})$. Since $c \in J(f)$ we must have $C_0(c)M^n r^{d(c)} > r$, so the lemma follows with constant $\kappa := \max_{c\in\mathscr{C}} C_0(c)^{-1}$. □

LEMMA 3.5. – *Let f be a rational map satisfying ExpShrink. Then there are constants $r_1 > 0$ and $A_1 > 0$ such that for every $c \in \mathscr{C}$, every $r \in (0, r_1)$, every $n \geqslant 1$ and every connected component W of $f^{-n}(B(c,r))$,*

$$\mathrm{dist}(c, W) \leqslant r \textit{ implies } n \geqslant A_1 \ln 1/r.$$

Proof. – Let C_0, $r_0 > 0$ and $\theta_0 \in (0,1)$ be given by Lemma 3.3. So, assuming $r \in (0, r_0)$ we have $\mathrm{diam}(W) \leqslant C_0 r^{\theta_0}$. Since W contains a n-th preimage of c we have by Lemma 3.4,

$$\kappa M^{-n} \leqslant r + \mathrm{diam}(W) \leqslant 2C_0 r^{\theta_0}.$$

Hence the lemma follows with any constant $A_1 \in (0, \theta_0/\ln M)$, by taking r_1 sufficiently small. □

LEMMA 3.6. – *Let f be a rational map satisfying ExpShrink and let $N \geqslant 1$ be the number of critical points of f in $J(f)$. Then there are constants $r_1 > 0$, $\xi_1 > 1$ and $\nu_1 > 0$ such that for every $x \in J(f)$, every $r \in (0, r_1)$ and every sequence of successive pull-backs $W_1, W_2, \ldots$ of $W_0 := B(x,r)$ by f, the following assertions hold.*

1. *The sequence $0 \leqslant m_0 < m_1 < \cdots$ of all integers m such that W_m contains a critical point satisfies,*

$$m_i \geqslant \nu_1 \xi_1^i \ln 1/r, \quad \textit{for every } i > N.$$

2. *For every $A > 0$ there is a constant $D_1 := D_1(A) \geqslant 1$ such that if $n \leqslant A \ln 1/r$ then the degree of*

$$f^n : W \to B(x,r)$$

is at most D_1.

Proof. – Note that part 2 is a direct consequence of part 1.

Fix $\lambda \in (1, \lambda_{\mathrm{Exp}})$ and let $r_0 > 0$, $C_0 > 0$ and $\theta_0 \in (0,1)$ be as in Lemma 3.3, so $\mathrm{diam}(W_i) \leqslant C_0 \lambda^{-i} r^{\theta_0}$ for $i \geqslant 0$. In particular we may suppose that W_{m_i} contains a unique critical point $c_i \in \mathscr{C}$.

Fix $i \geqslant N$. Then there is a critical point c such that

$$k = \#\{j \in \{0, \ldots, i\} \mid c_j = c\} \geqslant (i+1)/N > 1;$$

let $m_0 \leqslant n_0 < n_1 < \cdots < n_{k-1} \leqslant m_i$ be all the integers such that $c \in W_{n_j}$. Thus for every $j = 1, \ldots, k-1$ the set W_{n_j} contains a $(n_j - n_{j-1})$-th preimage of c. Then Lemma 3.4 implies that,

$$\kappa M^{-(n_j - n_{j-1})} \leqslant \operatorname{diam}(W_{n_j}) \leqslant C_0 \lambda^{-n_j} r^{\theta_0}.$$

So letting $\xi := \frac{\ln M}{\ln(M/\lambda)} > 1$ and $\nu := \frac{1}{2} \frac{\theta_0}{\ln(M/\lambda)}$ and assuming $r_1 > 0$ small enough, we have $n_j \geqslant \xi n_{j-1} + \nu \ln 1/r$. Therefore,

$$m_i \geqslant n_k \geqslant \nu \xi^{k-2} \ln 1/r \geqslant (\nu \xi^{-2})(\xi^{1/N})^i \ln 1/r,$$

and the lemma follows with $\xi_1 := \xi^{1/N}$ and $\nu_1 := \nu \xi^{-2}$. □

3.4. Proof of Proposition 3.1

Let f be a rational map satisfying condition ExpShrink with constants $\lambda_{\mathrm{Exp}} > 1$ and $r_0 > 0$, and let $C_0 > 0$ and $\theta_0 \in (0,1)$ be the constants given by part 1 of Lemma 3.3. Moreover let $r_1 > 0$ and $A_1 > 0$ be the constants given by Lemma 3.5. Denote by N the cardinality of $\mathscr{C}$ and choose $\theta_2 \in (0, \theta_0)$, $\lambda_0 \in (1, \lambda_{\mathrm{Exp}})$, $\lambda \in (0, \lambda_0)$, and

$$\alpha_2 \in \big(0, \min\{(A_1 \theta_2 / 4N) \ln(\lambda_0/\lambda), (\theta_0 - \theta_2)/2N\}\big).$$

Fix $\delta > 0$ small enough so that $\delta^{\theta_2} < r_1$. Let $c_0 \in \mathscr{C}$ and consider a sequence of successive pull-backs $W_1, W_2, \ldots$ of $W_0 := B(c_0, \delta)$ by f. For $j \geqslant 0$ define $c_j \in \mathscr{C}$, $\delta_j > 0$ and $m_j \geqslant 1$ inductively as follows. Put $\delta_0 := \delta$ and $m_0 := 0$. Once c_{j-1}, δ_{j-1} and m_{j-1} are defined, let $m_j > m_{j-1}$ be the least integer such that for some $c_j \in \mathscr{C}$ we have

$$\delta_j := \lambda^{m_j} \delta^{-j\alpha_2} \operatorname{diam}(W_{m_j}) > \operatorname{dist}(c_j, W_{m_j}).$$

Notice that for every $j \geqslant 0$ we have $W_{m_j} \subset B(c_j, 2\delta_j)$.

LEMMA 3.7. – *There is a constant $A_2 > 0$ independent of δ, such that the following assertions hold.*

1. *For every $j \geqslant 0$ we have $\delta_j \leqslant \delta^{\theta_2}$.*
2. *If $j \geqslant 1$ is such that $m_j \geqslant A_2 \ln 1/\delta$, then $\delta_j \leqslant \delta$.*

Proof. – Parts 1 and 2 of the lemma will be shown in parts 3 and 4 below.

1. If $\delta > 0$ is sufficiently small, then for every $j = 0, \ldots, 2N$ we have

$$\begin{aligned}\delta_j = \lambda^{m_j} \delta^{-j\alpha_2} \operatorname{diam}(W_{m_j}) &\leqslant \lambda^{m_j} \delta^{-j\alpha_2} C_0 \lambda_0^{-m_j} \delta^{\theta_0} \\ &\leqslant C_0 \delta^{\theta_0 - 2N\alpha_2} < \delta^{\theta_2}/2.\end{aligned}$$

2. Consider $j' > j \geqslant 0$ such that $c_{j'} = c_j$ and such that

$$\delta' := \max\{\delta_j, \delta_{j'}\} \leqslant \delta^{\theta_2}/2 < r_1/2.$$

Then $W_{m_j}, W_{m_{j'}} \subset B(c_j, 2\delta')$ and the pull-back W of $B(c_j, 2\delta')$ by $f^{m_{j'}-m_j}$ containing $W_{m_{j'}}$ intersects $B(c_j, 2\delta')$. So by Lemma 3.5

$$m_{j'} - m_j \geqslant A_1 \ln 1/(2\delta') \geqslant A_1\theta_2 \ln 1/\delta.$$

3. Given $k \geqslant 2N$ suppose by induction that $\delta_j \leqslant \delta^{\theta_2}$ for $j = 0, \ldots, k$. By part 1 this holds for $k = 2N$. Notice that there must be a critical point $c \in \mathscr{C}$ such that,

$$\#\big\{j \in \{1, \ldots, k\} \mid c_j = c\big\} \geqslant k/N.$$

Since $k \geqslant 2N$ it follows by part 2 that

$$m_k \geqslant (k/N - 1)A_1\theta_2 \ln 1/\delta \geqslant (A_1\theta_2/2N)k \ln 1/\delta. \tag{4}$$

By definition $\alpha_2 < (A_1\theta_2/4N)\ln(\lambda_0/\lambda)$, so $(\lambda_0/\lambda)^{m_k} > \delta^{-2k\alpha_2}$. Considering that $m_{k+1} > m_k$ we have

$$\operatorname{diam}(W_{m_{k+1}}) \leqslant C_0\lambda_0^{-m_{k+1}}\delta^{\theta_0} \leqslant C_0\lambda^{-m_{k+1}}\delta^{2k\alpha_2}\delta^{\theta_0} < \lambda^{-m_{k+1}}\delta^{(k+1)\alpha_2}\delta^{\theta_0}, \tag{5}$$

therefore

$$\delta_{k+1} = \lambda^{m_{k+1}}\delta^{-(k+1)\alpha_2} \operatorname{diam}(W_{m_{k+1}}) \leqslant \delta^{\theta_0} < \delta^{\theta_2}.$$

4. To prove part 2 of the lemma, let $A_0(\lambda_0, 0)$ be the constant given by part 2 of Lemma 3.3 with $\lambda := \lambda_0$ and $\beta := 0$ and put

$$A_2 := \max\big\{A_0(\lambda_0, 0), A_1\theta_2\big\}.$$

Let $j \geqslant 1$ be such that $m_j \geqslant A_2 \ln 1/\delta$. If $k := j - 1 \geqslant 2N$, then by Lemma 3.3 and by the same reasoning as in part 3, we conclude that inequality (5) holds with δ^{θ_0} replaced by δ. So in this case we have $\delta_j = \delta_{k+1} \leqslant \delta$. If $k := j - 1 < 2N$, then by definition of A_2 we have $m_k \geqslant A_1\theta_2 \ln 1/\delta \geqslant (A_1\theta_2/2N)k \ln 1/\delta$. So again inequality (5) holds with δ^{θ_0} replaced by δ and therefore $\delta_j = \delta_{k+1} \leqslant \delta$. □

Proof of Proposition 3.1. – For each $c \in \mathscr{C}$ put

$$\delta(c) := \sup_{W_0, W_1, \ldots,\ c_j = c} \lambda^{m_j}\delta^{-j\alpha_2} \operatorname{diam}(W_j),$$

where the supremum is taken over all $c_0 \in \mathscr{C}$ and over all successive pull-backs $W_1, W_2, \ldots$ of $W_0 := B(c_0, \delta)$ by f, and where c_j, δ_j and m_j are defined as above.

Put $\theta := \theta_2$ and fix $\alpha \in (0, \alpha_2)$. In what follows we will prove that, if $\delta > 0$ is sufficiently small, then the assertion of Proposition 3.1 holds for this choice θ, α and $\delta(c)$.

1. Taking $c_0 = c$ and $j = 0$ in the definition of $\delta(c)$, we have $\delta(c) \geqslant \delta$. Moreover by part 1 of Lemma 3.7 we have $\delta(c) \leqslant \delta^{\theta_2}$. On the other hand by part 2 of the same lemma, $\delta_j > \delta$ implies that $m_j \leqslant A_2 \ln 1/\delta$. So the supremum defining $\delta(c)$ is attained.

2. Let c, $c' \in \mathscr{C}$, $n \geqslant 1$ and let W be a pull-back of $B(c, \delta^{-\alpha}\delta(c))$ by f^n such that $\operatorname{dist}(c', W) \leqslant \delta(c')$.

By part 2 of Lemma 3.3 it follows that there is $A_0 > 0$ such that, if $n \geqslant A_0 \ln 1/\delta$, then $\operatorname{diam}(W) \leqslant \lambda^{-n}\delta \leqslant \lambda^{-n}\delta(c')$. So we assume that $n \leqslant A_0 \ln 1/\delta$. By part 1 there are $c_0 \in \mathscr{C}$

and a sequence of successive pull-backs $W_1, \ldots, W_{m_k}$ of $W_0 := B(c_0, \delta)$ by f, such that for some $k \geqslant 0$ we have $c_k = c$ and

$$\delta(c) = \delta_k = \lambda^{m_k} \delta^{-k\alpha_2} \operatorname{diam}(W_{m_k}) > \operatorname{dist}(c, W_{m_k}).$$

Note that $W_{m_k} \subset B(c, 2\delta(c)) \subset B(c, \delta^{-\alpha}\delta(c))$. Consider the sequence of successive pull-backs $W_{m_k+1}, \ldots, W_{m_k+n}$ of W_{m_k}, such that $W_{m_k+n} \subset W$. Let c_j, δ_j and m_j be as in Lemma 3.7 for the sequence of successive pull-backs $W_1, \ldots, W_{m_k+n}$ of W_0. Then for some $\ell > k$ we have $c_\ell = c'$ and $m_\ell = m_k + n$. Hence by definition of $\delta(c')$,

$$\operatorname{diam}(W_{m_k+n}) \leqslant \lambda^{-(m_k+n)} \delta^{\ell\alpha_2} \delta(c'). \tag{6}$$

3. Let $\widehat{W}$ be the pull-back of $B(c, 2\delta^{-\alpha}\delta(c))$ by f^n that contains W. Since $n \leqslant A_0 \ln 1/\delta$ it follows by Lemma 3.6 that the degree of $f^n : \widehat{W} \to B(c, 2\delta^{-\alpha}\delta(c))$ is bounded by a constant depending on A_0 only. Hence by Lemma 3.2 there is a constant $K > 1$ such that,

$$\operatorname{diam}(W)/\operatorname{diam}(W_{m_k+n}) \leqslant K\delta^{-\alpha}\delta(c)/\operatorname{diam}(W_{m_k}) = K\delta^{-\alpha}\lambda^{m_k}\delta^{-k\alpha_2}.$$

So by (6) and assuming $\delta > 0$ small enough we have,

$$\operatorname{diam}(W) \leqslant K\delta^{-\alpha}\lambda^{-n}\delta^{(\ell-k)\alpha_2}\delta(c') < \lambda^{-n}\delta(c').$$

This ends the proof of Proposition 3.1. □

4. Induced maps

Given a rational map satisfying the TCE condition, the purpose of this section is to construct an induced map that is hyperbolic in the sense that its derivative is exponentially big with respect to the return time, and that it satisfies some additional properties (Theorem E). The construction of this induced map is based on the construction of "nice couples" in [40]. We first recall the definition of nice sets (§4.1) and of nice couples (§4.2), and then we explain how to associate to each nice couple an induced map (§4.3). The statement and proof of Theorem E are in §4.4.

4.1. Nice sets

Let f be a complex rational map. We will say that a neighborhood V of $\mathscr{C}$ that is disjoint from the forward orbits of critical points not in $\mathscr{C}$ is a *nice set for f*, if it satisfies the following properties: The set V is the union of sets V^c, for $c \in \mathscr{C}$, such that V^c is a simply-connected neighborhood of c, such that the closures of the sets V^c are pairwise disjoint, and such that for every $n \geqslant 1$ we have $f^n(\partial V) \cap \overline{V} = \emptyset$.

The last condition, that for every $n \geqslant 1$ we have $f^n(\partial V) \cap \overline{V} = \emptyset$, is easily seen to be equivalent to the condition that for every pull-back W of V we have either

$$\overline{W} \cap \overline{V} = \emptyset \quad \text{or} \quad \overline{W} \subset V.$$

It follows that if W and W' are distinct pull-backs of a nice set V of f, then we have either,

$$\overline{W} \cap \overline{W'} = \emptyset, \quad \overline{W} \subset W' \quad \text{or} \quad \overline{W'} \subset W.$$

For a pull-back W of a nice set V of f we denote by $c(W)$ the critical point in $\mathscr{C}$ and by $m_W \geqslant 0$ the integer such that $f^{m_W}(W) = V^{c(W)}$. Moreover we put,

$$K(V) = \left\{ z \in \overline{\mathbb{C}} \mid \text{for every } n \geqslant 0 \text{ we have } f^n(z) \notin V \right\}.$$

Note that $K(V)$ is a compact and forward invariant set and for each $c \in \mathscr{C}$ the set V^c is a connected component of $\overline{\mathbb{C}} \setminus K(V)$. Moreover, if W is a connected component of $\overline{\mathbb{C}} \setminus K(V)$ different from the V^c, then $f(W)$ is again a connected component of $\overline{\mathbb{C}} \setminus K(V)$. It follows that W is a pull-back of V and that f^{m_W} is univalent on W.

4.2. Nice couples

A *nice couple for* f is a pair of nice sets $(\widehat{V}, V)$ for f such that $\overline{V} \subset \widehat{V}$ and such that for every $n \geqslant 1$ we have $f^n(\partial V) \cap \overline{\widehat{V}} = \emptyset$. Equivalently, a pair of nice sets $(\widehat{V}, V)$ is a nice couple if for every pull-back $\widehat{W}$ of $\widehat{V}$ we have either

$$\overline{\widehat{W}} \cap \overline{V} = \emptyset \quad \text{or} \quad \overline{\widehat{W}} \subset V.$$

Let $(\widehat{V}, V)$ be a nice couple for f. Then for each pull-back W of V we denote by $\widehat{W}$ the corresponding pull-back of $\widehat{V}$, in such a way that $W \subset \widehat{W}$, $m_{\widehat{W}} = m_W$ and $c(\widehat{W}) = c(W)$. If W and W' are disjoint pull-backs of V such that the sets $\widehat{W}$ and $\widehat{W'}$ intersect, then we have either

$$\widehat{W} \subset \widehat{W'} \setminus W' \quad \text{or} \quad \widehat{W'} \subset \widehat{W} \setminus W. \tag{7}$$

If W is a connected component of $\overline{\mathbb{C}} \setminus K(V)$, then for every $j = 0, \ldots, m_W - 1$, the set $f^j(W)$ is a connected component of $\overline{\mathbb{C}} \setminus K(V)$ different from the V^c, and $\widehat{f^j(W)}$ is disjoint from V. It follows that $\widehat{f^j(W)}$ does not contain critical points of f and that f^{m_W} is univalent on $\widehat{W}$.

4.3. The canonical induced map associated to a nice couple

Let f be a complex rational map and let $(\widehat{V}, V)$ be a nice couple for f. We will say that an integer $m \geqslant 1$ is a *good time for a point* z in V, if $f^m(z) \in V$ and if the pull-back of $\widehat{V}$ by f^m to z is univalent. Let D be the set of all those points in V having a good time and for $z \in D$ denote by $m(z) \geqslant 1$ the least good time of z. Then the map $F : D \to V$ defined by $F(z) := f^{m(z)}(z)$ is called *the canonical induced map associated to* $(\widehat{V}, V)$. We denote by $J(F)$ the maximal invariant set of F.

As V is a nice set, it follows that each connected component W of D is a pull-back of V. Moreover, f^{m_W} is univalent on $\widehat{W}$ and for each $z \in W$ we have $m(z) = m_W$. Similarly, for each positive integer n, each connected component W of the domain of definition of F^n is a pull-back of V and f^{m_W} is univalent on $\widehat{W}$. Conversely, if W is a pull-back of V contained in V and such that f^{m_W} is univalent on $\widehat{W}$, then there are $c \in \mathscr{C}$ and a positive integer n such that F^n is defined on W and $F^n(W) = V^c$. In fact, in this case m_W is a good time for each element of W and therefore $W \subset D$. Thus, either we have $F(W) = V^{c(W)}$, and then W is a connected component of D, or $F(W)$ is a pull-back of V contained in V such that $f^{m_{F(W)}}$ is univalent on $\widehat{F(W)}$. Thus, repeating this argument we can show by induction that there is a positive integer n such that F^n is defined on W and that $F^n(W) = V^{c(W)}$.

LEMMA 4.1. – *For every rational map f there is $r>0$ such that if $(\widehat{V}, V)$ is a nice couple satisfying*

$$\max_{c\in\mathscr{C}} \operatorname{diam}\big(\widehat{V}^c\big) \leqslant r, \tag{8}$$

then the canonical induced map $F: D\to V$ associated to $(\widehat{V}, V)$ is topologically mixing on $J(F)$. Moreover there is $\tilde{c}\in\mathscr{C}$ such that the set

$$\big\{m_W \mid W \text{ c.c. of } D \text{ contained in } V^{\tilde{c}} \text{ such that } F(W)=V^{\tilde{c}}\big\} \tag{9}$$

is non-empty and its greatest common divisor is equal to 1.

Proof. – Let p be a repelling periodic point of f. By the locally eventually onto property of Julia sets [7,27], for each critical point $c\in\mathscr{C}$ there is a backward orbit starting at c that is asymptotic to the backward periodic orbit of f starting at p. As our standing assumption is that no critical point is mapped into another critical point under forward iteration, this backward orbit does not contain critical points. Let $r>0$ be sufficiently small, so that $B(\mathscr{C}, r)$ intersects each of these backward orbits only at its starting point in $\mathscr{C}$.

Let $(\widehat{V}, V)$ be a nice couple for f satisfying (8) and let F be the canonical induced map associated to $(\widehat{V}, V)$. Since the partition of $J(F)$ induced by the connected components of D is generating, it follows that for every open set U intersecting $J(F)$ there is $c\in\mathscr{C}$ and an integer n such that $F^n(U)$ contains V^c. Thus, to prove that F is topologically mixing we just have to show that for every $c, c'\in\mathscr{C}$ there is $n_0>0$ such that for every positive integer $n\geqslant n_0$ the set $F^n(V^{c'})$ contains V^c. For this, we will show that for every $c, c'\in\mathscr{C}$ there is a positive integer n such that the set $F^n(V^{c'})$ contains V^c, and then that there is $c_0\in\mathscr{C}$ such that $F(V^{c_0})$ contains V^{c_0}.

By (8) it follows that for each $c\in\mathscr{C}$, the periodic point p is accumulated by a sequence $(\widehat{W}_n)_{n\geqslant 1}$ of connected components of $\overline{\mathbb{C}}\setminus K(\widehat{V})$ that are pull-backs of $\widehat{V}^c$. By the locally eventually onto property of Julia sets, it follows that for each $c'\in\mathscr{C}$ there are an integer $n(c')$ and $q(c')\in V^{c'}$ such that $f^{n(c')}(q(c'))=p$. As for large n the set $\widehat{W}_n$ is disjoint from $f^{n(c')}(\mathscr{C})$, it follows that for large n the pull-back of $\widehat{W}_n$ by $f^{n(c')}$ near $q(c')$ is a univalent pull-back of $\widehat{V}^c$. This shows that for every $c, c'\in\mathscr{C}$ there is a positive integer n such that $F^n(V^{c'})$ contains V^c.

Let us prove now that there is $c_0\in\mathscr{C}$ such that $F(V^{c_0})$ contains V^{c_0}. Choose an arbitrary $c'\in\mathscr{C}$ and let $q(c')$ be as before. Let $n\geqslant 0$ be the largest integer such that $q:=f^n(q(c'))\in V$ and let $c_0\in\mathscr{C}$ be such that $q\in V^{c_0}$. Then q is an iterated preimage of p such that $f(q)\in K(V)$. As before, for every $c\in\mathscr{C}$ the point q is accumulated by univalent pull-backs of $\widehat{V}^c$. As $f(q)\in K(V)$, it follows that the corresponding pull-backs of V^c are contained in the domain of F and thus that $F(V^{c_0})$ contains V^c. In particular $F(V^{c_0})$ contains V^{c_0}.

To prove the final statement we will use the fact that every rational map has a repelling periodic point of each sufficiently large period. This follows from the result of [2, Theorem 6.2.2, p. 102], that every rational map has a periodic point of a given (minimal) period greater than or equal to 4, and from the fact that a rational map possesses at most finitely many non-repelling periodic points. For the proof of the final statement, notice that we can take the sequence $(\widehat{W}_n)_{n\geqslant 1}$ above, in such a way that $(m_{\widehat{W}_n})_{n\geqslant 1}$ is an arithmetic progression for which the difference between 2 consecutive terms is equal to the minimal period of p. This shows that the set (9) contains a set of the form $\{a+nb \mid n\geqslant 0\}$, where b is the minimal period of p. Repeating the argument with $\#\mathscr{C}+1$ repelling periodic points whose periods are pairwise distinct prime numbers, we can find $\tilde{c}\in\mathscr{C}$ for which the set (9) contains sets of the form $\{a_0+nb_0 \mid n\geqslant 0\}$ and $\{a_1+nb_1 \mid n\geqslant 0\}$, where b_0 and b_1 are distinct prime numbers. This implies that the greatest common divisor of the set (9) is equal to 1. $\square$

4.4. Constructing nice couples

The purpose of this section is to prove the following result.

THEOREM E. – *Let f be a rational map satisfying ExpShrink with constant $\lambda_{\mathrm{Exp}} > 1$. Then for every $\lambda \in (1, \lambda_{\mathrm{Exp}})$, $\mathbf{m} > 0$ and $r > 0$ there is a nice couple $(\widehat{V}, V)$ such that,*

$$\min_{c \in \mathscr{C}} \operatorname{mod}\big(\widehat{V}^c \setminus \overline{V^c}\big) \geqslant \mathbf{m}, \qquad \max_{c \in \mathscr{C}} \operatorname{diam}\big(\widehat{V}^c\big) \leqslant r, \tag{10}$$

and such that the canonical induced map $F : D \to V$ associated to $(\widehat{V}, V)$ satisfies the following property: For every $z \in D$ we have $|F'(z)| \geqslant \lambda^{m(z)}$.

In view of Proposition 3.1, this theorem is a direct consequence of the following proposition.

PROPOSITION 4.2. – *Let f be a rational map satisfying ExpShrink with constant $\lambda_{\mathrm{Exp}} > 1$ and choose $\lambda \in (1, \lambda_{\mathrm{Exp}})$ and $\tau \in (0, \frac{1}{4})$. For $\delta > 0$ small and $c \in \mathscr{C}$, let $\delta(c) \geqslant \delta$ be given by Proposition 3.1. Then for every $\delta > 0$ sufficiently small there is a nice couple $(\widehat{V}, V)$ for f such that for each $c \in \mathscr{C}$ we have*

$$B\left(c, \frac{1}{2}\delta(c)\right) \subset \widehat{V}^c \subset B\big(c, \delta(c)\big) \quad \text{and} \quad B\big(c, \tau\delta(c)\big) \subset V^c \subset B\big(c, 2\tau\delta(c)\big). \tag{11}$$

The proof of this proposition is a repetition of [40, Proposition 6.6]. We include it here for completeness. It depends on the following lemma.

LEMMA 4.3. – *Given $\delta > 0$ small, put $\widetilde{V}_0 = \bigcup_{c \in \mathscr{C}} B(c, \delta(c))$ and*

$$K\big(\widetilde{V}_0\big) = \big\{z \in \overline{\mathbb{C}} \mid \text{for every } n \geqslant 0 \text{ we have } f^n(z) \notin \widetilde{V}_0\big\}.$$

If $\delta > 0$ is sufficiently small, then for each $c \in \mathscr{C}$ the connected component $\widetilde{V}^c$ of $\overline{\mathbb{C}} \setminus K(\widetilde{V}_0)$ that contains c satisfies

$$B\big(c, \delta(c)\big) \subset \widetilde{V}^c \subset B\big(c, 2\delta(c)\big).$$

Proof. – Let $\alpha > 0$ be given by Proposition 3.1 and suppose that $\delta > 0$ is sufficiently small so that $\delta^{-\alpha} > 2$.

Given $c \in \mathscr{C}$ and an integer $n \geqslant 0$, let $\widetilde{V}_n^c$ be the connected component of $\bigcup_{j=0,\ldots,n} f^{-j}(\widetilde{V}_0)$ that contains c. Note that $\widetilde{V}_0^c = B(c, \delta(c))$ and that $\widetilde{V}_0 = \bigcup_{c \in \mathscr{C}} \widetilde{V}_0^c$. Moreover, note that $\widetilde{V}_n^c$ is increasing with n and that $\widetilde{V}^c = \bigcup_{n \geqslant 0} \widetilde{V}_n^c$. To prove the lemma is enough to show that for every integer $n \geqslant 0$ we have $\widetilde{V}_n^c \subset B(c, 2\delta(c))$.

We will proceed by induction in n. The case $n = 0$ being trivial, suppose by induction hypothesis that the assertion holds for some $n \geqslant 0$ and fix $c \in \mathscr{C}$. We will show that the assertion holds for $n+1$. For every point $z \in \widetilde{V}_{n+1}^c$ there are an integer $m \in \{0, \ldots, n+1\}$ and $c_0 \in \mathscr{C}$ such that $f^m(z) \in B(c_0, \delta(c_0))$; let $m(z)$ be the least of such integers. Let X be a connected component of $\widetilde{V}_{n+1}^c \setminus B(c, \delta(c))$ and let $z \in X$ for which $m(z)$ is minimal among points in X. Let $c_0 \in \mathscr{C}$ be such that $f^{m(z)}(z) \in B(c_0, \delta(c_0))$. Considering that $m(z) > 0$, we have by induction hypothesis

$$f^{m(z)}(X) \subset \widetilde{V}_n^{c_0} \subset B\big(c_0, 2\delta(c_0)\big).$$

Then Proposition 3.1 implies that $\operatorname{diam}(X) < \delta(c)$. Thus $X \subset B(c, 2\delta(c))$ and $\widetilde{V}_{n+1}^c \subset B(c, 2\delta(c))$. This completes the induction step and the proof of the lemma. $\square$

Proof of Proposition 4.2. – Let $\theta \in (0,1)$ and $\alpha > 0$ be given by Proposition 3.1 and choose $\delta > 0$ sufficiently small so that the conclusions of Proposition 3.1 hold and so that $\delta^{-\alpha} > 2$. Furthermore, we assume that $\delta > 0$ is sufficiently small so that the least positive integer $L \geqslant 1$ such that $f^L(\mathscr{C})$ intersects $B(\mathscr{C}, \delta^\theta)$, satisfies $\lambda^{-L} < \tau$.

For $c \in \mathscr{C}$ let $\widetilde{V}^c$ be the connected component of $\overline{\mathbb{C}} \setminus K(\widetilde{V}_0)$ that contains c and put $\widetilde{V} = \bigcup_{c\in\mathscr{C}} \widetilde{V}^c$. Then Lemma 4.3 implies that for each $c \in \mathscr{C}$ we have $\widetilde{V}^c \subset B(c, 2\delta(c)) \subset B(c, \delta^{-\alpha}\delta(c))$.

For each $c \in \mathscr{C}$ we will construct sets $\widehat{V}^c$ and V^c satisfying (11) and such that in addition $\overline{\widehat{V}^c} \subset \widetilde{V}^c$ and $\partial\widehat{V}^c, \partial V^c \subset f^{-1}(K(\widetilde{V}_0))$. It follows that $\widehat{V} := \bigcup_{c\in\mathscr{C}} \widehat{V}^c$ and $V := \bigcup_{c\in\mathscr{C}} V^c$ are nice sets for f and that $(\widehat{V}, V)$ is a nice couple for f. We will just construct $\widehat{V}^c$; the construction of V^c is analogous. Note that Proposition 3.1 implies that for every $c \in \mathscr{C}$ and every pull-back W of $\widetilde{V}$ by f that intersects $B(c, \delta(c))$, we have $\operatorname{diam}(W) < \tau\delta(c)$. For each $c \in \mathscr{C}$ denote by $\ddot{V}^c$ the union of $B(c, \frac{1}{2}\delta(c))$ and of all the connected components of $\overline{\mathbb{C}} \setminus f^{-1}(K(\widetilde{V}_0))$ intersecting $B(c, \frac{1}{2}\delta(c))$. We have

$$B\left(c, \frac{1}{2}\delta(c)\right) \subset \ddot{V}^c \subset B\left(c, \left(\frac{1}{2} + \tau\right)\delta(c)\right) \quad \text{and} \quad \partial\ddot{V}^c \subset f^{-1}\big(K\big(\widetilde{V}_0\big)\big).$$

Now denote by $\widehat{V}^c$ the union of $\ddot{V}^c$ and of all the connected components W of $\overline{\mathbb{C}} \setminus \ddot{V}^c$ such that $\operatorname{diam}(W) < \operatorname{diam}(\ddot{V}^c)$. It follows that $\widehat{V}^c$ is simply-connected, that

$$B\left(c, \frac{1}{2}\delta(c)\right) \subset \widehat{V}^c \subset B\left(c, \left(\frac{1}{2} + \tau\right)\delta(c)\right) \text{ and that } \partial\widehat{V}^c \subset \partial\ddot{V}^c \subset f^{-1}\big(K\big(\widetilde{V}_0\big)\big).$$

Observe finally that $\overline{\widehat{V}^c} \subset \widetilde{V}^c$. $\square$

5. The density $\|\cdot\|_\alpha$

Throughout all this section we fix $\alpha > 0$.

5.1. Families of subsets of $\overline{\mathbb{C}}$ and the density $\|\cdot\|_\alpha$

Given a family $\mathfrak{F}$ of subsets of $\overline{\mathbb{C}}$ put

$$\operatorname{supp}(\mathfrak{F}) := \bigcup_{W\in\mathfrak{F}} W \quad \text{and} \quad \|\mathfrak{F}\|_\alpha := \sup_\varphi \left(\sum_{W\in\mathfrak{F}} \operatorname{diam}\big(\varphi(W)\big)^\alpha\right),$$

where the supremum is taken over all Möbius transformations φ. For two such families $\mathfrak{F}$ and $\mathfrak{F}'$ we have

$$\|\mathfrak{F} \cup \mathfrak{F}'\|_\alpha \leqslant \|\mathfrak{F}\|_\alpha + \|\mathfrak{F}'\|_\alpha.$$

In the case $\mathfrak{F} \subset \mathfrak{F}'$ we also have $\|\mathfrak{F}\|_\alpha \leqslant \|\mathfrak{F}'\|_\alpha$.

LEMMA 5.1. – *For every family $\mathfrak{F}$ of subsets of $\overline{\mathbb{C}}$ we have*

$$\sum_{W\in\mathfrak{F}} \operatorname{diam}(W)^\alpha \leqslant 4^\alpha \cdot \operatorname{diam}\big(\operatorname{supp}(\mathfrak{F})\big)^\alpha \|\mathfrak{F}\|_\alpha.$$

Proof. – Recall that we identify the Riemann sphere $\overline{\mathbb{C}}$ with $\mathbb{C} \cup \{\infty\}$.

When $\operatorname{diam}(\operatorname{supp}(\mathfrak{F})) \geqslant \frac{1}{4}$ there is nothing to prove, so assume that $\operatorname{diam}(\operatorname{supp}(\mathfrak{F})) < \frac{1}{4}$. After an isometric change of coordinates assume $0 \in \operatorname{supp}(\mathfrak{F})$. The hypothesis $\operatorname{diam}(\operatorname{supp}(\mathfrak{F})) < \frac{1}{4}$ implies that $\operatorname{supp}(\mathfrak{F})$ is contained in $\{z \in \mathbb{C} \mid |z| < 1\}$. Moreover, letting $\Delta := 2\operatorname{diam}(\operatorname{supp}(\mathfrak{F}))$, we have that the set $\operatorname{supp}(\mathfrak{F})$ is contained in the ball $B = \{z \in \mathbb{C} \mid |z| \leqslant \Delta\}$, because the density of the spherical metric with respect to the Euclidean metric on $\mathbb{C}$ is at least $\frac{1}{2}$ on $\{z \in \mathbb{C} \mid |z| \leqslant 1\}$.

On the other hand, note that the distortion of the Möbius map $\varphi(z) := \Delta^{-1}z$ on B is bounded by 2. So we have

$$\sum_{W \in \mathfrak{F}} \operatorname{diam}(W)^\alpha \leqslant 2^\alpha \Delta^\alpha \sum_{W \in \mathfrak{F}} \operatorname{diam}\big(\varphi(W)\big)^\alpha \leqslant 4^\alpha \operatorname{diam}\big(\operatorname{supp}(\mathfrak{F})\big)^\alpha \|\mathfrak{F}\|_\alpha. \qquad \square$$

5.2. Univalent pull-backs

The following assertion is an easy consequence of Koebe Distortion Theorem. For every $\mathbf{m} > 0$ there is a constant $C(\mathbf{m}) > 0$ such that the following property holds. Let U and $\widehat{U}$ be simply-connected subsets of $\overline{\mathbb{C}}$ such that $\overline{U} \subset \widehat{U}$ and such that $\widehat{U} \setminus \overline{U}$ is an annulus of modulus at least $\mathbf{m}$. Then for every univalent holomorphic map $h : \widehat{U} \to \overline{\mathbb{C}}$ there exists a Möbius map φ such that the distortion of $h \circ \varphi$ on U is bounded by $C(\mathbf{m})$.

The following lemma is a direct consequence of the property above.

LEMMA 5.2. – *For each $\mathbf{m} > 0$ there is a constant $C_0(\mathbf{m}) > 0$ such that the following properties hold. Let $\widehat{U}$, $\widehat{V}$ be simply-connected subsets of $\overline{\mathbb{C}}$ and let $h : \widehat{U} \to \widehat{V}$ be an univalent holomorphic map. Let $\mathfrak{F}$ be a family of subsets of $\overline{\mathbb{C}}$ such that* $\operatorname{supp}(\mathfrak{F})$ *is contained in a simply-connected subset V of $\widehat{V}$, in such a way that $\widehat{V} \setminus \overline{V}$ is an annulus of modulus at least* $\mathbf{m}$. *Then,*

$$\big\|\{h^{-1}(W) \mid W \in \mathfrak{F}\}\big\|_\alpha \leqslant C_0(\mathbf{m}) \|\mathfrak{F}\|_\alpha.$$

5.3. Modulus and diameter

Let A be an open subset of $\overline{\mathbb{C}}$ homeomorphic to an annulus. When the complement of A in $\overline{\mathbb{C}}$ contains at least 3 points, there is a unique $R \in (1, +\infty]$ such that A is conformally equivalent to $\{z \in \mathbb{C} \mid 1 < |z| < R\} \in (0, +\infty]$. In this case we put $\operatorname{mod}(A) = \ln R$. When the complement of A in $\overline{\mathbb{C}}$ consists of two points, we put $\ln A = +\infty$.

A *round annulus* is by definition the complement in $\overline{\mathbb{C}}$ of 2 disjoint closed balls or the complement of a closed ball in an open ball containing it, that is different from $\overline{\mathbb{C}}$.

LEMMA 5.3. – *There is a universal constant $\mathbf{m}_0 > 0$ such that for every subset A of $\overline{\mathbb{C}}$ that is homeomorphic to an annulus, the connected components B and B' of $\overline{\mathbb{C}} \setminus A$ satisfy*

$$\operatorname{diam}(B) \cdot \operatorname{diam}(B') \leqslant \exp\big(-\big(\operatorname{mod}(A) - \mathbf{m}_0\big)\big).$$

Proof. – It is easy to check by direct computation that there is a constant $\mathbf{m}_1 > 0$ so that property above holds for every round annulus, with $\mathbf{m}_0$ replaced by $\mathbf{m}_1$. On the other hand, there is a universal constant $\mathbf{m}_2 > 0$ such that every annulus in $\overline{\mathbb{C}}$ of modulus $\mathbf{m} \in (\mathbf{m}_1, +\infty)$ contains an essential round annulus of modulus $\mathbf{m} - \mathbf{m}_1$. So the assertion of the lemma holds with constant $\mathbf{m}_0 := \mathbf{m}_1 + \mathbf{m}_2$. $\square$

5.4. Unicritical pull-backs

Given simply-connected subsets U and V of $\overline{\mathbb{C}}$, we say that a holomorphic and proper map $h : U \to V$ is *unicritical* if it has a unique critical point. Note that the degree of a unicritical map as a ramified covering is equal to the local degree at its unique critical point. Given an unicritical map $h : U \to V$ and a family $\mathfrak{F}$ of connected subsets of $\overline{\mathbb{C}}$ such that $\mathrm{supp}(\mathfrak{F}) \subset V$, put

$$h^{-1}(\mathfrak{F}) := \big\{\text{connected component of } h^{-1}(W), \text{ for some } W \in \mathfrak{F}\big\}.$$

We say that a family $\mathfrak{F}$ of simply-connected subsets of $\overline{\mathbb{C}}$ is $\mathbf{m}$-*shielded* by a family $\widehat{\mathfrak{F}}$, if for every $W \in \mathfrak{F}$ there is an element $\widehat{W}$ of $\widehat{\mathfrak{F}}$ containing W and such that $\widehat{W} \setminus \overline{W}$ is an annulus of modulus at least $\mathbf{m}$.

LEMMA 5.4. – *Given $\mathbf{m} > 0$ and $d \geqslant 2$ there is a constant $C_1(\mathbf{m}, d) > 0$ such that the following properties hold. Let $h : \widehat{U} \to \widehat{V}$ be a unicritical map with critical point c and let $\mathfrak{F}$ be a family of simply-connected subsets of $\overline{\mathbb{C}}$ that is $\mathbf{m}$-shielded by a family $\widehat{\mathfrak{F}}$ such that $h(c) \notin \mathrm{supp}(\widehat{\mathfrak{F}})$ and such that $\mathrm{supp}(\widehat{\mathfrak{F}})$ is contained in a simply-connected subset V of $\widehat{V}$, so that $\widehat{V} \setminus \overline{V}$ is an annulus of modulus at least $\mathbf{m}$. Then*

$$\big\|h^{-1}(\mathfrak{F})\big\|_\alpha \leqslant C_1(\mathbf{m}, d) \cdot \|\mathfrak{F}\|_\alpha.$$

Proof. – Let γ be the Jordan curve that divides the annulus $\widehat{V} \setminus \overline{V}$ into two annuli of modulus equal to a half of the modulus of $\widehat{V} \setminus \overline{V}$. Denote by $\widetilde{V}$ the open disk in $\widehat{V}$ bounded by γ. There are two cases.

Case 1. $h(c) \notin \widetilde{V}$. Then the preimage of $\widetilde{V}$ by h has d connected components and on each of these h is univalent. As by definition of $\widetilde{V}$ the set $\widetilde{V} \setminus \overline{V}$ is an annulus of modulus at least $\frac{1}{2}\mathbf{m}$, in this case the assertion follows from Lemma 5.2, with constant $C_1(\mathbf{m}, d) := d \cdot C_0(\frac{1}{2}\mathbf{m})$.

Case 2. $h(c) \in \widetilde{V}$. In this case the set $\widetilde{U} := \varphi^{-1}(\widetilde{V})$ is connected and simply-connected. Let $\varphi : \mathbb{D} \to \widetilde{V}$ be a conformal uniformization such that $\varphi(0) = h(c)$. As the modulus of the annulus $\widetilde{V} \setminus \overline{V}$ is at least $\frac{1}{2}\mathbf{m}$, there is $r_0 \in (0,1)$ only depending on $\mathbf{m}$ such that V is contained in $\varphi(\{|z| \leqslant r_0\})$. By hypothesis for each $W \in \mathfrak{F}$ there is $\widehat{W} \in \widehat{\mathfrak{F}}$ contained in $V \setminus \{h(c)\}$, such that $\widehat{W} \setminus \overline{W}$ is an annulus of modulus at least $\mathbf{m}$. It follows that the diameter of W with respect to the hyperbolic metric of $V \setminus \{h(c)\}$ is bounded from above in terms of $\mathbf{m}$ only. Therefore there is $\rho \in (0,1)$ such that for every $r \in (0, \rho r_0)$, every element W of $\mathfrak{F}$ intersecting $\varphi(\{|z| = r\})$ must be contained in the annulus $\varphi(\{\rho r \leqslant |z| \leqslant \rho^{-1} r\})$. For $j \geqslant 1$ put

$$\widetilde{V}_j := \varphi\big(\big\{r_0\rho^{j+1} \leqslant |z| \leqslant r_0\rho^{j-1}\big\}\big) \quad \text{and} \quad \mathfrak{F}_j := \big\{W \in \mathfrak{F} \mid W \subset \widetilde{V}_j\big\}.$$

By definition of ρ we have $\mathfrak{F} = \bigcup_{j \geqslant 1} \mathfrak{F}_j$. Moreover put $\widetilde{U}_j := h^{-1}(\widetilde{V}_j)$. By §5.2 there is a constant $C' > 0$ only depending on r_0 and on the degree d of h, such that for every $j \geqslant 1$ we have

$$\big\|h^{-1}(\mathfrak{F}_j)\big\|_\alpha \leqslant C' \|\mathfrak{F}_j\|_\alpha \leqslant C' \|\mathfrak{F}\|_\alpha. \tag{12}$$

Let $j_0 \geqslant 1$ be the least integer such that for every $j \geqslant j_0$ we have $\mathrm{diam}(\widetilde{U}_j) < \frac{1}{2}$. Note that for every $j > j_0$ the set

$$h^{-1} \circ \varphi\big(\big\{z \in \overline{\mathbb{C}} \mid r_0\rho^{j-1} < |z| < r_0\rho^{j_0-1}\big\}\big)$$

is an annulus of modulus $(j - j_0)\ln \rho^{-1}/d$. So there are constants $C'' > 0$ and $\eta \in (0,1)$ only depending on ρ and d, such that for every $j \geqslant j_0$ we have $\mathrm{diam}(\widetilde{U}_j) \leqslant C''\eta^{(j-j_0)}$. Therefore, letting $C''' := 4^\alpha C' C'' (1-\eta)^{-1}$, Lemma 5.1 implies that

$$\sum_{W \in \bigcup_{j \geqslant j_0} h^{-1}(\mathfrak{F}_j)} \operatorname{diam}(W)^\alpha \leqslant C''' \|\mathfrak{F}\|_\alpha.$$

Let $\mathbf{m}_0$ be the constant given by Lemma 5.3 and let N be the smallest positive integer such that $\exp(-(Nd^{-1}\ln\rho^{-1} - \mathbf{m}_0)) < \frac{1}{4}$. Note that N depends on $\mathbf{m}$ only. By definition of j_0 we have $\operatorname{diam}(\widetilde{U}_{j_0-1}) \geqslant \frac{1}{2}$. As for every $j \leqslant j_0 - (N+3)$ the sets $\widetilde{U}_j$ and $\widetilde{U}_{j_0-1}$ are separated by an annulus of modulus $(j_0 - j - 3)d^{-1}\ln\rho^{-1} \geqslant Nd^{-1}\ln\rho^{-1}$, Lemma 5.3 implies that $\operatorname{diam}(\widetilde{U}_j) < \frac{1}{2}$ and as before we have

$$\sum_{W \in \bigcup_{j=1,\dots,j_0-(N+3)} h^{-1}(\mathfrak{F}_j)} \operatorname{diam}(W)^\alpha \leqslant C''' \|\mathfrak{F}\|_\alpha.$$

Therefore,

$$\begin{aligned}\sum_{W \in h^{-1}(\mathfrak{F})} \operatorname{diam}(W)^\alpha &\leqslant \sum_{j=j_0-(N+2),\dots,j_0-1} \left\|h^{-1}(\mathfrak{F}_j)\right\|_\alpha + 2C''' \|\mathfrak{F}\|_\alpha \\ &\leqslant \left(C'(N+2) + 2C'''\right)\|\mathfrak{F}\|_\alpha.\end{aligned}$$

As our hypotheses are coordinate free, this estimate proves the assertion of the lemma with constant $C_1(\mathbf{m}, d) := C'(N+2) + 2C'''$. □

6. Nice sets and the density $\|\cdot\|_\alpha$

Fix a complex rational map f of degree at least 2. Recall that the *hyperbolic Hausdorff dimension* of f is by definition

$$\mathrm{HD}_{\mathrm{hyp}}(f) := \sup_X \mathrm{HD}(X),$$

where the supremum is taken over all forward invariant subsets X of $\overline{\mathbb{C}}$ on which f is uniformly expanding.

The purpose of this section is to prove the following proposition. See §4.1 for the definition of $K(V)$.

PROPOSITION 6.1. – *Let f be a rational map such that for every neighborhood V' of $\mathscr{C}$ the map f is uniformly expanding on the set*

$$\left\{ z \in J(f) \mid \textit{for every } n \geqslant 0 \textit{ we have } f^n(z) \notin V' \right\}. \tag{13}$$

Then for every nice couple $(\widehat{V}, V)$ for f there exists $\alpha \in (0, \mathrm{HD}_{\mathrm{hyp}}(f))$ such that the collection $\mathfrak{D}_V$ of connected components of $\overline{\mathbb{C}} \setminus K(V)$ satisfies $\|\mathfrak{D}_V\|_\alpha < +\infty$.

After some preliminary lemmas, the proof of this proposition is given at the end of this subsection.

For a subset X of $\overline{\mathbb{C}}$ we denote by $\overline{\mathrm{BD}}(X)$ the upper box dimension of X.

LEMMA 6.2. – *Let f be a rational map as in the statement of Proposition* 6.1. *If K' is a compact subset of $\overline{\mathbb{C}}$ such that $f(K') \subset K'$ and such that f is uniformly expanding on K', then*

$$\mathrm{HD}(K') \leqslant \overline{\mathrm{BD}}(K') < \mathrm{HD}_{\mathrm{hyp}}\big(J(f)\big).$$

Proof. – We will show first that we can reduce to the case when K' is compact and forward invariant ($f(K') = K'$). Let K be the closure of $\bigcap_{n=0}^{\infty} f^n(K')$. Clearly K is compact and forward invariant. Thus there is a compact neighborhood U of K such that U is contained in the interior of $f(U)$. Let $\widetilde{K}$ be the maximal invariant set of f contained in U. This set is compact, forward invariant and, reducing U if necessary, f is uniformly expanding on $\widetilde{K}$. By definition of K it follows that there is a sufficiently large integer N such that $f^N(K')$ is contained in U, and hence in $\widetilde{K}$. It follows that $\overline{\mathrm{BD}}(K') \leqslant \overline{\mathrm{BD}}(\widetilde{K})$. Thus, replacing K' by $\widetilde{K}$ if necessary, we can reduce to the case in which K' is compact and invariant.

The first inequality is a general fact. As f is uniformly expanding on K', it follows that there is a uniformly expanding set K'' containing K', which has a Markov partition, see [37]. (The idea, taken from Bowen's construction, is to pick a δ-net in K' and shadow by forward f-trajectories all ϵ-trajectories of points in the net. The set K'' is defined as the union of these trajectories and it contains K', provided $\delta \ll \epsilon$.) Hence $\overline{\mathrm{BD}}(K'') = \mathrm{HD}(K'')$. To get this one uses an equilibrium state for the potential $-\mathrm{HD}(K'')\ln|f'|$ on an invariant topologically transitive part of the related topological Markov shift. Next, for a sufficiently small neighborhood V' of $\mathscr{C}$ one can construct in a similar way an expanding set K''' containing the set (13), and that has a Markov partition that is a substantial extension of the previous one. Hence we have

$$\overline{\mathrm{BD}}(K') \leqslant \overline{\mathrm{BD}}(K'') = \mathrm{HD}(K'') < \mathrm{HD}(K''') \leqslant \mathrm{HD}_{\mathrm{hyp}}\big(J(f)\big). \qquad \square$$

LEMMA 6.3. – *Let f, $(\widehat{V}, V)$ and $\mathfrak{D}_V$ be as in Proposition* 6.1. *Then there are constants $\alpha \in (0, \mathrm{HD}_{\mathrm{hyp}}(f))$ and $C_0 > 0$ such that*

$$\sum_{W \in \mathfrak{D}_V} \mathrm{diam}(W)^{\alpha} < +\infty, \tag{14}$$

and such that for every ball B of $\overline{\mathbb{C}}$ we have

$$\sum_{W \in \mathfrak{D}_V,\, W \subset B} \mathrm{diam}(W)^{\alpha} \leqslant C_0\, \mathrm{diam}(B)^{\alpha}. \tag{15}$$

Proof. – As by hypothesis f is uniformly expanding on $K := K(V) \cap J(f)$, it follows that for every ball B of $\overline{\mathbb{C}}$ intersecting K there is an integer $n \geqslant 0$ such that $f^n(B)$ has definite size and such that the distortion of f^n on B is bounded independently of B. So the existence of the constant $C_0 > 0$ for which (15) holds for every ball B follows from (14). In what follows we will prove that (14) holds for some appropriated choice of α.

1. Let V' be a neighborhood of $\mathscr{C}$ that is contained in V and such that for every $c \in \mathscr{C}$ the set K' defined by (13) intersects V^c. It follows that every element of $\mathfrak{D}_V$ intersects K'. Given $W \in \mathfrak{D}_V$ choose a point z_W in $W \cap K'$ and put $B_W := \{z \in \overline{\mathbb{C}} \mid \mathrm{dist}(z, W) < \mathrm{diam}(W)\}$, so that $W \subset B_W$.

By Lemma 6.2 we have $\overline{\mathrm{BD}}(K') < \mathrm{HD}_{\mathrm{hyp}}(f)$.

2. Fix $\alpha \in (\overline{\mathrm{BD}}(K'), \mathrm{HD}_{\mathrm{hyp}}(f))$. For $r > 0$ let $N(r)$ be the least number of balls of $\overline{\mathbb{C}}$ of radius r centered at some point of K', that are necessary to cover K'. It follows that for every $r > 0$ the cardinality of a collection of pairwise disjoint balls of radius at least r and centered at points in K' is at most $N(r)$. On the other hand, as

$$\alpha > \overline{\mathrm{BD}}(K') = \limsup_{r\to 0} \frac{\ln N(r)}{\ln(r^{-1})},$$

it follows that

$$\sum_{n\geqslant 0} N(2^{-n})2^{-n\alpha} < +\infty. \tag{16}$$

3. Since for every $W \in \mathfrak{D}_V$ the map f^{m_W} extends univalently to $\widehat{W}$, it follows that there is a constant $A_1 > 0$ such that for every $W \in \mathfrak{D}_V$ we have $\mathrm{Area}(W) \geqslant A_1 \operatorname{diam}(W)^2$. Therefore there is an integer A_2 such that for every $W \in \mathfrak{D}_V$ there are at most A_2 elements $W' \in \mathfrak{D}_V$ such that

$$\frac{1}{2}\operatorname{diam}(W) \leqslant \operatorname{diam}(W') \leqslant 2\operatorname{diam}(W),$$

and such that $B_{W'}$ intersects B_W.

So, if for $r > 0$ we denote by $N'(r)$ the number of elements of $\mathfrak{D}_V$ whose diameter belongs to $(r, 2r)$, then we can find a collection $\mathfrak{F}$ of such sets whose cardinality is at least $N'(r)/(A_2+1)$ and such that the balls B_W, for $W \in \mathfrak{F}$, are pairwise disjoint. As the centers of the balls B_W belong to K', it follows that

$$N'(r) \leqslant (A_2+1)N(r).$$

So by (16) we have

$$\sum_{W\in\mathfrak{D}_V} \operatorname{diam}(W)^\alpha \leqslant \sum_{n\geqslant 0} N'(2^{-n})2^{-n\alpha} < +\infty. \qquad \square$$

LEMMA 6.4. – *Let f be a complex rational map and let $(\widehat{V}, V)$ be a nice couple for f. Then there is a constant $C_1 > 0$ such that for every ball B of $\overline{\mathbb{C}}$ there is at most one connected component W of $\overline{\mathbb{C}} \setminus K(V)$ intersecting B and such that*

$$\operatorname{diam}(W) \geqslant C_1 \operatorname{diam}(B).$$

Proof. – Let $(\widehat{V}, V)$ be a nice couple for f. It follows from (7) that if W and W' are distinct elements of $\mathfrak{D}_V$, then we have either $\widehat{W} \cap W' = \emptyset$ or $W \cap \widehat{W'} = \emptyset$. So to prove the lemma is enough to show that there is a constant $C_1 > 0$ such that if W is an element of $\mathfrak{D}_V$ and if B is a ball intersecting W such that $\operatorname{diam}(W) \geqslant C_1 \operatorname{diam}(B)$, then $B \subset \widehat{W}$.

Note that for each $W \in \mathfrak{D}_V$ the set $\widehat{W}$ is disjoint from $K(\widehat{V})$ and $\widehat{W} \setminus \overline{W}$ is an annulus of modulus at least

$$\mathbf{m} := \min_{c\in\mathscr{C}} \operatorname{mod}\big(\widehat{V}^c \setminus \overline{V^c}\big).$$

As the set $K(\widehat{V})$ contains at least 2 points (because it contains ∂V), it follows that there is $\varepsilon > 0$ such that for every $W \in \mathfrak{D}_V$ the set

$$W(\varepsilon) = \big\{z \in \overline{\mathbb{C}} \mid \operatorname{dist}(z, W) \leqslant \varepsilon \operatorname{diam}(W)\big\}$$

is contained in $\widehat{W}$. Clearly the constant $C_1 := \varepsilon^{-1}$ has the desired property. $\square$

Recall that we identify $\overline{\mathbb{C}}$ with $\mathbb{C} \cup \{\infty\}$ and that the spherical metric is normalized in such a way that its density with respect to the Euclidean metric on $\mathbb{C}$ is given by $z \mapsto (1+|z|^2)^{-1}$. Moreover, balls, distances, diameters and derivatives are all taken with respect to the spherical metric, see the Preliminaries.

LEMMA 6.5. – *For each Möbius transformation φ the following properties hold.*

1. *There exists a pair of antipodal points in $\overline{\mathbb{C}}$ whose image by φ is a pair of antipodal points.*
2. *There exists $\lambda \geqslant 1$ such that, after an isometric change of coordinates on the domain and on the target, φ is of the form $z \mapsto \lambda z$.*

Proof. – We will use several times that the group of isometric Möbius transformations acts transitively on $\overline{\mathbb{C}}$.

1. After an isometric change of coordinates we assume that φ fixes ∞. It follows that there are $a \in \overline{\mathbb{C}} \setminus \{0\}$ and $b \in \mathbb{C}$ such that for each $z \in \overline{\mathbb{C}}$ we have $\varphi(z) = az + b$. If $b = 0$ then φ maps $\{0, \infty\}$ onto itself and the assertion is proved in this case. So we assume that $b \neq 0$. Then we are looking $\zeta \in \overline{\mathbb{C}} \setminus \{0\}$ such that the image by φ of the antipodal of ζ is equal to the antipodal of $\varphi(\zeta)$. Equivalently, we are looking for a solution $\zeta \in \overline{\mathbb{C}} \setminus \{0\}$ of the equation

$$\varphi\big(-\overline{\zeta}^{-1}\big) = -\overline{\varphi(\zeta)}^{-1}.$$

A direct computation shows that the solutions of this equation are precisely the roots of the quadratic polynomial $\zeta \mapsto -a\overline{b}\zeta^2 + (|a|^2 - |b|^2 - 1)\zeta + \overline{a}b$.

2. By part 1, after an isometric change of coordinates on the domain and on the target we may assume that $\varphi(\{0,\infty\}) = \{0,\infty\}$. Post-composing φ with the isometry $z \mapsto z^{-1}$ if necessary, we assume that φ fixes 0 and ∞. It follows that there is $\lambda \in \overline{\mathbb{C}} \setminus \{0\}$ such that for every $z \in \mathbb{C}$ we have $\varphi(z) = \lambda z$. Conjugating φ by the isometry $z \mapsto z^{-1}$ if necessary, we assume that $|\lambda| \geqslant 1$, and post-composing φ with the isometry $z \mapsto (|\lambda|/\lambda)z$ if necessary we assume that λ is real and positive. □

Proof of Proposition 6.1. – Let α, C_0 and C_1 be the constants given by Lemmas 6.3 and 6.4. Choose $\rho \in (0,1)$ sufficiently small so that for every $r \in (0,1]$ we have

$$\operatorname{dist}\big(\{|z| = r\}, \{|z| \leqslant \rho r\}\big) \geqslant C_1 \operatorname{diam}\big(\{|z| \leqslant \rho r\}\big).$$

Let φ be a given Möbius transformation. By part 2 of Lemma 6.5 we may assume that φ is of the form $\varphi(z) = \lambda z$, with λ real and satisfying $\lambda \geqslant 1$. Let $N \geqslant 1$ be the least integer satisfying $\rho^N < \lambda^{-1/2}$ and put

$$A_0 := \big\{|z| > \rho\lambda^{-1/2}\big\} \cup \{\infty\},$$
$$A_n := \big\{\rho^{n+1}\lambda^{-1/2} < |z| < \rho^{n-1}\lambda^{-1/2}\big\},$$

for $n = 1, \ldots, N-1$, and put

$$A_N := \big\{|z| < \rho^{N-1}\lambda^{-1/2}\big\}.$$

For $n = 0, \ldots, N$ we denote by $\mathfrak{F}_n$ the sub-collection of $\mathfrak{D}_V$ of sets contained in A_n and we denote by $\mathfrak{F}$ the sub-collection of $\mathfrak{D}_V$ of sets not contained in any of the A_n. By definition we have $\mathfrak{F} \cup \mathfrak{F}_0 \cup \cdots \cup \mathfrak{F}_N = \mathfrak{D}_V$.

In parts 1, 2 and 3 below we estimate the sum $\sum \operatorname{diam}(\varphi(W))^\alpha$, where W runs through $\mathfrak{F}$, $\mathfrak{F}_0$ and $\mathfrak{F}_1 \cup \cdots \cup \mathfrak{F}_N$, respectively. These estimates are independent of λ, so the lemma follows from them.

1. By definition each element of $\mathfrak{F}$ intersects at least two of the sets $\{|z| = \rho^n \lambda^{-1/2}\}$, for $n = 0, \ldots, N$. For $W \in \mathfrak{F}$ denote by $n(W)$ the largest integer $n = 0, \ldots, N$ such that W intersects $\{|z| = \rho^n \lambda^{-1/2}\}$. So for each $W \in \mathfrak{F}$ we have

$$\varphi(W) \subset \varphi\big(\big\{|z| \geqslant \rho^{n(W)+1}\lambda^{-1/2}\big\} \cup \{\infty\}\big) \subset \big\{|z| \geqslant \rho^{n(W)+2-N}\big\} \cup \{\infty\},$$

and $\operatorname{diam}(\varphi(W)) \leqslant (2\rho^{-2})\rho^{n(W)-N}$. By Lemma 6.4 and by the choice of ρ, for distinct $W, W' \in \mathfrak{F}$ the integers $n(W)$ and $n(W')$ are distinct. Therefore we have

$$\sum_{W \in \mathfrak{F}} \operatorname{diam}\big(\varphi(W)\big)^\alpha \leqslant \big(2\rho^{-2}\big)^\alpha \big(1 + \rho^\alpha + \cdots + \rho^{N\alpha}\big) \leqslant \big(2\rho^{-2}\big)^\alpha / \big(1 - \rho^\alpha\big).$$

2. As $\sup\{|\varphi'(z)| \mid z \in A_0\} \leqslant \rho^{-2}$, we have

$$\sum_{W \in \mathfrak{F}_0} \operatorname{diam}\big(\varphi(W)\big)^\alpha \leqslant \rho^{-2\alpha} \sum_{W \in \mathfrak{F}_0} \operatorname{diam}(W)^\alpha \leqslant \rho^{-2\alpha} \sum_{W \in \mathfrak{D}_V} \operatorname{diam}(W)^\alpha.$$

By Lemma 6.3 this last quantity is finite.

3. Note that there is a constant $C_2 > 0$ only depending on ρ, that bounds the distortion of φ on each of the sets A_n, for $n = 1, \ldots, N$. On the other hand, note that for each $n = 1, \ldots, N$ and $z \in \mathbb{C}$ such that $|z| = \rho^n \lambda^{-1/2}$, we have $z \in A_n$ and

$$\big|\varphi'(z)\big| = \lambda \frac{1 + \rho^{2n}\lambda^{-1}}{1 + \rho^{2n}\lambda} \leqslant 2\rho^{-2n}.$$

Therefore we have

$$\sum_{W \in \mathfrak{F}_n} \operatorname{diam}\big(\varphi(W)\big)^\alpha \leqslant (2C_2)^\alpha \rho^{-2n\alpha} \sum_{W \in \mathfrak{F}_n} \operatorname{diam}(W)^\alpha.$$

Since $\operatorname{supp}(\mathfrak{F}_n) \subset A_n \subset \{|z| \leqslant \rho^{n-1}\lambda^{-1/2}\} \subset \{|z| \leqslant \rho^{n+N-2}\}$, so

$$\sum_{W \in \mathfrak{F}_n} \operatorname{diam}(W)^\alpha \leqslant C_0 \operatorname{diam}(A_n)^\alpha \leqslant C_0\big(2\rho^{-2}\big)^\alpha \rho^{(n+N)\alpha}.$$

So, letting $C_3 := (2C_2)^\alpha C_0 (2\rho^{-2})^\alpha$ we have

$$\sum_{W \in \mathfrak{F}_n} \operatorname{diam}\big(\varphi(W)\big)^\alpha \leqslant C_3 \rho^{(N-n)\alpha} \quad \text{and} \quad \sum_{W \in \mathfrak{F}_1 \cup \cdots \cup \mathfrak{F}_N} \operatorname{diam}\big(\varphi(W)\big)^\alpha \leqslant C_3/(1 - \rho^\alpha). \qquad \square$$

7. Key Lemma

The purpose of this section is to prove the following lemma.

KEY LEMMA. – *Let f be a rational map satisfying ExpShrink. Then for every $\mathbf{m} > 0$ there exists $r > 0$ such that if $(\widehat{V}, V)$ is a nice couple satisfying,*

$$\min_{c \in \mathscr{C}} \operatorname{mod}\big(\widehat{V}^c \setminus \overline{V^c}\big) \geqslant \mathbf{m} \quad \textit{and} \quad \max_{c \in \mathscr{C}} \operatorname{diam}\big(\widehat{V}^c\big) \leqslant r, \tag{17}$$

then the canonical induced map $F : D \to V$ associated to $(\widehat{V}, V)$ satisfies the following properties.

1. *We have* $\mathrm{HD}((J(f)\setminus J(F))\cap V)<\mathrm{HD}(J(f))$. *In particular, for every* $c\in\mathscr{C}$ *we have* $\mathrm{HD}(J(F)\cap V^c)=\mathrm{HD}(J(f))$.
2. *There exists* $\alpha\in(0,\mathrm{HD}(J(f)))$ *such that*

$$\sum_{W \ c.c.\ of\ D}\mathrm{diam}(W)^\alpha<+\infty,$$

where the sum is over all connected components W *of* D.

After some preliminary considerations in §7.1, we prove parts 1 and 2 of the Key Lemma in §7.2 and in §7.3, respectively.

7.1. Bad pull-backs

Let f be a rational map and let $\widehat{V}$ be a nice set for f. For an integer $n\geqslant 1$ we will say that a connected component $\widetilde{W}$ of $f^{-n}(\widehat{V})$ is a *bad pull-back of* $\widehat{V}$ *of order* n, if f^n is not univalent on $\widetilde{W}$ and if for every $m=1,\ldots,n-1$ such that $f^m(\widetilde{W})\subset\widehat{V}$, the map f^m is not univalent on the connected component of $f^{-m}(\widehat{V})$ containing $\widetilde{W}$. Note that every bad pull-back of $\widehat{V}$ contains a critical point of f in $\mathscr{C}$.

The proof of the following lemma is similar to that of Lemma A.2 of [35].

LEMMA 7.1. – *Let* f *be a rational map, let* $\widehat{V}$ *be a nice set for* f*, and let* $L\geqslant 1$ *be the least integer such that* $f^L(\mathscr{C})$ *intersects* $\widehat{V}$*. Then for each positive integer* n*, there are at most* $(2L\#\mathscr{C})^{2n/L}$ *bad pull-backs of* $\widehat{V}$ *of order* n*.*

Proof. – **1**. For a bad pull-back $\widetilde{W}$ of $\widehat{V}$ of order n, let $\ell(\widetilde{W})$ be the largest integer ℓ in $\{0,\ldots,n-1\}$ such that $f^\ell(\widetilde{W})$ intersects $\mathscr{C}$. As $\widehat{V}$ is a nice set we have $f^{\ell(\widetilde{W})}(\widetilde{W})\subset\widehat{V}$. On the other hand, as $\widetilde{W}$ is a bad pull-back of $\widehat{V}$, when $\ell(\widetilde{W})$ is positive the connected component of $f^{-\ell(\widetilde{W})}(\widehat{V})$ containing $\widetilde{W}$ is a bad pull-back of $\widehat{V}$ of order $\ell(\widetilde{W})$.

2. Fix an integer $n\geqslant 1$. For a given bad pull-back $\widetilde{W}$ of order n define a strictly decreasing sequence of integers $(\ell_0,\ldots,\ell_k)$, by induction as follows. Define $\ell_0:=n$ and suppose that for some $j\geqslant 0$ the integer ℓ_j is already defined and that we have $f^{\ell_j}(\widetilde{W})\subset\widehat{V}$. If $\ell_j=0$, then define $k:=j$ and stop. If $\ell_j>0$, then by induction hypothesis $f^{\ell_j}(\widetilde{W})$ is contained in $\widehat{V}$ and therefore the connected component $\widetilde{W}'$ of $f^{-\ell_j}(\widehat{V})$ containing $\widetilde{W}$ is a bad pull-back of $\widehat{V}$ of order ℓ_j. Then define $\ell_{j+1}:=\ell(\widetilde{W}')$. As remarked in part 1, in this case we have $f^{\ell_{j+1}}(\widetilde{W})\subset f^{\ell_{j+1}}(\widetilde{W}')\subset\widehat{V}$, so the induction hypothesis is satisfied.

3. To each bad pull-back of $\widehat{V}$ of order n we associate a strictly decreasing sequence $(\ell_0,\ldots,\ell_k)$, as in part 2, so that $\ell_0=n$ and $\ell_k=0$. Note that for each $j=1,\ldots,k$, the pull-back of $\widehat{V}$ by $f^{\ell_{j-1}-\ell_j}$ containing $f^{\ell_j}(\widetilde{W})$ contains a critical point in $\mathscr{C}$. As for each $c\in\mathscr{C}$ and each integer m there are at most $\#\mathscr{C}$ connected components of $f^{-m}(\widehat{V}^c)$ containing an element of $\mathscr{C}$, it follows that there are at most $(\#\mathscr{C})^{k+1}$ bad pull-backs of order n with the same associated sequence.

On the other hand, by definition of L it follows that for every $j=1,\ldots,k$ we have $\ell_{j-1}-\ell_j\geqslant L$. So, $k\leqslant n/L$ and for each integer $m=1,\ldots,n$ there is at most one integer $r\in\{0,1,\ldots,L-1\}$ such that $m+r$ is one of the ℓ_j. It follows that there are at most $(L+1)^{2n/L}$ such decreasing sequences.

We conclude that the number of bad pull-backs of $\widehat{V}$ of order n is at most

$$(\#\mathscr{C})^{k+1}(L+1)^{2n/L}\leqslant(2L\#\mathscr{C})^{2n/L}.\qquad\square$$

7.2. Proof of part 1 of the Key Lemma

Let f be a rational map satisfying condition ExpShrink with constants $\lambda_{\mathrm{Exp}} > 1$ and $r_0 > 0$. We will show that part 1 of the Key Lemma holds for every nice couple $(\widehat{V}, V)$ for which $\max_{c \in \mathscr{C}} \operatorname{diam}(V^c)$ is sufficiently small. In fact, we will prove that for such a nice couple $(\widehat{V}, V)$, we have

$$\mathrm{HD}\big(\big(V \cap J(f)\big) \setminus D\big) < \mathrm{HD}\big(J(f)\big). \tag{18}$$

As every inverse branch of F is Lipschitz, this implies that $\mathrm{HD}((J(f) \setminus J(F)) \cap V) < \mathrm{HD}(J(f))$, as desired.

Given a nice couple $(\widehat{V}, V)$, denote by R_V the first return map to V and denote by $J(R_V)$ the subset of V of those points for which R_V^n is defined for every integer $n \geqslant 1$. Equivalently, $J(R_V)$ is the set of those points that return infinitely often to V under forward iteration of f. Note that $J(F) \subset J(R_V)$ and that

$$\mathrm{HD}\big(\big(V \cap J(f)\big) \setminus J(R_V)\big) \leqslant \mathrm{HD}\big(K(V) \cap J(f)\big).$$

As f is uniformly expanding on $K(V) \cap J(f)$ (this follows easily from condition ExpShrink), Lemma 6.2 implies that,

$$\mathrm{HD}\big(\big(V \cap J(f)\big) \setminus J(R_V)\big) < \mathrm{HD}\big(J(f)\big).$$

So, the following lemma implies that (18) holds for every nice couple $(\widehat{V}, V)$ for which $\max_{c \in \mathscr{C}} \operatorname{diam}(V^c)$ is sufficiently small, by choosing $\varepsilon \in (0, \mathrm{HD}(J(f)))$.

For a given $r > 0$ let $L(r) \geqslant 1$ be the smallest integer such that for some $c \in \mathscr{C}$ the point $f^{L(r)}(c)$ is at distance at most r from $\mathscr{C}$. As our standing convention is that no critical point in $\mathscr{C}$ is mapped to a critical point under forward iteration, we have that $L(r) \to +\infty$ as $r \to 0$.

LEMMA 7.2. – *Given $\varepsilon > 0$ choose $r \in (0, r_0)$ sufficiently small so that*

$$\big(2L(r)\#\mathscr{C}\big)^{2/L(r)} \lambda_{\mathrm{Exp}}^{-\varepsilon} < 1. \tag{19}$$

Then for every nice couple $(\widehat{V}, V)$ such that $\max_{c \in \mathscr{C}} \operatorname{diam}(\widehat{V}^c) \leqslant r$, the Hausdorff dimension of $J(R_V) \setminus D$ is at most ε.

Proof. – For a point z in $J(R_V) \setminus D$ there are arbitrarily large integers $n \geqslant 1$ such that $f^n(z) \in V$. Moreover, for every such n the pull-back $\widetilde{W}$ of $\widehat{V}$ to z by f^n is not univalent. It follows that $\widetilde{W}$ is a bad pull-back of $\widehat{V}$. So, if for $n \geqslant 1$ we denote by $\mathfrak{D}_n$ the collection of all bad pull-backs of $\widehat{V}$ of order n, then for every $n_0 \geqslant 1$ we have

$$J(R_V) \setminus D \subset \bigcup_{n \geqslant n_0} \bigcup_{\widetilde{W} \in \mathfrak{D}_n} \widetilde{W}.$$

As

$$\sum_{\widetilde{W} \in \mathfrak{D}_n} \operatorname{diam}(\widetilde{W})^{\varepsilon} \leqslant \big(2L(r)\#\mathscr{C}\big)^{2n/L(r)} \lambda_{\mathrm{Exp}}^{-n\varepsilon},$$

it follows from (19) that this sum is exponentially small with n. This implies the assertion of the lemma. □

7.3. Proof of part 2 of the Key Lemma

Let f, λ_{Exp}, $r_0 > 0$ and $L(r)$ be as in the previous subsection. For $c \in \mathscr{C}$ let $d(c) > 1$ be the local degree of f at c. Let $\mathbf{m} > 0$ be given and for an integer $d \geqslant 2$ let $C_1(\mathbf{m}, d) > 0$ be the constant given by Lemma 5.4 and put

$$C_1(\mathbf{m}) := \max_{c \in \mathscr{C}} C_1\bigl(\mathbf{m}, d(c)\bigr).$$

Let $r \in (0, r_0)$ be sufficiently small so that

$$\bigl(2L(r)(\#\mathscr{C})\bigr)^{2/L(r)} \lambda_{\mathrm{Exp}}^{-\mathrm{HD}(J(f))} C_1(\mathbf{m})^{1/L(r)} < 1. \tag{20}$$

We will prove that part 2 of the Key Lemma holds for this choice of r. So let $(\widehat{V}, V)$ be a nice couple satisfying,

$$\min_{c \in \mathscr{C}} \mathrm{mod}\bigl(\widehat{V}^c \setminus \overline{V^c}\bigr) \geqslant \mathbf{m} \quad \text{and} \quad \max_{c \in \mathscr{C}} \mathrm{diam}\bigl(\widehat{V}^c\bigr) \leqslant r.$$

Denote by $\mathfrak{D}_V$ the collection of all connected components of $\overline{\mathbb{C}} \setminus K(V)$ and let $\alpha \in (0, \mathrm{HD}_{\mathrm{hyp}}(J(f)))$ be given by Proposition 6.1, so that $\|\mathfrak{D}_V\|_\alpha < +\infty$ (it is easy to see that the hypotheses of this proposition are satisfied for maps satisfying property ExpShrink). Taking α closer to $\mathrm{HD}(J(f)) = \mathrm{HD}_{\mathrm{hyp}}(J(f))$ if necessary, we assume that (20) holds with $\mathrm{HD}(J(f))$ replaced by α.

Recall that we denote by D the subset of V of those points having a good time, see §4.3. Denote by $\mathfrak{D}$ the collection of the connected components of D. As V is a nice set, it is easy to see that every $W \in \mathfrak{D}$ is a pull-back of V and that for every $z \in W$ we have $m(z) = m_W$.

LEMMA 7.3. – *Let $\mathfrak{D}_0$ be the sub-collection of $\mathfrak{D}$ of those W such that m_W is the first return time of W to V. Then $\|\mathfrak{D}_0\|_\alpha < +\infty$.*

Proof. – Let $\mathfrak{D}'_V$ be the sub-collection of $\mathfrak{D}_V$ of sets W' in $\mathfrak{D}_V$ such that $\widehat{W'}$ is disjoint from the critical values of f. We have $\|\mathfrak{D}'_V\|_\alpha \leqslant \|\mathfrak{D}_V\|_\alpha < +\infty$ and $\mathfrak{D}_0 \subset f^{-1}(\mathfrak{D}'_V)$. Then the assertion of the lemma follows from Lemma 5.4 applied to $h := f$. □

LEMMA 7.4. –
1. *Every $W \in \mathfrak{D} \setminus \mathfrak{D}_0$ is contained in a bad pull-back $\widetilde{W}$ of $\widehat{V}$ such that $m_{\widetilde{W}} < m_W$ and $f^{m_{\widetilde{W}}}(W) \in \mathfrak{D}_0$.*
2. *For every bad pull-back $\widetilde{W}$ of $\widehat{V}$, the collection $\mathfrak{D}_{\widetilde{W}}$ of all $W \in \mathfrak{D}$ contained in $\widetilde{W}$ and such that $f^{m_{\widetilde{W}}}(W) \in \mathfrak{D}_0$, satisfies*

$$\|\mathfrak{D}_{\widetilde{W}}\|_\alpha \leqslant C_1(\mathbf{m})^{m_{\widetilde{W}}/L(r)} \|\mathfrak{D}_0\|_\alpha.$$

Proof. – **1**. Let $W \in \mathfrak{D} \setminus \mathfrak{D}_0$ be given and let n be the largest integer in $\{0, \dots, m_W - 1\}$ such that $f^n(W)$ intersects V. We have $n > 0$ because by assumption $W \notin \mathfrak{D}_0$. Then $W' := f^n(W) \subset V$ and, as f^{m_W} is univalent on $\widehat{W}$, we have that $f^{m_{W'}}$ is univalent on $\widehat{W'}$. Moreover, by maximality of n it follows that $f^{m_{W'}}$ coincides with the first return map to V on W', so that $W' \in \mathfrak{D}_0$. To finish the proof is enough to show that the pull-back $\widetilde{W}$ of $\widehat{V}$ by f^n containing W is bad. If $\widetilde{W}$ is not a bad pull-back of $\widehat{V}$, then there is $m \leqslant m_{\widetilde{W}} = n < m_W$ such that $f^m(W) \subset V$ and such that the pull-back of $\widehat{V}$ by f^m containing W is univalent. But this contradicts the fact that W is a connected component of D.

2. We keep the notation of parts 1 and 2 of the proof of Lemma 7.1. Note that we have $\ell_0 = m_{\widetilde{W}}$ and for each $j = 1, \ldots, k$ we have $\ell_{j-1} - \ell_j \geqslant L(r)$, so that $k \leqslant m_{\widetilde{W}}/L(r)$. For each $j = 0, \ldots, k$ denote by c_j the critical point such that $f^{\ell_j}(\widetilde{W}) \subset \widehat{V}^{c_j}$ and let $\widehat{U}_j$ be the connected component of $f^{-(\ell_j - \ell_{j+1})}(\widehat{V}^{c_j})$ containing c_{j+1}. It follows that $h_j : \widehat{U}_j \to \widehat{V}^{c_j}$ defined by $h_j := f^{\ell_j - \ell_{j+1}}|_{\widehat{U}_j}$ is a unicritical map with critical point c_{j+1} and degree $d(c_{j+1})$.

Define families $\mathfrak{F}_0, \ldots, \mathfrak{F}_k$ of pull-backs of V inductively as follows. Put $\mathfrak{F}_0 := \mathfrak{D}_0$ and for $j = 0, \ldots, k-1$ suppose that the family $\mathfrak{F}_j$ is already defined and that $\operatorname{supp}(\mathfrak{F}_j) \subset f^{\ell_j}(\widetilde{W}) \subset \widehat{V}^{c_j}$. Let $\mathfrak{F}'_j$ be the sub-family of $\mathfrak{F}_j$ of those W such that $\widehat{W}$ is disjoint from $h_j(c_{j+1})$ and define

$$\mathfrak{F}_{j+1} := h_j^{-1}(\mathfrak{F}'_j).$$

Clearly $\operatorname{supp}(\mathfrak{F}_{j+1}) \subset \widehat{U}_j \subset V^{c_{j+1}} \subset \widehat{V}^{c_{j+1}}$, so the induction hypothesis is satisfied.

It is easy to see that $\mathfrak{F}_k = \mathfrak{D}_{\widetilde{W}}$. To prove the assertion of the lemma, note that the family $\mathfrak{F}'_j$ is $\mathbf{m}$-shielded by the family $\widehat{\mathfrak{F}}'_j = \{\widehat{W} \mid W \in \mathfrak{F}'_j\}$ and that by definition of $\mathfrak{F}_{j+1}$ we have $h(c_{j+1}) \notin \operatorname{supp}(\widehat{\mathfrak{F}}'_j)$. Moreover, by induction in j we have that $\operatorname{supp}(\widehat{\mathfrak{F}}'_j) \subset V^{c_j}$. So Lemma 5.4 implies that

$$\|\mathfrak{F}_{j+1}\|_\alpha \leqslant C_1\big(\mathbf{m}, d(c_{j+1})\big)\|\mathfrak{F}_j\|_\alpha \leqslant C_1(\mathbf{m})\|\mathfrak{F}_j\|_\alpha.$$

Therefore we have

$$\|\mathfrak{D}_{\widetilde{W}}\|_\alpha = \|\mathfrak{F}_k\|_\alpha \leqslant C_1(\mathbf{m})^k\|\mathfrak{F}_0\|_\alpha \leqslant C_1(\mathbf{m})^{m_{\widetilde{W}}/L(r)}\|\mathfrak{D}_0\|_\alpha. \qquad \square$$

To prove part 2 of the Key Lemma, note that by part 1 of Lemma 7.4 we have

$$\mathfrak{D} = \mathfrak{D}_0 \cup \Big(\bigcup_{\widetilde{W}} \mathfrak{D}_{\widetilde{W}}\Big),$$

where the union is over all bad pull-backs $\widetilde{W}$ of $\widehat{V}$. Let $n \geqslant 1$ be an integer and let $\widetilde{W}$ be a bad pull-back of order n. Then we have $\operatorname{diam}(\widetilde{W}) \leqslant \lambda_{\mathrm{Exp}}^{-n}$. By part 2 of Lemma 7.4 and by Lemma 5.1 we have

$$\sum_{W \in \mathfrak{D}_{\widetilde{W}}} \operatorname{diam}(W)^\alpha \leqslant 4^\alpha \lambda_{\mathrm{Exp}}^{-n\alpha} C_1(\mathbf{m})^{n/L(r)}\|\mathfrak{D}_0\|_\alpha.$$

As the number of bad pull-backs of order n is bounded by $(2L(r)(\#\mathscr{C}))^{2n/L(r)}$, letting

$$\eta = \big(2L(r)(\#\mathscr{C})\big)^{2/L(r)}\lambda_{\mathrm{Exp}}^{-\alpha}C_1(\mathbf{m})^{1/L(r)},$$

we have

$$\sum_{\widetilde{W}} \sum_{W \in \mathfrak{D}_{\widetilde{W}}} \operatorname{diam}(W)^\alpha \leqslant 4^\alpha \eta^n \|\mathfrak{D}_0\|_\alpha,$$

where the sum is over all bad pull-backs $\widetilde{W}$ of $\widehat{V}$ of order n. Recall that we have chosen $r > 0$ sufficiently small and α sufficiently close to $\mathrm{HD}(J(f))$, so that (20) is satisfied with $\mathrm{HD}(J(f))$

replaced by α. As $\alpha \in (0, \mathrm{HD}(J(f)))$ it follows that $\eta \in (0,1)$ and that

$$\sum_{W \in \mathfrak{D}} \mathrm{diam}(W)^{\alpha} \leqslant 4^{\alpha}(1-\eta)^{-1} \|\mathfrak{D}_0\|_{\alpha} < +\infty.$$

8. Conformal and invariant measures

In this section we prove the main results of this paper.

8.1. Proof of Theorem A

Let f be a rational map satisfying the ExpShrink condition with constant $\lambda_{\mathrm{Exp}} > 1$, and let $\lambda \in (1, \lambda_{\mathrm{Exp}})$ and $\mathbf{m} > 0$ be given. Let $r > 0$ be given by the Key Lemma for this choice of $\mathbf{m}$ and let $(\widehat{V}, V)$ be a nice couple for f given by Theorem E for this choice of λ, $\mathbf{m}$ and r. It follows that the nice couple $(\widehat{V}, V)$ satisfies the conclusions of the Key Lemma. Reducing $r > 0$ if necessary, we assume that the canonical induced map associated to $(\widehat{V}, V)$ is topologically mixing (Lemma 4.1). Then Theorem A is a direct consequence of Theorem 2 in Appendix B, and of the fact that the equality $\alpha(f) = \mathrm{HD}(J(f))$ holds for maps satisfying ExpShrink, see [34].

8.2. Proof of Theorems B and C

The uniqueness part of Theorem B follows from Proposition B.1, and from the fact that the unique conformal probability measure of minimal exponent of f is supported on the conical Julia set (Theorem A). All the remaining statements will be obtained from some results of Young [46], that we recall now.

Let $(\Delta_0, \mathcal{B}_0, \mathfrak{m}_0)$ be a measurable space and let $T_0 : \Delta_0 \to \Delta_0$ be a measurable map for which there is a countable partition $\mathscr{P}_0$ of Δ_0, such that for each element Δ' of $\mathscr{P}_0$ the map $T_0 : \Delta' \to \Delta_0$ is a bijection. Moreover we assume that the partition $\mathscr{P}_0$ generates, in the sense that each element of the partition $\bigvee_{n=0}^{\infty} T_0^{-n} \mathscr{P}_0$ is a singleton. It follows that for every pair of points $x, y \in \Delta_0$ there is a non-negative integer s, such that $T_0^s(x)$ and $T_0^s(y)$ belong to different elements of the partition $\mathscr{P}_0$. We denote by $s_0(x,y)$ the least of such integers s and call it *the separation time* of x and y.

We assume furthermore that for each element Δ' of $\mathscr{P}_0$ the map $(T_0|_{\Delta'})^{-1}$ is measurable and that the Jacobian $\mathrm{Jac}(T_0)$ of T_0 is well defined and positive on a set of full measure of Δ_0. Moreover, we require that there are constants $C > 0$ and $\beta \in (0,1)$, such that for almost every $x, y \in \Delta_0$ that belong to the same element of $\mathscr{P}_0$, we have

$$\left| \mathrm{Jac}(T_0)(x) / \mathrm{Jac}(T_0)(y) - 1 \right| \leqslant C \beta^{s_0(T_0(x), T_0(y))}.$$

Let R be a measurable function defined on Δ_0, taking positive integer values, and constant on each element of $\mathscr{P}_0$. Moreover we assume that the greatest common divisor of the values of R is equal to 1. Then put

$$\Delta = \left\{ (z,n) \in \Delta_0 \times \{0,1,\ldots\} \mid n < R(z) \right\},$$

and endow Δ with the measure $\mathfrak{m}$, such that for each $n = 0, 1, \ldots$ its restriction to $\Delta_n := \{(z,n) \mid z \in \Delta_0, (z,n) \in \Delta\}$ is equal to the pull-back of $\mathfrak{m}_0$ by the map $(z,n) \mapsto z$. Moreover we define the map $T : \Delta \to \Delta$ by

$$T(z,n)=\begin{cases}(z,n+1) & \text{if } n+1<R(z),\\ (T_0(z),0) & \text{if } n+1=R(z).\end{cases}$$

Recall that "exponential mixing" and "Central Limit Theorem" were defined in §1.3.

THEOREM (L.-S. Young [46]). – *With the previous considerations, the following properties hold.*

1. *If $\int R\,d\mathfrak{m}_0<+\infty$, then the map $T:\Delta\to\Delta$ admits an invariant probability measure ρ that is absolutely continuous with respect to $\mathfrak{m}$. Moreover, the measure ρ is ergodic, mixing, and its density with respect to $\mathfrak{m}$ is almost everywhere bounded from below by a positive constant.*
2. *If $\mathfrak{m}_0(\{z\in\Delta_0\mid R(z)>m\})$ decreases exponentially fast with m, then the measure ρ is exponentially mixing and the Central Limit Theorem holds for ρ.*

We will also need the following general lemma, whose proof is below.

LEMMA 8.1. – *Let f be a rational map and let $(\widehat{V},V)$ be a nice couple for f such that the corresponding canonical induced map $F:D\to V$ satisfies the conclusions of the Key Lemma. Given $\tilde{c}\in\mathscr{C}$, let $\widetilde{F}:\widetilde{D}\to V^{\tilde{c}}$ be the first return map of F to $V^{\tilde{c}}$ and denote by $J(\widetilde{F})$ the maximal invariant set of $\widetilde{F}$. Then we have*

$$\mathrm{HD}\big(\big(J(f)\cap V^{\tilde{c}}\big)\setminus J\big(\widetilde{F}\big)\big)<\mathrm{HD}\big(J(f)\big),$$
$$\mathrm{HD}\Bigg(J(f)\setminus\bigcup_{n\geqslant 0}f^{-n}\big(J\big(\widetilde{F}\big)\big)\Bigg)<\mathrm{HD}\big(J(f)\big)$$

and there is $\tilde{\alpha}\in(0,\mathrm{HD}(J(f)))$ such that

$$\sum_{W\ c.c.\ of\ \widetilde{D}}\operatorname{diam}(W)^{\tilde{\alpha}}<+\infty. \tag{21}$$

To prove Theorems B and C, let f be a rational map satisfying the ExpShrink condition and let μ be the conformal probability measure of exponent $\alpha(f)=\mathrm{HD}(J(f))$ for f, given by Theorem A, so that μ is supported on the conical Julia set of f and $\mathrm{HD}(\mu)=\mathrm{HD}(J(f))$. We will use several times that the measure μ does not charge sets whose Hausdorff dimension is strictly less than $\mathrm{HD}(J(f))$. To see this, suppose by contradiction that there is a set X such that $\mu(X)>0$ and $\mathrm{HD}(X)<\mathrm{HD}(J(f))$. Then the invariant set $Y=\bigcup_{n\geqslant 0}f^n(X)$ has full measure with respect to μ (Proposition B.1) and we have that $\mathrm{HD}(Y)=\mathrm{HD}(X)<\mathrm{HD}(J(f))$. But this contradicts $\mathrm{HD}(\mu)=\mathrm{HD}(J(f))$.

Let $(\widehat{V},V)$ be a nice couple for f satisfying the conclusions of Lemma 4.1, Theorem E and of the Key Lemma, as in the proof of Theorem A in §8.1. Let $F:D\to V$ be the canonical induced map associated to $(\widehat{V},V)$. Moreover, let $\tilde{c}\in\mathscr{C}$ be given by Lemma 4.1 and let $\widetilde{F}:\widetilde{D}\to V^{\tilde{c}}$ be the first return map F to $V^{\tilde{c}}$. We denote by R the return time function of $\widetilde{F}$ with respect to f, so that $\widetilde{F}\equiv f^R$ on $\widetilde{D}$.

Put $\Delta_0=J(\widetilde{F})$, $\mathfrak{m}_0=\mu|_{J(\widetilde{F})}$, $T_0=\widetilde{F}|_{J(\widetilde{F})}$ and $\mathscr{P}_0=\{W\cap J(\widetilde{F})\mid W \text{ c.c. of } \widetilde{D}\}$. Notice that the measure $\mathfrak{m}_0$ is non-zero, because $\mathrm{HD}((J(F)\cap V^{\tilde{c}})\setminus J(\widetilde{F}))<\mathrm{HD}(J(f))$ (Lemma 8.1) and hence $\mu(J(\widetilde{F}))=\mu(J(F)\cap V^c)>0$. We will now verify that all the hypotheses of Young's theorem are satisfied for this choice of Δ_0, T_0, $\mathscr{P}_0$ and R.

First notice that $\widetilde{F}$ is an induced map of f in the sense of Appendix A. Hence, the bounded distortion property and the fact that the partition $\mathscr{P}_0$ is generating for $T_0=\widetilde{F}|_{J(\widetilde{F})}$, follow from §A.1. On the other hand, the final statement of Lemma 4.1 implies that the greatest common divisor of the values of R is equal to 1. It remains to prove the "tail estimate" in part 2 of Young's theorem. By the bounded distortion property of $\widetilde{F}$, it follows that there is a constant $C_0 > 1$ such that for every connected component W of $\widetilde{D}$ we have

$$C_0^{-1}\operatorname{diam}(W)^{\alpha(f)} \leqslant \mu(W) \leqslant C_0 \operatorname{diam}(W)^{\alpha(f)}.$$

Note that for each connected component W of $\widetilde{D}$ and for every $z \in W$ we have $R(z) = m_W$. Moreover, since for each $z \in D$ we have $|F'(z)| \geqslant \lambda^{m(z)}$, it follows that for every connected component W of $\widetilde{D}$ we have that $\operatorname{diam}(W) \leqslant \lambda^{-m_W} \operatorname{diam}(V^{c(W)})$. As $\tilde{\alpha} < \mathrm{HD}(J(f)) = \alpha(f)$, for each positive integer m we have

$$\begin{aligned}
\mu\big(\big\{z \in \widetilde{D} \mid R(z) \geqslant m\big\}\big) &= \sum_{W \text{ c.c. } \widetilde{D}, m_W \geqslant m} \mu(W) \\
&\leqslant C_0 \sum_{W \text{ c.c. } \widetilde{D},\, m_W \geqslant m} \operatorname{diam}(W)^{\alpha(f)} \\
&\leqslant C_1 \sum_{W \text{ c.c. } \widetilde{D},\, m_W \geqslant m} \big(\lambda^{-(\alpha(f)-\tilde{\alpha})}\big)^{m_W} \operatorname{diam}(W)^{\tilde{\alpha}} \\
&\leqslant C_1 \big(\lambda^{-(\alpha(f)-\tilde{\alpha})}\big)^{m} \sum_{W \text{ c.c. } \widetilde{D}} \operatorname{diam}(W)^{\tilde{\alpha}},
\end{aligned}$$

where $C_1 := C_0(\max_{c\in\mathscr{C}} \operatorname{diam}(V^c))^{\alpha(f)-\tilde{\alpha}}$. By (21) it follows that there is a constant $C_2 > 0$ such that, letting $\theta = \lambda^{-(\alpha(f)-\tilde{\alpha})} \in (0,1)$, we have

$$\mu\big(\big\{z \in D \mid R(z) \geqslant m\big\}\big) \leqslant C_2 \theta^m.$$

Thus we have verified all the hypotheses of Young's theorem. Let ρ be the invariant measure given by Young's theorem and consider the projection $\pi : \Delta \to \overline{\mathbb{C}}$ defined by $\pi(z,n) = f^n(z)$. We have $f \circ \pi = \pi \circ T$ and therefore the measure $\pi_*\rho$ is invariant by f, it is exponentially mixing and the Central Limit Theorem holds for this measure. It remains to show that this measure is absolutely continuous with respect to μ and that its density is almost everywhere bounded from below by a positive constant. First notice that by definition of $\mathfrak{m}$ we have $\pi_*\mathfrak{m}|_{J(\widetilde{F})\times\{0\}} = \mu|_{J(\widetilde{F})}$ and that for every connected component W of $\widetilde{D}$ and every $n \in \{0,\dots,m_W - 1\}$ the measure

$$\mu_{W,n} := \pi_*\mathfrak{m}|_{(W\cap J(\widetilde{F}))\times\{n\}} = (f^n)_*\mu|_{W\cap J(\widetilde{F})}$$

is absolutely continuous with respect to μ, with density $J_{W,n} := |(f^n)'|^{-\alpha}$ on $f^n(W)$, and 0 on the rest of $\overline{\mathbb{C}}$. When we sum these measures over all possible W and n, we obtain a series of measures $\sum_{W,n}\mu_{W,n}$, that converges to $\pi_*\mathfrak{m}$ as linear functionals on continuous functions defined on $J(f)$. If $u : J(f) \to \mathbb{R}$ is the constant function equal to 1, then by the Monotone Convergence Theorem we have

$$+\infty > \pi_*\mathfrak{m}\big(J(f)\big) = \sum_{W,n} \int J_{W,n}\, d\mu = \int \sum_{W,n} J_{W,n}\, d\mu.$$

Hence the function $\sum_{W,n} J_{W,n}$ is μ-integrable. Using the Monotone Convergence Theorem again we have that for every continuous function $u : J(f) \to \mathbb{R}$,

$$\int u\, d\pi_*\mathfrak{m} = \sum_{W,n} \int u\, d\mu_{W,n} = \sum_{W,n} \int u J_{W,n}\, d\mu = \int \sum_{W,n} u J_{W,n}\, d\mu = \int u \Bigl(\sum_{W,n} J_{W,n} \Bigr)\, d\mu.$$

Thus, it follows that $\pi_*\mathfrak{m}$ is absolutely continuous with respect to μ, with density $\sum_{W,n} J_{W,n}$. As ρ is absolutely continuous with respect to $\mathfrak{m}$, we have that $\pi_*\rho$ is absolutely continuous with respect to μ. Let h be the density of $\pi_*\rho$ with respect to μ. Since $\pi_0\mathfrak{m}|_{J(\widetilde{F})\times\{0\}} = \mu|_{J(\widetilde{F})}$, it follows by Young's theorem that there is a constant $c > 0$, such that we have $h > c$ almost everywhere on $J(\widetilde{F})$. As $\mu(V^{\tilde{c}} \setminus J(\widetilde{F})) = 0$ (because the Hausdorff dimension of this set is strictly less than that of $J(f)$), it follows that we have $h > c$ almost everywhere on $V^{\tilde{c}}$. Let N be a positive integer such that $J(f) \subset f^N(V^{\tilde{c}})$. By the invariance of μ, for almost every $z \in J(f)$ we have

$$h(z) = \sum_{f^N(y)=z} h(y)\bigl|(f^N)'(y)\bigr|^{-\alpha(f)} \geqslant c\Bigl(\sup_{\overline{\mathbb{C}}}\bigl|(f^N)'\bigr|\Bigr)^{-\alpha(f)}.$$

This finishes the proof of Theorems B and C.

Remark 8.2. – When $\mathscr{C}$ contains exactly one element Lemma 8.1 is not necessary, because in this case $\widetilde{F} = F$. When $\mathscr{C}$ contains more than one element we cannot apply Young's theorem directly to $\Delta_0 = J(F)$, $T_0 = F$ and $\mathscr{P}_0 = \{W \cap J(F) \mid W \text{ c.c. of } D\}$, because in this case the image by F of an element of $\mathscr{P}_0$ is of the form $V^c \cap \Delta_0$, for some $c \in \mathscr{C}$, and it is not equal to Δ_0. However Young's theorem extends to this more general setting, as shown in [15, Théorème 2.3.6 and Remarque 2.3.7], and we could apply these results directly to F.

Proof of Lemma 8.1. – The conclusions of the Key Lemma imply that $\mathrm{HD}(J(F)) = \mathrm{HD}(J(f))$ and that there is $\alpha \in (0, \mathrm{HD}(J(f)))$ at which the pressure function of F is finite. It follows by (22) in §A.3 that F is strongly regular and that its pressure function vanishes at $t = \mathrm{HD}(J(f))$.

1. Let G be the restriction of F to $(D \setminus V^{\tilde{c}}) \setminus F^{-1}(V^{\tilde{c}})$ and notice that G is an induced map of f in the sense of Appendix A. We will show now that $\mathrm{HD}(J(G)) < \mathrm{HD}(J(F))$ and that there is $\tilde{\alpha} \in (\alpha, \mathrm{HD}(J(F)))$ at which the pressure function of G is negative. As F is topologically mixing (Lemma 4.1) the domain of G is strictly smaller than $D \setminus V^{\tilde{c}}$. As F is strongly regular, it follows by Theorem 4.3.10 of [26] that $\mathrm{HD}(J(G)) < \mathrm{HD}(J(F))$. Moreover notice that the pressure function of G is finite at α and that, by formula (22) applied to G, it is negative at $\mathrm{HD}(J(F))$. It follows that there is $\tilde{\alpha} \in (\alpha, \mathrm{HD}(J(F)))$ at which the pressure function of G is negative.

2. Clearly the set $(J(F) \cap V^{\tilde{c}}) \setminus J(\widetilde{F})$ is equal to the preimage of $J(G)$ by $F|_{V^{\tilde{c}}}$. As each inverse branch of F is Lipschitz, it follows that

$$\mathrm{HD}\bigl(\bigl(J(F) \cap V^{\tilde{c}}\bigr) \setminus J\bigl(\widetilde{F}\bigr)\bigr) \leqslant \mathrm{HD}\bigl(J(G)\bigr) < \mathrm{HD}\bigl(J(F)\bigr) = \mathrm{HD}\bigl(J(f)\bigr)$$

and that $\mathrm{HD}((J(f) \cap V^{\tilde{c}}) \setminus J(\widetilde{F})) < \mathrm{HD}(J(f))$. As $\mathrm{HD}(J(f) \cap K(V)) < \mathrm{HD}(J(f))$ (Lemma 6.2), it also follows that $\mathrm{HD}(J(f) \setminus \bigcup_{n\geqslant 0} f^{-n}(J(\widetilde{F}))) < \mathrm{HD}(J(f))$.

3. Let D' be the subset of D of all those points z for which there is a positive integer m such that F^m is defined at z and such that $F^m(z) \in V^{\tilde{c}}$. For each $z \in D'$ we denote by $m'(z)$ the least value of such m, so that $\widetilde{F} \equiv F^{m'}$ on $\widetilde{D}$. Note that $D' \cap V^{\tilde{c}} = \widetilde{D}$. The function m' is constant on each connected component of D'. For a connected component W of D' we denote by m'_W the common value of m' on W. If W is a connected component of D' such that $m'_W = 1$, then $F(W) = V^{\tilde{c}}$. In the case $m'_W > 1$, the set $F(W)$ is a connected component of D' and

$m'_{F(W)} = m'_W - 1$. For each $c \in \mathscr{C} \setminus \{\tilde{c}\}$ fix a point $z_c \in V^c$. Thus, if we denote by $C_0 > 0$ the distortion constant of F, then we have

$$\sum_{W \text{ c.c. of } D' \setminus V^{\tilde{c}}} \operatorname{diam}(W)^{\tilde{\alpha}} \leqslant C_0^{\tilde{\alpha}} \left(\sum_{W \text{ c.c. of } D' \setminus V^{\tilde{c}}, m'_W = 1} \operatorname{diam}(W)^{\tilde{\alpha}} \right) \cdot \left(\sum_{m \geqslant 1} \sum_{c \in \mathcal{C} \setminus \{\tilde{c}\}} \sum_{y \in (G)^{-m}(z_c)} \left| (F^m)'(y) \right|^{-\tilde{\alpha}} \right).$$

Note that the first factor is finite by part 2 of the Key Lemma and that the second factor is finite because the pressure function of G is negative at $\tilde{\alpha}$. Now, if W' is a connected component of D that is contained in $V^{\tilde{c}}$ and whose image by F is not equal to $V^{\tilde{c}}$, then we have $F(W' \cap \widetilde{D}) = D' \cap V^{c(W')}$ and the sum

$$\sum_{W \text{ c.c. of } W' \cap \widetilde{D}} \operatorname{diam}(W)^{\tilde{\alpha}}$$

is less than a distortion constant times,

$$\operatorname{diam}(W')^{\tilde{\alpha}} \cdot \left(\sum_{W \text{ c.c. of } D' \cap V^{c(W')}} \operatorname{diam}(W)^{\tilde{\alpha}} \right).$$

Then the estimate (21) follows from part 2 of the Key Lemma. □

8.3. Proof of Theorem D

The proof relies on the following lemma.

LEMMA 8.3. – *Let f be a rational map of degree at least 2 and let ν be an ergodic invariant measure whose Lyapunov exponent is equal to 0. Then for every $\varepsilon > 0$ there is a set of full measure with respect to ν of points x such that for every sufficiently large integer n we have*

$$f^n\big(B\big(x, \exp(-2\varepsilon n)\big)\big) \subset B\big(f^n(x), \exp(-\varepsilon n)\big).$$

Proof. – By Birkhoff's ergodic theorem there is a set of full measure of x for which there is an integer n_0 such that for every $n \geqslant n_0$ we have

$$\exp\left(-\frac{1}{3}\varepsilon n\right) \leqslant \left|(f^n)'(x)\right| \leqslant \exp\left(\frac{1}{3}\varepsilon n\right).$$

Fix such x with the corresponding n_0. Let $C_0 > 0$ be such that for every $z \in \overline{\mathbb{C}}$ we have $|f'(z)| \leqslant C_0 \operatorname{dist}(z, \operatorname{Crit}(f))$. It follows that for every $n \geqslant n_0$ we have

$$\operatorname{dist}\big(f^n(x), \operatorname{Crit}(f)\big) \geqslant C_0^{-1} \exp\left(-\frac{2}{3}\varepsilon(n+1)\right).$$

Thus there is n_1 such that for every $n \geqslant n_1$ the distortion of f on the ball $B(f^n(x), \exp(-\varepsilon n))$ is at most $\exp(\frac{1}{3}\varepsilon)$.

Let $n_2 \geqslant n_1$ be sufficiently large so that $\exp(\frac{1}{3}\varepsilon n_2) \geqslant 2$ and so that f^{n_2} is univalent on $B(x, \exp(-2\varepsilon n_2))$ with distortion bounded by $2 \leqslant \exp(\frac{1}{3}\varepsilon n_2)$. In particular for every $n \geqslant n_2$

we have

$$f^{n_2}\big(B\big(x,\exp(-2\varepsilon n)\big)\big)\subset B\left(f^{n_2}(x),\exp(-2\varepsilon n)\exp\left(\frac{1}{3}\varepsilon n_2\right)\right)\big|(f^{n_2})'(x)\big|.$$

Given $n\geqslant n_2$ we will show by induction that for every $j=n_2,\dots,n$ the inclusion above holds with n_2 replaced by j. The conclusion of the lemma is obtained by taking $j=n$. Suppose then that for some $j\in\{n_2,\dots,n-1\}$ the inclusion above holds with n_2 replaced by j. As the distortion of f on $B(f^j(x),\exp(-\varepsilon j))$ is bounded by $\exp(\frac{1}{3}\varepsilon)$ it follows that

$$f^{j+1}\big(B\big(x,\exp(-2\varepsilon n)\big)\big)\subset B\left(f^{j+1}(x),\exp(-2\varepsilon n)\exp\left(\frac{1}{3}\varepsilon(j+1)\right)\big|\big(f^{j+1}\big)'(x)\big|\right). \qquad \square$$

For the proof of Theorem D, let f be a rational map having an exponentially mixing invariant measure ν, that is absolutely continuous with respect to a conformal measure μ of f. In particular ν is ergodic.

As ν is exponentially mixing, the Fatou–Sullivan classification of connected components of the Fatou set [2,27,7] implies that ν is either supported on $J(f)$, or on a fixed point of f outside $J(f)$. As conformal measures cannot charge fixed points outside $J(f)$, the second case does not occur. Thus ν must be supported on $J(f)$. Suppose first that ν is supported on a point p in $J(f)$. Then p is a fixed point of f and μ charges p. As μ is a conformal measure, it follows that p is indifferent, so the theorem is verified in this case. From now on we assume that the topological support of ν contains at least 2 points.

1. Let $\delta>0$ be sufficiently small so that there exists $\varepsilon>0$ such that for every x in the topological support of ν we have $\nu(B(x,2\delta))\leqslant 1-\varepsilon$. Moreover let $C>0$ and $\rho\in(0,1)$ be as in the inequality (1) of the exponential mixing property of ν. We will show now that there is a constant $C_0>0$ such that for every $x\in\overline{\mathbb{C}}$, $r>0$ and $n\geqslant 1$ such that $f^n(B(x,2r))\subset B(f^n(x),\delta)$, we have

$$r\nu\big(B(x,r)\big)\leqslant C_0\rho^n.$$

Let $\varphi:\overline{\mathbb{C}}\to[0,1]$ be a continuous function that is constant equal to 0 outside $B(f^n(x),2\delta)$ and that is constant equal to 1 on $B(f^n(x),\delta)$. Moreover, let $\psi:\overline{\mathbb{C}}\to[0,1]$ be a Lipschitz function that is constant equal to 0 outside $B(x,2r)$ and that is constant equal to 1 on $B(x,r)$. We define ψ in such a way that there is a constant $C_1>0$ independent of r and x such that $\|\psi\|_{\mathrm{Lip}}\leqslant C_1r^{-1}$. Then we have

$$C\|\psi\|_{\mathrm{Lip}}\rho^n\geqslant\int(\varphi\circ f^n)\psi\,d\nu-\int\varphi\,d\nu\int\psi\,d\nu=\int\psi\,d\nu\left(1-\int\varphi\,d\nu\right)\geqslant\varepsilon\nu\big(B(x,r)\big).$$

This shows the desired inequality with $C_0:=\varepsilon^{-1}CC_1$.

2. We will show now that the Lyapunov exponent of ν is positive. Suppose by contradiction that the Lyapunov exponent of ν is not positive. As ν is supported on the Julia set of f it follows that the Lyapunov exponent of ν is equal to 0 [30]. Since ν is exponentially mixing, it follows that it is ergodic, so ν satisfies the hypothesis of Lemma 8.3. Given $\alpha>\mathrm{HD}(\nu)$ choose $\varepsilon>0$ sufficiently small so that $\exp(-2\varepsilon(1+\alpha))>\rho$. By Frostman's Lemma, the set of points x for which there exists a sequence $(r_j)_{j\geqslant 0}$ converging to 0 such that for every $j\geqslant 1$ we have $\nu(B(x,r_j))\geqslant r_j^{\alpha}$, has full measure with respect to ν. Fix such a point x, that also satisfies the conclusions of Lemma 8.3. Thus, if j is sufficiently large and n is the largest integer such that $r_j\leqslant\exp(-2\varepsilon n)$, then we have

$$f^n\big(B\big(x,\exp(-2\varepsilon n)\big)\big)\subset B\big(f^n(x),\exp(-\varepsilon n)\big)\subset B\big(f^n(x),\delta\big).$$

Thus, part 1 with $r = \exp(-2\varepsilon n)$ gives

$$\exp\bigl(-2\varepsilon(1+\alpha)(n+1)\bigr) \leqslant \exp(-2\varepsilon n) r_j^{\alpha} \leqslant \exp(-2\varepsilon n)\nu\bigl(B\bigl(x, \exp(-2\varepsilon n)\bigr)\bigr) \leqslant C_0 \rho^n.$$

Since we can take n arbitrarily large and since $\exp(-2\varepsilon(1+\alpha)) > \rho$, we get a contradiction that proves the assertion.

3. We will show that there is a constant $\lambda > 1$ such that for every positive integer n and every repelling periodic point p of f of period n we have $|(f^n)'(p)| \geqslant \lambda^n$. By [35] this last property is equivalent to the TCE condition, so this proves the theorem.

As ν is ergodic and its Lyapunov exponent is positive, it follows by [13] that ν is equivalent to μ and that there is a constant $c > 0$ such that the density of ν with respect to μ is almost everywhere larger than c.

The rest of the proof is similar to that of [28, Lemma 8.2]; we include it here for completeness. Let n be a positive integer and let p be a repelling periodic point of f of period n. Let ϕ be a local inverse of f^n at p that fixes p. Let $r > 0$ be sufficiently small such that ϕ is defined on the ball $B(p, r)$ and such that $\phi(B(p, r)) \subset B(p, r)$. By Koebe Distortion Theorem there is a constant $\eta > 1$ such that for every positive integer k there is $r' > 0$ such that

$$\eta^{-1} r \bigl|\bigl(f^{kn}\bigr)'(p)\bigr|^{-1} < r' < \eta r \bigl|\bigl(f^{kn}\bigr)'(p)\bigr|^{-1},$$

and

$$B(p, r') \subset \phi^k\bigl(B(p, \varepsilon r)\bigr) \subset B(p, \eta r').$$

By the conformality of μ it also follows that, if we denote by α the exponent of μ, then we can take $\eta > 1$ large enough so that $\mu(B(p, \frac{1}{2}r')) \geqslant \eta^{-1}|(f^{kn})'(p)|^{-\alpha}$. Then, by part 1 we have with $r = \frac{1}{2}r'$, we obtain

$$\begin{aligned} c\eta^{-1}\bigl|\bigl(f^{kn}\bigr)'(p)\bigr|^{-\alpha} &\leqslant c\mu\left(B\left(p, \frac{1}{2}r'\right)\right) \leqslant \nu\left(B\left(p, \frac{1}{2}r'\right)\right) \leqslant 2(r')^{-1} C_0 \rho^{kn} \\ &\leqslant 2C_0 \eta r^{-1} \bigl|\bigl(f^{kn}\bigr)'(p)\bigr| \rho^{kn}. \end{aligned}$$

Letting $C'' = (c^{-1} 2C_0 \eta^2 r^{-1})^{\frac{1}{1+\alpha}}$ we then have

$$\bigl|(f^n)'(p)\bigr|^k = \bigl|\bigl(f^{kn}\bigr)'(p)\bigr| \geqslant C''\bigl(\rho^{-\frac{1}{1+\alpha}}\bigr)^{kn}.$$

As this holds for every positive integer k, it follows that $|(f^n)'(p)| \geqslant (\rho^{-\frac{1}{1+\alpha}})^n$. This shows the desired assertion with $\lambda := \rho^{-\frac{1}{1+\alpha}}$.

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Appendix A. Induced maps

In this appendix we study a class of induced maps of a given rational map. We show in particular that these maps fall into the category of maps studied in [26], and gather in Theorem 1 several results of this book.

Throughout all this section we fix a rational map f.

A.1. Definition and general properties of induced maps

Recall that for a nice set $V = \bigcup_{c\in\mathscr{C}} V^c$ for f and for each pull-back W of V, we denote by $c(W) \in \mathscr{C}$ the critical point and by $m_W \geqslant 0$ the integer such that $f^{m_W}(W) = V^{c(W)}$, see §4.1. Moreover, if $(\widehat{V}, V)$ is a nice couple for f, then for each pull-back W of V we denote by $\widehat{W}$ the unique pull-back of $\widehat{V}$ that contains W and such that $m_{\widehat{W}} = m_W$.

DEFINITION A.1. – Let f be a rational map and let $(\widehat{V}, V)$ be a nice couple for f. We will say that a map $F : D \to V$, with $D \subset V$, is *induced by* f, if the following properties hold: Each connected component W of D is a pull-back of V, f^{m_W} is univalent on $\widehat{W}$ and $F|_W = f^{m_W}|_W$.

It is straightforward to check that every induced map satisfies the following properties.

(I_1) *Markov property*. For every connected component W of D, the map $F|_W$ is a biholomorphism between W and $V^{c(W)}$.

(I_2) *Univalent extension*. For each $c \in \mathscr{C}$ and for every connected component W of D satisfying $c(W) = c$, the inverse of $F|_W$ extends to a biholomorphism between $\widehat{V}^c$ and $\widehat{W}$.

(I_3) *Strong separation*. The connected components of D have pairwise disjoint closures.

Denote by $\mathfrak{D}$ the collection of connected components of D and for each $c \in \mathscr{C}$ denote by $\mathfrak{D}^c$ the collection of all elements of $\mathfrak{D}$ contained in V^c, so that $\mathfrak{D} = \bigsqcup_{c\in\mathscr{C}} \mathfrak{D}^c$. A word on the alphabet $\mathfrak{D}$ will be called *admissible* if for every pair of consecutive letters $W, W' \in \mathfrak{D}$ we have $W \in \mathfrak{D}^{c(W')}$. For a given integer $n \geqslant 1$ we denote by E^n the collection of all admissible words of length n and we set $E^* = \bigsqcup_{n\geqslant 1} E^n$. Moreover we denote by E^∞ the collection of all infinite admissible words of the form $W_1 W_2 \ldots$.

Given an integer $n \geqslant 1$ and an infinite word $\underline{W} = W_1 W_2 \ldots \in E^\infty$ or a finite word $\underline{W} = W_1 \ldots W_m \in E^*$ of length $m \geqslant n$, put $\underline{W}|_n = W_1 \ldots W_n$. Given $W \in \mathfrak{D}$, denote by ϕ_W the holomorphic extension to $\widehat{V}^{c(W)}$, of the inverse of $F|_W$, that is given by property (I_2). For a finite word $\underline{W} = W_1 \ldots W_n \in E^*$ put $c(\underline{W}) := c(W_n)$. Note that the composition

$$\phi_{\underline{W}} := \phi_{W_1} \circ \cdots \circ \phi_{W_n}$$

is well defined and univalent on $\widehat{V}^{c(\underline{W})}$ and takes images into V. Moreover, put

$$D_{\underline{W}} := \phi_{\underline{W}}(V^c), \quad \widehat{D}_{\underline{W}} := \phi_{\underline{W}}\big(\widehat{V}^c\big) \quad \text{and} \quad A_{\underline{W}} := \widehat{D}_{\underline{W}} \setminus D_{\underline{W}}.$$

Note that $A_{\underline{W}}$ is an annulus of the same modulus as the annulus $\widehat{V}^{c(\underline{W})} \setminus \overline{V^{c(\underline{W})}}$ and that, when $\underline{W} := W \in \mathfrak{D}$, we have $D_W = W$ and $\widehat{D}_W = \widehat{W}$.

Bounded distortion. For each $\underline{W} \in E^*$ the map $\phi_{\underline{W}}$ is defined and univalent on $\widehat{V}^{c(\underline{W})}$. It follows by Koebe Distortion Theorem that the distortion of $\phi_{\underline{W}}$ on $V^{c(\underline{W})}$ is bounded independently of $\underline{W}$.

Given $n \geqslant 0$ note that the map F^n is well defined on $\bigsqcup_{\underline{W} \in E^n} D_{\underline{W}}$. Moreover, for each $\underline{W} \in E^n$ the restriction of F to $D_{\underline{W}}$ is equal to the inverse of $\phi_{\underline{W}}$ on this set. It follows that the distortion of F^n on each $D_{\underline{W}}$ is bounded independently of n and of $\underline{W}$.

Expansion. For each $c \in \mathscr{C}$ endow $\widehat{V}^c$ with the corresponding hyperbolic metric. The restriction of this metric to V^c is comparable to the spherical metric on $\overline{\mathbb{C}}$. By Schwarz–Pick Lemma it follows that for each $W \in \mathfrak{D}$ the holomorphic map $\phi_W : \widehat{V}^{c(W)} \to V$ decreases the hyperbolic metric by a factor $s \in (0,1)$, independent of c and W. So there is a constant $C_D > 0$ such that for every finite word $\underline{W} \in E^n$ the spherical diameter of $D_{\underline{W}}$ is at most $C_D \cdot s^n$.

Maximal invariant set. For every infinite word $\underline{W} \in E^\infty$ and every $n \geqslant 1$, we have

$$\overline{D}_{\underline{W}}|_{n+1} \subset D_{\underline{W}}|_n \quad \text{and} \quad \operatorname{diam}(D_{\underline{W}}|_n) \leqslant C_D \cdot s^n.$$

It follows that the intersection $\bigcap_{n \geqslant 1} D_{\underline{W}}|_n$ is a singleton. We denote the corresponding point by $\pi(\underline{W})$. So π defines a map from E^∞ to $\overline{\mathbb{C}}$; we denote by $J(F) := \pi(E^\infty)$ the image of π. Note that the set $J(F)$ is equal to the maximal invariant set of F.

It follows from condition (I_3) that for every integer $n \geqslant 1$ and for distinct $W, W' \in E^n$, the closure of the sets $D_{\underline{W}}$ and $D_{\underline{W}'}$ are disjoint. Therefore π induces a bijection between E^∞ and $J(F)$.

The following technical property is important to use the results of [26].

PROPOSITION A.2. – *Every induced map satisfies the following property*

(I_4) *There is a constant $C_M > 0$ such that for every $\kappa \in (0,1)$ and every ball B of $\overline{\mathbb{C}}$, the following property holds. Every collection of pairwise disjoint sets of the form $D_{\underline{W}}$, with $\underline{W} \in E^*$, intersecting B and with diameter at least $\kappa \cdot \operatorname{diam}(B)$, has cardinality at most $C_M \kappa^{-2}$.*

The proof of this proposition is based on the following lemma.

LEMMA A.3. – *Let f be a rational map and let $F : D \to V$ be a map induced by f. Then the following properties hold.*

1. *There exists a constant $C_0 > 0$ such that if $\underline{W}$ and $\underline{W}' \in E^*$ are such that $D_{\underline{W}}$ and $D_{\underline{W}'}$ are disjoint, then*

$$\operatorname{dist}(D_{\underline{W}'}, D_{\underline{W}}) \leqslant C_0 \cdot \operatorname{diam}(D_{\underline{W}}), \text{ implies } \widehat{D}_{\underline{W}'} \subset A_{\underline{W}}.$$

2. *Let B be a ball. Then for every pair $\underline{W}, \underline{W}' \in E^*$ such that the sets $D_{\underline{W}}$ and $D_{\underline{W}'}$ are disjoint and intersect B, we have either*

$$\operatorname{diam}(D_{\underline{W}}) < C_0^{-1} \operatorname{diam}(B) \quad \textit{or} \quad \operatorname{diam}(D_{\underline{W}'}) < C_0^{-1} \operatorname{diam}(B).$$

Proof. – **1**. Observe that for every $\underline{W} \in E^*$ the set $A_{\underline{W}}$ is an annulus whose modulus is at least

$$\min_{c \in \mathscr{C}} \operatorname{mod}\big(\widehat{V}^c \setminus \overline{V^c}\big) > 0.$$

It follows that there is a constant $C_0 > 0$ such that every point whose distance to $D_{\underline{W}}$ is at most $C_0 \cdot \operatorname{diam}(D_{\underline{W}})$ is contained in $\widehat{D}_{\underline{W}}$. So, if $\underline{W}'$ is an element of E^* such that $D_{\underline{W}'}$ is disjoint from $D_{\underline{W}}$ and at distance at most $C_0 \cdot \operatorname{diam}(D_{\underline{W}})$ from $D_{\underline{W}}$, then $D_{\underline{W}'}$ intersects $\widehat{D}_{\underline{W}}$. As the sets $\widehat{D}_{\underline{W}'}$ and $\widehat{D}_{\underline{W}}$ are both pull-backs of $\widehat{V}$, it follows that in this case we have $\widehat{D}_{\underline{W}'} \subset \widehat{D}_{\underline{W}} \setminus \overline{D}_{\underline{W}} = A_{\underline{W}}$.

2. Assume by contradiction that there are such $\underline{W}$ and $\underline{W}'$ in E^* for which $\operatorname{diam}(D_{\underline{W}})$ and $\operatorname{diam}(D_{\underline{W}'})$ are both bigger than or equal to $C_0^{-1}\operatorname{diam}(B)$. Then

$$\operatorname{dist}(D_{\underline{W}}, D_{\underline{W}'}) \leqslant \operatorname{diam}(B) \leqslant C_0 \cdot \min\bigl\{\operatorname{diam}(D_{\underline{W}}), \operatorname{diam}(D_{\underline{W}'})\bigr\},$$

and part 1 implies that $\widehat{D}_{\underline{W}} \subset A_{\underline{W}'}$ and that $\widehat{D}_{\underline{W}'} \subset A_{\underline{W}}$. So we get a contradiction that proves the assertion. □

Proof of Proposition A.2. – By the bounded distortion property there is a constant $C > 0$ such that for every $\underline{W} \in E^*$ we have

$$\operatorname{Area}(D_{\underline{W}}) \geqslant C^{-1}\operatorname{diam}(D_{\underline{W}})^2.$$

Taking C larger if necessary we assume that for every ball B in $\overline{\mathbb{C}}$ we have $\operatorname{Area}(B) \leqslant C\operatorname{diam}(B)^2$.

To prove property (I_4) let $\kappa \in (0,1)$ and let B be a ball. Moreover let $\mathfrak{F}$ be a finite collection of elements of E^*, such that the sets $D_{\underline{W}}$, for $\underline{W} \in \mathfrak{F}$, are pairwise disjoint and such that for every $\underline{W} \in \mathfrak{F}$ we have $\operatorname{diam}(D_{\underline{W}}) \geqslant \kappa \cdot \operatorname{diam}(B)$. It follows that for every $\underline{W} \in \mathfrak{F}$ we have $\operatorname{Area}(D_{\underline{W}}) \geqslant C^{-1}\kappa^2 \cdot \operatorname{diam}(D_{\underline{W}})$. By part 2 of Lemma A.3 it follows that there is at most one element $\underline{W}$ of $\mathfrak{F}$ such that $\operatorname{diam}(D_{\underline{W}}) \geqslant C_0^{-1}\operatorname{diam}(B)$. Let $\widetilde{B}$ be the ball with the same center as B and with radius equal to $C_0^{-1}\operatorname{diam}(B)$. So for every $\underline{W} \in \mathfrak{F}$ we have $D_{\underline{W}} \subset \widetilde{B}$, with at most one exception. Therefore

$$\#\mathfrak{F} \leqslant 1 + \operatorname{Area}\bigl(\widetilde{B}\bigr)/\bigl(C^{-1}\kappa^2 \cdot \operatorname{diam}(B)^2\bigr) \leqslant 1 + 4C_0^{-2}C^2\kappa^{-2} \leqslant \bigl(1 + 4C_0^{-2}C^2\bigr)\kappa^{-2}. \qquad \square$$

A.2. Pressure function

Fix an induced map $F : D \to V$ of f. Then for $t \geqslant 0$ we define

$$Z_n(t) := \sum_{\underline{W} \in E^n} \bigl(\sup\bigl\{\bigl|\phi'_{\underline{W}}(z)\bigr| \mid z \in V^{c(\underline{W})}\bigr\}\bigr)^t.$$

The proof of the following lemma is standard, see for example Lemmas 2.1.1 and 2.1.2 of [26].

LEMMA A.4. – *For every $t \geqslant 0$ the sequence $(\ln Z_n(t))_{n\geqslant 1}$ is sub-additive and*

$$P(t) := \lim_{n\to+\infty} \frac{1}{n}\ln Z_n(t) = \inf\left\{\frac{1}{n}\ln Z_n(t) \mid n \geqslant 1\right\}.$$

The function $P : [0,+\infty) \to [-\infty,+\infty]$ so defined is called the *pressure function*. It is easy to see that for every $t \geqslant 0$ the sequence $(\frac{1}{n}\ln Z_n(t))_{n\geqslant 1}$ is uniformly bounded from below. In particular the function P does not take the value $-\infty$. Note however that if D has infinitely many connected components, then $P(0) = +\infty$. The proof of the following lemma is straightforward.

LEMMA A.5. – *Put $\theta(F) := \inf\{t \geqslant 0 \mid P(t) < +\infty\}$. Then the pressure function P is finite, continuous and strictly decreasing to $-\infty$ on $(\theta(F), +\infty)$.*

It follows from this lemma that there is at most one value of $t \geqslant 0$ for which the pressure function P vanishes. Following the terminology of [26] we say that F is *strongly regular* if there is $t > \theta(F)$ at which P vanishes.

A.3. Hausdorff dimension and conformal measures

Fix an induced map $F: D \to V$ of f. For a given $t > 0$, a finite Borel measure μ will be called *conformal with exponent* t for F, if μ is supported on $J(F)$ and if for every $W \in \mathfrak{D}$ and every Borel set U contained in W, we have

$$\mu\big(F(U)\big) = \int_U |F'|^t \, d\mu.$$

In the following theorem we gather several results of [26], applied to our particular setting. In fact, the collection of maps $\Phi = \{\phi_W \mid W \text{ c.c. } D\}$ is a *Conformal Graph Directed Markov System* (*CGDMS* for short), as defined in §4.2 of [26], except for the fact that the *Cone Condition* (4d) of [26] is replaced here by the weaker condition (I_4). However the Cone Condition is only used in [26] to guaranty that the conclusion of Lemma 4.2.6 holds, and this is our condition (I_4). Finally note that if F is topologically mixing, then Φ is finitely primitive in the sense of [26].

We first note that by Theorem 2.4.13 of [26] we have

$$\mathrm{HD}\big(J(F)\big) = \inf\big\{t \geqslant 0 \mid P(t) < 0\big\}. \tag{22}$$

THEOREM 1 [26]. – *Let F be a topologically mixing and strongly regular induced map and let $h \geqslant 0$ be the unique zero of the pressure function of F. Then we have $h = \mathrm{HD}(J(F))$, there is a unique conformal probability measure μ of exponent h for F and this measure satisfies $\mathrm{HD}(\mu) = \mathrm{HD}(J(f))$.*

Proof. – The equality $h = \mathrm{HD}(J(F))$ follows from (22). The existence and uniqueness of the conformal measure of exponent h for F is given by Theorem 4.2.9 of [26]. Finally the equality $\mathrm{HD}(\mu) = h$ is given by Corollary 4.4.6 of [26]. □

Appendix B. Conformal measures via inducing

Fix throughout all this section a rational map f and a nice couple $(\widehat{V}, V)$ for f. In this appendix we study the conformal measures of f, through the canonical induced map F associated to $(\widehat{V}, V)$. In particular we give a sufficient condition on F, for f to have a conformal measure supported on the conical Julia set. See §B.1 and §B.2 for the definition of conformal measure and of conical Julia set, respectively.

This appendix is dedicated to the proof of the following result. See §B.1 for the definition of $\alpha(f)$.

THEOREM 2. – *Let f be a rational map of degree at least 2, let $(\widehat{V}, V)$ be a nice couple for f and let $F: D \to V$ be the canonical induced map associated to $(\widehat{V}, V)$. Assume that F is topologically mixing and that the following properties hold.*

1. *For every $c \in \mathscr{C}$ we have $\mathrm{HD}(J(F) \cap V^c) = \alpha(f)$.*
2. *There is $\alpha \in (0, \alpha(f))$ such that*

$$\sum_{W \text{ c.c. of } D} \mathrm{diam}(W)^\alpha < +\infty. \tag{23}$$

Then the canonical induced map F is strongly regular in the sense of [26] *and there is a unique conformal probability measure of exponent $\alpha(f)$ for f. Moreover this measure is non-atomic,* *ergodic, its Hausdorff dimension is equal to $\alpha(f)$ and it is supported on the conical Julia set of f.*

Observe that $J(F)$ is clearly contained in the conical Julia set $J_{\mathrm{con}}(f)$ of f. In [11,30] it is shown that $\alpha(f) = \mathrm{HD}(J_{\mathrm{con}}(f))$ (see also [25]), so by the inclusion $J(F) \subset J_{\mathrm{con}}(f)$, we have

$$\mathrm{HD}\big(J(F)\big) \leqslant \mathrm{HD}\big(J_{\mathrm{con}}(f)\big) = \alpha(f).$$

So the first hypothesis of the theorem requires, in fact, that for each $c \in \mathscr{C}$ the Hausdorff dimension of $J(F) \cap V^c$ is as large as possible

B.1. Conformal measures

For a given $t \geqslant 0$, we say that a non-zero Borel measure μ is *conformal of exponent* t for f, if for every Borel subset U of $\overline{\mathbb{C}}$ where f is injective, we have

$$\mu\big(f(U)\big) = \int_U |f'|^t \, d\mu.$$

By the locally eventually onto property it is easy to see that if the topological support of a conformal measure is contained in the Julia set $J(f)$ of f, then the support is in fact equal to $J(f)$. Note that a conformal measure of exponent $t = 0$ must be supported on the exceptional set of f. So the exponent of a conformal measure supported on $J(f)$ is positive.

It was shown by Sullivan that every rational map admits a conformal measure supported on the Julia set [43]. So the infimum

$$\alpha(f) := \inf\big\{t > 0 \mid \text{ there exists a conformal measure of exponent } t \text{ supported on } J(f)\big\}$$

is well defined and it is easy to see that it is attained. It follows that $\alpha(t)$ is positive, as there is no conformal measure of exponent $t = 0$ supported on the Julia set.

B.2. The conical Julia set and sub-conformal measures

The *conical Julia set* of f, denoted by $J_{\mathrm{con}}(f)$, is by definition the set of all those points x in $J(f)$ for which there exist $\rho(x) > 0$ and arbitrarily large positive integers n, such that the pull-back of the ball $B(f^n(x), \rho(x))$ to x by f^n is univalent. This set is also called *radial Julia set*.

We will need the following general result, which is a strengthened version of [25, Theorem 5.1], [9, Theorem 1.2], with the same proof. Given $t \geqslant 0$ we will say that a Borel measure μ is *sub-conformal of exponent t for f*, if for every subset U of $\overline{\mathbb{C}}$ on which f is injective we have

$$\int_U |f'|^t \, d\mu \leqslant \mu\big(f(U)\big). \tag{24}$$

PROPOSITION B.1. – *If μ is a sub-conformal measure for f supported on $J_{\mathrm{con}}(f)$, whose exponent is at least $\alpha(f)$, then μ is conformal of exponent $\alpha(f)$ and every other conformal measure of exponent $\alpha(f)$ is proportional to μ. Moreover μ is non-atomic and every subset X of $\overline{\mathbb{C}}$ such that $f(X) \subset X$ and $\mu(X) > 0$, has full measure with respect to μ. In particular μ admits at most one absolutely continuous invariant probability measure.*

B.3. Induced maps and conformal measures

For a nice set V for f denote by R_V the first return map to V.

PROPOSITION B.2. – *Let F be the canonical induced map associated to a nice couple $(\widehat{V}, V)$ for f. Then every conformal measure of F of exponent greater than or equal to $\alpha(f)$ is the restriction to V of a conformal measure of f supported on $J_{\mathrm{con}}(f)$. Moreover these measures have the same Hausdorff dimension.*

The proof of this proposition is provided below, it depends on some lemmas. Fix a nice couple $(\widehat{V}, V)$ for f and denote by $F : D \to V$ the canonical induced map associated to it.

LEMMA B.3. – *Denote by $\mathfrak{D}_V$ the collection of connected components of $\overline{\mathbb{C}} \setminus K(V)$ and for $W \in \mathfrak{D}_V$ denote by $\phi_W : \widehat{V}^{c(W)} \to \widehat{W}$ the inverse of $f^{m_W}|_{\widehat{W}}$. Given a conformal measure μ for F of exponent t, for each $W \in \mathfrak{D}_V$ let μ_W be the measure supported on W, defined by*

$$\mu_W(X) = \int_{f^{m_W}(X)} |\phi_W'|^t \, d\mu.$$

Then the measure $\sum_{W\in\mathfrak{D}_V} \mu_W$ is supported on $J_{\mathrm{con}}(f)$. If moreover $t \geqslant \alpha(f)$, then this measure is finite.

Proof. – Clearly $J(F) \subset J_{\mathrm{con}}(f)$, so μ and each of the measures μ_W, for $W \in \mathfrak{D}_V$, are supported on $J_{\mathrm{con}}(f)$. To prove the second assertion, let $\hat{\mu}$ be a conformal measure for f of exponent $\alpha(f)$. It follows by Koebe distortion property that for every $W \in \mathfrak{D}_V$ we have $\hat{\mu}(W) \sim \mathrm{diam}(W)^{\alpha(f)}$ and $\mu_W(\overline{\mathbb{C}}) \sim \mathrm{diam}(W)^t$. When $t \geqslant \alpha(f)$, it follows that the measure $\sum_{W\in\mathfrak{D}_V} \mu_W$ is finite. □

LEMMA B.4. – *The canonical induced map F satisfies the following property*:

(C) *For every connected component W of D we have either $R_V(W) = V^{c(W)}$ or $R_V(W) \subset D$.*

Proof. – Observe that for every $z \in D$ and every $r = 1, \ldots, m(z) - 1$, we have that $m := m(z) - r$ is a good time for $f^r(z)$. So, if W is a connected component of D such that $F \not\equiv R_V$ on W, then letting r be such that R_V is equal to f^r on W, we have that $m := m_W - r$ is a good time for every element of $R_V(W)$. So $R_V(W) \subset D$ in this case. □

Proof of Proposition B.2. – By Lemma B.3 the measure $\hat{\mu} := \sum_{W\in\mathfrak{D}_V} \mu_W$ is finite and supported on $J_{\mathrm{con}}(f)$. So by Proposition B.1 we just need to prove that $\hat{\mu}$ is a sub-conformal measure for f. So, let U be a subset of $\overline{\mathbb{C}}$ on which f is injective. We have to prove that the inequality (24) holds. Clearly the general case follows from the following special cases.

Case 1. *U is contained in an element W of $\mathfrak{D}_V$, distinct from the V^c.* Then $W' := f(W) \in \mathfrak{D}_V$, $m_{W'} = m_W - 1$, $c(W') = c(W)$ and $\phi_{W'} = f|_{\widehat{W}} \circ \phi_W$. So

$$\hat{\mu}_{W'}(f(U)) = \int_{f^{m_{W'}}(f(U))} |\phi_{W'}'|^h \, d\mu = \int_{f^{m_W}(U)} |f' \circ \phi_W|^h \cdot |\phi_W'|^h \, d\mu = \int_U |f'|^h \, d\hat{\mu}_W.$$

Case 2. *U is contained in a connected component W of D.* Let W' be the element of $\mathfrak{D}_V$ containing $f(W)$. By property (C) we have $R_V(W) \subset D$. Let W'' be the connected component of D that contains $R_V(W)$. If $R_V(W) = W''$ then f is equal to $\phi_{W'} \circ F$ on W, and then (24) holds with equality. If $R_V(W)$ is strictly contained in W'', then an induction argument using property (C) shows that there is an integer n such that F^n is well defined on W'' and that it coincides with $f^{m_{W''}}$ on this set. In this case f is equal to $\phi_{W'} \circ (F^n|_{W''})^{-1} \circ F$ on W and therefore (24) holds with equality.

Case 3. $U \subset K(V) \cup (V \setminus D)$. As by definition $\hat{\mu}(K(V) \cup (V \setminus D)) = 0$, there is nothing to prove in this case. □

Remark B.5. – The proof of Proposition B.2 does not give directly that the measure $\hat{\mu}$ is conformal. In fact, a priori the set $f(V \setminus D)$ might have positive measure with respect to $\hat{\mu}$, so in case 3 of the proof it is not immediate that we have (24) with equality.

Proof of Theorem 2. – By the first hypothesis we have $\mathrm{HD}(J(F)) = \alpha(f)$ and by the second one we have that there is $\alpha \in (0, \mathrm{HD}(J(F)))$ at which P is finite. Then the formula (22) implies that $P(\alpha) > 0$ and that F is strongly regular. Moreover, Theorem 1 implies that F admits a conformal measure μ of exponent $\alpha(f)$ whose Hausdorff dimension is equal to $\alpha(f)$. It follows that μ is the restriction to V of a conformal measure for f that is supported on $J_{\mathrm{con}}(f)$ and whose Hausdorff dimension is equal to $\alpha(f)$ (Proposition B.2). By Proposition B.1 this measure is non-atomic, ergodic and every other conformal measure of exponent $\alpha(f)$ is proportional to it. □

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