ON A FULLY NONLINEAR YAMABE PROBLEM

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Dedicated to Professor W.Y. Ding on the occasion of his 60th birthday

ABSTRACT. – We solve the σ_2-Yamabe problem for a nonlocally conformally flat manifold of dimension $n > 8$.

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RÉSUMÉ. – On résout le problème de σ_2-Yamabe pour des variétés riemanniennes compactes sans bord non localement conformément plates de dimension $n > 8$.

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1. Introduction

Let (M, g_0) be a compact, oriented Riemannian manifold with metric g_0 and $[g_0]$ the conformal class of g_0. Let Ric_g and R_g be the Ricci tensor and scalar curvature of g respectively. The Schouten tensor of the metric g is defined by

$$S_g = \frac{1}{n-2}\left(\mathrm{Ric}_g - \frac{R_g}{2(n-1)} \cdot g\right).$$

The Schouten tensor plays an important role in conformal geometry. Let σ_k be the kth elementary symmetric function. For a symmetric $n \times n$ matrix A, set $\sigma_k(A) = \sigma_k(\Lambda)$, where $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is the set of eigenvalues of A. The σ_k-*scalar curvature* of g is defined by

$$\sigma_k(g) := \sigma_k\big(g^{-1} \cdot S_g\big),$$

where $g^{-1} \cdot S_g$ is locally defined by $(g^{-1} \cdot S_g)^i_j = \sum_k g^{ik}(S_g)_{kj}$, see [38]. Note that $\sigma_1(g) = \frac{1}{2(n-1)}R_g$. It is an interesting question to find a metric g in a given conformal class $[g_0]$ such that

$$\sigma_k(g) = c \tag{1}$$

for some constant c. Since the Schouten tensors S_g and S_{g_0} of conformal metrics $g = e^{-2u}g_0$ and g_0 have the following relation

$$S_g = \nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0},$$

Eq. (1) is equivalent to the following fully nonlinear equation

$$(2) \qquad \sigma_k \left(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0} \right) = c e^{-2ku},$$

for some constant c. When $k = 1$, it is the well-known Yamabe equation.

Let

$$\Gamma_k^+ = \left\{ \Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n \mid \sigma_j(\Lambda) > 0, \ \forall j \leqslant k \right\}$$

be Gårding's cone. A metric g is said to be k-positive or simply $g \in \Gamma_k^+$ if $\sigma_j(g)(x) > 0$ for any $j \leqslant k$ and at every point $x \in M$. If $g = e^{-2u} g_0$, we say u is k-admissible if g is k-positive. In this paper we consider the following

σ_k-*Yamabe problem.* – Let $g_0 \in \Gamma_k^+$. Find a conformal metric $g \in [g_0] \cap \Gamma_k^+$ such that

$$\sigma_k(g) = c$$

for some constant c.

The study of the fully nonlinear equations (1) was initiated by Viaclovsky. Since then there is a lot of work concerning these equations. Here, we just mention some results directly related to the existence of the σ_k-Yamabe problem. This problem has been solved in the following cases. When $k = n$, under a sufficient condition, Viaclovsky proved the existence in [40]. When $n = 2k = 4$, which is an important case, Chang–Gursky–Yang solved the problem in [7]. See also [6,23]. When the underlying manifold is locally conformally flat, this problem was solved by Guan and Wang [19] and Li and Li [29] independently. See also [4]. Note that when the underlying manifold (M, g_0) is locally conformally flat and $g \in \Gamma_k^+$ with $k \geqslant n/2$, the universal cover of M is conformally equivalent to a spherical space form [17]. When $k > n/2$, the σ_k-Yamabe problem was solved by Gursky–Viaclovsky in [24]. See also their earlier work [22].

In this paper, we consider the case $k = 2$. In this case, Eq. (2) is a variational problem, which was observed by Viaclovsky in [38]. This is crucial for our method presented here. Our main result in this paper is

THEOREM 1. – *Let (M^n, g_0) be a compact, oriented Riemannian manifold with $g_0 \in \Gamma_2^+$. When $n > 8$ and the Weyl tensor $W_{g_0} \neq 0$, then there is a conformal metric $g \in [g_0] \cap \Gamma_2^+$ such that*

$$\sigma_2(g) = c$$

for some constant c.

Combining the results of [19,29], the σ_2-Yamabe problem is solvable if $n > 8$. Like the ordinary Yamabe problem, there is a well-known difficulty—the loss of compactness of Eq. (1). Another more difficult problem is the fully nonlinearity of (1). Our result here is an analogue of the result of Aubin [2] for the ordinary Yamabe problem. Even the ideas of proof are quite similar. However the techniques to realize these ideas become more delicate due to the fully nonlinearity.

Set $\mathcal{C}_2 = \{ g \in [g_0] \mid g \in \Gamma_2^+ \}$ and define a Yamabe type constant by

$$Y_2 \big(M, [g_0] \big) = \begin{cases} \inf_{g \in \mathcal{C}_2} \widetilde{\mathcal{F}}_2(g), & \text{if } \mathcal{C}_2 \neq \emptyset, \\ +\infty, & \text{if } \mathcal{C}_2 = \emptyset, \end{cases}$$

where $\widetilde{\mathcal{F}}_2(g) = \mathrm{vol}(g)^{-\frac{n-4}{n}} \int_M \sigma_2(g)\, d\,\mathrm{vol}(g)$. This is a natural generalization of the Yamabe constant that was considered in [20] in the fully nonlinear context.

We first prove the following proposition.

PROPOSITION 1. – *Let (M^n, g_0) be a compact, oriented Riemannian manifold of dimension $n > 4$ with $g_0 \in \Gamma_2^+$. The σ_2-Yamabe problem is solvable, provided that*

$$(3) \qquad\qquad Y_2\big(M, [g_0]\big) < Y_2\big(\mathbb{S}^n\big).$$

The idea to prove the proposition is a "blow-up" analysis, which is a typical tool in the field of semilinear equations. The observation that the fully nonlinear equation (1) also admits a blow-up analysis was made in [18]. Inspired by Yamabe's approach (see e.g. [42,3]), we first prove the existence of solutions to a "subcritical" equation (6) for any small $\varepsilon > 0$. To prove the existence of solutions of (6), we use a fully nonlinear flow (8). We show that this flow globally converges to a solution u_ε of the subcritical equation (6). In fact, u_ε is a minimizer for a corresponding functional. This is one of the crucial points of this paper. Then we consider the sequence u_ε as $\varepsilon \to 0$. Using the blow-up analysis developed in [18] and the classification of "bubbles" in [8] or [29], we can show that the sequence u_ε subconverges to a solution of (2) under the condition (3). The flow method to attack the existence of fully nonlinear equations was used by many mathematicians, see for instance [9,41,37,10]. In the fully nonlinear conformal equations, it was used in [19,20].

Then we show

PROPOSITION 2. – *Let (M^n, g_0) be a compact, oriented Riemannian manifold with $g_0 \in \Gamma_2^+$. When $n > 8$ and the Weyl tensor $W_{g_0} \neq 0$,*

$$Y_2\big(M, [g_0]\big) < Y_2\big(\mathbb{S}^n\big).$$

The proof of this proposition is a delicate gluing argument. We need to construct suitable test metrics as in [2,32] for the ordinary Yamabe problem. A subtle point in the gluing is that all metrics we constructed should lie in Γ_2^+. Recall that in the ordinary Yamabe problem, the test metrics constructed by Aubin and Schoen have negative scalar curvature somewhere. To overcome this difficulty, we adopt a method of Gromov–Lawson in their construction of metrics of positive scalar curvature. A similar method was also used in [31] for metrics of positive isotropic curvature and [15] for metrics of positive Γ_k-curvatures on locally conformally flat manifolds. See also [25,33]. We believe that by a similar, but more delicate construction one can prove Proposition 2 for $n = 8$. For $n = 5, 6, 7$, this problem becomes delicate. We will consider these cases later.

By-products of our work for flow (8) are the Poincaré type inequality and Sobolev inequality for the operator $\sigma_2(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0})$. (In Section 4, we provide another proof.)

PROPOSITION 3. – *Let (M, g_0) be a compact, oriented Riemannian manifold with $g_0 \in \Gamma_2^+$ and the dimension $n > 4$. Then there exists a positive constant $\lambda_1 > 0$ depending only on (M, g_0) such that for any C^2 function u with $e^{-2u} g_0 \in \mathcal{C}_2([g_0])$ we have*

$$\int_M \sigma_2\big(e^{-2u} g_0\big)\, d\,\mathrm{vol}\big(e^{-2u} g_0\big) \geqslant \lambda_1 \int e^{4u}\, d\,\mathrm{vol}\big(e^{-2u} g_0\big).$$

Equivalently, for such a function u we have

$$\int_M e^{(4-n)u} \sigma_2\bigg(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0}\bigg)\, d\,\mathrm{vol}(g_0) \geqslant \lambda_1 \int e^{(4-n)u}\, d\,\mathrm{vol}(g_0).$$

THEOREM 2. – *Let (M, g_0) be a compact, oriented Riemannian manifold with $g_0 \in \Gamma_2^+$ and the dimension $n > 4$. Then there exists a positive constant $C > 0$ depending only on (M, g_0) such that for any C^2 function u with $e^{-2u} g_0 \in \mathcal{C}_2([g_0])$ we have*

$$\int_M \sigma_2\big(e^{-2u} g_0\big) \, d \operatorname{vol}\big(e^{-2u} g_0\big) \geqslant C \operatorname{vol}\big(e^{-2u} g_0\big)^{\frac{n-4}{n}}.$$

Equivalently, for such a function u we have

$$\int_M e^{(4-n)u} \sigma_2 \left(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0} \right) d \operatorname{vol}(g_0) \geqslant C \left(\int_M e^{-nu} \, d \operatorname{vol}(g_0) \right)^{\frac{n-4}{n}}.$$

The Sobolev inequality and other geometric inequalities, the Moser–Trudinger inequality and a conformal quermassintegral inequality for $\sigma_k(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0})$ for a locally conformally flat manifold were established in [20]. See also [17,13].

The method presented here works for a conformal quotient equation

$$\frac{\sigma_2(g)}{\sigma_1(g)} = c,$$

on a general manifold [12]. See other results for conformal quotient equations in [20,16,24].

The paper is organized as follows. In Section 2, we discuss various fully nonlinear flows and we prove local estimates for these flows in Section 3. In Section 4, we establish the Poincaré and Sobolev inequalities. We prove the global convergence of these fully nonlinear flows and Proposition 1 in Section 5. In Section 6, we prove Proposition 2, and hence Theorem 1.

2. Various flows and ideas of proof

Consider the following functional

$$(4) \qquad\qquad\qquad \mathcal{F}_k(g) = \int_M \sigma_k(g) \, d \operatorname{vol}(g)$$

and its normalization $\widetilde{\mathcal{F}}_k$

$$(5) \qquad\qquad\qquad \widetilde{\mathcal{F}}_k(g) = \operatorname{vol}(g)^{-\frac{n-2k}{n}} \int_M \sigma_k(g) \, d \operatorname{vol}(g).$$

When $k = 2$ or the underlying manifold is locally conformally flat, Viaclovsky proved that critical points of $\widetilde{\mathcal{F}}_2$ are solutions of (1). Therefore, in these cases, (1) is a variational problem. The case when the underlying manifold is locally conformally flat was studied in [19,29], as mentioned in the introduction. See also [4]. In this paper we only consider the case $k = 2$. Since the case $k = 2$ and $n \leqslant 4$ was solved in [6,22,24], we focus on the case $k = 2$ and $n > 4$.

Recall that $\mathcal{C}_2 = \{ g \in [g_0] \mid g \in \Gamma_2^+ \}$ and the Yamabe type constant is defined by

$$Y_2\big(M, [g_0]\big) = \begin{cases} \inf_{g \in \mathcal{C}_2} \widetilde{\mathcal{F}}_2(g), & \text{if } \mathcal{C}_2 \neq \emptyset, \\ \infty, & \text{if } \mathcal{C}_2 = \emptyset. \end{cases}$$

Our main aim of this paper is to show that $Y_2(M, [g_0])$ is achieved for nonlocally conformally flat manifolds when $\mathcal{C}_2([g_0]) \neq \emptyset$. In order to achieve our aim, we will first consider *subcritical* equations

$$(6) \qquad \sigma_2^{1/2}\left(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0}\right) = ce^{(\varepsilon-2)u},$$

for $\varepsilon \in (0, 2]$ and the positive constant c. Its corresponding functional is

$$(7) \qquad \widetilde{\mathcal{F}}_{2,\varepsilon}(g) = V_\varepsilon(g)^{-\frac{n-4}{n-2\varepsilon}} \int_M \sigma_2(g)\, d\operatorname{vol}(g),$$

where

$$V_\varepsilon(g) := \int_M e^{2\varepsilon u}\, d\operatorname{vol}(g) = \int_M e^{(2\varepsilon-n)u}\, d\operatorname{vol}(g_0),$$

for $g = e^{-2u}g_0$. It is clear that $V_0(g) = \operatorname{vol}(g)$, the volume of g and $V_2(g) = \int e^{(4-n)u}\, d\operatorname{vol}(g)$. Set

$$Y_\varepsilon(M, [g_0]) = \inf_{g \in \mathcal{C}_2} \widetilde{\mathcal{F}}_{2,\varepsilon}(g).$$

We will show that $Y_\varepsilon(M, [g_0])$ is achieved at u_ε, which is clearly a solution of (6). To prove this we consider the following fully nonlinear flow

$$\begin{aligned}
(8) \quad 2\frac{du}{dt} &= -g^{-1} \cdot \frac{d}{dt}g \\
&= \left(h\left(e^{-2u}\sigma_2^{1/2}(g)\right) - h\left(r_\varepsilon^{1/2}(g)e^{(\varepsilon-2)u}\right)\right) - s_\varepsilon(g) \\
&= h\left(\sigma_2^{1/2}\left(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0}\right)\right) - h\left(r_\varepsilon^{1/2}(g)e^{(\varepsilon-2)u}\right) - s_\varepsilon(g),
\end{aligned}$$

with initial value $u(0) = 1$, where $r_\varepsilon(g)$ and $s_\varepsilon(g)$ are given by for any $\varepsilon \in [0, 2]$

$$r_\varepsilon(g) := \frac{\int_M \sigma_2(g)\, d\operatorname{vol}(g)}{\int_M e^{2\varepsilon u}\, d\operatorname{vol}(g)},$$

$$s_\varepsilon(g) := \frac{\int_M e^{2\varepsilon u}\left(h(e^{-2u}\sigma_2^{1/2}(g)) - h(r_\varepsilon^{1/2}(g)e^{(\varepsilon-2)u})\right) d\operatorname{vol}(g)}{\int_M e^{2\varepsilon u}\, d\operatorname{vol}(g)}$$

and $h: \mathbb{R}_+ \to \mathbb{R}$ is smooth concave function with $h'(t) \geqslant 1$ for $t \in \mathbb{R}_+$ satisfying

$$h(s) = \begin{cases} 2\log s & \text{if } t \leqslant 1, \\ s & \text{if } t \geqslant 2. \end{cases}$$

Flow (8) preserves V_ε and nonincreases $\mathcal{F}_2$.

LEMMA 1. – *For any $\varepsilon \in [0, 2]$, the flow (8) preserves the functional V_ε and nonincreases $\mathcal{F}_2$. In fact, we have*

$$(9) \qquad \frac{d}{dt}\mathcal{F}_2(g) = -\frac{n-4}{2}\int_M \left(h\left(e^{-2u}\sigma_2^{1/2}(g)\right) - h\left(r_\varepsilon^{1/2}(g)e^{(\varepsilon-2)u}\right)\right)\left(\sigma_2(g) - r_\varepsilon e^{2\varepsilon u}\right) d\operatorname{vol}(g).$$

Moreover, r_ε is bounded.

Proof. – We note that

$$\frac{d}{dt}\mathcal{F}_2(g) = \frac{n-4}{2}\int_M \left(g^{-1}\cdot\frac{d}{dt}g\right)\sigma_2(g)\,d\operatorname{vol}(g)$$

and

$$\frac{d}{dt}V_\varepsilon(g) = \frac{n-2\varepsilon}{2}\int_M \left(g^{-1}\cdot\frac{d}{dt}g\right)e^{2\varepsilon u}\,d\operatorname{vol}(g) = 0.$$

See the proof in [19]. It is clear that V_ε is preserved along the flow. On the other hand, a direct computation gives

$$\begin{aligned}
(10)\qquad \frac{d}{dt}\mathcal{F}_2(g) &= \frac{n-4}{2}\int_M \left(g^{-1}\cdot\frac{d}{dt}g\right)\big(\sigma_2(g) - r_\varepsilon e^{2\varepsilon u}\big)\,d\operatorname{vol}(g)\\
&= -\frac{n-4}{2}\int_M \big(h\big(e^{-2u}\sigma_2^{1/2}(g)\big) - h\big(r_\varepsilon^{1/2}(g)e^{(\varepsilon-2)u}\big)\big)\\
&\qquad\times \big(\sigma_2(g) - r_\varepsilon e^{2\varepsilon u}\big)\,d\operatorname{vol}(g),
\end{aligned}$$

where in the second equality we have used the fact

$$\int_M \big(\sigma_2(g) - r_\varepsilon e^{2\varepsilon u}\big)\,d\operatorname{vol}(g) = 0.$$

Now it is easy to see that r_ε is bounded.   □

In fact, flow (8) strictly decreases the functional $\mathcal{F}_2$ except at the solutions of Eq. (6). When $\varepsilon = 0$ Eq. (6) is just (2). When $\varepsilon = 2$ Eq. (6) is a corresponding equation for a nonlinear eigenvalue problem, which was considered in [21]. See also Section 4.

Since $g_0 \in \Gamma_2^+$, flow (8) is parabolic near $t = 0$. By the standard implicit function theorem we have the following short-time existence result.

PROPOSITION 4. – *For any $g_0 \in C^\infty(M)$ with $g_0 \in \Gamma_2^+$, there exists a positive constant $T^* \in (0,\infty]$ such that flow (8) exists and is parabolic for $t \in [0,T^*)$, and $\forall T < T^*$,*

$$g \in C^{3,\alpha}\big([0,T]\times M\big), \quad \forall 0 < \alpha < 1, \quad and \quad g(t) \in \Gamma_2^+.$$

We assume that T^* is the largest number, for which Proposition 4 holds. We first show the global convergence of flow (8) when $\varepsilon = 2$. The global convergence implies a Poincaré type inequality. Then, using this inequality and the divergence free of the first Newton transformation of the Schouten tensor, which was an observation due to Viaclovsky, we obtain an optimal Sobolev inequality. By establishing a flow version of local gradient estimates, which was proved in [18], we show that flow (8) globally converges to a solution u_ε of (6) for any $\varepsilon \in (0,2]$. With the help of the local estimate obtained in [18] and a classification in [29] or [8], we show that u_ε subconverges to a solution u_0 of (2), provided that

$$(11)\qquad\qquad\qquad Y_2\big(M,[g_0]\big) < Y_2\big(\mathbb{S}^n\big).$$

In this case, it is clear that u_0 is the minimum of $\widetilde{\mathcal{F}}_2$.

3. Local estimates

In this section, we will establish a local estimate for solutions of (8), which is a parabolic version of a local estimate for solutions of (2) obtained in [18].

THEOREM 3. – *Let u be a solution of (8) with $\varepsilon \in [0,2]$ in a geodesic ball $B_r \times [0,T]$ for $T < T^*$ and $r < r_0$, the injectivity radius of M. There is a constant $C > 0$ depending only on (B_r, g_0) such that for any $(x,t) \in B_{r/2} \times [0,T]$*

$$(12) \qquad |\nabla u|^2 + \left|\nabla^2 u\right| \leqslant C\big(1 + e^{-(2-\varepsilon)\inf_{(x,t)\in B_r\times[0,T]} u(x,t)}\big).$$

Proof. – The proof follows [18] closely. We only point out the different places. Without loss of generality, we assume $r = 1$. Let $\rho \in C_0^\infty(B_1)$ be a test function defined as in [18] such that

$$(13) \qquad \begin{aligned} \rho &\geqslant 0, & &\text{in } B_1, \\ \rho &= 1, & &\text{in } B_{1/2}, \\ \left|\nabla\rho(x)\right| &\leqslant 2b_0\rho^{1/2}(x), & &\text{in } B_1, \\ \left|\nabla^2\rho\right| &\leqslant b_0, & &\text{in } B_1. \end{aligned}$$

Here $b_0 > 1$ is a constant. Set $H(x,t) = \rho|\nabla u|^2$. Let (x_0, t_0) be the maximum of H in $M \times [0,T]$. Without loss of generality, we assume $t_0 > 0$. We have at (x_0, t_0) that

$$(14) \qquad 0 \leqslant H_t = 2\rho \sum_l u_l u_{lt},$$

$$(15) \qquad 0 = H_j = \rho_j|\nabla u|^2 + 2\rho \sum_l u_l u_{lj},$$

$$(16) \qquad 0 \geqslant (H_{ij}).$$

Let $W = (w_{ij})$ be an $n \times n$ matrix with $w_{ij} = \nabla^2_{ij} u + u_i u_j - \frac{|\nabla u|^2}{2}(g_0)_{ij} + (S_{g_0})_{ij}$. Here u_i and u_{ij} are the first and second derivatives of u with respect to the background metric g_0. By choosing suitable normal coordinates, we may assume that W is diagonal at (x_0, t_0), and hence we have at (x_0, t_0),

$$(17) \qquad \begin{aligned} w_{ii} &= u_{ii} + u_i^2 - \frac{1}{2}|\nabla u|^2 + (S_{g_0})_{ii}, \\ u_{ij} &= -u_i u_j - (S_{g_0})_{ij}, & &\forall i \neq j. \end{aligned}$$

In view of (13), (15) and (17), we have at (x_0, t_0)

$$(18) \qquad \left|\sum_{l=1}^n u_{il} u_l\right| \leqslant b_0 \rho^{-1/2}|\nabla u|^2.$$

We may assume that

$$H(x_0, t_0) \geqslant A_0^2 b_0^2,$$

i.e., $\rho^{-1/2} \leqslant \frac{1}{A_0 b_0}|\nabla u|$, and

$$|\nabla S_{g_0}| + |S_{g_0}| \leqslant A_0^{-1}|\nabla u|^2,$$

where $A_0 > 1$ is a large, but fixed number to be chosen later, otherwise we are done. Thus, from (18) we have

$$\text{(19)} \qquad \left| \sum_{l=1}^n u_{il} u_l \right| \leqslant \frac{|\nabla u|^3}{A_0} (x_0, t_0).$$

Set $F = h(\sigma_2^{1/2}(W))$ and

$$F^{ij} = \frac{\partial F(W)}{\partial w_{ij}}.$$

Note that flow (8) is equivalent to $2u_t = F - h(r_\varepsilon^{1/2} e^{(\varepsilon-2)u}) - s_\varepsilon$ and F^{ij} is diagonal at (x_0, t_0). Since matrix (F^{ij}) is positive definite, from (14) and (16) we have

$$\text{(20)} \qquad 0 \geqslant \sum_{i,j} F^{ij} H_{ij} - 2H_t$$

$$= \sum_{i,j} F^{ij} \left\{ \left(-2\frac{\rho_i \rho_j}{\rho} + \rho_{ij} \right) |\nabla u|^2 + 2\rho \sum_l u_{lij} u_l + 2\rho \sum_l u_{il} u_{jl} \right\}$$

$$- 4\rho \sum_l u_l u_{lt}.$$

We need to estimate the term $\sum_{i,j,l} F^{ij} u_{lij} u_l - 2\sum_l u_l u_{lt}$. Since changing the order of derivatives only causes a low order term, we have

$$\text{(21)} \qquad \sum_{i,j,l} F^{ij} u_{lij} u_l - 2\sum_l u_l u_{lt}$$

$$\geqslant \sum_{i,j,l} F^{ij} u_{ijl} u_l - 2\sum_l u_l u_{lt} - c\sum_i F^{ii} |\nabla u|^2$$

$$\geqslant \sum_{i,j,l} F^{ij} (w_{ij})_l u_l - \sum_{i,l} F^{ii} \left(u_i^2 - \frac{1}{2} |\nabla u|^2 \right)_l u_l - 2\sum_l u_l u_{lt}$$

$$- c\sum_i F^{ii} |\nabla u|^2 - \sum_{i,l} F^{ii} \nabla_l (S_{g_0})_{ii} u_l$$

$$\geqslant \sum_l (F_l - 2u_{tl}) u_l - cA_0^{-1} \sum_i F^{ii} |\nabla u|^4$$

$$\geqslant (\varepsilon - 2)h' \left(r_\varepsilon^{1/2} e^{(\varepsilon-2)u} \right) r_\varepsilon^{1/2} e^{(\varepsilon-2)u} |\nabla u|^2 - cA_0^{-1} \sum_i F^{ii} |\nabla u|^4,$$

where we have used (8) and (19). Here c is a constant independent of u, but it may vary from line to line. The term $\sum_{i,j} F^{ij} (-2\frac{\rho_i \rho_j}{\rho} + \rho_{ij}) |\nabla u|^2$ is bounded from below by $-10b_0^2 |\nabla u|^2 \sum_j F^{jj}$. For the term $F^{ij} u_{il} u_{jl}$ we have the following crucial lemma.

LEMMA 2 [18]. – *There is a constant A_0 sufficient large (depending only on n, and $\|g_0\|_{C^3(B_1)}$), such that,*

$$\text{(22)} \qquad \sum_{i,j,l} F^{ij} u_{il} u_{jl} \geqslant A_0^{-3/4} |\nabla u|^4 \sum_{i \geqslant 1} F^{ii}.$$

Altogether gives us

$$(23) \qquad \big(A_0^{-3/4} - cA_0^{-1}\big)\rho|\nabla u|^4 \sum_i F^{ii} \leqslant 10b_0^2|\nabla u|^2 \sum_i F^{ii} + c\rho\big(1 + e^{(\varepsilon-2)u}\big)|\nabla u|^2.$$

By the Newton–McLaurin inequality and the fact that $h'(t) \geqslant 1$ for any $t \geqslant 0$, it is easy to check that

$$\sum_i F^{ii} \geqslant 1,$$

which, together with (23), proves the local gradient estimate

$$|\nabla u|^2 \leqslant C\big(1 + e^{(2-\varepsilon)\inf_{(x,t)\in B_r\times[0,T]} u(x,t)}\big),$$

for some constant $C > 0$ depending only on (B_r, g_0).

Now we show the local estimates for second order derivatives. Since $e^{-2u}g_0 \in \Gamma_2^+$, to bound $|\nabla^2 u|$ we only need bound Δu from above. This is a well-known fact, see for instance [18]. Set

$$G = \rho\big(\Delta u + |\nabla u|^2\big),$$

where ρ is defined as above. Let (y_0, t_0) be a maximum point of G in $M \times [0, T]$. Without loss of generality, we assume $G(y_0, t_0) > 1 + 2\max H(x, t)$, $t_0 > 0$ and (u_{ij}) is diagonal at (y_0, t_0). Recall that $H = \rho|\nabla u|^2$. Hence we have

$$0 < \rho\Delta u(y_0, t_0) \leqslant G(y_0, t_0) \leqslant 2\rho\Delta u(y_0, t_0).$$

At (y_0, t_0), we have

$$(24) \qquad 0 \leqslant G_t = \rho\sum_l (u_{llt} + 2u_l u_{lt}),$$

$$(25) \qquad 0 = G_j = \frac{\rho_j}{\rho}G + \rho\sum_{l\geqslant 1}(u_{llj} + 2u_l u_{lj}), \quad \text{for any } j,$$

$$(26) \qquad 0 \geqslant \left(G_{ij} = \frac{\rho\rho_{ij} - 2\rho_i\rho_j}{\rho^2}G + \rho\sum_{l\geqslant 1}(u_{llij} + 2u_{li}u_{lj} + 2u_l u_{lij})\right).$$

Recall that $F^{ij} = \frac{\partial}{\partial w_{ij}}F$ is nonnegative definite. Hence, we have

$$0 \geqslant \sum_{i,j\geqslant 1} F^{ij}G_{ij} - 2G_t$$

$$\geqslant \sum_{i,j\geqslant 1} F^{ij}\frac{\rho\rho_{ij} - 2\rho_i\rho_j}{\rho^2}G + \rho\sum_{i,j,l\geqslant 1} F^{ij}(u_{ijll} + 2u_{li}u_{lj} + 2u_l u_{lij})$$

$$- 2\rho\sum_l(u_{llt} + 2u_l u_{lt}) - C\rho\sum_i\big(|u_{ii}| + |u_i|\big)\sum_{i,j}\big|F^{ij}\big|,$$

where the last term comes from the commutators related to the curvature tensor of g_0 and its derivatives. From the definition of ρ, we have

$$\sum_{i,j\geqslant 1} F^{ij}\frac{\rho\rho_{ij} - 2\rho_i\rho_j}{\rho^2}G \geqslant -C\sum_{i,j\geqslant 1}\big|F^{ij}\big|\frac{1}{\rho}G.$$

By the concavity of $\sigma_2^{1/2}$, we have

$$(27) \quad \sum_{i,j,l \geqslant 1} F^{ij} u_{ijll} = \sum_{i,j,l \geqslant 1} F^{ij} w_{ijll} - \sum_{i,j,l \geqslant 1} F^{ij} \left(u_i u_j - \frac{1}{2} |\nabla u|^2 (g_0)_{ij} + (S_{g_0})_{ij} \right)_{ll}$$

$$\geqslant \sum_l F_{ll} - \sum_{i,j,l \geqslant 1} F^{ij} \left(u_i u_j - \frac{1}{2} |\nabla u|^2 (g_0)_{ij} + (S_{g_0})_{ij} \right)_{ll}.$$

We also have

$$(28) \quad \sum_{i,j,l} F^{ij} u_l u_{lij} = \sum_{i,j,l} F^{ij} u_l (w_{ij})_l - \sum_{i,j,l} F^{ij} u_l \left(u_i u_j - \frac{1}{2} |\nabla u|^2 (g_0)_{ij} + (S_{g_0})_{ij} \right)_l$$

$$+ \sum_{i,j,l} F^{ij} u_l (u_{lij} - u_{ijl})$$

$$= \sum_l F_l u_l - \sum F^{ij} u_l \left(u_i u_j - \frac{1}{2} |\nabla u|^2 (g_0)_{ij} + (S_{g_0})_{ij} \right)_l$$

$$+ \sum_{i,j,l} F^{ij} u_l (u_{lij} - u_{ijl}).$$

Hence, we have

$$(29) \quad \sum_{i,j,l \geqslant 1} F^{ij} \left(u_{ijll} + 2 u_{li} u_{lj} + 2 u_l u_{lij} \right)$$

$$\geqslant \sum_l (F_{ll} + 2 F_l u_l) - 2 \sum_{i,j,l} F^{ij} u_i u_{jll} + \sum_{j,k,l} F^{jj} u_k u_{kll}$$

$$- 2 \sum_{i,j,l} F^{ij} u_l \left(u_i u_j - \frac{1}{2} |\nabla u|^2 (g_0)_{ij} + (S_{g_0})_{ij} \right)_l$$

$$+ \sum_{i,k,l} F^{ii} (u_{kl})^2 - C \left(1 + \frac{G}{\rho} \right) \sum_{i,j} |F^{ij}|.$$

From (24) and Eq. (8), we have

$$(30) \quad \rho \sum_l (F_{ll} + 2 F_l u_l) \geqslant 2 \rho \sum_l (u_{tll} + 2 u_l u_{tl}) - C(2 - \varepsilon) G \left(1 + e^{(\varepsilon - 2)u} \right).$$

The term $-2 \sum_{i,j,l} F^{ij} u_i u_{jll} + \sum_{j,k,l} F^{jj} u_k u_{kll}$ can be controlled as in [18] with the help of (25). And the other terms in (29) can easily be estimated. On the other hand, it follows from the positivity of (F^{ij}) that

$$\sum_{i,j} |F^{ij}| \leqslant C \sum_i F^{ii}.$$

This completes the proof of the theorem. $\quad \square$

From the local estimates, we have

COROLLARY 1. – *If "bubble" occurs, i.e.,* $\inf_{M \times [0, T^*)} u = -\infty$, *then there is a positive constant* $c_0 > 0$ *such that*

$$\lim_{\delta \to 0} \lim_{t \to T^*} V_\varepsilon(g, B_\delta) > c_0.$$

4. A Poincaré inequality and a Sobolev inequality

The Sobolev inequality is a very important analytic tool in many problems arising from analysis and geometry. It plays a crucial role in the resolution of the Yamabe problem, which was solved completely by Yamabe [42], Trudinger [36], Aubin [2] and Schoen [32]. See various optimal Sobolev inequalities in [26]. In this section we are interested in a similar type inequality for the class of fully nonlinear conformal operators $\sigma_k(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0})$. In [20], the Sobolev inequality was generalized to the operator $\sigma_k(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0})$ for $k < n/2$, if the underlying manifold is locally conformally flat. Namely,

THEOREM 4 [18]. – *Let (M^n, g_0) be a compact, oriented Riemannian manifold with $g_0 \in \Gamma_k^+$ and $k < n/2$. Assume that (M, g_0) is locally conformally flat, then there exists a positive constant $C > 0$ depending only on n, k and (M, g_0) such that for any C^2 function u with $e^{-2u} g_0 \in \mathcal{C}_k([g_0])$ we have*

$$(31) \qquad \int_M \sigma_k\left(e^{-2u} g_0\right) d\operatorname{vol}\left(e^{-2u} g_0\right) \geqslant C \operatorname{vol}\left(e^{-2u} g_0\right)^{\frac{n-2k}{n}}.$$

Equivalently, for such a function u we have

$$(32) \qquad \int_M e^{(2k-n)u} \sigma_k\left(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0}\right) d\operatorname{vol}(g_0)$$

$$\geqslant C\left(\int_M e^{-nu} d\operatorname{vol}(g_0)\right)^{\frac{n-2k}{n}}.$$

When $k = 1$, inequality (33) is just the Sobolev inequality. The proof of Theorem 4 uses a Yamabe type flow. See also the work of [13].

In this section, we establish the Sobolev inequality for $k = 2$ without the flatness condition.

THEOREM 5. – *Let (M, g_0) be a compact, oriented Riemannian manifold with $g_0 \in \Gamma_2^+$ and the dimension $n > 4$. Then there exists a positive constant $C > 0$ depending only on (M, g_0) such that for any C^2 function u with $e^{-2u} g_0 \in \mathcal{C}_2([g_0])$ we have*

$$(33) \qquad \int_M \sigma_2\left(e^{-2u} g_0\right) d\operatorname{vol}\left(e^{-2u} g_0\right) \geqslant C \operatorname{vol}\left(e^{-2u} g_0\right)^{\frac{n-4}{n}}.$$

Equivalently, for such a function u we have

$$(34) \qquad \int_M e^{(4-n)u} \sigma_2\left(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0}\right) d\operatorname{vol}(g_0)$$

$$\geqslant C\left(\int_M e^{-nu} d\operatorname{vol}(g_0)\right)^{\frac{n-4}{n}}.$$

First we prove a Poincaré type inequality, which will be used in the proof of our Sobolev inequality. The usual Poincaré type inequality is associated to the first eigenvalue problem. In our case, there is a nonlinear eigenvalue problem, which was studied in [21].

PROPOSITION 5. – *Let (M, g_0) be a compact manifold with $g_0 \in \Gamma_k^+$. Then there is a function u with $e^{-2u} g_0 \in \Gamma_k^+$ satisfying*

$$(35) \qquad \sigma_k\left(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0}\right) = \lambda_1 > 0.$$

Moreover the constant λ_1 is unique and the solution is unique up to a constant.

An elliptic method was used in the proof, which was motivated by a method introduced in [30]. See also [41] for a Hessian operator. In view of Proposition 5, one may guess that

$$(36) \qquad \int e^{2ku} \sigma_k\left(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0}\right) d\operatorname{vol}\left(e^{-2u} g_0\right)$$
$$\geqslant \lambda_1 \int e^{2ku} d\operatorname{vol}\left(e^{-2u} g_0\right),$$

for any u with $e^{-2u} g_0 \in \Gamma_k^+$. It is easy to see that when $k = 1$ inequality (36) holds. In fact it is the Poincaré inequality. In this section, we show that (36) holds for $k = 2$ by flow (8) with $\varepsilon = 2$.

PROPOSITION 6. – *Let (M, g_0) be a compact, oriented Riemannian manifold with $g_0 \in \Gamma_2^+$ and the dimension $n > 4$. Then for any C^2 function u with $e^{-2u} g_0 \in \mathcal{C}_2([g_0])$ we have*

$$(37) \qquad \int_M \sigma_2\left(e^{-2u} g_0\right) d\operatorname{vol}\left(e^{-2u} g_0\right) \geqslant \lambda_1 \int e^{4u} d\operatorname{vol}\left(e^{-2u} g_0\right).$$

Equivalently, for such a function u we have

$$(38) \qquad \int_M e^{(4-n)u} \sigma_2\left(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0}\right) d\operatorname{vol}(g_0)$$
$$\geqslant \lambda_1 \int e^{(4-n)u} d\operatorname{vol}(g_0).$$

Proof. – To prove the proposition, we consider flow (8) with $\varepsilon = 2$. We want to show that the flow converges globally to a solution obtained in Proposition 5. By Theorem 3, we have

$$(39) \qquad \left|\nabla^2 u\right| + |\nabla u|^2(x, t) \leqslant C,$$

where C is a constant independent of $(x, t) \in M \times [0, T^*)$. Since the flow preserves the functional V_2, in view of (39) we have that $|u| \leqslant C$, for some constant $C > 0$. Now following the method in [19] we can show that

$$\sigma_2(g) > c_0,$$

for some constant c_0 independent of t. See the proof in the next section. Hence, this flow exists globally and is uniformly elliptic. By the result of Krylov, $g(t) \in C^{4+\alpha, 2+\alpha}$. Since the flow satisfies (9), one can show that for any sequence of $\{t_i\}$ with $t_i \to \infty$ there is a subsequence, still denoted by $\{t_i\}$, such that $g(t_i)$ converges strongly to g^*, which satisfies (35). On the other hand, $V_2(g^*) \equiv V_2(g(t))$. By the uniqueness in Proposition 5, one can show that the flow globally converges to g^*. Since the flow preserves V_2 and decreases $\mathcal{F}_2$, we have

$$\mathcal{F}_2(g) \geqslant \mathcal{F}_2(g^*),$$

for any $g \in \mathcal{C}_2$. This is the Poincaré inequality that we want to prove. $\quad\square$

Proof of Theorem 5. – Let $g = e^{-2u}g_0$. We have

$$2\sigma_2 = \sum_{i,j} T^{ij} S_{ij},$$

where $T(g)^{ij} = \sigma_1(g)g^{ij} - S(g)^{ij}$ is the so-called first Newton transformation. We will use the following formulas

$$(40) \qquad S(g)_{ij} = u_{ij} + u_i u_j - \frac{1}{2}|\nabla u|_{g_0}^2 (g_0)_{ij} + S(g_0)_{ij}$$

and

$$(41) \qquad \widetilde{\nabla}_{ij}^2 u = u_{ij} + 2u_i u_j - |\nabla u|_{g_0}^2 (g_0)_{ij},$$

where $\widetilde{\nabla}$ are the derivatives w.r.t. g. Thus,

$$(42) \qquad 2\sigma_2(g) = \sum_{i,j} T(g)^{ij} \widetilde{\nabla}_{ij}^2 u - \sum_{i,j} T(g)^{ij} u_i u_j$$
$$+ \frac{n-1}{2}\sigma_1(g)|\widetilde{\nabla}u|_g^2 + \sum_{i,j} T(g)^{ij} S(g_0)_{ij}.$$

Here we have used $\operatorname{tr} T(g) = (n-1)\sigma_1(g)$. Note that

$$(43) \qquad \sum_{i,j} T(g)^{ij} S(g_0)_{ij} > 0,$$

thanks to Gårding's inequality

$$\sum_{i,j} T(g)^{ij} S(g_0)_{ij} \geqslant 2e^{2u}\sigma_2^{1/2}(g)\sigma_2^{1/2}(g_0).$$

Due to an observation of Viaclovsky, $\sum_i \widetilde{\nabla}_i T(g)^{ij} = 0$, we have

$$(44) \qquad 2\int \sigma_2(g) = -\int \sum_{i,j} T(g)^{ij} u_i u_j \, d\operatorname{vol}(g) + \frac{n-1}{2}\int \sigma_1(g)|\widetilde{\nabla}u|_g^2 \, d\operatorname{vol}(g)$$
$$+ \int \sum_{i,j} T(g)^{ij} S(g_0)_{ij} \, d\operatorname{vol}(g).$$

Recall that $T(g) = \sigma_1(g)g - S(g)$. We have

$$(45) \qquad -\int \sum_{i,j} T(g)^{ij} u_i u_j \, d\operatorname{vol}(g)$$
$$= -\int \sigma_1(g)|\widetilde{\nabla}u|_g^2 \, d\operatorname{vol}(g) + \int \sum_{i,j} S(g)^{ij} u_i u_j \, d\operatorname{vol}(g)$$
$$= -\int \sigma_1(g)|\widetilde{\nabla}u|_g^2 \, d\operatorname{vol}(g) + \int \sum_{i,j} \widetilde{\nabla}^{ij} u \, u_i u_j \, d\operatorname{vol}(g)$$

$$-\frac{1}{2}\int|\widetilde{\nabla}u|_g^4\,d\operatorname{vol}(g)+\int\sum_{i,j}S(g_0)^{ij}u_iu_j\,d\operatorname{vol}(g)$$

and

$$(46)\qquad \int\sum_{i,j}\widetilde{\nabla}^{ij}uu_iu_j\,d\operatorname{vol}(g)$$

$$=\frac{1}{2}\int\sum_i\widetilde{\nabla}^i\big(|\widetilde{\nabla}u|_g^2\big)u_i\,d\operatorname{vol}(g)=-\frac{1}{2}\int|\widetilde{\nabla}u|_g^2\operatorname{tr}\big(\widetilde{\nabla}^2u\big)\,d\operatorname{vol}(g)$$

$$=-\frac{1}{2}\int\sigma_1(g)|\widetilde{\nabla}u|_g^2+\frac{n-2}{4}\int|\widetilde{\nabla}u|_g^4+\int\frac{1}{2}\sigma_1(g_0)|\widetilde{\nabla}u|_g^2e^{2u}\,d\operatorname{vol}(g).$$

Hence

$$(47)\quad -\int\sum_{i,j}T(g)^{ij}u_iu_j\,d\operatorname{vol}(g)$$

$$=-\frac{3}{2}\int\sigma_1(g)|\widetilde{\nabla}u|_g^2+\frac{n-4}{4}\int|\widetilde{\nabla}u|_g^4+\int\sum_{i,j}S(g_0)^{ij}u_iu_j+\frac{1}{2}\int\sigma_1(g_0)|\widetilde{\nabla}u|_g^2e^{2u},$$

where all integrals are w.r.t. g. (44) and (47) give us

$$(48)\qquad 2\int\sigma_2(g)\,d\operatorname{vol}(g)$$

$$=\frac{n-4}{2}\int\sigma_1(g)|\widetilde{\nabla}u|_g^2\,d\operatorname{vol}g+\frac{n-4}{4}\int|\widetilde{\nabla}u|_g^4\,d\operatorname{vol}(g)$$

$$+\int\sum_{i,j}T^{ij}S(g_0)_{ij}\,d\operatorname{vol}(g)+\int\sum_{i,j}S(g_0)^{ij}u_iu_j\,d\operatorname{vol}(g)$$

$$+\frac{1}{2}\int\sigma_1(g_0)|\widetilde{\nabla}u|_g^2e^{2u}\,d\operatorname{vol}(g).$$

Finally, we obtain

$$(49)\qquad 2\int\sigma_2(g)\,d\operatorname{vol}(g)$$

$$=\frac{n-4}{2}\int\sigma_1(g)|\nabla u|_{g_0}^2e^{2u}\,d\operatorname{vol}g+\frac{n-4}{4}\int|\nabla u|_{g_0}^4e^{4u}\,d\operatorname{vol}(g)$$

$$+\int\sum_{i,j}T^{ij}S(g_0)_{ij}\,d\operatorname{vol}(g)+\int\sum_{i,j}S(g_0)^{ij}u_iu_j\,d\operatorname{vol}(g)$$

$$+\frac{1}{2}\int\sigma_1(g_0)|\nabla u|_{g_0}^2e^{4u}\,d\operatorname{vol}(g).$$

Recall (43) and positivity of $\sigma_1(g)$ and $\sigma_1(g_0)$. Using the estimates

$$(50)\qquad \sum_{i,j}S(g_0)^{ij}u_iu_j\geqslant-c|\nabla u|_{g_0}^2e^{4u}\geqslant-\frac{n-4}{8}|\nabla u|_{g_0}^4e^{4u}-\frac{2c^2}{n-4}e^{4u}$$

we deduce

$$(51)\qquad 2\int\sigma_2(g)\,d\operatorname{vol}(g)\geqslant\frac{n-4}{8}\int|\nabla u|_{g_0}^4e^{4u}\,d\operatorname{vol}(g)-c\int e^{4u}\,d\operatorname{vol}(g).$$

In view of the Poincaré inequality (37), the Sobolev inequality (33) follows from (51). □

We remark that a similar method was used to obtain Sobolev inequalities on locally conformally flat manifolds by González in [13]. The argument given in the next section will provide another proof of Theorem 5.

5. Global convergence of flow (8) when $\varepsilon > 0$

PROPOSITION 7. – *For any $\varepsilon \in (0, 2]$, flow (8) converges globally to u_ε, which satisfies (6).*

Proof. – For any $t \in [0, T^*)$, set

$$m(t) = \min_{(x,s) \in M \times [0,t]} u(x,s).$$

If $\inf_{t \in [0,T^*)} m(t) > -\infty$, then by estimates given in Section 3, we have a uniform bound of $|\nabla u|^2 + |\nabla^2 u|$. Since flow (8) preserves the functional V_ε, we have a uniform C^2 bound. Now we claim that there is a constant $c > 0$ such that

$$(52) \qquad F(x,t) \geqslant c > 0, \quad \text{for any } (x,t) \in M \times [0,T^*).$$

Recall that $F = \sigma_2^{1/2}(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0})$. We will prove the claim at the end of the proof. (52) implies that flow (8) is uniformly elliptic in $M \times [0,T^*)$. Hence, by Krylov's result, u has a uniform bound for higher order derivatives, which implies first that $T^* = \infty$, the global existence. The global convergence of (8) with $\varepsilon \in (0, 2]$ follows now closely the argument presented in [19], which, in turn, follows closely the argument given in [35] and [1]. Therefore, to prove the proposition, we only need to exclude that

$$(53) \qquad \inf_{t \in [0,T^*)} m(t) = -\infty.$$

We assume by contradiction that $\inf_{t \in [0,T^*)} m(t) = -\infty$. Let T_i be a sequence tending to T^* with $m(T_i) \to -\infty$ as $i \to \infty$. Let $(x_i, t_i) \in M \times [0, T_i]$ with $u(x_i, t_i) = m(T_i)$. Fixing $\delta \in (\frac{2}{5}, \frac{1}{2})$, we consider $r_i = \frac{\varepsilon}{2}|m(T_i)|e^{(1-\delta)m(T_i)}$. Clearly, we have $r_i \to 0$. It follows from Theorem 3 that for sufficiently large i

$$
\begin{aligned}
u(x, t_i) &\leqslant m(T_i) + |\nabla u| r_i \\
&\leqslant m(T_i) + Ce^{(\frac{\varepsilon}{2}-1)m(T_i)} \frac{\varepsilon}{2} |m(T_i)| e^{(1-\delta\varepsilon)m(T_i)} \\
&= m(T_i) + C\frac{\varepsilon}{2} |m(T_i)| e^{\varepsilon(1/2-\delta)m(T_i)} \\
&\leqslant (1-\kappa)m(T_i), \quad \forall x \in B(x_i, r_i),
\end{aligned}
$$

for some $\kappa \in (0, (\delta - \frac{2}{n})\varepsilon)$. Note that $\delta - \frac{2}{n} > 0$, for $n \geqslant 5$. Therefore, we obtain

$$\int_{B(x_i,r_i)} e^{2\varepsilon u} \, d\text{vol}(g) \geqslant \int_{B(x_i,r_i)} e^{(2\varepsilon-n)m(T_i)(1-\kappa)} \, d\text{vol}(g_0) \geqslant Ce^{(2\varepsilon-n)m(T_i)(1-\kappa)} r_i^n$$

$$\geqslant C\left(\frac{|m(T_i)|\varepsilon}{2}\right)^n \to \infty.$$

Hence, this fact contradicts the boundedness of V_ε.

Now it remains to prove claim (52). For any $0 < T < T^*$, set

$$T_1 := \inf\{T' \in [0, T] \mid \forall (x, t) \in M \times [T', T], \; r_\varepsilon^{1/2}(g(t)) e^{(\varepsilon-2)u(x,t)} < 1/2\}.$$

It is clear that $\forall t \in [0, T_1]$ we have $r_\varepsilon(g(t)) > C$ for positive constant $C > 0$ independent of T. Let us consider a function $H : M \times [0, T_1]$ defined by

$$H := \frac{1}{2}\left(h\left(\sigma_2^{1/2}\left(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0}\right) - h(r_\varepsilon^{1/2}(g)e^{(\varepsilon-2)u})\right) - e^{-u}\right.$$

$$= u_t + \frac{1}{2}s_\varepsilon(g) - e^{-u}.$$

We first compute the evolution equation for σ_2. A direct computation, see for instance Lemma 2 in [19], gives

$$\frac{d}{dt}\sigma_2 = 2\sigma_2 g \cdot \frac{d}{dt}(g^{-1}) + \text{tr}\left\{T_1(S_g)g^{-1}\frac{d}{dt}S_g\right\}$$

$$= 4\sigma_2(g)u_t + \text{tr}\{T_1(S_g)g^{-1}\widetilde{\nabla}_g^2(u_t)\},$$

where $\widetilde{\nabla}$ is the derivatives with respect to the evolved metric g. Without loss of generality, we assume that the minimum of H is achieved at $(x_0, t_0) \in M \times (0, T_1]$. We will show that there is a constant $c_0 > 0$ independent of T such that

$$(54) \qquad\qquad\qquad \sigma_2(g)(x_0, t_0) > c_0.$$

Since $|u|$ has a uniform bound, without loss of generality we may assume that at (x_0, t_0)

$$e^{-2u}\sigma_2^{1/2}(g) < 1/2.$$

Recall that $h(t) = 2\log t$ for $t < 1$. Hence, in a small neighborhood of (x_0, t_0)

$$H = \log(e^{-2u}\sigma_2^{1/2}(g)) - \frac{1}{2}h(r_\varepsilon^{1/2}(g)e^{(\varepsilon-2)u}) - e^{-u}.$$

Let us use $O(1)$ to denote terms with a uniform bound. We have near (x_0, t_0)

$$(55) \qquad \frac{d}{dt}H = \frac{1}{2\sigma_2(g)}\text{tr}\{T_1(S_g)g^{-1}\widetilde{\nabla}_g^2(u_t)\}$$

$$- \frac{1}{4}h'(r_\varepsilon^{1/2}(g)e^{(\varepsilon-2)u})\frac{dr_\varepsilon(g)}{dt}r_\varepsilon^{-1/2}(g)e^{(\varepsilon-2)u}$$

$$+ \left[e^{-u} + \frac{2-\varepsilon}{2}h'(r_\varepsilon^{1/2}(g)e^{(\varepsilon-2)u})r_\varepsilon^{1/2}(g)e^{(\varepsilon-2)u}\right]u_t$$

$$\geqslant \frac{1}{2\sigma_2(g)}\text{tr}\{T_1(S_g)g^{-1}\widetilde{\nabla}_g^2(H)\} + \frac{1}{2\sigma_2(g)}\text{tr}\{T_1(S_g)g^{-1}\widetilde{\nabla}_g^2(e^{-u})\}$$

$$+ \left[e^{-u} + \frac{2-\varepsilon}{2}h'(r_\varepsilon^{1/2}(g)e^{(\varepsilon-2)u})r_\varepsilon^{1/2}(g)e^{(\varepsilon-2)u}\right]u_t,$$

where we have used $\frac{dr_\varepsilon(g)}{dt} \leqslant 0$. See Lemma 1. Let $F = \log \sigma_2(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0})$ and $F^{ij} = \frac{\partial}{\partial w_{ij}} F$. Since (x_0, t_0) is the minimum of H in $M \times [0, T_1]$, at this point, we have

$$\frac{dH}{dt} \leqslant 0,$$

$$0 = H_l = \frac{1}{2} \sum_{ij} F^{ij} w_{ijl} + \left(e^{-u} + \frac{2-\varepsilon}{2} h'\big(r_\varepsilon^{1/2}(g)e^{(\varepsilon-2)u}\big) r_\varepsilon^{1/2}(g)e^{(\varepsilon-2)u} \right) u_l = 0 \quad \forall l$$

and

$$(H_{ij}) \quad \text{is nonnegative definite.}$$

Note that

$$\big(\widetilde{\nabla}_g^2\big)_{ij} H = H_{ij} + u_i H_j + u_j H_i - \sum_l u_l H_l \delta_{ij} = H_{ij},$$

at (x_0, t_0), where H_j and H_{ij} are the first and second derivatives with respect to the background metric g_0. We write $k(x,t) := (e^{-u} + \frac{2-\varepsilon}{2} h'(r_\varepsilon^{1/2}(g)e^{(\varepsilon-2)u}) r_\varepsilon^{1/2}(g)e^{(\varepsilon-2)u})$. We can check that $\forall (x,t) \in M \times [0, T_1]$ we have $C_1 > k(x,t) > C_2$. Here the positive constants C_1 and C_2 are independent of T.

From the positivity of (F^{ij}) and (54), we have

$$(56) \qquad 0 \geqslant H_t - \frac{1}{2} \sum_{i,j} F^{ij} H_{ij}$$

$$\geqslant \frac{1}{2\sigma_2(g)} \operatorname{tr}\big\{ T_1(S_g)\widetilde{\nabla}_g^2\big(e^{-u}\big)\big\} + k(x,t)u_t$$

$$= \frac{1}{2} \sum_{i,j} F^{ij} \big\{ \big(e^{-u}\big)_{ij} + u_i\big(e^{-u}\big)_j + u_j\big(e^{-u}\big)_i - u_l\big(e^{-u}\big)_l \delta_{ij} \big\} + k(x,t)u_t$$

$$= \frac{1}{2} e^{-u} \sum_{i,j} F^{ij} \big\{ -u_{ij} - u_i u_j + |\nabla u|^2 \delta_{ij} \big\} + k(x,t)u_t$$

$$= \frac{1}{2} e^{-u} \sum_{i,j} F^{ij} \left\{ -w_{ij} + S(g_0)_{ij} + \frac{1}{2}|\nabla u|^2 \delta_{ij} \right\} + k(x,t)u_t$$

$$\geqslant \frac{1}{2} e^{-u} \sum_{i,j} F^{ij} \big\{ -w_{ij} + S(g_0)_{ij} \big\} + k(x,t)u_t$$

$$= \frac{1}{2} e^{-u} \sum_{i,j} F^{ij} S(g_0)_{ij} + O(1) \log \sigma_2(g) + O(1)$$

$$\quad - \frac{1}{2} k(x,t)\big(h\big(r_\varepsilon^{1/2}(g)e^{(\varepsilon-2)u}\big) + s_\varepsilon(g)\big).$$

Here we have used $\sum_{i,j} F^{ij} w_{ij} = \sum_{i,j} \frac{1}{\sigma_2(g)} \frac{\partial \sigma_2(g)}{\partial w_{ij}} w_{ij} = 2$. Since $g_0 \in \Gamma_2^+$, by Gårding's inequality [11],

$$(57) \qquad \sum_{i,j} F^{ij} S(g_0)_{ij} = \sum_{i,j} \frac{1}{\sigma_2(g)} \frac{\partial \sigma_2(g)}{w_{ij}} S(g_0)_{ij} \geqslant 2e^{2u} \frac{\sigma_2^{1/2}(g_0)}{\sigma_2^{1/2}(g)}.$$

On the other hand, one can check $h(r_\varepsilon^{1/2}(g)e^{(\varepsilon-2)u}) + s_\varepsilon(g)$ is bounded from above, for $\|u\|_{C^2}$ is uniformly bounded. Now from (56) and (57), we have

$$0 \geqslant e^u \frac{\sigma_2^{1/2}(g_0)}{\sigma_2^{1/2}(g)} + O(1)\log \sigma_2(g) + O(1)$$

$$\geqslant \frac{c_1}{\sigma_2^{1/2}(g)} + c_2 \log \sigma_2(g) - c_3,$$

for positive constants c_1, c_2 and c_3 independent of T. Clearly, this inequality implies that there is a constant $c_0 > 0$ independent of T such that (54) holds. Namely

$$\sigma_2(g) \geqslant c_0,$$

at point (x_0, t_0). Hence, we have for any point $(x, t) \in M \times [0, T_1]$

$$\log\left(e^{-2u(x,t)}\sigma_2^{1/2}(g)(x,t)\right) - \frac{1}{2}h\left(r_\varepsilon^{1/2}(g)e^{(\varepsilon-2)u(x,t)}\right) - e^{-u(x,t)}$$

$$= H(x, t)$$

$$\geqslant H(x_0, t_0)$$

$$= \log\left(e^{-2u(x_0,t_0)}\sigma_2^{1/2}(g)(x_0,t_0)\right) - \frac{1}{2}h\left(r_\varepsilon^{1/2}(g)e^{(\varepsilon-2)u(x_0,t_0)}\right) - e^{-u(x_0,t_0)}$$

$$\geqslant \log C_1 - e^C,$$

provided $e^{-2u(x,t)}\sigma_2^{1/2}(g)(x,t) < 1$. It follows that $\sigma_2(g)(x,t) \geqslant C_2 > 0$ for some positive constant independent of T.

On $M \times [T_1, T]$, we consider a function $H : M \times [T_1, T]$ defined by $H = \log(e^{-\varepsilon u}\sigma_2^{1/2}(g)) - e^{-u}$. By the same argument, there is a constant $c > 0$ independent of T such that $F(x,t) \geqslant c > 0$ for any $(x,t) \in M \times [T_1, T]$. Hence, we deduce the claim (52). This finishes the proof of the proposition. $\quad\square$

We remark that the Sobolev inequality, Theorem 2, implies that r_ε has a positive lower bound. In the proof of claim (52), we avoided to use the Sobolev inequality.

Proof of Proposition 1. – By local estimates established in [18] (in fact similar local estimates as in Theorem 3 hold), we can use the argument given in the proof of Proposition 7 to show that the set of solutions of (6) with the bounded $\mathcal{F}_2$ and $V_\varepsilon(e^{-2u}g_0) = 1$ is compact for $\varepsilon \in (0, 2]$. Hence, Proposition 7 implies that Y_ε is achieved by a function u_ε, which clearly is a solution of (6). We may assume that u_ε satisfies $V_\varepsilon(e^{-2u_\varepsilon}g_0) = 1$ and

$$(58) \qquad \sigma_2\left(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0}\right) = ce^{2(\varepsilon-2)u},$$

where $c = Y_\varepsilon$. For any fixed metric g, the function $\widetilde{\mathcal{F}}_{2,\varepsilon}(g)$ is continuous on ε so that Y_ε is semi-continuous from above on ε. On the other hand, it follows from the Hölder's inequality, Y_ε is semi-continuous from below on ε. Hence, Y_ε is continuous and we have

$$\lim_{\varepsilon \to 0} Y_\varepsilon = Y_2\left(M, [g_0]\right) < Y_2\left(\mathbb{S}^n\right).$$

If $\inf u_\varepsilon$ has a uniform lower bound, then the estimates established in [18] imply that $\|u_\varepsilon\|_{C^2}$ is uniformly bounded. By the result of Evans–Krylov, $\|u_\varepsilon\|_{C^{2,\alpha}}$ is uniformly bounded for any

$\alpha \in (0,1)$. Hence u_ε, by taking a subsequence, converges strongly in $C^{2,\alpha}$ to u_0, which is a solution of (1). Moreover, u_0 is a minimizer. Now suppose $\varliminf_{\varepsilon \to 0} \inf u_\varepsilon = -\infty$. Let $(x_\varepsilon) \in M$ such that $u_\varepsilon(x_\varepsilon) = \min_{x \in M} u_\varepsilon(x)$. We consider a new function

$$v_\varepsilon(y) = u(\exp_{x_\varepsilon} \delta_\varepsilon y) - u_\varepsilon(x_\varepsilon)$$

and defined on $B_{\delta_\varepsilon^{-1}}$ with a pull-back metric $g_\varepsilon := (\exp_{x_\varepsilon} \delta_\varepsilon \cdot)^* g_0$, where $\delta_\varepsilon = e^{(1-\frac{\varepsilon}{2})u_\varepsilon(x_\varepsilon)}$. Since $u_\varepsilon(x_\varepsilon) \to -\infty$, $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$. And one can check that $B_{\delta_\varepsilon^{-1}}$ tends to $\mathbb{R}^n$ and g_ε to the standard Euclidean metric in any compact set in $\mathbb{R}^n$ for any C^k norm. We can check that v_ε satisfies the same Eq. (58) on $B_{\delta_\varepsilon^{-1}}$ with S_{g_0} replaced by S_{g_ε}. By the local estimates in [18], (v_ε) is uniformly bounded in C^2 on any fixed compact set. From the result of Evans–Krylov, it follows that (v_ε) is uniformly bounded in $C^{2,\alpha}$ on any fixed compact set. Hence, v_ε converges in any compact domain of $\mathbb{R}^n$ to an entire solution u of the following equation on $\mathbb{R}^n$

$$(59) \qquad \sigma_2\left(\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g_{\mathbb{R}^n}\right) = c_0 e^{-4u},$$

with $c_0 = Y_2(M, [g_0])$. It is easy to check that $\int_{\mathbb{R}^n} e^{-nu} \, d\operatorname{vol}(g_{\mathbb{R}^n}) \leqslant 1$. In fact, we have

$$\int_{B_{\delta_\varepsilon^{-1}}} e^{(2\varepsilon - n)v_\varepsilon} \, d\operatorname{vol}(g_\varepsilon)$$

$$= \delta_\varepsilon^{-n} e^{(n-2\varepsilon)u_\varepsilon(x_\varepsilon)} \int_{B(x_\varepsilon, 1)} e^{(2\varepsilon - n)u_\varepsilon} \, d\operatorname{vol}(g_0)$$

$$= e^{(n/2-2)\varepsilon u_\varepsilon(x_\varepsilon)} \int_{B(x_\varepsilon, 1)} e^{(2\varepsilon - n)u_\varepsilon} \, d\operatorname{vol}(g_0) \leqslant V_\varepsilon\left(e^{-2u_\varepsilon} g_0\right) = 1.$$

Letting $\varepsilon \to 0$, the claim yields. By the classification of (59) given in [29] or [8], we have $c_0 \geqslant Y_2(\mathbb{S}^n) > Y_2(M, [g_0])$, which contradicts $c_0 = Y_2(M, [g_0])$. $\quad\square$

6. Existence

In this section, we will construct a conformal metric $\tilde{g}$ such that $\widetilde{\mathcal{F}}_2(\tilde{g}) < Y_2(\mathbb{S}^n)$ and $\tilde{g} \in \Gamma_2^+$. Our construction is inspired from Aubin's work [2]. See also [32]. The basic idea is to construct a suitable test function. But the more delicate point in our case, as already mentioned in the Introduction, is to keep the conformal metric in the admissible class Γ_2^+ as in [20]. Fix a point $P \in M$. Assume $n \geqslant 5$. It follows from the work by Lee–Parker that there exists a conformal metric g_1 on $\mathcal{M}$ such that in a normal coordinate system for g_1 at P

$$(60) \qquad R = O\left(r^2\right),$$

$$(61) \qquad \Delta R = -\frac{1}{6}\left|W(P)\right|^2,$$

$$(62) \qquad \operatorname{Ric}(P) = 0,$$

$$(63) \qquad \sqrt{\det g_1} = 1 + O\left(r^5\right),$$

where $r = |x|$. We denote

$$(64) \qquad g_v = v^{-2} g_1,$$

where

$$v(x) = \begin{cases} \lambda + r^2, & \text{if } x \in B(0, r_0), \\ \lambda + r_0^2, & \text{else.} \end{cases}$$

We first need some estimates.

LEMMA 3. – *Assume*

$$A = g_1^{-1}\left(\frac{\nabla_{g_1}^2 v}{v} - \frac{1}{2}\frac{|\nabla_{g_1} v|^2}{v^2}g_1 + S_{g_1}\right),$$

where

$$S_{g_1} = \frac{1}{n-2}\left(\text{Ric}_{g_1} - \frac{R}{2(n-1)}g_1\right).$$

Then we have

(65)
$$\text{tr}(A) = \frac{2n\lambda}{(\lambda + r^2)^2} + \frac{O(r^5)}{\lambda + r^2} + \frac{R}{2(n-1)}$$

and

(66)
$$\text{tr}(A^2) = \frac{4n\lambda^2}{(\lambda + r^2)^4} + \frac{2R\lambda}{(n-1)(\lambda + r^2)^2} - \frac{\text{Ric}(\nabla_{g_1} v, \nabla_{g_1} v)}{(\lambda + r^2)^2} + O(r).$$

Proof. – By definition, we have

(67)
$$\sigma_2(g_v) = v^4 \sigma_2\left(g_1^{-1}\left(\frac{\nabla_{g_1}^2 v}{v} - \frac{1}{2}\frac{|\nabla_{g_1} v|^2}{v^2}g_1 + S_{g_1}\right)\right).$$

It is clear that

$$\begin{aligned}\text{tr}(A) &= \frac{\Delta_{g_1} v}{v} - \frac{n}{2}\frac{|\nabla_{g_1} v|^2}{v^2} + \text{tr}\left(g_1^{-1}S_{g_1}\right) \\ &= \frac{\Delta_{g_1} v}{v} - \frac{n}{2}\frac{|\nabla_{g_1} v|^2}{v^2} + \frac{R}{2(n-1)},\end{aligned}$$

where

$$\Delta_{g_1} = \frac{1}{\sqrt{\det g_1}}\sum_{i,j}\frac{\partial}{\partial x^i}\left(\sqrt{\det g_1}\,g_1^{ij}\frac{\partial}{\partial x^j}\right).$$

Recall

(68)
$$|\nabla_{g_1} v|^2 = 4r^2.$$

In view of (63), we have

(69)
$$\Delta_{g_1} v = 2n + O(r^5).$$

Therefore,

$$\begin{aligned}\text{tr}(A) &= \frac{2n + O(r^5)}{\lambda + r^2} - \frac{n}{2}\frac{4r^2}{(\lambda + r^2)^2} + \frac{R}{2(n-1)} \\ &= \frac{2n\lambda}{(\lambda + r^2)^2} + \frac{O(r^5)}{\lambda + r^2} + \frac{R}{2(n-1)}.\end{aligned}$$

It is also easy to check that

$$\operatorname{tr}(A^2) = \frac{|\nabla^2_{g_1} v|^2}{v^2} + \frac{n|\nabla_{g_1} v|^4}{4v^4} + \operatorname{tr}\left(\left(g_1{}^{-1}S_{g_1}\right)^2\right)$$
$$- \frac{|\nabla_{g_1} v|^2 \Delta_{g_1} v}{v^3} + 2\operatorname{tr}\left(\frac{g_1{}^{-1}\nabla^2_{g_1} v g_1{}^{-1}S_{g_1}}{v}\right) - \frac{|\nabla_{g_1} v|^2}{v^2}\operatorname{tr}\left(g_1{}^{-1}S_{g_1}\right).$$

A simple calculation gives us

$$(70) \qquad \frac{n|\nabla_{g_1} v|^4}{4v^4} = \frac{4nr^4}{(\lambda + r^2)^4},$$

$$(71) \qquad \operatorname{tr}\left(\left(g_1{}^{-1}S_{g_1}\right)^2\right) = O(r^2),$$

$$(72) \qquad -\frac{|\nabla_{g_1} v|^2 \Delta_{g_1} v}{v^3} = -\frac{8nr^2}{(\lambda + r^2)^3} + O(r),$$

$$(73) \qquad -\frac{|\nabla_{g_1} v|^2}{v^2}\operatorname{tr}\left(g_1{}^{-1}S_{g_1}\right) = -\frac{2r^2 R}{(n-1)(\lambda + r^2)^2},$$

$$(74) \qquad g_1{}^{-1}\nabla^2_{g_1} v = 2I + O(r^2),$$

$$(75) \qquad \operatorname{tr}\left(g_1{}^{-1}\nabla^2_{g_1} v g_1{}^{-1}S_{g_1}\right) = 2\operatorname{tr}\left(g_1{}^{-1}S_{g_1}\right) + O(r^3) = \frac{R}{n-1} + O(r^3),$$

$$(76) \qquad 2\operatorname{tr}\left(\frac{g_1{}^{-1}\nabla^2_{g_1} v g_1{}^{-1}S_{g_1}}{v}\right) = \frac{2R}{(n-1)(\lambda + r^2)} + O(r).$$

To handle $\frac{|\nabla^2_{g_1} v|^2}{v^2}$, we recall the Bochner's formula

$$(77) \qquad \langle \nabla(\Delta v), \nabla v\rangle = -\left|\nabla^2_{g_1} v\right|^2 + \frac{1}{2}\Delta\left(|\nabla_{g_1} v|^2\right) - \operatorname{Ric}(\nabla_{g_1} v, \nabla_{g_1} v).$$

Hence, we have

$$(78) \qquad \left|\nabla^2_{g_1} v\right|^2 = -\langle \nabla(\Delta v), \nabla v\rangle + \frac{1}{2}\Delta\left(|\nabla_{g_1} v|^2\right) - \operatorname{Ric}(\nabla_{g_1} v, \nabla_{g_1} v)$$
$$= -\langle \nabla\left(2n + O(r^5)\right), \nabla v\rangle + \frac{1}{2}\Delta\left(4r^2\right) - \operatorname{Ric}(\nabla_{g_1} v, \nabla_{g_1} v)$$
$$= 4n - \operatorname{Ric}(\nabla_{g_1} v, \nabla_{g_1} v) + O(r^5).$$

From (70), (76) and (78), we have

$$\operatorname{tr}(A^2) = \frac{4n}{(\lambda + r^2)^2} + \frac{4nr^4}{(\lambda + r^2)^4} - \frac{8nr^2}{(\lambda + r^2)^3} + \frac{2R}{(n-1)(\lambda + r^2)}$$
$$- \frac{2Rr^2}{(n-1)(\lambda + r^2)^2} - \frac{\operatorname{Ric}(\nabla_{g_1} v, \nabla_{g_1} v)}{(\lambda + r^2)^2} + O(r)$$
$$= \frac{4n\lambda^2}{(\lambda + r^2)^4} + \frac{2R\lambda}{(n-1)(\lambda + r^2)^2} - \frac{\operatorname{Ric}(\nabla_{g_1} v, \nabla_{g_1} v)}{(\lambda + r^2)^2} + O(r). \qquad \square$$

LEMMA 4. – *Assume $\beta \in (\frac{1}{2}, \frac{1}{4})$. We have $\sigma_1(g_v) > 0$ and $\sigma_2(g_v) > 0$ in $B(0, \lambda^\beta)$. Moreover, if $n \geqslant 9$, we have*

$$(79) \qquad \int_{B(0,\lambda^\beta)} \sigma_2(g_v)\, d\operatorname{vol}(g_v) = \lambda^{-\frac{n}{2}+2}\left\{2n(n-1)B + C\Delta R(0)\lambda^2\right.$$

$$+ O\big(\lambda^{5/2} + \lambda^{n(1/2-\beta)} + \lambda^{2+(n-8)(1/2-\beta)}\big)\big\}$$

and

(80)
$$\int_{B(0,\lambda^\beta)} d\,\mathrm{vol}(g_v) = \lambda^{-\frac{n}{2}}\big[B + O\big(\lambda^{5/2} + \lambda^{n(1/2-\beta)}\big)\big],$$

where the constants B, C are given by

(81)
$$B = \int_{\mathbb{R}^n} \frac{1}{(1+|x|^2)^n}\,dx,$$

(82)
$$C = \int_{\mathbb{R}^n} \left(\frac{|x|^2}{2n(1+|x|^2)^{n-2}} + \frac{2|x|^4}{n(n+2)(1+|x|^2)^{n-2}}\right) dx > 0.$$

Proof. – It follows directly from (65) and (66)

$$\sigma_1(g_v) = v^2\left(\frac{2n\lambda}{(\lambda+r^2)^2} + \frac{R}{2(n-1)} + O\big(r^3\big)\right),$$

(83)
$$\sigma_2(g_v) = \frac{v^4}{2}\left[\frac{4n(n-1)\lambda^2}{(\lambda+r^2)^4} + \frac{2\lambda R}{(\lambda+r^2)^2} + \frac{\mathrm{Ric}(\nabla_{g_1}v, \nabla_{g_1}v)}{(\lambda+r^2)^2} + O(r)\right].$$

Thus, the first part of the lemma is clear. On the other hand, we obtain

(84)
$$\int_{B(0,\lambda^\beta)} \sigma_2(g_v)\,d\,\mathrm{vol}(g_v)$$

$$= \int_{B(0,\lambda^\beta)} \frac{1}{(\lambda+r^2)^n}\bigg\{2n(n-1)\lambda^2 + R\lambda\big(\lambda+r^2\big)^2$$

$$+ \frac{1}{2}\mathrm{Ric}(\nabla_{g_1}v, \nabla_{g_1}v)\big(\lambda+r^2\big)^2 + O\big(r\big(\lambda+r^2\big)^4\big)\bigg\}\big(1 + O\big(r^5\big)\big)\,dx.$$

We can calculate

$$\mathrm{Ric}(\nabla_{g_1}v, \nabla_{g_1}v) = 4\sum_{i,j} R_{ij}(x)x^i x^j$$

$$= 4\sum_{i,j}\left(R_{ij}(0) + \sum_k R_{ij,k}(0)x^k + \sum_{k,l}\frac{1}{2}R_{ij,kl}(0)x^k x^l\right)x^i x^j + O\big(r^5\big)$$

$$= 2\sum_{i,j,k,l} R_{ij,kl}(0)x^k x^l x^i x^j + O\big(r^5\big).$$

It is known that (see [2])

$$\frac{1}{r^4 w_{n-1}}\int_{S(r)}\sum_{i,j,k,l} R_{ij,kl}(0)x^k x^l x^i x^j\,d\Omega = \frac{2}{n(n+2)}\Delta R(0)$$

and

$$R(x) = \frac{1}{2}\sum_{i,j} R_{,ij}(0)x^i x^j + O\big(r^3\big),$$

where $S(r)$ is the geodesic sphere of radius equal to r, $d\Omega$ is the volume element on the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ and w_{n-1} is the volume of the unit sphere $\mathbb{S}^{n-1}$. Therefore,

$$\int\limits_{B(0,\lambda^\beta)} \sigma_2(g_v) \, d\operatorname{vol}(g_v)$$

$$= \int\limits_{B(0,\lambda^\beta)} \frac{2n(n-1)\lambda^2}{(\lambda+r^2)^n} + \frac{\lambda r^2 \Delta R(0)}{2n(\lambda+r^2)^{n-2}} \, dx$$

$$+ \int\limits_{B(0,\lambda^\beta)} \left(\frac{2r^4 \Delta R(0)}{n(n+2)(\lambda+r^2)^{n-2}} + \frac{O(r)}{(\lambda+r^2)^{n-4}} \right) dx$$

$$= \lambda^{-\frac{n}{2}+2} \int\limits_{B(0,\lambda^{\beta-1/2})} \left(\frac{2n(n-1)}{(1+r^2)^n} + a(x)\lambda^2 \Delta R(0) \right) dx + O\big(\lambda^{-\frac{n}{2}+4+1/2}\big),$$

where

$$a(x) = \frac{|x|^2}{2n(1+|x|^2)^{n-2}} + \frac{2|x|^4}{n(n+2)(1+|x|^2)^{n-2}}.$$

Thus, (79) yields. Similarly, we can estimate

$$\int\limits_{B(0,\lambda^\beta)} d\operatorname{vol}(g_v) = \int\limits_{B(0,\lambda^\beta)} v^{-n}\sqrt{\det g_1} \, dx$$

$$= \int\limits_{B(0,\lambda^\beta)} \frac{1+O(r^5)}{(\lambda+r^2)^n} \, dx$$

$$= \lambda^{-\frac{n}{2}} \int\limits_{B(0,\lambda^{\beta-1/2})} \frac{dx}{(1+r^2)^n} + O\big(\lambda^{-\frac{n}{2}+5/2}\big)$$

$$= \lambda^{-\frac{n}{2}} \big[B + O\big(\lambda^{5/2} + \lambda^{n(1/2-\beta)}\big) \big].$$

Therefore, we finish the proof. $\quad\square$

LEMMA 5. – *Let g_1 as above and $\gamma \in (0,2)$ be given. Assume $n \geqslant 9$. For sufficiently small $\delta > 0$ such that $\lambda^{1/4} \gg \delta \gg \lambda^{1/2}$, there exist a constant $1 > \delta_1 > \delta$ and a function $u : B_{\delta_1} \to \mathbb{R}$ satisfying*:
  (0) $\delta_1^{\frac{n-4}{2}} = (\frac{2}{\gamma}-1)\lambda^{-1}\delta^{\frac{n}{2}}(1+o(1))$;
  (1) *the metric $\tilde{g} = e^{-2u}g_1$ has positive Γ_2-curvatures*;
  (2) $u = \log(\lambda + |x|^2) + b_0$ *for $|x| \leqslant \delta$*;
  (3) $u = \gamma \log|x|$ *for $|x| \geqslant \delta_1$*;
  (4) $\operatorname{vol}(B_{\delta_1} \backslash B_\delta, \tilde{g}) \leqslant C(\delta^{\frac{n+4-n\gamma}{2(2-\gamma)}}/\lambda)^{2n(2-\gamma)/(n-4)}$;
  (5) $\int_{B_{\delta_1}\backslash B_\delta} \sigma_2(\tilde{g}) \, d\operatorname{vol}(\tilde{g}) \leqslant C\delta^{4+n(1-\gamma)}\lambda^{-3+2\gamma}$,
where b_0 satisfies (94) below.

Proof. – We want to find a function u with $u'(r) = \frac{\alpha(r)}{r}$. The Schouten tensor of $\tilde{g} = e^{-2u}g_1$ is

$$(85) \qquad S(\tilde{g})_{ij} = \nabla^2_{ij}u + \nabla_i u \nabla_j u - \frac{|\nabla u|^2}{2}(g_1)_{ij} + S(g_1)_{ij}$$

$$= \frac{2\alpha}{2r^2}\delta_{ij} - \frac{\alpha^2}{2r^2}(g_1)_{ij} + \left(\frac{\alpha'}{r} + \frac{\alpha^2 - 2\alpha}{r^2}\right)\frac{x_i x_j}{r^2} + S(g_1)_{ij} + O(r^2)\frac{\alpha}{r^2},$$

so that

$$(86) \qquad S(\tilde{g})^i_j = \frac{2\alpha - \alpha^2}{2r^2}\delta_{ij} + \left(\frac{\alpha'}{r} + \frac{\alpha^2 - 2\alpha}{r^2}\right)\frac{x_i x_j}{r^2} + S(g_1)^i_j + O(r^2)\frac{\alpha}{r^2},$$

since it follows from the Gauss Lemma that $\sum_i (g_1)^{ij} x_i = x_j$. We look for a function $\alpha(r) \in (\gamma, 2)$ for all $r \in (\delta, \delta_1)$. Hence one can find a fixed constant $A > 0$ independent of λ such that

$$(87) \qquad S(\tilde{g})^i_j \geqslant \frac{2\alpha - \alpha^2 - Ar^2\alpha}{2r^2}\delta_{ij} + \left(\frac{\alpha'}{r} + \frac{\alpha^2 - 2\alpha}{r^2}\right)\frac{x_i x_j}{r^2}$$

and

$$(88) \qquad S(\tilde{g})^i_j \leqslant \frac{2\alpha - \alpha^2 + Ar^2\alpha}{2r^2}\delta_{ij} + \left(\frac{\alpha'}{r} + \frac{\alpha^2 - 2\alpha}{r^2}\right)\frac{x_i x_j}{r^2}.$$

Consequently, we obtain

$$\sigma_2(\tilde{g}) \geqslant e^{4u}\frac{(n-1)}{2}\left(\frac{2\alpha - \alpha^2 - Ar^2\alpha}{2r^2}\right)^2\left(n - 4 + 4\frac{r\alpha' - Ar^2\alpha}{2\alpha - \alpha^2 - Ar^2\alpha}\right)$$

and

$$\sigma_1(\tilde{g}) \geqslant e^{2u}\left(\frac{2\alpha - \alpha^2 - Ar^2\alpha}{2r^2}\right)\left(n - 2 + 2\frac{r\alpha' - Ar^2\alpha}{2\alpha - \alpha^2 - Ar^2\alpha}\right).$$

We want to find an α satisfying

$$\alpha = \begin{cases} \frac{2r^2}{\lambda + r^2}, & \text{if } |x| \leqslant \delta, \\ \text{solution of (89)}, & \text{if } |x| \in (\delta, \delta_1), \\ \gamma, & \text{if } |x| \geqslant \delta_1. \end{cases}$$

Such a function can be found as follows. First we solve the following equation

$$(89) \qquad \frac{n-4}{4} + \frac{r\alpha' - Ar^2\alpha}{2\alpha - \alpha^2 - Ar^2\alpha} = 0.$$

Recall this is the Bernoulli differential equation. One can find a general solution of (89) as follows

$$\frac{1}{\alpha} = r^{\frac{n-4}{2}}e^{-\frac{nAr^2}{8}}\left(\int_1^r \frac{4-n}{4}\frac{1}{t^{\frac{n-2}{2}}}e^{\frac{nAt^2}{8}}\,dt + c\right).$$

Set

$$H(r) = -\frac{An}{8}\int_1^r \frac{1}{t^{\frac{n-6}{2}}}e^{\frac{nAt^2}{8}}\,dt.$$

We have

$$\alpha = \frac{2}{1 + 2a_1 r^{\frac{n-4}{2}} e^{-\frac{nAr^2}{8}} + 2H(r)r^{\frac{n-4}{2}} e^{-\frac{nAr^2}{8}}}$$

$$= \frac{2}{1 + 2a_1 r^{\frac{n-4}{2}} + 2G(r)},$$

where

$$G = a_1 r^{\frac{n-4}{2}} \left(e^{-\frac{nAr^2}{8}} - 1 \right) + H(r)r^{\frac{n-4}{2}} e^{-\frac{nAr^2}{8}}.$$

Here the constant a_1 is determined by

$$\alpha(\delta) = \frac{2\delta^2}{\lambda + \delta^2}.$$

We have the estimate

(90) $$a_1 = \frac{\lambda}{2\delta^{\frac{n}{2}}}\big(1 + o(1)\big),$$

since we use the fact $\lambda^{1/4} \gg \delta$. Define δ_1 by $\alpha(\delta_1) = \gamma$. We have

(91) $$\delta_1^{\frac{n-4}{2}} = \left(\frac{2}{\gamma} - 1\right)\lambda^{-1}\delta^{\frac{n}{2}}\big(1 + o(1)\big)$$

so that $1 \gg \delta_1 \gg \delta$. Note that $n \geqslant 9$. Hence, for all $r \in (\delta, \delta_1)$ we have

(92) $$G(r) = O(1)r^2,$$

so that for all $r \in (\delta, \delta_1)$

(93) $$u(r) = \frac{4}{4-n}\log\big(r^{\frac{4-n}{2}} + 2a_1\big) + a_2 + O(r^2),$$

where

$$a_2 = (\gamma - 2)\log\delta_1 - \frac{4}{4-n}\log\frac{2}{\gamma} + o(1).$$

For $r < \delta$ we have

$$u(r) = \log\big(\lambda + r^2\big) + b_0,$$

where

(94) $$b_0 = (\gamma - 2)\log\delta_1 + O(1),$$

where we use $\delta \gg \lambda^{1/2}$. In view of (89), we have

(95) $$\big(2\alpha - \alpha^2 + Ar^2\alpha\big)\left(n - 4 + 4\frac{r\alpha' + Ar^2\alpha}{2\alpha - \alpha^2 + Ar^2\alpha}\right) = 2nAr^2\alpha = O(1)r^2.$$

We also have for all $r \in (\delta, \delta_1)$

(96) $$\alpha(r) \in (\gamma, 2)$$

and

(97) $$2 - \alpha + Ar^2 = \frac{4a_1 r^{\frac{n-4}{2}}}{1 + 2a_1 r^{\frac{n-4}{2}}} + O\big(r^2\big).$$

Now we can check that

$$\int\limits_{B_{\delta_1}\setminus B_\delta} \sigma_2(\tilde g)\, d\operatorname{vol}(\tilde g)$$

$$\leqslant C(n)\int\limits_{B_{\delta_1}\setminus B_\delta} e^{(4-n)u}\left(\frac{2\alpha-\alpha^2+Ar^2\alpha}{2r^2}\right)^2\left(n-4+4\frac{r\alpha'+Ar^2\alpha}{2\alpha-\alpha^2+Ar^2\alpha}\right)d\operatorname{vol}(g_1)$$

$$\leqslant O(1)\int\limits_\delta^{\delta_1} e^{(4-n)a_2}\big(r^{\frac{4-n}{2}}+2a_1\big)^4\frac{1}{r^2}\big(2-\alpha+Ar^2\big)r^{n-1}\,dr$$

$$\leqslant O(1)\int\limits_\delta^{\delta_1} \delta_1^{(n-4)(2-\gamma)}r^{5-n}a_1 r^{\frac{n-4}{2}}\big(1+2a_1 r^{\frac{n-4}{2}}\big)^4\,dr$$

$$\leqslant O(1)\int\limits_\delta^{\delta_1} \delta_1^{(n-4)(2-\gamma)}a_1 r^{3-\frac{n}{2}}\,dr$$

$$\leqslant O(1)\delta_1^{(n-4)(2-\gamma)}\delta^{4-\frac{n}{2}}a_1 = O(1)\delta^{n(1-\gamma)+4}\lambda^{-3+2\gamma}$$

and

$$\operatorname{vol}(B_{\delta_1}\setminus B_\delta,\tilde g)=\int\limits_{B_{\delta_1}\setminus B_\delta} e^{-nu}\,d\operatorname{vol}(g_1)$$

$$\leqslant O(1)\int\limits_\delta^{\delta_1} e^{-na_2}\big(r^{\frac{4-n}{2}}+2a_1\big)^{4n/(n-4)}r^{n-1}\,dr$$

$$= O(1)\int\limits_\delta^{\delta_1} \delta_1^{n(2-\gamma)}r^{-1-n}\big(1+2a_1 r^{\frac{n-4}{2}}\big)^{4n/(n-4)}\,dr$$

$$\leqslant O(1)\delta_1^{n(2-\gamma)}\delta^{-n} = O(1)\left(\frac{\delta^{\frac{n+4-n\gamma}{2(2-\gamma)}}}{\lambda}\right)^{2n(2-\gamma)/(n-4)}.$$

Therefore, after smoothing u, we get a desired u. $\quad\square$

We write $g_0 = e^{-2u_0}g_1$. In the following result, we try to connect the initial metric g_0 to some tube object. More precisely, we prove the following lemma.

LEMMA 6. – *Let $g_0 \in \Gamma_2^+$ and the geodesic ball $B(0,r_0)$ as above. Assume that $n \geqslant 5$. For any given $\gamma \in (0,2)$, then there is a conformal metric $\tilde g = e^{-2u}g_1$ of positive Γ_2-curvatures on $B(0,r_0)\setminus\{0\}$ satisfying:*
(1) *The metric $\tilde g = e^{-2u}g_1$ has positive Γ_2-curvatures;*
(2) $u = \gamma\log|x|$ *for $|x| \leqslant r_2$;*
(3) $u = u_0(x) + b_1$ *for $|x| \geqslant r_1$;*
where $r_2 < r_1 < r_0$ and b_1 is a constant.

Proof. – We write $u(x) = w(r) + \xi(r)u_0(x)$ where $\xi(r)$ is some cut-off function equal to 1 near r_0 and to 0 near 0, and w with $w'(r) = \frac{\alpha(r)}{r}$, where α is equal to 0 near r_0. As before, the Schouten tensor of $\tilde g = e^{-2u}g_1$ is

$$(98) \qquad S(\tilde{g})_{ij} = \nabla^2_{ij} w + \nabla_i w \nabla_j w + \nabla_i w \nabla_j (\xi u_0) + \nabla_i (\xi u_0) \nabla_j w$$

$$- \left(\frac{|\nabla w|^2}{2} + \langle \nabla w, \nabla(\xi u_0) \rangle \right)(g_1)_{ij} + S\big(e^{-2\xi u_0} g_1\big)_{ij}$$

so that

$$(99) \qquad S(\tilde{g})^i_j = \frac{2\alpha - \alpha^2}{2r^2} \delta_{ij} + \left(\frac{\alpha'}{r} + \frac{\alpha^2 - 2\alpha}{r^2} \right) \frac{x_i x_j}{r^2}$$

$$+ S\big(e^{-2\xi u_0} g_1\big)^i_j + O\big(r + |\nabla(\xi u_0)|\big) \frac{\alpha}{r}.$$

Fix $\varepsilon \in (0, \frac{2-\gamma}{5})$ and let C_1 bound the term $O(r + |\nabla u_0|)$. Set $r_4 = \min(\frac{r_0}{2}, \frac{1}{2}, \frac{\varepsilon}{2(1+C_1)})$. For some small r_3 to be fixed later, we want α to decrease from γ to 0 in (r_3, r_4) and $\xi \equiv 1$ in (r_3, r_0). In $B_{r_0} \setminus B_{r_3}$, we write $A = S(\tilde{g}) - S(g_0)$. Therefore

$$\sigma_2(\tilde{g}) = e^{4(w+u_0)} \sigma_2\big(A + S(g_0)\big).$$

We want $A + S(g_0) \in \Gamma^+_2$ in $B_{r_0} \setminus B_{r_3}$. It is clear in $B_{r_4} \setminus B_{r_3}$

$$A > \left(\frac{2\alpha - \alpha^2 - \varepsilon\alpha}{2r^2} \delta_{ij} + \left(\frac{\alpha'}{r} + \frac{\alpha^2 - 2\alpha}{r^2} \right) \frac{x_i x_j}{r^2} \right).$$

This gives

$$\sigma_2(A) > e^{4u} \frac{(n-1)}{2} \left(\frac{2\alpha - \alpha^2 - \varepsilon\alpha}{2r^2} \right)^2 \left(n - 4 + 4 \frac{r\alpha' - \varepsilon\alpha}{2\alpha - \alpha^2 - \varepsilon\alpha} \right)$$

and

$$\sigma_1(A) > e^{2u} \left(\frac{2\alpha - \alpha^2 - \varepsilon\alpha}{2r^2} \right) \left(n - 2 + 2 \frac{r\alpha' - \varepsilon\alpha}{2\alpha - \alpha^2 - \varepsilon\alpha} \right).$$

We see that for all small $\delta > 0$,

$$(100) \qquad \alpha(r) = \frac{(2 - 5\varepsilon)\delta}{\delta + r^{\frac{1}{2} - \frac{5}{4}\varepsilon}}$$

solves the equation

$$(101) \qquad \frac{1}{4}\big(2\alpha - \alpha^2 - \varepsilon\alpha\big) = -r\alpha' + \varepsilon\alpha.$$

We choose some $r_5 < r_4$ and a nonincreasing function α in (r_5, r_4) such that $\alpha(r_4) = 0$, $\alpha(r_5) > 0$ and $\tilde{g} \in \Gamma^+_2$ in $B_{r_4} \setminus B_{r_5}$ by openness of Γ^+_2. Now we choose a suitable δ in (100) and take a small $r_6 < r_5$ such that $\alpha(r_6) = \gamma$. From the above calculations, we see that $A \in \Gamma^+_2$ in $B_{r_5} \setminus B_{r_6}$ so that $S(\tilde{g}) \in \Gamma^+_2$ in $B_{r_5} \setminus B_{r_6}$. Now we set $\alpha(r) = \gamma$ for all $r < r_6$ and $r_3 = r_6$. We see that there exists some cut-off function ξ such that $\xi(r) = 1 \; \forall r > r_7$, $\xi(r) = 0 \; \forall r < r_8$ and $r^2 S(e^{-2\xi u_0} g_1)$ is small in $B_{r_7} \setminus B_{r_8}$ where $r_8 < r_7 < r_6$. Thus we can choose such suitable cut-off function such that $\tilde{g}$ in Γ^+_2. Now it is sufficient to choose some $r_2 < r_8$ and $r_1 = r_4$. Finally, we obtain the desired u by smoothing it. $\square$

The construction of such metrics of positive Γ_2-curvatures is motivated by the method introduced by Gromov and Lawson [14] in their study of metrics of positive scalar curvature. See also for the constructions of other positive metrics in [31,15]. Now we can prove the main result in this section.

THEOREM 6. – *Let (M, g_0) be a compact, oriented Riemannian manifold with $g_0 \in \Gamma_2^+$. Assume that $n \geqslant 9$. Then there exists $\tilde{g} \in [g_0]$ such that*

$$\tag{102} \tilde{g} \in \Gamma_2^+$$

and

$$\tag{103} \widetilde{\mathcal{F}}_2(\tilde{g}) < Y_2(\mathbb{S}^n).$$

Proof. – We fix some $\gamma \in (1, 2)$ and let the geodesic ball $B(0, r_0)$ w.r.t. g_1 as above. We define a conformal metric $\tilde{g}$ as follows. Let $r_2 < r_1 < r_0$ as in Lemma 6 and set $\delta = \lambda^\beta$ with $\beta \in (\frac{1}{4}, \frac{1}{2})$ for any small λ. Find δ_1 as in Lemma 5. Now for any small λ with $\delta_1 < r_2$, define $\tilde{g}$ on B_{δ_1} by Lemma 5 and on $B_{r_0} \backslash B_{r_2}$ by Lemma 6. And on $M \backslash B(0, r_0)$, $\tilde{g} = e^{-2b_1} g_0$, where the constant b_1 is given in Lemma 6. Since on $B_{r_2} \backslash B_{\delta_1}$ the metrics constructed in Lemmas 5 and 6 are the same, $\tilde{g}$ is smooth. From Lemmas 5 and 6, we know (102) holds. In the following, we keep the notations of the geodesic ball with respect to the background metric g_1. By the Lemmas 4, 5 and 6, we can estimate

$$\tag{104} \int_{B_{\delta_1} \backslash B_\delta} \sigma_2(\tilde{g}) \, d\,\mathrm{vol}(\tilde{g}) \leqslant C \delta_1^{(n-4)(2-\gamma)} \delta^{4-\frac{n}{2}} a_1,$$

$$\tag{105} \int_{M \backslash B_{\delta_1}} \sigma_2(\tilde{g}) \, d\,\mathrm{vol}(\tilde{g}) \leqslant C \delta_1^{(n-4)(2-\gamma)} \delta_1^{4-n},$$

$$\tag{106} \int_{B_\delta} \sigma_2(\tilde{g}) \, d\,\mathrm{vol}(\tilde{g}) = \delta_1^{(n-4)(2-\gamma)} \lambda^{-\frac{n}{2}+2} \Big[2n(n-1)B + C\Delta R(0)\lambda^2$$

$$+ O\big(\lambda^{5/2} + \lambda^{n(1/2-\beta)} + \lambda^{2+(n-8)(1/2-\beta)} \big) \Big],$$

$$\tag{107} \mathrm{vol}(M, \tilde{g}) \geqslant \mathrm{vol}(B_\delta, \tilde{g}) = \delta_1^{n(2-\gamma)} \lambda^{-\frac{n}{2}} \big[B + O\big(\lambda^{5/2} + \lambda^{n(1/2-\beta)} \big) \big].$$

We choose some $\beta \in (\frac{1}{4}, \frac{n-4}{2n})$ so that we obtain

$$\delta^{4-\frac{n}{2}} a_1 = o\big(\lambda^{-\frac{n}{2}+4} \big) \quad \text{and} \quad \delta_1^{4-n} = o\big(\lambda^{-\frac{n}{2}+4} \big).$$

As a consequence, we get

$$\widetilde{\mathcal{F}}_2(\tilde{g}) \leqslant B^{\frac{4-n}{n}} \big[2n(n-1)B + C\Delta R(0)\lambda^2 + o\big(\lambda^2 \big) \big].$$

Recall $2n(n-1)B^{\frac{4}{n}} = Y_2(\mathbb{S}^n)$ and $\Delta R(0) < 0$. Therefore, we deduce (103) provided λ is sufficiently small. Hence, we finish the proof. $\quad \square$

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found in [34], in which they showed that (3) is true for $n > 4$. Hence, the σ_2-Yamabe problem is completely solved.

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